On the Hochschild homology of hypersurfaces as a mixed complex

Volume 1

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Abstract

In this thesis we describe Hochschild homology over k of quotients of polynomial algebras $k[x_1, \ldots, x_n]/f$ for certain polynomials f in $n \leq 2$ variables, as an object of the ∞-category of mixed complexes $\mathcal{M}\mathrm{ixed}$, where k is a commutative ring in which 2 is invertible.

In 1992, the Buenos Aires Cyclic Homology Group [BACH] constructed, for any n and any commutative ring k, a quasiisomorphism between the standard Hochschild complex over k of $k[x_1, \ldots, x_n]/f$ and a quite small chain complex, under the assumption that f is monic with respect to a chosen monomial order. This result was improved upon by Larsen in 1995 [Lar95] by taking the mixed structure into account as well, though only considering polynomials f in $n = 2$ variables that are monic with respect to one of the variables.

Assuming a conjectural description of Hochschild homology of polynomial rings, we extend these previous results by constructing, for a large subset of the polynomials f considered in [BACH], a strict mixed structure on the chain complex described in [BACH] and showing that it represents the Hochschild homology over k of $k[x_1, \ldots, x_n]/f$ as an object in the ∞-category of mixed complexes. We also verify the conjecture in some cases, leading to unconditional results for $n \leq 2$ variables, as long as 2 is invertible in k.

The results of this thesis do not rely on the two aforementioned prior results, but instead use the modern approach to Hochschild homology based on ∞-categorical methods. Along the way, to be able to state and prove our result in this setting, we prove some results that may be of independent interest.

Resumé

I denne afhandling beskriver vi Hochschild homologi over k for kvotienter af polynomiumsalgebraer $k[x_1, \dots, x_n]/f$ for visse polynomier f i $n \leq 2$ variable, som et objekt i ∞-kategorien $\mathcal{M}$ixed af såkaldte blandede komplekser, for k en kommutativ ring, hvori 2 er invertibel.

I 1992 konstruerede Buenos Aires Cyclic Homology gruppen [BACH] en kvasiisomorfi mellem standardhochschildkomplekset over k af $k[x_1, \dots, x_n]/f$ og et lille kædekompleks, under antagelsen, at f er monisk med hensyn til en valgt monomisk ordning, men for alle n og alle kommutative ring k. Denne resultat blev forbedret af Larsen i 1995 [Lar95], som også betragtede den blandede struktur, dog kun for polynomier f i $n = 2$ variabler som er monisk med hensyn til én af de to variable.

Under antagelsen af en formodete beskrivelse af Hochschild homologi af polynomiumsalgebraer generaliserer vi disse tidligere resultater ved at konstruere, for en stor delmængde af de polynomier f studeret i [BACH], en strengt blandet struktur på kædekomplekset beskrevet i [BACH] og at vise, at det repræsenterer Hochschild homologi over k af $k[x_1, \dots, x_n]/f$ som objekt i ∞-kategorien af blandede komplekser. Vi også verificerer formodningen i nogle tilfælde, og får dermed ubetingede resultater for $n \leq 2$ variable, forudsat, at 2 er invertibel i k.

Resultaterne i denne afhandling afhænger ikke af de to førnævnte arbejder, men bruger derimod den moderne tilgang til Hochschild homologi baseret på ∞-kategoriske metoder. Undervejs til at kunne beskrive og bevise vores resultat i denne ramme beviser vi nogle resultater som kan have selvstændig interesse.

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Chapter 1

Introduction

In this thesis we evaluate Hochschild homology over a commutative ring k of quotients of polynomial algebras $k[x_1, \ldots, x_n]/f$ for certain polynomials f, as an object of the ∞-category of mixed complexes $\mathcal{M}\mathrm{ixed}$, assuming a conjectural description of Hochschild homology of polynomial algebras. We do this by giving an explicit, and quite small, strict mixed complex representing $\mathrm{HH}(k[x_1, \ldots, x_n]/(f))$. We verify the conjecture in some cases, leading to unconditional results in the case of $n \leq 2$ variables as long as 2 is invertible in k. This result improves upon prior work by Larsen [Lar95] where stronger conditions on f are imposed[1], and by the Buenos Aires Cyclic Homology Group [BACH], where only the underlying chain complex was considered. The results of this thesis do not rely on the two aforementioned prior results, but use a different approach, employing the modern framework for Hochschild homology in the setting of ∞-categories.

The motivation for calculating Hochschild homology as a mixed complex stems from its usefulness to calculations of algebraic K-theory. The modern framework for topological cyclic homology by Nikolaus–Scholze [NikSch] opened up the possibility of obtaining calculations of algebraic K-theory using trace methods with only Hochschild homology as a mixed complex as input, via a method developed by Speirs [Spe18; Spe20; Spe21] and Hesselholt–Nikolaus [HN20]. In this modern setting, Hochschild homology is a functor of ∞-categories

$$\mathrm{HH}_{\mathbb{T}} \colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

assigning to each associative algebra in the derived category of k an object of $\mathcal{D}(k)$ equipped with an action by the circle group $\mathbb{T}$. The ∞-category $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ is equivalent to the underlying ∞-category $\mathcal{M}\mathrm{ixed}$ of a model category Mixed of strict mixed complexes[2], and we denote the composition of $\mathrm{HH}_{\mathbb{T}}$ with this equivalence by $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$. We now formulate the main result of this thesis, and will explain the meaning of the conditions on f and the notation used in the formula for d later in this introduction.

[1] But no assumption is made on invertibility of 2 in k.

[2] A strict mixed complex is a chain complex with an additional operator d increasing degree by 1 and satisfying $\partial\mathrm{d} + \mathrm{d}\partial = 0$ and $\mathrm{d}^2 = 0$.

Theorem A. *Let k be a commutative ring in which 2 is invertible*[3]*, $n \leq 2$ a positive integer, and $\preceq$ a monomial order (for n variables). Let f be a monic (with respect to $\preceq$) polynomial in n variables, and assume that furthermore the following property holds for any $\overrightarrow{i} \in \mathbb{Z}^n_{\geq 0}$ such that the coefficient of the monomial $x^{\overrightarrow{i}}$ in f is non-zero: If $1 \leq j \leq n$ and $\deg_{\preceq}(f)_j \neq 0$, then $\overrightarrow{i}_j \leq \deg_{\preceq}(f)_j$. In other words, we require that every monomial appearing in f divides the leading monomial, after replacing by 1 those variables that do not appear in the leading monomial of f.*

Then there is an equivalence[4]

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f) \simeq \gamma_{\mathsf{Mixed}}(k[x_1,\ldots,x_n]/f \otimes \Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n) \otimes \Gamma(t))$$

in $\mathcal{M}\mathrm{ixed}$*, where*

$$k[x_1,\ldots,x_n]/f \otimes \Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n) \otimes \Gamma(t)$$

is a strict mixed complex with underlying $\mathbb{Z}$-graded k-module[5] *as indicated, with x_i of degree 0, $\mathrm{d}\,x_i$ of degree 1 and t of degree 2. The boundary operator ∂ is defined by extending the following formulas*[6] *by k-linearity and the Leibniz rule, where $P \in k[x_1,\ldots,x_n]/f$, $1 \leq i \leq n$, and $m \geq 0$.*

$$\partial(P) = 0, \qquad \partial(\mathrm{d}\,x_i) = 0, \qquad \partial\left(t^{[m]}\right) = -p(\mathrm{d}\,f)t^{[m-1]}$$

The differential d is defined by extending by k-linearity the following formula for a polynomial $P \in k[x_1,\ldots,x_n]$, $\overrightarrow{\epsilon} \in \{0,1\}^n$, and $m \geq 0$.

$$\mathrm{d}\left(p(P)\,\mathrm{d}\,x^{\overrightarrow{\epsilon}}t^{[m]}\right) := \left(p\left(\mathrm{d}\left(r^0_f(P)\right)\right) + mp\left(q^1_f\left(\mathrm{d}\,f \cdot r^0_f(P)\right)\right)\right)\mathrm{d}\,x^{\overrightarrow{\epsilon}}t^{[m]} \qquad \heartsuit$$

A proof of Theorem A can be found on Page 590. Most of the steps in the proof of Theorem A do not require the assumption that $n \leq 2$ and that 2 is invertible in k. We however need Conjecture D to hold for f. Conjecture D will be formulated and verified for $n \leq 2$ as long as 2 is invertible in k in Section 7.5.

Let us now give an overview over the remainder of this chapter. We begin in Section 1.1 by describing our motivation for studying Hochschild homology as a mixed complex, which arises from its relevance in the methods used in calculations of algebraic K-theory groups in [Spe20], [Spe21], and [HN20].

In Section 1.2 we explain how $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$, the main object of study, as well as the ∞-category $\mathcal{M}\mathrm{ixed}$ and 1-category Mixed are defined.

[3] The assumption that 2 is invertible in k is not needed when $n \leq 1$.

[4] γ_{Mixed} is a functor from the category of strict mixed complexes with cofibrant underlying chain complex to $\mathcal{M}\mathrm{ixed}$ and will be discussed in Section 1.2.2.

[5] We will use the commutative $\mathbb{Z}$-graded k-algebra structure to write elements and describe ∂, but we warn that d does *not* satisfy the Leibniz rule, so this is not an algebra in strict mixed complexes.

[6] We denote by p the quotient morphism $p\colon k[x_1,\ldots,x_n] \to k[x_1,\ldots,x_n]/f$.

We will then turn towards describing the proof of Theorem A, which splits up naturally into two main steps. We describe the first main step in Section 1.3, which involves writing the quotient $k[x_1,\ldots,x_n]/f$ as a relative tensor product $k[x_1,\ldots,x_n]\otimes_{k[t]}k$, and then using that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserves relative tensor products. This yields a strict mixed complex X_f of medium size representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$. Finding a smaller sub-mixed-complex such that the inclusion into X_f is a quasiisomorphism is the content of the second main step in the proof of Theorem A and will be described in Section 1.4. Along the way we will introduce the definitions of concepts and notation used in the formulation of Theorem A.

In Section 1.5 we then give an overview over the content of the individual chapters and appendices of this thesis, and in Section 1.6 we describe some directions for future work and questions left open by this thesis.

1.1 Motivation

The project that eventually became this thesis started with the goal of determining the structure of the algebraic K-theory groups

$$\mathrm{K}_*(k[x_1,\ldots,x_n]/(x_1\cdots x_n),(x_1,\ldots,x_n))$$

for k a perfect field of positive characteristic, with the polynomial $x_1\cdots x_n$ geometrically corresponding to the union of the coordinate hyperplanes. A method recently made possible by the Nikolaus–Scholze framework for topological cyclic homology [NikSch], and used by Speirs in the case of truncated polynomial algebras [Spe20][7] and the union of coordinate axes [Spe21][8], and by Hesselholt–Nikolaus for cuspidal curves [HN20], makes attacking such questions significantly easier.

In all these cases, what is determined are algebraic K-theory groups

$$\mathrm{K}_*(k[x_1,\ldots,x_n]/(f_1,\ldots,f_m),(x_1,\ldots,x_n))$$

for k a perfect field of positive characteristic, n and m positive integers, and $f_1,\ldots,f_m$ specific polynomials in n variables with coefficients in $\mathbb{Z}$. This is done by employing trace methods, and the input ultimately required for this method circles around $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1,\ldots,x_n]/(f_1,\ldots,f_m))$, though there are variations between [Spe21], [Spe20], and [HN20] in what precisely is used as input. The following table is an overview.

[7] The relevant K-theory groups had first been evaluated by Hesselholt–Madsen [HM97], but the calculation was significantly simplified by Speirs.

[8] Generalizing results by Hesselholt [Hes07] from the two-dimensional case.

	n	$(f_1, \ldots, f_m)$	Input used
[Spe21]	$n \geq 1$	$(x_i x_j)_{i \neq j}$	$\mathrm{B}^{\mathrm{cyc}}(\Pi)$[9]as an object of $\mathcal{S}_*^{\mathrm{B}\,\mathbb{T}}$
[Spe20]	1	(x_1^a), for $a \geq 1$ an integer	Homotopy groups of $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1]/(x_1^a))$ together with Connes' operator
[HN20]	2	$x_1^a - x_2^b$ for $a, b \geq 2$ relatively prime	$\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1, x_2]/(x_1^a - x_2^b))$ as an object of $\mathcal{D}(\mathbb{Z})^{\mathrm{B}\,\mathbb{T}}$

In [Spe21], Speirs uses that $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1, \ldots, x_n]/(x_i x_j)_{i \neq j})$ is the $\mathbb{Z}$-linearization of a space with $\mathbb{T}$-action $\mathrm{B}^{\mathrm{cyc}}(\Pi)$, and manages to even determine the $\mathbb{T}$-equivariant homotopy type of $\mathrm{B}^{\mathrm{cyc}}(\Pi)$, rather than only its $\mathbb{Z}$-linearization. In general we would however expect that it will be easier to only determine $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m))$ itself, which is all that is required.

In contrast, in [Spe20] Speirs manages to get by with even less information than $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1]/(x_1^a))$ as an object of $\mathcal{D}(\mathbb{Z})^{\mathrm{B}\,\mathbb{T}}$, only using its homology as well as Connes' operator (induced by the circle action), and extracting e. g. the homotopy groups of the $\mathbb{T}$-fixed points using the fixed points spectral sequence. In this particular case, this is made feasible due to $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1]/(x_1^a))$ decomposing into pieces whose homology is concentrated in only two successive degrees, making the relevant spectral sequences easy enough to evaluate. In more complicated cases we can however not expect to (in general) be able to fully evaluate those spectral sequences without additional information.

Thus, in order to expand the results of [Spe20], [Spe21], and [HN20] to similar algebras, it seems reasonable to start by evaluating the relevant Hochschild homology $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m))$ as an object of $\mathcal{D}(\mathbb{Z})^{\mathrm{B}\,\mathbb{T}}$.

1.2 Hochschild homology as a mixed complex

1.2.1 Hochschild homology as an object with circle action

Having motivated our interested in $\mathrm{HH}_{\mathbb{T}}$, we will now give an idea of how it is defined. As $\mathrm{HH}_{\mathbb{T}}$ is a special case of the cyclic bar construction, we begin in somewhat greater generality.

Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category. Then the *cyclic bar construction* for $\mathcal{C}$ is a functor

$$\mathrm{B}^{\mathrm{cyc}} \colon \operatorname{Alg}(\mathcal{C}) \to \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$$

that associates to every associative algebra R in $\mathcal{C}$ an object with $\mathbb{T}$-action $\mathrm{B}^{\mathrm{cyc}}(R)$ in $\mathcal{C}$. To construct the underlying object in $\mathcal{C}$ of $\mathrm{B}^{\mathrm{cyc}}(R)$, one pro-

[9] $\mathrm{B}^{\mathrm{cyc}}(\Pi)$ denotes the cyclic bar construction of the pointed monoid

$$\Pi = \{0, 1, x_1, x_1^2, \ldots, x_2, x_2^2, \ldots, x_n, x_n^2, \ldots\}.$$

ceeds in two steps. One first constructs out of R a simplicial object $\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)$ in $\mathcal{C}$ such that $\mathrm{B}^{\mathrm{cyc}}_{n}(R)$ is given by $R^{\otimes(n+1)}$ and the structure morphisms $d_i\colon R^{\otimes n} \to R^{\otimes(n-1)}$ and $s_i\colon R^{\otimes n} \to R^{\otimes(n+1)}$ can be described as follows.

1. If $i \leq n-2$, then d_i is $\mathrm{id}_R^{\otimes i} \otimes \mu \otimes \mathrm{id}_R^{\otimes(n-2-i)}$, where $\mu\colon R \otimes R \to R$ is the multiplication morphism.

2. d_{n-1} is the postcomposition of the symmetry isomorphism that brings the last tensor factor to the front with $\mu \otimes \mathrm{id}_R^{\otimes(n-2)}$.

3. s_i is $\mathrm{id}_R^{i+1} \otimes \iota \otimes \mathrm{id}_R^{\otimes(n-i-1)}$, where $\iota\colon \mathbb{1}_{\mathcal{C}} \to R$ is the unit morphism.

Defining a simplicial object in $\mathcal{C}$, i.e. a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, also requires data for higher morphisms; for a full definition of the functor

$$\mathrm{B}^{\mathrm{cyc}}_{\bullet}\colon \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$$

see Section 6.1.2. The underlying object of $\mathrm{B}^{\mathrm{cyc}}(R)$ is then given by the geometric realization[10] of $\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)$. The circle action on $\mathrm{B}^{\mathrm{cyc}}(R)$ is constructed by first using cyclic permutations of the tensor factors to upgrade $\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)$ to a *cyclic object* in $\mathcal{C}$, i.e. lift the functor $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ to a functor to $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$, where $\mathbf{\Lambda}$ is Connes' cyclic category. The additional structure encoded by $\mathbf{\Lambda}$ equips the geometric realization of a cyclic object with the action of the circle group, so that composing $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ with the geometric realization functor for cyclic objects yields a functor $\mathrm{B}^{\mathrm{cyc}}\colon \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$. For a more detailed account of the construction of $\mathrm{B}^{\mathrm{cyc}}$ we refer to Chapter 6.

In the special case $\mathcal{C} = \mathcal{S}\mathrm{p}$, the ∞-category of spectra, the functor $\mathrm{B}^{\mathrm{cyc}}$ is denoted by THH, and if $\mathcal{C}$ is $\mathcal{D}(k)$, the derived ∞-category of a commutative ring k, we denote the functor $\mathrm{B}^{\mathrm{cyc}}$ by $\mathrm{HH}_{\mathbb{T}}(-/k)$ and call $\mathrm{HH}_{\mathbb{T}}(R/k)$ the Hochschild homology of R over k. We will from now on fix a commutative ring k and just write $\mathrm{HH}_{\mathbb{T}}(-)$ instead of $\mathrm{HH}_{\mathbb{T}}(-/k)$.

1.2.2 Mixed complexes

Our goal is to determine $\mathrm{HH}_{\mathbb{T}}(R)$ for specific k-algebras R. However it is somewhat difficult to write down and manipulate objects of $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ directly, so we use strict mixed complexes instead. The situation can be summarized by the following diagram.

$$\begin{tikzcd} & \mathsf{Mixed}_{\mathrm{cof}} \arrow[d, "\gamma_{\mathsf{Mixed}}"] \\ \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[r, dash, "\simeq"] & \mathcal{M}\mathrm{ixed} \end{tikzcd} \tag{1.1}$$

[10]So the underlying object of $\mathrm{B}^{\mathrm{cyc}}(R)$ is $\left|\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)\right| \coloneqq \mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)$.

The horizontal functor is an equivalence between $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ and the *∞-category of mixed complexes*, which the functor γ_{Mixed} exhibits as the underlying ∞-category of the 1-category with weak equivalences $\mathsf{Mixed}_{\mathrm{cof}}$ of *strict mixed complexes* (with cofibrant underlying chain complexes)[11].

We begin explaining diagram (1.1) with the 1-category Mixed. A strict mixed complex consists of an underlying chain complex of k-modules X (with boundary operator ∂ *decreasing* degree) together with an additional operator d, that we sometimes call the differential, *increasing* degree by 1, and satisfying the following identities.

$$\mathrm{d} \circ \mathrm{d} = 0 \qquad \text{and} \qquad \mathrm{d} \circ \partial + \partial \circ \mathrm{d} = 0$$

A morphism of strict mixed complexes is a morphism of underlying chain complexes that commutes with the respective differentials d. The strict mixed complexes and their morphisms define a 1-category Mixed.

There is also another description of Mixed: It is isomorphic to the category of left modules in $\mathsf{Ch}(k)$ over the differential graded algebra $\mathrm{D} = k[\mathrm{d}]/(\mathrm{d}^2)$, where d is of chain degree 1. Under this isomorphism $\mathsf{Mixed} \cong \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$, the action by the element d of D corresponds to the differential d. This suggests the following definition of the ∞-category of mixed complexes.

$$\mathcal{M}\mathrm{ixed} \coloneqq \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$$

The symmetric monoidal functor[12] $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ exhibiting $\mathcal{D}(k)$ as the underlying ∞-category of $\mathsf{Ch}(k)$ then induces a functor

$$\gamma_{\mathsf{Mixed}}\colon \mathsf{Mixed}_{\mathrm{cof}} \to \mathcal{M}\mathrm{ixed}$$

where $\mathsf{Mixed}_{\mathrm{cof}}$ refers to the subcategory of Mixed spanned by those strict mixed complexes whose underlying chain complex is cofibrant with respect to the projective model structure[13].

We can make $\mathsf{Mixed}_{\mathrm{cof}}$ into a category with weak equivalences, where a morphism is a weak equivalence if and only if the underlying morphism of chain complexes is a quasiisomorphism, and it turns out that γ_{Mixed} then exhibits $\mathcal{M}\mathrm{ixed}$ as the ∞-category obtained from $\mathsf{Mixed}_{\mathrm{cof}}$ by inverting weak equivalences. We will discuss both Mixed as well as $\mathcal{M}\mathrm{ixed}$ in greater detail in Chapter 4.

The equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ is the composition of two different equivalences. There first is an equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k))$, where $k\boxtimes\mathbb{T}$ is the *k-linear circle*. The remaining equivalence

$$\mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \simeq \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) = \mathcal{M}\mathrm{ixed}$$

[11] The reason why we do not just say that $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ is exhibited as the underlying ∞-category of $\mathsf{Mixed}_{\mathrm{cof}}$ by the composition is that, while both $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ and $\mathcal{M}\mathrm{ixed}$ carry symmetric monoidal structures, the equivalence is only shown to be $\mathbb{E}_1$-monoidal. We should thus be careful to distinguish $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ and $\mathcal{M}\mathrm{ixed}$ whenever $\mathbb{E}_2$-monoidal structures may become relevant.

[12] The superscript cof refers to the subcategory of cofibrant objects.

[13] See Fact 4.1.3.1 for a definition.

is then induced by an equivalence $k \boxtimes \mathbb{T} \simeq \mathrm{D}$ in $\mathrm{Alg}(\mathcal{D}(k))$. We discuss these equivalences in detail in Chapter 5.

1.3 The first step in the proof of the main result

As mentioned before, we define $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ to be the composition of $\mathrm{HH}_{\mathbb{T}}$ with a specific equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ sketched above. Theorem A then sets the task before us to define a strict mixed complex that is mapped by γ_{Mixed} to an object in $\mathcal{M}\mathrm{ixed}$ that is equivalent to $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$.

The proof of Theorem A proceeds in two main steps. The idea of the first main step is to use that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is compatible with relative tensor products and that the quotient $k[x_1, \ldots, x_n]/f$ can be written as a relative tensor product of polynomial algebras[14].

Before going into more detail about why $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is compatible with relative tensor products, let us first describe the monoidal structure on Mixed. Given strict mixed complexes X and Y, we define the underlying chain complex of $X \otimes Y$ to be the tensor product in $\mathsf{Ch}(k)$ of the underlying chain complexes. The differential d is then defined using the Leibniz rule, so $\mathrm{d}(x \otimes y) = \mathrm{d}(x) \otimes y + (-1)^{\deg_{\mathsf{Ch}}(x)} x \otimes \mathrm{d}(y)$. Taking the perspective that a strict mixed complex is a left-D-module as described above, this symmetric monoidal structure arises from a bialgebra structure on D, where the comultiplication maps d to $\mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}$. Chapter 3 constructs monoidal structures on ∞-categories of left modules over bialgebras in a functorial way, so that we can upgrade $\gamma_{\mathsf{Mixed}} \colon \mathsf{Mixed}_{\mathrm{cof}} \to \mathcal{M}\mathrm{ixed}$ to a monoidal functor.

That $\mathrm{HH}_{\mathbb{T}}$ is a symmetric monoidal functor essentially follows from the fact that $\boldsymbol{\Delta}^{\mathrm{op}}$ is sifted and the tensor product in $\mathcal{D}(k)$ preserves colimits separately in each variable; we roughly obtain equivalences

$$\left|R^{\bullet+1}\right| \otimes \left|S^{\bullet+1}\right| \simeq \left|R^{\bullet+1} \otimes S^{\bullet+1}\right| \simeq \left|(R \otimes S)^{\bullet+1}\right|$$

that should make plausible that $\mathrm{HH}_{\mathbb{T}}$ is symmetric monoidal. $\mathrm{HH}_{\mathbb{T}}$ also preserves sifted colimits, and hence preserves relative tensor products. For more details see Chapter 6.

To then deduce that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ also preserves relative tensor products it remains to show that $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ preserves relative tensor products. As an equivalence, it is clear that this functor preserves sifted colimits, but that it is $\mathbb{E}_1$-monoidal is not obvious, relying on a longer argument[15] carried out in Section 5.1, showing that D and $k \boxtimes \mathbb{T}$ are equivalent not only as associative algebras in $\mathcal{D}(k)$, but as $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebras[16] .

[14] This idea was suggested by Thomas Nikolaus.
[15] The strategy for this argument was suggested by Achim Krause.
[16] I. e. as commutative and coassociative bialgebras.

The quotient $k[x_1, \ldots, x_n]/f$ is isomorphic to the relative tensor product

$$k[x_1, \ldots, x_n] \otimes_{k[t]} k$$

in $\mathrm{Alg}(\mathrm{LMod}_k(\mathsf{Ab}))$, where t acts by multiplication with f on $k[x_1, \ldots, x_n]$ and by multiplication with 0 on k. Under the assumptions made for f in Theorem A, this ordinary relative tensor product calculates the *derived* one, so that we obtain an equivalence

$$k[x_1, \ldots, x_n]/f \simeq k[x_1, \ldots, x_n] \otimes_{k[t]} k$$

in $\mathrm{Alg}(\mathcal{D}(k))$ as well, inducing an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]) \otimes_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t])} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

in $\mathcal{M}\mathrm{ixed}$.

To proceed we require a description of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n])$ as well as $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$ as modules over $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t])$ in $\mathcal{M}\mathrm{ixed}$. The following conjecture provides such a description in terms of the mixed complexes of de Rham forms.

Conjecture D. *Let $n \geq 0$ be an integer and f an element of $k[x_1, \ldots, x_n]$. Denote by $F\colon k[t] \to k[x_1, \ldots, x_n]$ the morphism of commutative k-algebras that maps t to f and by $G\colon k[t] \to k$ the morphism of commutative k-algebras that maps t to 0. Then there exists a commutative diagram*

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k/k}\right) \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t]) \arrow[u, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(G)"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] \arrow[r, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[t]/k}\right) \arrow[u, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{G/k}\right)"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{F/k}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[x_1, \ldots, x_n]/k}\right)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ *such that the horizontal morphisms are equivalences.*

We will often refer to the existence of such a commutative diagram for a specific f as "Conjecture D holds for f". ♧

Conjecture D will be discussed in Section 7.5, where we will also show that it holds if $n \leq 1$ or $n = 2$ and 2 is invertible in k.

Assuming that Conjecture D holds for f, we then obtain an equivalence[17]

$$\begin{aligned}&\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]) \otimes_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t])} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)\\ \simeq\ &\gamma_{\mathsf{Mixed}}(k[x_1,\ldots,x_n]\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n)) \otimes_{\gamma_{\mathsf{Mixed}}(k[t]\otimes\Lambda(\mathrm{d}\,t))} \gamma_{\mathsf{Mixed}}(k)\end{aligned}$$

where x_i and t are in degree 0, $\mathrm{d}\,x_i$ and $\mathrm{d}\,t$ are in degree 1, and t acts by multiplication with f on $k[x_1,\ldots,x_n]\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n)$ and trivially on k. As alluded to by the naming, the differential of the respective mixed complexes maps x_i to $\mathrm{d}\,x_i$ and t to $\mathrm{d}\,t$, and is defined on the other elements by k-linearity and the Leibniz rule, while all three underlying chain complexes have zero boundary operator.

To obtain a strict mixed complex that represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$ we thus have to calculate the derived tensor product in Mixed over $k[t]\otimes\Lambda(\mathrm{d}\,t)$ of $k[x_1,\ldots,x_n]\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n)$ with k. To do so, we need to replace k with a sufficiently cofibrant replacement as a module over $k[t]\otimes\Lambda(\mathrm{d}\,t)$ in Mixed. Such a replacement is given by a strict complex A whose underlying graded k-module is given by the tensor product[18]

$$k[t]\otimes\Lambda(\mathrm{d}\,t)\otimes\Lambda(s)\otimes\Gamma(\mathrm{d}\,s)$$

where t is of degree 0, $\mathrm{d}\,t$ and s are of degree 1, and $\mathrm{d}\,s$ is of degree 2. The boundary operator ∂ and differential d are k-linear and satisfy the Leibniz rule, and are thus determined by the following formulas.

$$\begin{aligned}&\partial(t)=0, &&\partial(\mathrm{d}\,t)=0, &&\partial(s)=t, &&\partial\Big(\mathrm{d}\,s^{[m]}\Big)=-\,\mathrm{d}\,t\,\mathrm{d}\,s^{[m-1]}\\ &\mathrm{d}(t)=\mathrm{d}\,t, &&\mathrm{d}(\mathrm{d}\,t)=0, &&\mathrm{d}(s)=\mathrm{d}\,s^{[1]}, &&\mathrm{d}\Big(\mathrm{d}\,s^{[m]}\Big)=0\end{aligned}$$

There is an obvious morphism of algebras in Mixed from $k[t]\otimes\Lambda(\mathrm{d}\,t)$ to A that maps t to t. In Section 8.2 it is shown that this makes A into a sufficiently cofibrant replacement for k as a left-$(k[t]\otimes\Lambda(\mathrm{d}\,t))$-module to calculate the derived relative tensor product discussed above as the ordinary relative tensor product

$$\begin{aligned}&(k[x_1,\ldots,x_n]\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n))\otimes_{k[t]\otimes\Lambda(\mathrm{d}\,t)}(k[t]\otimes\Lambda(\mathrm{d}\,t)\otimes\Lambda(s)\otimes\Gamma(\mathrm{d}\,s))\\ \cong\ &k[x_1,\ldots,x_n]\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n)\otimes\Lambda(s)\otimes\Gamma(\mathrm{d}\,s)=:X_f\end{aligned}$$

in Mixed. We thus obtain an equivalence

$$\begin{aligned}&\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)\\ \simeq\ &\gamma_{\mathsf{Mixed}}(k[x_1,\ldots,x_n]\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n)\otimes\Lambda(s)\otimes\Gamma(\mathrm{d}\,s))\end{aligned}$$

in $\mathcal{M}$ixed. The boundary operator ∂ and differential d satisfy the Leibniz rule on X_f, and $\partial(s)=f$.

[17]The notation Λ is used for the exterior algebra, see Section 2.3 (29).
[18]The notation Γ is used for the divided power algebra, see Section 2.3 (30).

1.4 The second step in the proof of the main result

With the strict mixed complex X_f as above we already have a reasonably small strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\dots,x_n]/f)$, but we still want to identify a smaller, quasiisomorphic, sub-mixed-complex. In particular, X_f is given by $k[x_1,\dots,x_n]$ in degree 0, while the homology is $k[x_1,\dots,x_n]/f$ in degree 0. We will thus try to find a small sub-mixed-complex quasiisomorphic to X_f such that the k-module in degree 0 is isomorphic – as a k-module – to $k[x_1,\dots,x_n]/f$.

Before we get started with this we first describe one of the assumptions we need to make on f, which is that f needs to be *monic* with respect to a chosen *monomial order*. A monomial order is a well-order $\preceq$ on the set of monomials in $x_1,\dots,x_n$, or equivalently on $\mathbb{Z}^n_{\geq 0}$, such that $\vec{a} \preceq \vec{b}$ implies $\vec{a}+\vec{c} \preceq \vec{b}+\vec{c}$ for $\vec{a}, \vec{b}, \vec{c} \in \mathbb{Z}^n_{\geq 0}$. From now on we fix a monomial order $\preceq$. We can then define f to be monic (with respect to $\preceq$) if the biggest (with respect to $\preceq$) monomial appearing[19] in f has coefficient 1. The *degree* of f (with respect to $\preceq$), denoted by $\deg_{\preceq}(f)$, is the element of $\mathbb{Z}^n_{\geq 0}$ that is maximal with respect to $\preceq$ such that the coefficient of $x^{\deg_{\preceq}(f)}$ in f is non-zero.

If f is monic, then it is possible to divide polynomials in $x_1,\dots,x_n$ by f with remainder. Specifically, if P is an element of $k[x_1,\dots,x_n]$, then there is a unique decomposition of P as $P = q^1_f(P)f + r^0_f(P)$ such that $r^0_f(P)$ is *f-reduced*, meaning that only monomials that are not divisible by the lead monomial of f may appear in $r^0_f(P)$. For more details on these notions for multivariable polynomials see Section 9.1.

One perspective on the just mentioned decomposition is that it means that there is a *unique* f-reduced representative in $k[x_1,\dots,x_n]$ for every element of $k[x_1,\dots,x_n]/f$. We can thus define a section ϱ (as morphisms of k-modules) of the quotient morphism $p\colon k[x_1,\dots,x_n] \to k[x_1,\dots,x_n]/f$ by defining $\varrho(p(P))$ to be $r^0_f(P)$. Along ϱ we can thus identify $k[x_1,\dots,x_n]/f$ as a k-module with the k-submodule of $k[x_1,\dots,x_n]$ spanned by the reduced polynomials, i. e. $\mathrm{Im}(\varrho)$.

We now start with the sub-graded-k-module $\mathrm{Im}(\varrho)$ of X_f, and discuss what additional generators we need to add to our sub-graded-k-module to satisfy the following three conditions.

(a) It needs to be closed under ∂, to define a subcomplex.

(b) It needs to be closed under d, to define a sub-mixed-complex.

(c) The inclusion into X_f must be a quasiisomorphism.

As we require closedness under d, we first enlarge to the sub-graded-k-module

$$\mathrm{Im}(\varrho) \otimes \Lambda(\mathrm{d}\, x_1,\dots,\mathrm{d}\, x_n)$$

[19] That is, having non-zero coefficient.

of X_f. Now there are however elements that are multiples of $\mathrm{d}\, f$ and which are cycles but not boundaries, while they *are* boundaries in X_f. In order to achieve (c) we will thus need to add elements whose boundary are the relevant multiples of $\mathrm{d}\, f$. Our first attempt might be to consider the sub-graded-k-module

$$\mathrm{Im}(\varrho) \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes k \cdot \{1, \mathrm{d}\, s^{[1]}\}$$

as $\partial(-\mathrm{d}\, s^{[1]}) = \mathrm{d}\, f$. As we have now created new multiples of both $\mathrm{d}\, f$ as well as $\mathrm{d}\, s^{[1]}$ that will be cycles but not boundaries as needed for (c), we actually keep going and consider the sub-graded-k-module

$$\mathrm{Im}(\varrho) \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(\mathrm{d}\, s)$$

of X_f.

Let us turn towards condition (a) and check whether this could be a sub-complex of X_f. For this, let R be an element of $\mathrm{Im}(\varrho)$. Then we obtain

$$\partial\Big(R\,\mathrm{d}\, s^{[1]}\Big) = -R\,\mathrm{d}\, f = -q_f^1(R\,\mathrm{d}\, f)f - r_f^0(R\,\mathrm{d}\, f)$$

For this to lie in our provisional sub-graded-k-module we need to have that $q_f^1(R\,\mathrm{d}\, f) = 0$, but unfortunately this will in general not be the case. To fix this, we should then modify $R\,\mathrm{d}\, s^{[1]}$ by adding another generator whose boundary will be $q_f^1(R\,\mathrm{d}\, f)f$. Such an element is given by $sq_f^1(R\,\mathrm{d}\, f)$, which leads us to the following definition. We define $\mathcal{J}_0$ as the set

$$\mathcal{J}_0 \coloneqq \left\{ \left(\vec{i}, \vec{\epsilon}, m\right) \in \mathbb{Z}_{\geq 0}^n \times \{0,1\}^n \times \mathbb{Z}_{\geq 0} \;\middle|\; x^{\vec{i}} \text{ is } f\text{-reduced} \right\}$$

and for $(\vec{i}, \vec{\epsilon}, m)$ an element of $\mathcal{J}_0$ we define

$$e_{\vec{i}, \vec{\epsilon}, m} \coloneqq x^{\vec{i}}\,\mathrm{d}\, x^{\vec{\epsilon}}\,\mathrm{d}\, s^{[m]} + sq_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\,\mathrm{d}\, s^{[m-1]}$$

as an element of X_f. We can then define $X_{f,0}^e$ to be the sub-graded-k-module of X_f spanned by the elements of the form $e_{\vec{i}, \vec{\epsilon}, m}$ for $(\vec{i}, \vec{\epsilon}, m)$ in $\mathcal{J}_0$.

It turns out that $X_{f,0}^e$ indeed satisfies conditions (a) and (c), but not in general (b). Thus the chain complex $X_{f,0}^e$ *does* represent the underlying object in $\mathcal{D}(k)$ of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ (this reproves the main result of [BACH] as long as Conjecture D is satisfied for f), but we need to make further assumptions to ensure that $X_{f,0}^e$ is a sub-*mixed*-complex of X_f.

In the formulation of Theorem A we use a sufficient condition for f that is very easy to check and that ensures that $X_{f,0}^e$ is a sub-mixed-complex of X_f. The strict mixed complex used in the statement is then obtained by merely renaming the basis of $X_{f,0}^e$, where the element $e_{\vec{i}, \vec{\epsilon}, m}$ of $X_{f,0}^e$ corresponds to the element $p(x^{\vec{i}})\,\mathrm{d}\, x^{\vec{\epsilon}} t^{[m]}$ in the strict mixed complex described in Theorem A.

1.5 Overview over the chapters of this thesis

This thesis tries to give a rigorous proof of Theorem A, so it was attempted to include a proof for every needed statement for which no proof could be found in the literature. By necessity this means that many statements and proofs will already have been known to the experts, and some may even have already appeared, spread throughout the literature. This holds particularly with regards to the material contained in the appendices, where we collect various required statements on various aspects of working in an ∞-categorical setting. We hope that this will help fill some gaps in the literature. A reader primarily interested in applying the result and already familiar with Hochschild homology and mixed complexes may thus wish to only read Chapter 9 containing the statement of the result and the notation and notions necessary to understand and apply it, as well as Chapter 10, which contains an example worked out in detail.

The material is ordered linearly; proofs in the appendices only depend on statements occurring earlier in the appendices, and proofs in the main text only depend on statements occurring earlier in the main text or in the appendices.

We now briefly summarize the content of the chapters of this thesis. Each chapter, and most sections and subsections, also begin with an introduction, so we refer there for more details.

In **Chapter 2** we list and explain the notation and conventions that we use, and discuss what we assume the reader is familiar with.

In **Chapter 3** we construct monoidal structures on ∞-categories of left modules over bialgebras. If C is a symmetric monoidal 1-category and A a (associative, coassociative) bialgebra in C, then the category of left-A-modules $\mathrm{LMod}_A(\mathsf{C})$ can be given a monoidal structure again, constructed from the coalgebra structure of A[20]. The underlying object in C of the tensor product of two left-A-modules X and Y is the tensor product in C of the underlying objects, with action of A defined via the composition

$$A \otimes X \otimes Y \xrightarrow{\Delta\otimes \mathrm{id}_X \otimes \mathrm{id}_Y} A \otimes A \otimes X \otimes Y \xrightarrow{\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_X} A \otimes X \otimes A \otimes Y \to X \otimes Y$$

where Δ is the comultiplication, τ is the symmetry isomorphism, and the last morphism is the tensor product of the action morphisms of A on X and Y.

In Chapter 3 we construct such monoidal structures on $\mathrm{LMod}_A(\mathcal{C})$, where $\mathcal{C}$ is now allowed to be an $\mathbb{E}_2$-monoidal ∞-category, and A an $\mathbb{E}_1, \mathbb{E}_1$-bialgebra in $\mathcal{C}$. Our construction will be functorial in both A as well as $\mathcal{C}$ and thus allow us to compare Mixed, $\mathcal{M}\mathrm{ixed}$, and $\mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k))$, which are all monoidal ∞-categories arising via this construction.

In **Chapter 4** we define the 1-category Mixed and ∞-category $\mathcal{M}\mathrm{ixed}$. Beyond what was already mentioned in Section 1.2.2, we also discuss model

[20]This monoidal structure should not be confused with the monoidal structure one can define using relative tensor products over A if A is commutative.

structures on both Mixed and $\mathrm{Alg}(\mathsf{Mixed})$, show that $\mathcal{M}\mathrm{ixed}$ and $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ are the respective underlying ∞-categories, and put the classical notion of strongly homotopy linear morphisms of strict mixed complexes into this context. That every algebra in $\mathcal{M}\mathrm{ixed}$ has a strict model will play a role in Chapter 7, when we discuss $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of polynomial algebras as an object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$.

In **Chapter 5** we construct a monoidal equivalence between $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ and $\mathcal{M}\mathrm{ixed}$, as discussed in Section 1.2.2 above.

In **Chapter 6** we define Hochschild homology, both in its modern incarnation as a symmetric monoidal functor of ∞-categories

$$\mathrm{HH}_{\mathbb{T}}\colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

as well as the classical model for Hochschild homology given by the standard Hochschild complex. In particular, we discuss how the standard Hochschild complex represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ as a mixed complex (by [Hoy18]) as well as HH of commutative rings as an object of $\mathrm{CAlg}(\mathcal{D}(k))$.

In **Chapter 7** we show that the mixed complex of de Rham forms is a model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of polynomial algebras in at most 2 variables as an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. Important input for this will be the comparison results discussed in Chapter 6 as well as the strictification result for algebras in $\mathcal{M}\mathrm{ixed}$ from Chapter 4. We also discuss compatibility with morphisms of polynomial algebras, by formulating Conjecture C and Conjecture D, and proving them in some cases.

In **Chapter 8** we perform the first step of the proof of Theorem A that we discussed in Section 1.3 above. The main result of Chapter 8 will be applicable in more generality, providing a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(y_1, \ldots, y_n))$ for R a commutative algebra in $\mathsf{Ch}(k)$, and $y_1, \ldots, y_n$ elements of R in degree 0, providing that the requirements of Proposition 8.3.0.1 are met, and we in particular are given a strict model of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ with sufficient structure.

Finally, we put everything together in **Chapter 9**. This chapter introduces the necessary notions for multivariable polynomials and carries out the second step of the proof of Theorem A that we discussed in Section 1.4 above.

For actual applications, we expect that the user of Theorem A will likely need to further simplify the resulting strict mixed complex. In **Chapter 10** we thus discuss the example $f = x_1^2 - x_2x_3$ in detail[21], identifying an even smaller strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}[x_1, x_2, x_3]/f)$ than the one given by Theorem A (conditional on Conjecture D holding for f). We take care to not only prove the end result, but to describe the steps in the order and manner that one would take them when trying to come up with such a simplification, and

[21] As this is an example in three variables, Theorem A only holds for f conditional on Conjecture D. However, it is an interesting example with which we can demonstrate the combinatorial notions used to formulate the result of Theorem A, and how the result can be further manipulated.

hope that this example will help the reader to similarly simplify the result of Theorem A for other concrete polynomials.

The appendices contain various material relating to working with various notions in an ∞-categorical setting that do not have a very strong thematic relation to the main content of this thesis, apart from being needed in it.

Appendix A and **Appendix D** contain some statements on basic notions of ∞-category theory, such as mapping spaces, undercategories, and adjunctions. The reason this material is split up into two appendices is in order to conserve linearity of the material in the appendices, as some material from Appendix A is needed in the intermediate appendices, from where Appendix D needs some results.

In **Appendix B** we discuss the notions of (fully) faithful functors of ∞-categories as well as monomorphisms in Cat_∞.

Appendix C collects a number of statements involving (co)cartesian fibrations. In particular, we discuss for functors of ∞-categories $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ the property of the cocartesian fibration classified by F that corresponds to $\mathcal{C}$ having all products and F preserving them.

In **Appendix E** we discuss various statements that relate to ∞-operads and their ∞-categories of algebras, such as the induced ∞-operad structures on ∞-categories of algebras, free algebras, and relative tensor products.

Appendix F discusses cartesian symmetric monoidal ∞-categories. If $\mathcal{C}$ is a cartesian symmetric monoidal ∞-category and $\mathcal{O}$ an ∞-operad, then the ∞-categories of $\mathcal{O}$-algebras and $\mathcal{O}$-monoids in $\mathcal{C}$ are equivalent. A large part of Appendix F is concerned with iterating this, i. e. applying $\mathrm{Alg}_{\mathcal{O}'}$ or $\mathrm{Mon}_{\mathcal{O}'}$ to $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ or $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ and comparing the resulting ∞-categories. The reason is that we not only need to know that there exist some equivalences between the various ∞-categories, but require concrete descriptions of specific equivalences.

1.6 Future directions

In this section we present some questions left open by this thesis and directions for future work. The most obvious open problem is the conjecture our main result depends on.

(1) Conjecture D is proven in Chapter 7 only for $n \leq 2$ variables, in the case $n = 2$ requiring an assumption on k. Showing this conjecture for polynomials in more variables would extend Theorem A.

The next possibility for future work we would like to mention is the application to calculations of algebraic K-theory.

(2) Let k be a perfect field of positive characteristic, n a positive integer, and f a polynomial in n variables satisfying the conditions of Theorem A. One can then try to determine the structure of the K-theory groups $\mathrm{K}_*(k[x_1, \ldots, x_n]/f, (x_1, \ldots, x_n))$ using the techniques of

[Spe20], [Spe21], and [HN20], using the strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ as the starting point.

The project that became this thesis was in fact started with the goal of determining the structure of

$$\mathrm{K}_*(k[x_1, \ldots, x_n]/(x_1 \cdots x_n), (x_1, \ldots, x_n))$$

i. e. of the K-theory groups of the union of hyperplanes. Another first test case to apply this to might be the cone $x_1^2 = x_2 x_3$, i. e. trying to determine the structure of $\mathrm{K}_*(k[x_1, \ldots, x_n]/(x_1^2 - x_2 x_3), (x_1, x_2, x_3))$. To obtain new unconditional results both of these would require first extending the validity of Theorem A by proving Conjecture D for the three-variable case.

There are also a number of questions directly left open in this thesis.

(3) In [Spe20] and [HN20] it is important that THH and $\mathrm{HH}_{\mathbb{T}}$ have a compatible decomposition as a sum, which arises from a grading on the polynomial ring with respect to which the polynomial divided out is homogeneous.

Before tackling (2) it will therefore be important to upgrade Theorem A to take into account such a grading.

(4) In Chapter 5 we show that there is an $\mathbb{E}_1$-monoidal equivalence between $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ and $\mathcal{M}\mathrm{ixed}$. Does there exist an $\mathbb{E}_2$-monoidal equivalence? One can also add some additional conditions, such as asking for a commutative triangle

$$\begin{tikzcd} \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[rr, no head, "\simeq"] \arrow[dr] & & \mathcal{M}\mathrm{ixed} \arrow[dl] \\ & \mathcal{D}(k) & \end{tikzcd}$$

of $\mathbb{E}_2$-monoidal functors, with the horizontal one being an equivalence, and where the two other functors are the forgetful ones.

(5) Theorem A is shown in Proposition 9.5.2.3, where, apart from Conjecture D needing to hold for f, the condition is actually that f needs to be monic and satisfy $\mathrm{logdim}_f(\mathrm{d}\, f) \leq 1$, rather then the condition used in the the formulation of Theorem A above, which by Corollary 9.4.2.6 implies $\mathrm{logdim}_f(\mathrm{d}\, f) \leq 1$. This leaves the question whether Corollary 9.4.2.6 is sharp. To be more precise, suppose $f \neq 1$ is a polynomial that is monic with respect to a monomial ordering $\preceq$ and such that $\mathrm{logdim}_f(\mathrm{d}\, f) \leq 1$. Then does it hold for every $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ such that the coefficient of the monomial $x^{\vec{i}}$ in f is non-zero that if $1 \leq j \leq n$ and $\deg_{\preceq}(f)_j \neq 0$, then $\vec{i}_j \leq \deg_{\preceq}(f)_j$?

(6) A related question to (5) is what kind of values $\operatorname{logdim}_f(\mathrm{d}\, f)$ can take. In particular, is there a monic polynomial f such that $\operatorname{logdim}_f(\mathrm{d}\, f)$ is finite, but bigger than 1?

(7) Is there a class of monic polynomials f with $\operatorname{logdim}_f(\mathrm{d}\, f) > 1$ and for which $X^e_{f,0}$ is not a sub-mixed-complex of X_f, but there is some other, intermediate sub-mixed-complex that is also equivalent to X_f? For example it may be that there exists such a sub-mixed-complex for some f in which the power of f is bounded[22,23], unlike in X.

It is possible that $\operatorname{logdim}_f(\mathrm{d}\, f)$ has already been studied (if so, likely under a different name), so perhaps there already exist answers to (5) and (6) in the literature.

1.7 Acknowledgments

There are many people who helped make this thesis possible. This project was suggested by my advisor, Lars Hesselholt, and he and my co-advisor, Jesper Grodal, helped keep me on track. The idea to approach Hochschild homology of quotients by writing quotients as relative tensor products was suggested to me by Thomas Nikolaus, and a sketch of the proof strategy used to show formality of the k-linear circle as an $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebra in Section 5.1 was suggested to me by Achim Krause.

In a earlier version there was a mistake in the proof of what is now Proposition 7.2.2.2, where it was claimed that ϵ is natural with respect to all morphisms of k-algebras, rather than only those that map variables to variables. This incorrect result was then used in what amounted to a proof of Conjecture B, Conjecture C, and Conjecture D for polynomial algebras in arbitrary many variables. The mistake was pointed out by Thomas Nikolaus.

I had useful mathematical discussions that left their marks on this thesis with many people, among them David Bauer, Elden Elmanto, Aras Ergus, Jesper Grodal, Lars Hesselholt, Kaif Hilman, Joshua Hunt, Achim Krause, Markus Land, Jonas McCandless, Thomas Nikolaus, Riccardo Pengo, Philipp Schmitt, Martin Speirs, and Robin Sroka, and received helpful feedback on earlier drafts from David Bauer, Aras Ergus, Jesper Grodal, Lars Hesselholt, and Martin Speirs. I am sure to have missed someone who should have been listed above, for which I apologize.

Helpful discussions with Achim Krause and Thomas Nikolaus were made possible by the hospitality of the University of Münster. This work was supported by the Danish National Research Foundation through the Centre for

[22] In the sense that it is generated as a graded k-module by elements $c_{\vec{i},l,\vec{\epsilon},m}$ and $e_{\vec{i},l,\vec{\epsilon},m}$ as in Definition 9.2.3.2 such that l is smaller than or equal to some $l_{\max}$.

[23] For example the power of f might be bounded by $n \cdot (\operatorname{logdim}_f(\mathrm{d}\, f) - 1)$ (as long as $\operatorname{logdim}_f(\mathrm{d}\, f) \geq 1$), which would yield the correct bound 0 for $\operatorname{logdim}_f(\mathrm{d}\, f) = 1$. This would of course not be helpful if it turned out that there exist no monic polynomials f with $1 < \operatorname{logdim}_f(\mathrm{d}\, f) < \infty$.

Symmetry and Deformation (DNRF92) and the Copenhagen Centre for Geometry and Topology (DNRF151).

Chapter 2

Notation and conventions

2.1 Prerequisites

We will work extensively in ∞-categorical settings, and thus reading this thesis will likely require a solid foundation in the theory of ∞-categories and higher algebra as developed in [HTT] and [HA]. We will however try to give references for any major statements that we use, and we refrain from using statements that are well-known to experts without giving a proof ourselves if no citable reference could be found in the literature – many such statements are thus collected in the appendices.

We assume that the reader is familiar with the basics of (homological) algebra, as well as the theory of model categories, for which we use [Hov99] and [HTT, A.2] as our main references. Wherever terminology differs between [Hov99] and [HTT, A.2], we follow the terminology of [HTT, A.2].

In contrast, it is not strictly necessary to have prior exposure to Hochschild homology or related concepts, as all the necessary definitions will be provided.

2.2 On how this thesis is structured

To make it easy to reference parts of this thesis we make liberal use of section subdivisions and encapsulate a large part of the material in various environments such as remarks, constructions, propositions, proofs, and similar.

To mark the end of such an environment we use several different symbols, which appear on the end of the last line of the respective environment, i. e. rightmost on the page. A square $\square$ is used to denote the end of a proof, as is usual. For statements that come with a proof we use a heart $\heartsuit$, and for statements that could come with a proof (facts, conjectures, etc.) but do not we use a club $\clubsuit$. Other environments, such as definitions, constructions, etc. are ended with a diamond $\diamondsuit$. The author first saw the idea to use card suits for environment end markers in Tashi Walde's Master's thesis.

The only types of mathematical statements with proof that we distinguish in the text are corollaries (for statements whose proof is a direct specialization of previous results) and propositions (for everything else). The only exception is Theorem A, which is stated in the introduction.

2.3 Various notations and conventions

In this section we state various conventions and notation that will be used throughout the thesis.

(1) We fix a commutative ring k for the entire thesis. If X and Y are k-modules, then $X \otimes Y$ refers to the tensor product over k unless something else is explicitly stated.

(2) With regards to ∞-categories, we try to work as model independently as possible, so by an ∞-category we mean an object in the $(\infty, 2)$-category of ∞-categories $\mathcal{C}\mathrm{at}_\infty$, not a representative in a specific model, such as quasicategories[1]. In particular, if we e. g. talk about a pullback of ∞-categories, then this refers to a pullback in the ∞-category of ∞-categories, not to a (categorical) pullback of quasicategories (simplicial sets).

(3) We denote by $\mathcal{C}\mathrm{at}_\infty$ the ∞-category of ∞-categories. If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then there exists an ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$, denoted by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. We will thus also consider $\mathcal{C}\mathrm{at}_\infty$ as an $(\infty, 2)$-category, though we will not require a general theory of $(\infty, 2)$-categories.

(4) We denote by Cat the $(\infty, 2)$-category[2] of 1-categories[3], as a full subcategory of $\mathcal{C}\mathrm{at}_\infty$. We will thus not use any notation to indicate the inclusion[4] of Cat into $\mathcal{C}\mathrm{at}_\infty$; if C is a 1-category, then C is in particular an ∞-category.

(5) We use different fonts to visually distinguish between 1-categories, ∞-categories, quasicategories, and other kinds of objects. Named 1-categories (like Ring rather than C) use the same font as unnamed 1-categories, for named ∞-categories we use a different calligraphic font than for unnamed ∞-categories.

We illustrate this with the following table.

[1] For the implications for (co)cartesian fibrations see the introduction to Appendix C.
[2] By [HTT, 2.3.4.8] Cat is actually a $(2, 2)$-category.
[3] For us, 1-categories are ∞-categories with discrete mapping spaces, compare [HTT, 2.3.4.1, 2.3.4.5, and 2.3.4.18].
[4] If we model ∞-categories by quasicategories, then this inclusion is given by the nerve construction, see [HTT, 1.1.2.6].

Type of object	Font description	Examples
1-category	sans-serif	C, D, E
Named 1-category	sans-serif	Cat, $\mathsf{Ch}(k)$, Mixed, sSet
∞-category	calligraphic	$\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$
Named ∞-category	calligraphic	$\mathcal{C}\mathrm{at}_\infty$, $\mathcal{D}(k)$, $\mathcal{M}\mathrm{ixed}$, $\mathcal{S}$
Quasicategories[5]	typewriter	$\mathtt{C}$, $\mathtt{f}$, $\mathtt{p}$
Other	serif and Greek	C, Φ, α, a

(6) The following table collects notation for some named 1-categories.

Notation	Description / ∞-category of	Reference
Set	sets	
Fin	finite sets	
sSet	simplicial sets	[HTT, A.2.7]
Top	nice[6]topological spaces	[Hov99, 2.4.21]
Ab	abelian groups	
$\mathsf{Ch}(k)$	chain complexes of k-modules	Definition 4.1.1.1
PoSet	partially ordered sets	Definition 6.1.1.2
$\mathbb{Z}\mathsf{PoSet}$	partially ordered sets with $\mathbb{Z}$-action	Definition 6.1.1.2
Mixed	strict mixed complexes	Definition 4.2.1.2

(7) The following table collects notation for some named ∞-categories.

Notation	Description / ∞-category of	Reference
$\mathcal{S}$	spaces	[HTT, 1.2.16]
$\mathcal{S}\mathrm{p}$	spectra	[HA, 1.4.3]
$\mathcal{D}(k)$	derived category of k	Prop. 4.3.2.1 (1)
$\mathcal{P}\mathrm{r}$	presentable ∞-categories, as a full subcategory of $\mathcal{C}\mathrm{at}_\infty$	[HTT, 5.5.0.1]
$\mathcal{P}\mathrm{r}^{\mathrm{L}}$	presentable ∞-categories, morphisms are functors preserving all small colimits, as a subcategory of $\mathcal{P}\mathrm{r}$	[HTT, 5.5.3.1]
$\mathcal{M}\mathrm{ixed}$	mixed complexes	Notation 4.4.0.2

(8) We generally follow the notation used in [HA] for ∞-operads that we use, though with a different font to be consistent with (4) and (5).

[5] Including morphisms.

[6] It is not really relevant for us if one takes k-spaces, compactly generated topological spaces, or another variant. What is important for us is that geometric realization and the singular simplicial set functor define a Quillen equivalence as follows.

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, shift left, "|-|"] \arrow[r, phantom, "\perp"] & \mathsf{Top} \arrow[l, shift left, "\mathrm{Sing}"] \end{tikzcd}$$

Notation	Notation in [HA]	Name	Reference
Comm or Fin_*	Comm or $\mathcal{F}\mathrm{in}_*$	commutative ∞-operad	[HA, 2.1.1.18]
Assoc	$\mathcal{A}\mathrm{ssoc}$	associative ∞-operad	[HA, 4.1.1.3]
Triv	$\mathcal{T}\mathrm{riv}$	trivial ∞-operad	[HA, 2.1.1.20]
LM	$\mathcal{LM}$	∞-operad of left modules	[HA, 4.2.1.7]
$\mathbb{E}_n$	$\mathbb{E}_n$	∞-operad of little n-cubes	[HA, 5.1.0.3 and 5.1.1.6]

By [HA, 5.1.0.7] there is an equivalence of ∞-operads $\mathbb{E}_1 \simeq \mathsf{Assoc}$. We will identify these two ∞-operads along this equivalence and use $\mathbb{E}_1$ and Assoc as interchangeable notation. The ∞-operad $\mathbb{E}_\infty$ is by definition equal to Comm.

(9) We sometimes use parenthesis to cover multiple cases at the same time to avoid repetitious language. For example we might write

> X is adjective$_1$ (adjective$_2$, adjective$_3$) if it satisfies property$_1$ (property$_2$, property$_3$).

which is to be interpreted as

> X is adjective$_1$ if it satisfies property$_1$. Furthermore, X is adjective$_2$ if it satisfies property$_2$. Finally, X is adjective$_3$ if it satisfies property$_3$.

A variant version of this convention is

> X is (adverb) adjective if it satisfies property$_1$ (property$_2$).

which is to be read as follows.

> X is adjective if it satisfies property$_1$. Furthermore, X is adverb adjective if it satisfies property$_2$.

(10) If $\mathcal{C}$ is an ∞-category, then we use the notation

$$\mathrm{Map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$

for the mapping space functor. Similarly, if C is a (Ab-enriched, or $\mathrm{LMod}_k(\mathsf{Ab})$-enriched) 1-category, then we denote by $\mathrm{Mor}_{\mathsf{C}}$ (by $\mathrm{Hom}_{\mathsf{C}}$) the morphism set functor (Hom functor) with codomain Set (Ab and $\mathrm{LMod}_k(\mathsf{Ab})$, respectively). If C is an object of $\mathcal{C}$, then we use use $\mathrm{Aut}_{\mathcal{C}}(C)$ as the notation for the automorphism space of C, i. e. the subspace of $\mathrm{Map}_{\mathcal{C}}(C,C)$ spanned by equivalences $C \to C$.

(11) We use $-$ as notation for an unnamed argument in order to describe functions (and functors etc.) without introducing unnecessary notation. For example, instead of defining the function that maps a real number to its square by

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^2$$

and then using f in some place where a function $\mathbb{R} \to \mathbb{R}$ is expected, we would just use the following notation.

$$-^2$$

If there is more than one argument we may subscript $-$, such as in the following example.

$$(-_1 + -_2)^2 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Finally, we also use $\bullet$ in a similar manner for "inner" functions. For example

$$\bullet^{-} \colon \mathbb{Z}_{\geq 1} \to \mathrm{Mor}_{\mathsf{Set}}(\mathbb{R}, \mathbb{R})$$

would refer to the map that sends n to the map that sends x to x^n.

(12) Let C be a model category with class of weak equivalences W. Then we denote by $\mathrm{Ho}_W(\mathsf{C})$ the homotopy category of C in the model-category sense. If $\mathcal{C}$ is an ∞-category, then we denote by $\mathrm{Ho}(\mathcal{C})$ the homotopy category of $\mathcal{C}$ as defined in [HTT, 1.2.3]. For the relationship between these two definitions, see Proposition A.1.0.1.

(13) Let C be a model category. Then we denote by $\mathsf{C}^{\mathrm{cof}}$ (by $\mathsf{C}^{\mathrm{fib}}$) the full subcategory of cofibrant (fibrant) objects of C. The model categories we consider admit functorial (co)fibrant replacement functors, which we will denote as follows.

$$-^{\mathrm{cof}} \colon \mathsf{C} \to \mathsf{C}^{\mathrm{cof}} \qquad \text{and} \qquad -^{\mathrm{fib}} \colon \mathsf{C} \to \mathsf{C}^{\mathrm{fib}}$$

(14) Let $\mathcal{C}$ be an ∞-category admitting products. If X and Y are objects of $\mathcal{C}$ and $X \times Y$ a product object of X and Y, then we denote by $\mathrm{pr}_1 \colon X \times Y \to X$ and $\mathrm{pr}_2 \colon X \times Y \to Y$ the morphisms that exhibit $X \times Y$ as a product of X and Y.

If $f_1 \colon X \to Y_1$ and $f_2 \colon X \to Y_2$ are two morphisms in $\mathcal{C}$, then we denote by

$$f_1 \times f_2 \colon X \to Y_1 \times Y_2$$

the induced morphism determined by equivalences $\mathrm{pr}_i \circ (f_1 \times f_2) \simeq f_i$.

If $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ are two morphisms in $\mathcal{C}$, then we will also denote by

$$f_1 \times f_2 \colon X_1 \times X_2 \to Y_1 \times Y_2$$

the induced morphism between the products, which is determined by equivalences $\mathrm{pr}_i \circ (f_1 \times f_2) \simeq f_i \circ \mathrm{pr}_i$. While this could in principle lead to confusion, we will always make clear in the context which of the two interpretations are intended.

Analogous notation is used for products over more factors, possibly indexed by a set.

(15) We say that a functor of ∞-categories *detects* something[7] if it both preserves and reflects it.

(16) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and $\mathcal{E}$ another ∞-category. Then we sometimes denote by F_* the induced functor

$$\mathrm{Fun}(\mathcal{E}, F)\colon\ \mathrm{Fun}(\mathcal{E}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{D})$$

and by F^* the following induced functor.

$$\mathrm{Fun}(F, \mathcal{E})\colon\ \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

We also use this notation in variant cases, such as induced functors on subcategories of functor categories, or ∞-categories of functors over another ∞-category.

(17) Let $p\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. We will often just say that $\mathcal{O}$ is an ∞-operad, dropping the $\otimes$ superscript, or even that $F\colon \mathcal{O} \to \mathcal{O}'$ is a morphism of ∞-operads when $\mathcal{O}'$ is another ∞-operad[8]. If we are referring to $\mathcal{O}^{\otimes}$ as an ∞-category, for example talking about an object of $\mathcal{O}^{\otimes}$, then we will however never drop the superscript. To make this convention consistent, the total ∞-category of a functor to Fin_* that we think of as an ∞-operad will always be denoted by a notation that includes a superscript $\otimes$. We hope that this will not lead to confusion in practice, but will instead make many terms more concise and readable.

(18) Consistent with (17), if $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{O}''$ are ∞-operads, then we use the notation $\mathrm{BiFunc}(\mathcal{O}, \mathcal{O}'; \mathcal{O}'')$ for the ∞-category of bifunctors of ∞-operads that is denoted by $\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{O}''^{\otimes})$ in [HA] – see [HA, 2.2.5.3].

(19) If $\mathcal{O}$ and $\mathcal{C}$ are ∞-operads, then we denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ the ∞-category of ∞-operad morphism from $\mathcal{O}$ to $\mathcal{C}$ [9]. If $\mathcal{O} = \mathsf{Assoc}$ we will also write $\mathrm{Alg}(\mathcal{C})$ instead, and if $\mathcal{O} = \mathsf{Comm}$ we will also write $\mathrm{CAlg}(\mathcal{C})$.

Similarly, if $\mathcal{O} = \mathsf{Assoc}$ we will just say "monoidal" and if $\mathcal{O} = \mathsf{Comm}$ we will say "symmetric monoidal" instead of "$\mathcal{O}$-monoidal".

(20) For $n \geq 1$ an integer and $1 \leq i \leq n$ we denote by $\rho^i\colon \langle n \rangle \to \langle 1 \rangle$ the morphism of Fin_* defined in [HA, 2.0.0.2], i. e. given by the following formula.

$$\rho^i(j) := \begin{cases} 1 & \text{if } i = j \\ * & \text{otherwise} \end{cases}$$

[7]For example equivalences or colimits.

[8]Where we of course already use this convention, so implicitly we introduced a functor $\mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$ exhibiting $\mathcal{O}'^{\otimes}$ as an ∞-operad, and F is actually to be a functor $\mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ over Fin_*.

[9]See [HA, 2.1.2.7] We will also use the related notation introduced in [HA, 2.1.3.1].

(21) Let $\mathcal{O}$ be an ∞-operad. Then we use $\oplus$ as notation for the operation defined and discussed in [HA, 2.1.1.15 and 2.2.4.6]. In particular, if X_i is an object in $\mathcal{O}$ for $1 \leq i \leq n$, then $X = X_1 \oplus \cdots \oplus X_n$ will be an object in $\mathcal{O}^{\otimes}_{\langle n \rangle}$, coming with inert morphisms $X \to X_i$ in $\mathcal{O}^{\otimes}$ lying over ρ^i, or equivalently equivalences $\rho^i_!(X) \simeq X_i$.

If we introduce an object $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ as $X \simeq X_1 \oplus \cdots \oplus X_n$ for X_i objects of $\mathcal{O}$, then we implicitly assume that X comes with inert morphisms $X \to X_i$ lying over ρ^i.

(22) If $p\colon \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration and $f\colon X \to Y$ a morphism in $\mathcal{D}$, then we usually denote the induced morphism on fibers[10] (see [HTT, 5.2.1]) by $f_!\colon \mathcal{C}_X \to \mathcal{C}_Y$ if the cocartesian fibration p is clear from context, and otherwise as $f^p_!$.

(23) Let $\mathcal{C}$ be an ∞-category. A *subcategory* of $\mathcal{C}$ is an ∞-category $\mathcal{C}'$ together with a monomorphism[11] $\iota\colon \mathcal{C}' \to \mathcal{C}$ in $\mathfrak{Cat}_\infty$. Up to equivalence a subcategory of $\mathcal{C}$ is given by specifying a replete subcategory of $\operatorname{Ho}\mathcal{C}$, see Section B.6.

(24) Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be ∞-categories. Then we denote by

$$\widehat{-}\colon \operatorname{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{C}, \operatorname{Fun}(\mathcal{D}, \mathcal{E}))$$

and

$$\check{-}\colon \operatorname{Fun}(\mathcal{C}, \operatorname{Fun}(\mathcal{D}, \mathcal{E})) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

the equivalences arising from the $\times$-Fun-adjunction[12]. We will use the same notation for the equivalences

$$\widehat{-}\colon \operatorname{Fun}(\mathcal{D} \times \mathcal{C}, \mathcal{E}) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{C}, \operatorname{Fun}(\mathcal{D}, \mathcal{E}))$$

and

$$\check{-}\colon \operatorname{Fun}(\mathcal{C}, \operatorname{Fun}(\mathcal{D}, \mathcal{E})) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{D} \times \mathcal{C}, \mathcal{E})$$

and will make clear from context which of the two variants is meant.

(25) Let $\mathcal{C}$ be an ∞-category. We denote by $\mathfrak{CFib}(\mathcal{C})$ the subcategory of $(\mathfrak{Cat}_\infty)_{/\mathcal{C}}$ spanned by the cartesian fibrations and morphisms of cartesian fibrations[13]. Similarly, we denote by $\operatorname{co}\mathfrak{CFib}(\mathcal{C})$ the subcategory of $(\mathfrak{Cat}_\infty)_{/\mathcal{C}}$ spanned by the cocartesian fibrations and morphisms of cocartesian fibrations.

[10] The notation $\mathcal{C}_X$ refers to the fiber of p over X, i. e. to the pullback object $\{X\} \times_{\mathcal{D}} \mathcal{D}$ of p along the inclusion of $\{X\}$.
[11] See Appendix B for more on monomorphisms in $\mathfrak{Cat}_\infty$.
[12] So if $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a functor, then $\widehat{F}\colon \mathcal{C} \to \operatorname{Fun}(\mathcal{D}, \mathcal{E})$ is its adjoint.
[13] See Appendix C for more on (co)cartesian fibrations.

(26) Let $\mathcal{C}$ be an ∞-category. We denote by

$$\mathrm{Gr}\colon \operatorname{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) \to \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{C})$$

the Grothendieck construction that maps a functor $F\colon \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ to the cocartesian fibration classified by F.

(27) Let S be a set. Then an S-graded k-module is an S-tuple of k-modules, or equivalently a functor $S \to \mathrm{LMod}_k(\mathsf{Ab})$ from the discrete category with set of objects S to the category of k-modules.

If $(X_s)_{s\in S}$ is an S-graded k-module, then we can form a k-module $X \coloneqq \bigoplus_{s\in S} X_s$, but one should not confuse the k-module X with the S-graded k-module $(X_s)_{s\in S}$, for example in the context of (28) directly below.

(28) The category of $\mathbb{Z}$-graded k-modules carries a symmetric monoidal structure defined just like for chain complexes, in which the symmetry isomorphism contains signs – see Definition 4.1.2.1. Commutative algebras in this symmetric monoidal category will then of course involve signs in their commutativity relations, so if x and y are elements of a commutative $\mathbb{Z}$-graded k-algebra A of degrees n and m, then this implies that $x \cdot y = (-1)^{nm} y \cdot x$. In some places in the literature this is referred to as "graded commutativity". However, as the mentioned symmetric monoidal structure on $\mathbb{Z}$-graded k-modules is the only one we define, there is no other, "non-graded commutativity" one could consider, so we do not use this terminology.

(29) Let M be a $\mathbb{Z}$-graded k-module that is concentrated in odd degrees. Then the tensor algebra $T(M)$ (or $T_k(M)$ if we want to make k explicit) of M is defined as

$$T(M) \coloneqq \bigoplus_{i\geq 0} M^{\otimes i}$$

where the tensor product of $\mathbb{Z}$-graded k-modules is as in (28). One can define a multiplication on $T(M)$ by k-linearly extending the formula

$$(m_1 \otimes \cdots \otimes m_i) \cdot (m'_1 \otimes \cdots \otimes m'_j) \coloneqq m_1 \otimes \cdots \otimes m_i \otimes m'_1 \otimes \cdots \otimes m'_j$$

for $i, j \geq 0$ and $m_1, \ldots, m_i, m'_1, \ldots, m'_j$ elements of M. This makes $T(M)$ into a $\mathbb{Z}$-graded k-algebra, with unit given by the element 1 of $k = M^{\otimes 0}$.

We define the *exterior $\mathbb{Z}$-graded k-algebra generated by M*, denoted by $\Lambda(M)$ or $\Lambda_k(M)$, to be the quotient of $T(M)$ by the two-sided ideal generated by elements of the form $m \cdot m$ for $m \in M$[14].

[14]This definition differs from the one given in [Lod98, A.1] if 2 is not invertible in k. In those cases the usage of the definition of [Lod98, A.1] is however incorrect with regards to the results we cite from [Lod98] relating to the mixed complex of de Rham forms – the definition we give here is the correct one. In particular, the proof of [Lod98, 3.2.2] implicitly assumes the definition we have given here.

The composition of the inclusion[15] of M into $T(M)$ with the quotient morphism to $\Lambda(M)$ is an injection, so that we can consider M as a sub-$\mathbb{Z}$-graded-k-module of $\Lambda(M)$, and elements of M generate $\Lambda(M)$ multiplicatively. For m and m' elements of M it holds in $\Lambda(M)$ that

$$m \cdot m' = (m + m') \cdot (m + m') - m' \cdot m - m \cdot m - m' \cdot m' = -m' \cdot m$$

so that $\Lambda(M)$ is in fact a *commutative* $\mathbb{Z}$-graded k-algebra.

Finally, let us note that we will also use the notation $\Lambda(x_1, \ldots, x_n)$ as a shorthand for $\Lambda(k \cdot \{x_1, \ldots, x_n\})$.

(30) For an even integer n we define a commutative $\mathbb{Z}$-graded k-algebra $\Gamma(x)$, called the *divided power $\mathbb{Z}$-graded k-algebra* generated by the variable x in degree n as follows.

The underlying $\mathbb{Z}$-graded k-module is given by

$$\Gamma(x) := k \cdot \{1, x^{[1]}, x^{[2]}, \ldots\}$$

with $x^{[i]}$ of degree $i \cdot n$, where we let $x^{[0]} = 1$. A multiplication on $\Gamma(x)$ is defined by k-linearly extending the formula

$$x^{[i]} \cdot x^{[j]} := \binom{i+j}{i} x^{[i+j]}$$

for $i, j \geq 0$, which makes $\Gamma(x)$ into a commutative $\mathbb{Z}$-graded k-algebra with multiplicative unit 1.

We furthermore define

$$\Gamma(x_1, \ldots, x_n) := \Gamma(x_1) \otimes \cdots \otimes \Gamma(x_n)$$

for all x_i of even degree.

(31) Elements for $\mathbb{Z}^n_{\geq 0}$ are tuples of nonnegative integers $(a_1, \ldots, a_n)$. We will often write such a tuple as $\vec{a}$, and use $\vec{e_i}$ as notation for the tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the single 1 is in the i-th slot. For $\vec{\epsilon} \in \{0, 1\}^n$ we furthermore make the following definition.

$$|\vec{\epsilon}| = \sum_{i=1}^{n} \epsilon_i$$

We use analogous notation for tuples indexed by a set set other than $\{1, \ldots, n\}$ for a natural number n.

[15] This refers to the inclusion of M as the summand $M^{\otimes 1}$.

(32) For $\vec{a} \in \mathbb{Z}^n_{\geq 0}$ we will write $x^{\vec{a}}$ for the monomial $x_1^{a_1} \cdots x_n^{a_n}$ in the polynomial algebra $k[x_1, \ldots, x_n]$. Vectors in $\mathbb{Z}^n_{\geq 0}$ are added pointwise, and we have e. g. $x^{\vec{a}+\vec{b}} = x^{\vec{a}} \cdot x^{\vec{b}}$.

We use analogous notation for exterior and divided power algebras. Concretely, we will for $\vec{\epsilon} \in \{0,1\}^n$ use the notation $\mathrm{d}\, x^{\vec{\epsilon}}$ to refer to

$$\mathrm{d}\, x^{\vec{\epsilon}} \coloneqq \mathrm{d}\, x_1^{\epsilon_1} \cdots \mathrm{d}\, x_n^{\epsilon_n}$$

and not to $\mathrm{d}(x^{\vec{\epsilon}})$. One can remember this as the convention that d binds stronger than exponentiation with a vector.

Similarly, for $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ we define

$$x^{[\vec{i}]} \coloneqq x_1^{[i_1]} \cdots x_n^{[i_n]}$$

in the divided power algebra $\Gamma(x_1, \ldots, x_n)$.

(33) If $f \in k[x_1, \ldots, x_n]$ is a polynomial and $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ a vector, then we let $f_{\vec{i}} \in k$ be the coefficient of the monomial $x^{\vec{i}}$ in f, i. e. the unique decomposition of f as a k-linear combination of monomials is as follows.

$$f = \sum_{\vec{i} \in \mathbb{Z}^n_{\geq 0}} f_{\vec{i}} x^{\vec{i}}$$

(34) If $n \geq 0$ is an integer, then we denote by Σ_n the *symmetric group on n elements*; it is the group of bijections of the set $\{1, \ldots, n\}$, also called permutations of $\{1, \ldots, n\}$. It will sometimes be convenient to extend an element σ of Σ_n to a bijection of $\{0, \ldots, n\}$ by setting $\sigma(0) = 0$, which we will do implicitly. If $n' > n$, then there exists an inclusion of Σ_n into $\Sigma_{n'}$ given by extending an element σ of Σ_n by $\sigma(i) = i$ for $n < i \leq n'$. We also usually not distinguish in notation between σ as an element of Σ_n and its extension as an element of $\Sigma_{n'}$.

Given a permutation σ on n elements and a subset S of $\{1, \ldots, n\}$, we say that σ *preserves the ordering of S* if for every pair of elements $i < i'$ in S it holds that $\sigma(i) < \sigma(i')$. We also use this terminology for other injective maps between totally ordered sets. Let $1 \leq i, j \leq n$. Then there is a unique element of Σ_n that maps i to j and preserves the ordering of $\{1, \ldots, i-1, i+1, \ldots, n\}$. We will call this element $\sigma_{i \to j}$. Note that if $n' > n$, then the extension of $\sigma_{i \to j}$ to a permutation of n' elements is again of the same form, which justifies that n is not part of the notation.

We define $\sigma_{\mathrm{cyc},n}$ to be the element $\sigma_{n \to 1}$ of Σ_n. If n is clear from context we will also denote $\sigma_{\mathrm{cyc},n}$ by σ_{cyc}. We denote by C_n the subgroup of Σ_n generated by $\sigma_{\mathrm{cyc},n}$.

We also need a manner of restricting permutations. Let σ be an element of Σ_n, and S a subset of $\{1,\dots,n\}$. Denote the set $\sigma(S)$ by S'. Then there are unique order-preserving bijections $\phi\colon \{1,\dots,|S|\} \to S$ and $\psi\colon S' \to \{1,\dots,|S|\}$. We define $r_S(\sigma)$ to be the element of $\Sigma_{|S|}$ that is given by the composition $\psi \circ \sigma\big|_S^{S'} \circ \phi$. This defines a map of sets $r_S\colon \Sigma_n \to \Sigma_{|S|}$. Note that in the above situation we have that if σ' is another element of Σ_n, then $r_S(\sigma' \circ \sigma) = r_{S'}(\sigma') \circ r_S(\sigma)$.

We can also add permutations as follows. Let $n, n' \geq 0$. Then there is a group homomorphism $- \amalg -\colon \Sigma_n \times \Sigma_{n'} \to \Sigma_{n+n'}$ given as follows. If σ is an element of Σ_n and σ' an element of $\Sigma_{n'}$, then we define $\sigma \amalg \sigma'$ as follows.

$$(\sigma \amalg \sigma')(i) := \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq n \\ \sigma'(i-n)+n & \text{if } n+1 \leq i \leq n+n' \end{cases}$$

Note that $r_{\{1,\dots,n\}} \circ (-\amalg-)$ and $r_{\{n+1,\dots,n+n'\}} \circ (-\amalg-)$ are the projection to the first and second factor, respectively.

Given a permutation σ on n elements and a subset S of $\{1,\dots,n\}$, we say that σ *cyclically preserves the ordering of* S if $r_S(\sigma)$ is an element of $C_{|S|}$. This terminology can easily be extended to more general maps. Let $f\colon X \to Y$ be an injective map between any finite totally ordered sets X and Y, and S a subset of X. Then there exist unique order-preserving bijections $\phi\colon \{1,\dots,|X|\} \to X$ and $\psi\colon \operatorname{Im}(f) \to \{1,\dots,|X|\}$, making $\sigma := \psi \circ f|^{\operatorname{Im}(f)} \circ \phi$ into an element of $\Sigma_{|X|}$. We say that f (cyclically) preserves the ordering of the subset S if σ (cyclically) preserves the ordering of the subset $\phi^{-1}(S)$.

(35) Formulations such as "$\mathcal{C}$ admits all colimits" mean that $\mathcal{C}$ admits all *small* colimits. We never refer to non-small (co)limits with generic formulations. See also Section 2.4 directly below.

2.4 Size issues

In Section 2.3 (4) we defined Cat as the 1-category[16] of all 1-categories. Taken directly as stated Cat would be an object of itself and we would run into the usual set-theoretic paradoxes, so we need to be more careful in defining Cat.

The usual way to deal with this issue is to postulate the existence of Grothendieck universes $\mathcal{U}_1 \in \mathcal{U}_2 \in \mathcal{U}_3$ (and possibly more if required), which are sets whose elements satisfy the usual axioms of set theory. Sets that are

[16] We defined Cat as a $(2,2)$-category, but to make our exposition here easier we only consider the underlying 1-category.

elements of $\mathcal{U}_i$ are called $\mathcal{U}_i$-*small*. We can then perform all the usual operations of set theory with $\mathcal{U}_i$-small sets, but now there exists e. g. a $\mathcal{U}_2$-small set of $\mathcal{U}_1$-small sets (namely $\mathcal{U}_1$).

For $i \geq j$ we could (this is ad hoc notation) define an (i, j)-small 1-category to be a 1-category C whose set of objects is $\mathcal{U}_i$-small and for which $\mathrm{Mor}_{\mathsf{C}}(X, Y)$ is $\mathcal{U}_j$-small for all objects X and Y of C. Let us use $\mathsf{Cat}^{i,j}$ as ad hoc notation for the 1-category of (i, j)-small 1-categories. What we usually consider as 1-categories are $(2, 1)$-small 1-categories, which then form the 1-category $\mathsf{Cat}^{2,1}$, which will however *not* be $(2, 1)$-small itself, though it is $(3, 2)$-small. For a more detailed discussion of Grothendieck universes and size issues in an ∞-categorical context, see [HTT, 1.2.15].

In this thesis we will very often use gadgets such as Cat or $\mathcal{C}\mathrm{at}_\infty$. To be completely rigorous we should thus always keep track of with respect to which universe the various objects we consider are small. In most of the thesis this would however cause significant notational bloat while being completely orthogonal to the rest of the content, so to make the exposition more accessible we will instead stay silent on size issues, while of course still taking care not to use inadmissible arguments. There will be one part of the thesis, Chapter 7, where a size issue is somewhat relevant for the argument, and there we will deal with this issue in an explicit manner.

In particular, we will not decorate $\mathcal{C}\mathrm{at}_\infty$ to keep track of sizes, and might e. g. define an ∞-category as a pullback in $\mathcal{C}\mathrm{at}_\infty$ of a diagram that involves the ∞-category $\mathcal{C}\mathrm{at}_\infty$. While in this notation it would then seem as though the two occurrences of $\mathcal{C}\mathrm{at}_\infty$ refer to the same gadget, a diligent adding of size decorations would distinguish them, and we will be careful not to make any arguments in which is not possible to do so consistently.

Chapter 3

Bialgebras and modules over them

Let C be a symmetric monoidal category and A an associative algebra in C. A left module in C over A consists of an object X in C together with a morphism $A \otimes X \to X$ satisfying some properties. If A is commutative, then any left-A-module can naturally be made into a A, A-bimodule, so that we can use the relative tensor product over A to define a monoidal structure on the category of left-A-modules $\mathrm{LMod}_A(\mathsf{C})$.

Now let A be a associative, coassociative bialgebra. Then there is also a way to define a tensor product on $\mathrm{LMod}_A(\mathsf{C})$, and in such a way that the underlying object in C of the tensor product of two left-A-modules X and Y is just given by the tensor product of the two underlying objects. To do this, we need to define an action morphism $A \otimes (X \otimes Y) \to X \otimes Y$, which we do as the composition

$$A \otimes (X \otimes Y) \xrightarrow{\Delta \otimes \mathrm{id}_{X \otimes Y}} (A \otimes A) \otimes (X \otimes Y) \cong (A \otimes X) \otimes (A \otimes Y) \to X \otimes Y$$

where Δ is the comultiplication on A, the middle isomorphism uses associativity and symmetry of the tensor product to swap the two middle tensor factors, and the last morphism is the tensor product of the action morphisms for X and Y. One can then check, that this makes $X \otimes Y$ into a left-A-module.

It is not only possible to construct the monoidal category $\mathrm{LMod}_A(\mathsf{C})$ for individual bialgebras A – this construction enjoys functoriality in both A and C: If $f\colon A \to B$ is a morphism of bialgebras in C, then there is a monoidal functor

$$\mathrm{LMod}_B(\mathsf{C}) \to \mathrm{LMod}_A(\mathsf{C})$$

that preserves the underlying object but restricts the action along f. If A is a bialgebra in C and $F\colon \mathsf{C} \to \mathsf{D}$ is a symmetric monoidal functor, then F induces a monoidal functor

$$\mathrm{LMod}_A(\mathsf{C}) \to \mathrm{LMod}_{F(A)}(\mathsf{D})$$

that sends a left-A-module with underlying object X to a left-$F(A)$-module with underlying object $F(X)$.

To encode this functoriality we can define a category $\mathsf{BiAlgOp}$ as follows. Objects are pairs (C, A) with C a symmetric monoidal category and A an associative and coassociative bialgebra in C. Morphisms from (C, A) to (D, B) are pairs (F, f), where $F\colon \mathsf{C} \to \mathsf{D}$ is a symmetric monoidal functor, and $f\colon B \to F(A)$ is a morphism of bialgebras in D. We can then upgrade the construction of $\mathrm{LMod}_A(\mathsf{C})$ to a functor

$$\mathrm{LMod}\colon \mathsf{BiAlgOp} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathsf{Cat})$$

where $\mathrm{Mon}_{\mathsf{Assoc}}(\mathsf{Cat})$ is the category of monoidal categories.

The goal of this section is to implement this idea for ∞-categories rather than just ordinary categories. In this setting, we want to construct an ∞-category $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ whose objects can be described as pairs $(\mathcal{C}, A)$, where $\mathcal{C}$ is an $\mathbb{E}_2$-monoidal ∞-category and A an $\mathbb{E}_1, \mathbb{E}_1$-bialgebra in $\mathcal{C}$. We then want to upgrade LMod to a functor

$$\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$$

that can be interpreted as functorially upgrading left module categories over $\mathbb{E}_1$ algebras to $\mathbb{E}_1$-monoidal ∞-categories in the way described above.

We now briefly describe our approach to constructing $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. Instead of trying to construct $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ directly, we will first construct an ∞-category $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ that can be described as having as objects pairs $(\mathcal{C}, A)$ where $\mathcal{C}$ is a $\mathbb{E}_1$-monoidal infinity category and A is an $\mathbb{E}_1$-algebra in $\mathcal{C}$, and where a morphism from $(\mathcal{C}, A)$ to $(\mathcal{D}, B)$ is given by a pair (F, f) with $F\colon \mathcal{C} \to \mathcal{D}$ an $\mathbb{E}_1$-monoidal functor and $f\colon B \to F(A)$ a morphism in $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D})$. The ∞-category $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ will turn out to have products, with the product of $(\mathcal{C}, A)$ and $(\mathcal{D}, B)$ given by $(\mathcal{C} \times \mathcal{D}, (A, B))$. We can thus consider monoids in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. A monoid in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ roughly consists of an object $(\mathcal{C}, A)$ in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ together with a coherently associative multiplication morphism $(\mathcal{C}, A) \times (\mathcal{C}, A) \to (\mathcal{C}, A)$. Such a morphism corresponds to an $\mathbb{E}_1$-monoidal functor $F\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a morphism $f\colon A \to F(A, A)$ in $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$. By the Eckmann-Hilton argument, $F(A, A)$ is equivalent to $A \otimes A$, so that we can identify f with a morphism $A \to A \otimes A$, which we can interpret as being the comultiplication of a coalgebra structure on A. We will later show that $\mathrm{Mon}_{\mathbb{E}_1}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p})$ indeed implements the discussed idea of what $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ should be.

Finally, the functor $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ sending a pair $(\mathcal{C}, A)$ to the ∞-category $\mathrm{LMod}_A(\mathcal{C})$ is product-preserving, so that we obtain an induced functor $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{C}\mathrm{at}_\infty)$.

Our approach is heavily inspired by [HA, 4.8]. The goal in [HA, 4.8.3] is to functorially encode the fact that the ∞-category of left-A-modules[1] $\mathrm{LMod}_A(\mathcal{C})$ can be upgraded to an ∞-category that is right-tensored over $\mathcal{C}$. The functoriality encoded is however not the same as the one we discussed

[1] Lurie actually considers right modules, but to keep our exposition consistent we will discuss Lurie's results in the analogous form for left modules.

above: Lurie's construction maps a morphism $A \to B$ of algebras in $\mathcal{C}$ to the functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_B(\mathcal{C})$ that sends a left-A-module X to the left-B-module $B \otimes_A X$. The functor Lurie constructs preserves products as well [HA, 4.8.5.16] and so induces a functor on $\mathbb{E}_1$-monoids. However, due to the covariant functoriality in algebras, this induced functor describes the $\mathbb{E}_1$-monoidal structure induced on $\mathrm{LMod}_A(\mathcal{C})$ by an $\mathbb{E}_2$-algebra A using the relative tensor product over A (see the discussion at the start of this section). Because of this, we will mostly follow the ideas in [HA, 4.8.3 and 4.8.5], making the changes that are needed to make the construction contravariant in algebras.

During preparation of this text, the preprint [Rak20] appeared, in which existence of constructions similar to the ones we discuss below is also claimed in analogy to Lurie's construction, though without proof, see [Rak20, 2.2 and in particular 2.2.6].

We now give a brief overview of the sections below. In Section 3.1 we will construct $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ as well as the functor $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$. We will also discuss how LMod interacts with presentability. For this we will construct a variant $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ whose objects can be interpreted as pairs $(\mathcal{C}, A)$ with $\mathcal{C}$ a presentable monoidal ∞-category and A an algebra in $\mathcal{C}$, and show that LMod lifts to a functor $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$.

In Section 3.2 we will show that LMod is product-preserving as a functor from $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ to $\mathcal{C}\mathrm{at}_\infty$ and hence induces a symmetric monoidal functor with respect to the respective cartesian symmetric monoidal structures. We will also construct an appropriate symmetric monoidal structure on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ and show that the functor $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ can be upgraded to a symmetric monoidal functor as well.

Bialgebras will be defined in Section 3.3, and in Section 3.4 we will then discuss how LMod induces functors $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ as well as the variant functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}) \to \mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)$, where $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)$ is the ∞-category of presentable $\mathcal{O}$-monoidal ∞-categories. We will furthermore make precise how we can interpret objects of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p})$ as pairs $(\mathcal{C}, A)$, where $\mathcal{C}$ is an $\mathcal{O} \otimes \mathsf{Assoc}$-monoidal ∞-category, and A is an $\mathsf{Assoc}, \mathcal{O}$-bialgebra in $\mathcal{C}$.

3.1 Modules over algebras

In this section we will construct a functor $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ implementing the idea described in the introduction to Chapter 3. To do so we first need to construct the ∞-category $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$, which is to have as objects pairs $(\mathcal{C}, A)$ with $\mathcal{C}$ a monoidal ∞-category and A an associative algebra in $\mathcal{C}$. We can thus interpret $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ as a sort of ∞-category of algebras not only in a single monoidal ∞-category, but a whole collection of them – in this case all of them. The notion that encapsulates the idea of a collection of monoidal ∞-categories is that of a *cocartesian family of monoidal ∞-categories*, which we will define in Section 3.1.1. The process of forming algebras in cocartesian families of monoidal ∞-categories is then defined and studied in Section 3.1.2, and everything is put together to construct $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ and LMod in Section 3.1.3.

3.1.1 Cocartesian families of monoidal ∞-categories

In this section we discuss the notion *cocartesian families of $\mathcal{O}$-monoidal ∞-categories* for ∞-operads $\mathcal{O}$. We start in Section 3.1.1.1 with the definition. In Section 3.1.1.2 we discuss an important example: The *universal* cocartesian family of $\mathcal{O}$-monoidal ∞-categories, which can be though of as the collection of *all* $\mathcal{O}$-monoidal ∞-categories. In particular, this will be the example that we will use to define $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ and LMod as discussed in the introduction to Chapter 3. We end the section with Section 3.1.1.3, in which we discuss the interaction between cocartesian families and products. This will be relevant later, when we want to argue that the functor to be defined $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ is compatible with products.

3.1.1.1 Definition

As we want to form ∞-categories like $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ in which objects are algebras not just in a single monoidal ∞-category, but in a whole collection of monoidal ∞-categories, we first need a definition that encapsulates the idea of combining a collection of monoidal ∞-categories into a single mathematical object.

If $\mathcal{O}$ is an ∞-operad, then by [HA, 2.4.2.4] a cocartesian fibration over $\mathcal{O}^\otimes$ is an $\mathcal{O}$-monoidal ∞-category if and only if the associated functor $\mathcal{O}^\otimes \to \mathcal{C}\mathrm{at}_\infty$ is an $\mathcal{O}$-monoid. We can thus consider a functor

$$F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$

for some ∞-category $\mathcal{C}$ as parametrizing a collection of $\mathcal{O}$-monoidal ∞-categories by $\mathcal{C}$. Composing with the inclusion of $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ into the functor category $\mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{C}\mathrm{at}_\infty)$, we obtain a functor

$$F'\colon \mathcal{C} \to \mathrm{Fun}\big(\mathcal{O}^\otimes, \mathcal{C}\mathrm{at}_\infty\big)$$

of which we can take the adjoint $\breve{F'}\colon \mathcal{O}^\otimes \times \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$. By passing to the cocartesian fibration classified by the functor $\breve{F'}$ we then obtain a cocartesian fibration $p\colon \mathcal{D}^\otimes \to \mathcal{O}^\otimes \times \mathcal{C}$. This cocartesian fibration will have extra properties that correspond to F' factoring over $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$. This leads us to the following proposition and definition.

Proposition 3.1.1.1. *Let $\mathcal{C}$ be an ∞-category, $\mathcal{O}$ an ∞-operad, and*

$$p\colon \mathcal{D}^\otimes \to \mathcal{O}^\otimes \times \mathcal{C}$$

a cocartesian fibration. Then the following are equivalent.

(1) The functor $F\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty})$ *that corresponds to* p *under the equivalence*

$$\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{O}^{\otimes} \times \mathcal{C}) \xleftarrow{\mathrm{Gr}} \mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{C}, \mathcal{C}\mathrm{at}_{\infty}) \xleftarrow{\breve{(-)}} \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}))$$

factors through $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$.

(2) For every object X *of* $\mathcal{C}$ *the restriction* $p_X : \mathcal{D}^{\otimes}_X \to \mathcal{O}^{\otimes}$ *is a cocartesian fibration of* ∞*-operads*[2]*.* ♡

Proof. Let $G \coloneqq \mathrm{Gr}^{-1}(p)$, let F be as in (1), and let X be an object of $\mathcal{C}$. Naturality of the Grothendieck construction[3] (see [GHN17, A.32]) implies that the cocartesian fibration p_X is classified by the restriction of G to $\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \times \{X\}$. [HA, 2.4.2.4] implies that p_X is a cocartesian fibration of ∞-operads if and only if this restriction is an $\mathcal{O}$-monoid. Using naturality of $\breve{(-)}$ we can reformulate this as follows: The cocartesian fibration p_X is a cocartesian fibration of ∞-operads if and only if $F(X)$ is an $\mathcal{O}$-monoid. As $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$ is defined as the full subcategory of $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty})$ of $\mathcal{O}$-monoids, this finishes the proof. □

Definition 3.1.1.2 ([HA, Definition 4.8.3.1])**.** Let $\mathcal{C}$ be an ∞-category and $\mathcal{O}$ an ∞-operad. A *cocartesian* $\mathcal{C}$*-family of* $\mathcal{O}$*-monoidal* ∞*-categories* is a cocartesian fibration $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ satisfying the conditions in Proposition 3.1.1.1.

We let $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{am}_{\mathcal{O}}(\mathcal{C})$ be the full subcategory of $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{O}^{\otimes} \times \mathcal{C})$ spanned by cocartesian $\mathcal{C}$-families of $\mathcal{O}$-monoidal ∞-categories. ♢

Remark 3.1.1.3 ([HA, 4.8.3.3])**.** Let $\mathcal{C}$ be an ∞-category and $\mathcal{O}$ an ∞-operad. Let ι be the inclusion of $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$ into $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty})$.

Then the equivalences Gr and $\breve{(-)}$ as in Proposition 3.1.1.1 restrict as in the following commutative diagram where the right vertical functor is the inclusion, and such that all horizontal functors are equivalences.

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty})) \arrow[r, "\breve{(-)}", "\simeq"'] & \mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{C}, \mathcal{C}\mathrm{at}_{\infty}) \arrow[r, "\mathrm{Gr}", "\simeq"'] & \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{O}^{\otimes} \times \mathcal{C}) \\
\mathrm{Fun}(\mathcal{C}, \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})) \arrow[u, "\iota_*"] \arrow[rr, "\simeq"'] & & \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{am}_{\mathcal{O}}(\mathcal{C}) \arrow[u]
\end{tikzcd}$$

Note that $\mathrm{Fun}(\mathcal{C}, \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}))$ is contravariantly functorial in $\mathcal{C}$ and[4] $\mathcal{O}$, so the construction of $\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{am}_{\mathcal{O}}(\mathcal{C})$ must be as well. Using naturality of $\breve{(-)}$

[2]See [HA, 2.1.2.13] for a definition

[3]Precomposing a functor into $\mathcal{C}\mathrm{at}_{\infty}$ by some functor ι corresponds to taking the base change along ι of the corresponding cocartesian fibration.

[4]If $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ is a morphism of ∞-operads, then it follows directly from the definition that the functor $\alpha^* \colon \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}) \to \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}_{\infty})$ restricts to a functor on monoids.

and Gr (see [GHN17, A.32] and [Maz19]) we can describe this functoriality explicitly as follows.

Let $G\colon \mathcal{C}' \to \mathcal{C}$ be a functor of ∞-categories, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ a functor corresponding under the above equivalence to a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories p. Then the composite functor

$$\mathcal{C}' \xrightarrow{G} \mathcal{C} \xrightarrow{F} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \xrightarrow{\alpha^*} \mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}\mathrm{at}_\infty)$$

corresponds under the above functor to the pullback p' of p along $\alpha \times G$, as in the following diagram.

$$\begin{tikzcd} \mathcal{D}'^{\otimes} \arrow[r] \arrow[d, "p'"'] & \mathcal{D}^{\otimes} \arrow[d, "p"] \\ \mathcal{O}'^{\otimes} \times \mathcal{C}' \arrow[r, "\alpha\times G"'] & \mathcal{O}^{\otimes} \times \mathcal{C} \end{tikzcd} \tag{3.1}$$

In particular, the pullback of a cocartesian family of monoidal ∞-categories along a functor of the form $\alpha \times G$ is again a cocartesian family of monoidal ∞-categories. ♢

3.1.1.2 The universal family

In this section we discuss the *universal* cocartesian family of $\mathcal{O}$-monoidal ∞-categories, from which we can obtain every other cocartesian family of $\mathcal{O}$-monoidal ∞-categories by pulling back. This will also be the main example that we will apply later constructions to.

Definition 3.1.1.4 ([HA, 4.8.3.3]). Let $\mathcal{O}$ be an ∞-operad.

We define
$$p^{\mathcal{O}}\colon \widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$
to be the cocartesian $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$-family of $\mathcal{O}$-monoidal ∞-categories that under the equivalence in Remark 3.1.1.3 corresponds to the identity functor $\mathrm{id}_{\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)}$. ♢

Remark 3.1.1.5 ([HA, 4.8.3.3]). Let $\mathcal{O}$ be an ∞-operad, let $\mathcal{C}$ be an ∞-category, and let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. Let $F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ be the functor corresponding to p under the equivalence in Remark 3.1.1.3. Then F factors as $F \simeq \mathrm{id}_{\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)} \circ F$, so by Remark 3.1.1.3 we can conclude that there is a pullback diagram as follows.

$$\begin{tikzcd} \mathcal{D}^{\otimes} \arrow[r] \arrow[d, "p"'] & \widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[d, "p^{\mathcal{O}}"] \\ \mathcal{O}^{\otimes} \times \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{O}^{\otimes}}\times F"'] & \mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \end{tikzcd}$$

♢

3.1.1.3 Compatibility of fibers with products

The property described in the following proposition and definition regarding a cocartesian family of monoidal ∞-categories' interaction with products will be needed later.

Proposition 3.1.1.6. *Let $\mathcal{C}$ be an ∞-category, $\mathcal{O}$ an ∞-operad,*

$$p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$$

a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories, and

$$F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$$

the functor corresponding to p as in Proposition 3.1.1.1. Assume that $\mathcal{C}$ admits all products. Then the following are equivalent.

(1) F preserves products.

(2) For every object O in $\mathcal{O}^{\otimes}$ the cocartesian fibration

$$p_O\colon \mathcal{D}^{\otimes}_O \coloneqq \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}\times\mathcal{C}} (\{O\} \times \mathcal{C}) \xrightarrow{\mathrm{pr}_2} \{O\} \times \mathcal{C} \xrightarrow{\simeq} \mathcal{C}$$

has fibers compatible with products in the sense of Definition C.2.0.1.

(3) For every object O in $\mathcal{O}$ the cocartesian fibration

$$p_O\colon \mathcal{D}^{\otimes}_O \coloneqq \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}\times\mathcal{C}} (\{O\} \times \mathcal{C}) \xrightarrow{\mathrm{pr}_2} \{O\} \times \mathcal{C} \xrightarrow{\simeq} \mathcal{C}$$

has fibers compatible with products in the sense of Definition C.2.0.1. ♡

Proof. Proof that (1) implies (2): Let ι denote the inclusion of the full subcategory $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$ into $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty})$, which preserves products by Proposition F.2.0.1. Let O be an object in $\mathcal{O}^{\otimes}$. As limits in in functor categories are computed pointwise by [HTT, 5.1.2.3], the evaluation functor $\mathrm{ev}_O\colon \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}) \to \mathfrak{Cat}_{\infty}$ preserves products as well, and thus the composite $\mathrm{ev}_O \circ \iota \circ F$ preserves products. By using naturality of the Grothendieck construction and $\widetilde{(-)}$ we can conclude that the cocartesian fibration p_O is classified by $\mathrm{ev}_O \circ \iota \circ F$, and hence p_O having fibers compatible with products follows from Remark C.2.0.2.

Proof that (2) implies (3): Clear.

Proof that (3) implies (1): Using notation from above, that p_O has fibers compatible with products for every object O in $\mathcal{O}$ implies by Remark C.2.0.2 that $\mathrm{ev}_O \circ \iota \circ F$ preserves products for every O in $\mathcal{O}$. Combining that products in functor categories are detected pointwise and that the composition

$$\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty}) \xrightarrow{\iota} \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}) \to \mathrm{Fun}(\mathcal{O}, \mathfrak{Cat}_{\infty})$$

detects products as well by Proposition F.2.0.1 we can conclude that F preserves products. □

Definition 3.1.1.7. Let $\mathcal{C}$ be an ∞-category, $\mathcal{O}$ an ∞-operad, and

$$p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$$

a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.

We say that p has the *product-fiber-property* if $\mathcal{C}$ admits all products and satisfies the equivalent conditions in Proposition 3.1.1.6. ♢

The product-fiber-property is preserved by taking the pullback as in Remark 3.1.1.3 of a cocartesian family of monoidal ∞-categories, as long as the functor G preserves products, as we record in the following proposition.

Proposition 3.1.1.8. *In the situation of diagram* (3.1) *of Remark 3.1.1.3, if p has the product-fiber-property, $\mathcal{C}'$ admits all products, and G preserves products, then p' has the product-fiber property as well.* ♡

Proof. Follows immediately from the definition in terms of condition (2) in Proposition 3.1.1.6 using that (induced maps on) fibers of p' can be identified with (induced maps on) fibers of p by Proposition C.1.1.1. □

Finally, we end this section by noting that the universal cocartesian family of $\mathcal{O}$-monoidal ∞-categories satisfies the product-fiber-property.

Proposition 3.1.1.9. *Let $\mathcal{O}$ be an ∞-operad. Then $p^{\mathcal{O}}$ has the product-fiber-property.* ♡

Proof. Follows immediately from the description Proposition 3.1.1.6 (1), as the functor corresponding to $p^{\mathcal{O}}$ is by definition the identity functor, which preserves products. □

3.1.2 Algebras in cocartesian families

Given a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$, Lurie defines[5] in [HA, Notation 4.8.3.11] an ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ whose objects can be described as being pairs (X, A) where X is an object of $\mathcal{C}$ (and hence determines a $\mathcal{O}$-monoidal ∞-category $\mathcal{D}^{\otimes}_X$) and A is an object of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}^{\otimes}_X)$. We will discuss a definition of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ in Section 3.1.2.1. Lurie's definition is not quite written down like the definition we present however, so we next show in Section 3.1.2.2 that the two definitions agree. We will then spend some time discussing various functorialities exhibited by this construction. Fixing $\mathcal{O}' \to \mathcal{O}$, we can vary the cocartesian family of $\mathcal{O}$-monoidal ∞-categories $\mathcal{D}$ by taking pullbacks along functors $\mathcal{C}' \to \mathcal{C}$. In fact, we showed in Remark 3.1.1.5 that every family of $\mathcal{O}$-monoidal ∞-categories can be obtained like this from the universal family of $\mathcal{O}$-monoidal ∞-categories $p^{\mathcal{O}}$. The main message of Section 3.1.2.3 is that we also do not obtain anything

[5] While the definition is only written down for $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = \mathsf{Assoc}^{\otimes}$ and $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = \mathsf{LM}^{\otimes}$, we present a straightforward generalization.

new when taking algebras: $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ can be obtained as a pullback of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty))$. More useful is functoriality when varying $\mathcal{O}'$, which we discuss in Section 3.1.2.4, and functoriality that is encoded by the family itself, which will be discussed in Section 3.1.2.5, and in which we we will show that there is a cocartesian fibration $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$. We end this section with Section 3.1.2.6, in which we discuss the interaction of this cocartesian fibration with products in $\mathcal{C}$.

3.1.2.1 Definition

Definition 3.1.2.1. Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. Then we define $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ together with $\mathrm{pr}_{\mathcal{C}}$ and $\mathrm{pr}_{\mathrm{Fun}}$ as the following pullback of ∞-categories.

$$\begin{tikzcd}
\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}"] \arrow[d, "\mathrm{pr}_{\mathcal{C}}"'] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}) \arrow[d, "p_*"] \\
\mathcal{C} \arrow[r, "\widehat{(\alpha\times\mathrm{id}_{\mathcal{C}})}"'] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C})
\end{tikzcd}$$

♢

Proposition 3.1.2.2. *Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.*

Let A be an object of $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$. Then the following are equivalent.

(1) The functor $\mathrm{pr}_{\mathrm{Fun}}(A)\colon \mathcal{O}'^{\otimes} \to \mathcal{D}^{\otimes}$ sends inert morphisms to p-cocartesian ones.

(2) The functor $A'\colon \mathcal{O}'^{\otimes} \to \mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(A)}$ over $\mathcal{O}^{\otimes}$ which corresponds to A under the equivalence

$$\begin{aligned}
&\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_{\mathrm{pr}_{\mathcal{C}}(A)}\\
&\simeq \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}\times\mathcal{C})} \mathcal{C} \times_{\mathcal{C}} \{\mathrm{pr}_{\mathcal{C}}(A)\}\\
&\simeq \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}\times\mathcal{C})} \{\mathrm{pr}_{\mathcal{C}}(A)\}\\
&\simeq \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}\times\mathcal{C})}\\
&\qquad\qquad \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \{\mathrm{pr}_{\mathcal{C}}(A)\}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes})} \{\alpha\}\\
&\simeq \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}\times\mathcal{C}} \big(\mathcal{O}^{\otimes} \times \{\mathrm{pr}_{\mathcal{C}}(A)\}\big)\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes})} \{\alpha\}\\
&\simeq \mathrm{Fun}_{\mathcal{O}^{\otimes}}\Big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(A)}\Big)
\end{aligned}$$

lies in the full subcategory $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_{\mathrm{pr}_{\mathcal{C}}(A)})$ of $\mathcal{O}'$-algebras in the $\mathcal{O}$-monoidal ∞-category $\mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(A)}$. ♡

Proof. Let A' be as in (2). The following commutative diagram summarizes the situation, where $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ is the canonical morphism of ∞-operads, ι is inclusion of $\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \times \{\mathrm{pr}_{\mathcal{C}}(A)\}$, and the square in the middle right is a pullback square.

$$\begin{tikzcd}
\mathcal{O}'^{\otimes} \arrow[r, "A'"] \arrow[dr, "\alpha"'] \arrow[rr, bend left, "\mathrm{pr}_{\mathrm{Fun}}(A)"] & \mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(A)} \arrow[r] \arrow[d, "p_{\mathrm{pr}_{\mathcal{C}}(A)}"] & \mathcal{D}^{\otimes} \arrow[d, "p"] \\
 & \mathcal{O}^{\otimes} \arrow[r, "\iota"] \arrow[d, "p_{\mathcal{O}}"] & \mathcal{O}^{\otimes} \times \mathcal{C} \\
 & \mathsf{Fin}_* &
\end{tikzcd}$$

By definition [HA, 2.1.2.7] A' lies in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_{\mathrm{pr}_{\mathcal{C}}(A)})$ if and only if A' carries inert morphisms to $p_{\mathcal{O}} \circ p_{\mathrm{pr}_{\mathcal{C}}(A)}$-cocartesian ones. As α is a morphism of ∞-operads, it sends inert morphisms to $p_{\mathcal{O}}$-cocartesian ones, so it follows from [HTT, 2.4.1.3 (3)] that A' lies in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_{\mathrm{pr}_{\mathcal{C}}(A)})$ if and only if it carries inert morphisms to $p_{\mathrm{pr}_{\mathcal{C}}(A)}$-cocartesian ones, which by Proposition C.1.1.1 is the case if and only if $\mathrm{pr}_{\mathrm{Fun}}(A)$ carries inert morphisms to p-cocartesian ones. □

Definition 3.1.2.3. Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.

Then we define $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ to be the full subcategory of $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ spanned by those objects satisfying the equivalent conditions in Proposition 3.1.2.2. ♢

Remark 3.1.2.4. In the situation of Definition 3.1.2.3 it follows immediately from Proposition 3.1.2.2 (2) that for any object C of $\mathcal{C}$ the fiber $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_C$ is naturally equivalent to $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_C)$. ♢

3.1.2.2 Comparison with Lurie's definition

Lurie's definition is not phrased quite like Definition 3.1.2.3, so we show below in Proposition 3.1.2.7 that Lurie's definition is equivalent to the one we used.

Definition 3.1.2.5 ([HA, 4.8.3.11])**.** Let $\mathtt{C}$ be a quasicategory representing an ∞-category $\mathcal{C}$, let $\mathtt{O}$ be a quasicategorical ∞-operad representing an ∞-operad $\mathcal{O}$, let $\mathtt{p}\colon \mathtt{D}^{\otimes} \to \mathtt{O}^{\otimes} \times \mathtt{C}$ be an inner fibration representing a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories, and let $\mathtt{a}\colon \mathtt{O}'^{\otimes} \to \mathtt{O}^{\otimes}$ be a morphism of quasicategorical ∞-operads representing a morphism of ∞-operads $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$.

We define an unnamed property for functors of quasicategories $\mathtt{q}\colon \mathtt{A} \to \mathtt{C}$, which is to hold if there is a natural bijection

$$\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{C}}}(-,\mathtt{q}) \cong \mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^{\otimes}\times\mathtt{C}}}(\mathtt{a}\times -,\mathtt{p})$$

of functors $\mathsf{sSet}_{/\mathtt{C}} \to \mathsf{Set}$. ♢

Remark 3.1.2.6. In the situation of Definition 3.1.2.5, the Yoneda lemma implies that if a $\mathtt{q}$ with the property exists, then it is unique up to canonical isomorphism as an object of $\mathsf{sSet}_{/\mathtt{C}}$. ♢

Proposition 3.1.2.7. *Let $\mathtt{C}$ be a quasicategory representing an ∞-category $\mathcal{C}$, let $\mathtt{O}$ be a quasicategorical ∞-operad representing an ∞-operad $\mathcal{O}$, let $\mathtt{p}\colon \mathtt{D}^{\otimes} \to \mathtt{O}^{\otimes}\times\mathtt{C}$ be an inner fibration of quasicategories representing a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories, and let $\mathtt{a}\colon \mathtt{O}'^{\otimes} \to \mathtt{O}^{\otimes}$ be a morphism of quasicategorical ∞-operads representing a morphism of ∞-operads $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$.*

Define $\mathtt{E}$ and $\mathtt{q}$ via the following categorical pullback square in sSet.

$$\begin{tikzcd}
\mathtt{E} \arrow[r] \arrow[d, "\mathtt{q}"'] & \mathrm{Fun}\big(\mathtt{O}'^{\otimes},\mathtt{D}^{\otimes}\big) \arrow[d, "\mathtt{p}_*"] \\
\mathtt{C} \arrow[r, "\widehat{(\mathtt{a}\times\mathrm{id}_{\mathtt{C}})}"'] & \mathrm{Fun}\big(\mathtt{O}'^{\otimes},\mathtt{O}^{\otimes}\times\mathtt{C}\big)
\end{tikzcd}$$

Then the following hold.

(1) *The map $\mathtt{q}$ satisfies the property defined in Definition 3.1.2.5.*

(2) *In particular, if $\mathtt{a} = \mathrm{id}_{\mathsf{Assoc}}$ and $\mathtt{a} = \mathrm{id}_{\mathsf{LM}}$, then $\mathtt{q}$ can be identified with the functors of quasicategories $\widetilde{\mathrm{Alg}}(\mathtt{D}) \to \mathtt{C}$ and $\widetilde{\mathrm{LMod}}(\mathtt{D}) \to \mathtt{C}$ as defined in [HA, 4.8.3.11], respectively.*

(3) *The pullback is a homotopy pullback with respect to the Joyal model structure.*

(4) *The pullback square represents the pullback square of ∞-categories in Definition 3.1.2.1 that defines $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$.*

(5) *If $\mathtt{a} = \mathrm{id}_{\mathsf{Assoc}}$, then $\mathrm{Alg}(\mathtt{D}) \to \mathtt{C}$ as defined in [HA, 4.8.3.11] represents $\mathrm{Alg}_{/\mathsf{Assoc}}(\mathcal{D})$ as defined in Definition 3.1.2.3. If $\mathtt{a} = \mathrm{id}_{\mathsf{LM}}$, then $\mathrm{LMod}(\mathtt{D}) \to \mathtt{C}$ as defined in [HA, 4.8.3.11] represents $\mathrm{Alg}_{/\mathsf{LM}}(\mathcal{D})$ as defined in Definition 3.1.2.3.* ♡

Proof. Proof of (1): Let $\mathtt{s}\colon \mathtt{K} \to \mathtt{C}$ be a map of simplicial sets. Then there is a sequence of bijections natural in $\mathtt{s}$ (as an object of $\mathsf{sSet}_{/\mathtt{C}}$) as follows.

$$\begin{aligned}
&\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{C}}}(\mathtt{s},\mathtt{q}) \\
\cong\ &\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K},\mathtt{E}) \times_{\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K},\mathtt{C})} \{\mathtt{s}\}
\end{aligned}$$

$$
\begin{aligned}
&\cong \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{K}, \mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{D}^{\otimes}\big)\big) \\
&\qquad \times_{\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K}, \mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}^{\otimes} \times \mathtt{C}))} \mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K}, \mathtt{C}) \times_{\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K}, \mathtt{C})} \{\mathtt{s}\} \\
&\cong \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{K}, \mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{D}^{\otimes}\big)\big) \times_{\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K}, \mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}^{\otimes} \times \mathtt{C}))} \Big\{\widehat{(\mathtt{a} \times \mathtt{s})}\Big\} \\
&\cong \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{O}'^{\otimes} \times \mathtt{K}, \mathtt{D}^{\otimes}\big) \times_{\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{O}'^{\otimes} \times \mathtt{K}, \mathtt{O}^{\otimes} \times \mathtt{C})} \{(\mathtt{a} \times \mathtt{s})\} \\
&\cong \mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^{\otimes} \times \mathtt{C}}}(\mathtt{a} \times \mathtt{s}, \mathtt{p})
\end{aligned}
$$

Proof of (2): Follows directly from the definition.

Proof of (3): By assumption, $\mathtt{p}$ is a cocartesian fibration in the sense of [HTT, 2.4.2.1], so that by [HTT, 3.1.2.1] the functor of quasicategories $\mathtt{p}_*$ is again a cocartesian fibration in the sense of [HTT, 2.4.2.1]. That the pullback square is a homotopy pullback square in the Joyal model structure follows now by applying [HTT, 3.3.1.4] (to the opposite diagram).

Proof of (4): Follows directly from (3).

Proof of (5): Immediate by unwrapping the definitions of the respective full subcategories. □

3.1.2.3 Functoriality when varying families

We next consider functoriality of ∞-categories of algebras of cocartesian families of monoidal ∞-categories when we vary the cocartesian family. We first discuss functoriality in the ∞-operad factor, for which the following proposition can be considered a generalization of Proposition E.2.0.2.

Remark 3.1.2.8. Let $\mathcal{C}$ be an ∞-category, let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ as well as $\beta\colon \mathcal{O}''^{\otimes} \to \mathcal{O}'^{\otimes}$ be morphisms of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. Assume that the following diagram is a pullback diagram in $\mathfrak{Cat}_\infty$.

$$
\begin{tikzcd}
\mathcal{D}'^{\otimes} \arrow[r, "G^{\otimes}"] \arrow[d, "p'"'] & \mathcal{D}^{\otimes} \arrow[d, "p"] \\
\mathcal{O}'^{\otimes} \times \mathcal{C} \arrow[r, "\alpha \times \mathrm{id}_{\mathcal{C}}"'] & \mathcal{O}^{\otimes} \times \mathcal{C}
\end{tikzcd}
\tag{3.2}
$$

By Remark 3.1.1.3 is p' is a cocartesian $\mathcal{C}$-family of $\mathcal{O}'$-monoidal ∞-categories.

Consider the following commutative diagram, where the square on the left is the pullback square of Definition 3.1.2.1 and the square on the right is induced by the pullback square (3.2) by applying $\mathrm{Fun}(\mathcal{O}''^{\otimes}, -)$ and hence also a pullback square.

$$\begin{tikzcd}
\widetilde{\mathrm{Alg}}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{D}') \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}"] \arrow[d, "\mathrm{pr}_{\mathcal{C}}"'] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{D}'^{\otimes}) \arrow[r, "G^{\otimes}_{*}"] \arrow[d, "p'_{*}"] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{D}^{\otimes}) \arrow[d, "p_{*}"] \\
\mathcal{C} \arrow[r, "\widehat{(\beta\times\mathrm{id}_{\mathcal{C}})}"'] \arrow[rr, bend right, "\widehat{(-)}((\alpha\circ\beta)\times\mathrm{id}_{\mathcal{C}})"'] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{O}'^{\otimes}\times\mathcal{C}) \arrow[r, "(\alpha\times\mathrm{id}_{\mathcal{C}})_{*}"'] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{O}^{\otimes}\times\mathcal{C})
\end{tikzcd}$$

By the pasting lemma for pullbacks [HTT, 4.4.2.1], the outer square is a pullback as well, so that we obtain a canonical identification as follows.

$$\widetilde{\mathrm{Alg}}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{D}') \simeq \widetilde{\mathrm{Alg}}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})$$

Furthermore, with the description of p'-cocartesian morphisms from Proposition C.1.1.1 it follows directly from Definition 3.1.2.3 in the form of Proposition 3.1.2.2 (1) that this equivalence restricts to an equivalence of ∞-categories of algebras as follows.

$$\mathrm{Alg}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{D}') \simeq \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}) \qquad \diamondsuit$$

We now turn to functoriality in the ∞-category that parametrizes our cocartesian family of monoidal ∞-categories.

Construction 3.1.2.9. Let $F\colon \mathcal{C}' \to \mathcal{C}$ be a functor of ∞-categories, let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads, and let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}\times\mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. Assume that the following diagram is a pullback diagram in $\mathcal{C}\mathrm{at}_{\infty}$.

$$\begin{tikzcd}
\mathcal{D}'^{\otimes} \arrow[r, "G^{\otimes}"] \arrow[d, "p'"'] & \mathcal{D}^{\otimes} \arrow[d, "p"] \\
\mathcal{O}^{\otimes}\times\mathcal{C}' \arrow[r, "\mathrm{id}_{\mathcal{O}^{\otimes}}\times F"'] & \mathcal{O}^{\otimes}\times\mathcal{C}
\end{tikzcd} \qquad (*)$$

Remark 3.1.1.3 implies that p' is a cocartesian $\mathcal{C}'$-family of $\mathcal{O}$-monoidal ∞-categories.

Then there is a commutative cube as follows

$$\begin{tikzcd}
& \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[rr, "\mathrm{pr}_{\mathrm{Fun}}"] \arrow[dl, dashed, "F_{*}"'] \arrow[dd, "\mathrm{pr}_{\mathcal{C}'}"] & & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{D}'^{\otimes}) \arrow[dl, "G^{\otimes}_{*}"] \arrow[dd, "p'_{*}"] \\
\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[rr, "\mathrm{pr}_{\mathrm{Fun}}"' near start] \arrow[dd, "\mathrm{pr}_{\mathcal{C}}"'] & & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}) \arrow[dd, "p_{*}" near start] & \\
& \mathcal{C}' \arrow[rr, "\widehat{(\alpha\times\mathrm{id}_{\mathcal{C}'})}"' near start] \arrow[dl, "F"'] & & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}\times\mathcal{C}') \arrow[dl, "(\mathrm{id}_{\mathcal{O}^{\otimes}}\times F)_{*}"] \\
\mathcal{C} \arrow[rr, "\widehat{(\alpha\times\mathrm{id}_{\mathcal{C}})}"'] & & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}\times\mathcal{C}) &
\end{tikzcd}$$

where the front and back squares are the respective defining pullback squares for $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ and $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}')$, and the dashed functor F_* is the induced one.

The square on the right is obtained by applying $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$ to the pullback square $(*)$ and is thus a pullback square as well. As the front square is also a pullback square, it follows that their composition, which we can identify with the composition of the left and back square, is a pullback square as well. As the back square is also a pullback square, if finally follows using the pasting law for pullbacks [HTT, 4.4.2.1] that the square

$$\begin{tikzcd} \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"] \arrow[d, "\mathrm{pr}_{\mathcal{C}'}"'] & \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d, "\mathrm{pr}_{\mathcal{C}}"] \\ \mathcal{C}' \arrow[r, "F"'] & \mathcal{C} \end{tikzcd} \tag{3.3}$$

is a pullback square. ♢

Proposition 3.1.2.10. *Let us assume we are in the situation of Construction 3.1.2.9. Then the pullback square* (3.3) *restricts to a pullback square in* $\mathcal{C}\mathrm{at}_\infty$ *as follows.*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"] \arrow[d, "\mathrm{pr}_{\mathcal{C}'}"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d, "\mathrm{pr}_{\mathcal{C}}"] \\ \mathcal{C}' \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

♡

Proof. It suffices to show that the dashed functor in the following commutative diagram (where the vertical functors are the canonical inclusions) exists and that the square is a pullback square in $\mathcal{C}\mathrm{at}_\infty$.

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, dashed, "F_*"] \arrow[d] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d] \\ \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"'] & \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \end{tikzcd}$$

Let A be an object in $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}')$. Then by Definition 3.1.2.3 and Proposition 3.1.2.2 (1), A is in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}')$ if and only if $\mathrm{pr}_{\mathrm{Fun}}(A)\colon \mathcal{O}'^{\otimes} \to \mathcal{D}'^{\otimes}$ sends inert morphisms to p'-cocartesian morphisms, which by Proposition C.1.1.1 is the case if and only if $G^{\otimes} \circ \mathrm{pr}_{\mathrm{Fun}}(A) \simeq \mathrm{pr}_{\mathrm{Fun}}(F_*(A))$ sends inert morphisms to p-cocartesian morphisms. Thus A is in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}')$ if and only if $F_*(A)$ is in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$. This shows that in the following commutative diagram of ∞-categories, where the small square is defined as a pullback square, the dashed functor making the outer square commute exists, and that the induced dotted functor is essentially surjective (this uses the description of $\mathcal{E}$ given in Proposition B.5.2.1).

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[drr, dotted] \arrow[ddrr, bend right] \arrow[drrr, dashed, bend left, "F_*"] & & & \\
& & \mathcal{E} \arrow[r] \arrow[d, "q"] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d] \\
& & \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"'] & \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})
\end{tikzcd}$$

By Proposition B.5.2.1 the functor q is fully faithful, so the dotted functor is fully faithful as well and hence an equivalence. It follows that the outer square is a pullback square because the inner square is. □

Remark 3.1.2.11. Let $\mathcal{C}$ be an ∞-category, let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads, let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories, and let $F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$ be the functor corresponding to p under the equivalence in Remark 3.1.1.3. By Remark 3.1.1.5 there is a pullback diagram as follows.

$$\begin{tikzcd}
\mathcal{D}^{\otimes} \arrow[r] \arrow[d, "p"'] & \widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[d, "p^{\mathcal{O}}"] \\
\mathcal{O}^{\otimes} \times \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{O}^{\otimes}} \times F"'] & \mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd}$$

Applying Proposition 3.1.2.10 we obtain a pullback diagram of algebra ∞-categories.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[r, "F_*"] \arrow[d, "\mathrm{pr}_{\mathcal{C}}"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}\Big(\widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})\Big) \arrow[d, "\mathrm{pr}_{\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})}"] \\
\mathcal{C} \arrow[r, "F"'] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd}$$

♢

3.1.2.4 Functoriality when varying the operad

In this section we discuss functoriality of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ when varying $\mathcal{O}'$.

Construction 3.1.2.12. Let $\mathcal{C}$ be an ∞-category, lat $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ as well as $\beta\colon \mathcal{O}''^{\otimes} \to \mathcal{O}'^{\otimes}$ be morphisms of ∞-operads, and let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.

Then the commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widehat{(\alpha\times\mathrm{id}_{\mathcal{C}})}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}"'] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}\times\mathcal{C}) \arrow[d, "\beta^*"] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}) \arrow[l, "p_*"'] \arrow[d, "\beta^*"] \\ \mathcal{C} \arrow[r, "\widehat{(-)}((\alpha\circ\beta)\times\mathrm{id}_{\mathcal{C}})"'] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{O}^{\otimes}\times\mathcal{C}) & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{D}^{\otimes}) \arrow[l, "p_*"] \end{tikzcd}$$

induces a functor on pullbacks as follows.

$$\beta^*\colon \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \widetilde{\mathrm{Alg}}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}) \qquad \diamondsuit$$

Remark 3.1.2.13. In the situation of Construction 3.1.2.12, if we are given another morphism of ∞-operads $\gamma\colon \mathcal{O}'''^{\otimes} \to \mathcal{O}''^{\otimes}$, then it is clear from the definition that the composition $\gamma^* \circ \beta^*$ is equivalent to $(\beta\circ\gamma)^*$. $\diamondsuit$

Proposition 3.1.2.14. *In the situation of Construction 3.1.2.12, the functor*

$$\beta^*\colon \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \widetilde{\mathrm{Alg}}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})$$

restricts to a functor on algebras as follows.

$$\beta^*\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}) \qquad \heartsuit$$

Proof. What we have to show is by Definition 3.1.2.3 in the form of Proposition 3.1.2.2 (1) that the functor

$$\beta^*\colon \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \to \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{D}^{\otimes}\big)$$

sends functors that send inert morphisms to p-cocartesian morphisms to functors with the same property. But this follows immediately from the fact that, as it is a morphism of ∞-operads, β preserves inert morphisms. $\square$

Remark 3.1.2.15. Assume we are in the situation of Construction 3.1.2.12, and let C be an object of $\mathcal{C}$. The functor $\beta^*\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})$ is a functor over $\mathcal{C}$ and thus induces a functor as follows.

$$\beta^*_C\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_C \to \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})_C$$

It follows directly from the definition together with Remark 3.1.2.4 that this functor can be identified with the following functor induced by β.

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_C) \to \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}_C) \qquad \diamondsuit$$

Remark 3.1.2.16. Assume we are in the situation of Construction 3.1.2.9 and we are given another morphism of ∞-operads $\beta\colon \mathcal{O}''^{\otimes} \to \mathcal{O}'^{\otimes}$. Then it follows from the respective constructions that β^* and F_* commute in the sense that there is a commutative diagram as follows.

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"] \arrow[d, "\beta^*"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d, "\beta^*"] \\ \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"'] & \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}) \end{tikzcd}$$

♢

3.1.2.5 Functoriality encoded by families

Let $\mathcal{C}$ be an ∞-category, let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads, and let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. In Section 3.1.2.1 we constructed a functor of ∞-categories

$$\mathrm{pr}_{\mathcal{C}}\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$$

and identified the fiber of $\mathrm{pr}_{\mathcal{C}}$ over an object C in $\mathcal{C}$ with $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_C)$, see Remark 3.1.2.4.

As was explained at the start of Section 3.1.1, we can interpret p as a collection of $\mathcal{O}$-monoidal ∞-categories that is indexed by $\mathcal{C}$. We will show below that $\mathrm{pr}_{\mathcal{C}}$ is again a cocartesian fibration, and thus classified by a functor $\mathcal{C} \to \mathfrak{Cat}_\infty$, which we can then interpret as encoding the functoriality of the construction $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(-)$ that produces the ∞-category of $\mathcal{O}'$-algebras out of an $\mathcal{O}$-monoidal ∞-category.

Proposition 3.1.2.17 ([HA, 4.8.3.13]). *Let $\mathcal{C}$ be an ∞-category and let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads. Let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. Then the following hold.*

(1) $\mathrm{pr}_{\mathcal{C}}\colon \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$ *is a cocartesian fibration and a morphism f is* $\mathrm{pr}_{\mathcal{C}}$*-cocartesian if and only if* $\mathrm{pr}_{\mathrm{Fun}}(f)(X)$ *is p-cocartesian for every object X of $\mathcal{O}'^{\otimes}$ (see Definition 3.1.2.1 for this notation).*

(2) $\mathrm{pr}_{\mathcal{C}}\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$ *is a cocartesian fibration and a morphism f in* $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ *is* $\mathrm{pr}_{\mathcal{C}}$*-cocartesian if and only if* $\mathrm{pr}_{\mathrm{Fun}}(f)(X)$ *is p-cocartesian for every object X of $\mathcal{O}'^{\otimes}$.*

(3) A morphism f in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ *is* $\mathrm{pr}_{\mathcal{C}}$*-cocartesian if and only if the morphism* $\mathrm{pr}_{\mathrm{Fun}}(f)(X)$ *is p-cocartesian for every object X of $\mathcal{O}'$.* ♡

Proof. Proof of (1): This is a combination of [HTT, 3.1.2.1] (preservation of cocartesian fibrations under application of $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$) with Proposition C.1.1.1 (preservation of cocartesian fibrations under pullbacks).

Proof of (2): It suffices to verify the assumption needed to apply the dual of Proposition C.1.2.1 to the restriction of $\mathrm{pr}_{\mathcal{C}}\colon \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$ to $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$.

So let A be an object in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ and $f\colon A \to B$ a $\mathrm{pr}_{\mathcal{C}}$-cocartesian morphism in $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$. We have to show that B also lies in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$. By definition this means that we need to show that $\mathrm{pr}_{\mathrm{Fun}}(B)\colon \mathcal{O}'^{\otimes} \to \mathcal{D}^{\otimes}$ sends inert morphisms to p-cocartesian morphisms. So let $\varphi\colon X \to Y$ be an inert morphism in $\mathcal{O}'^{\otimes}$. We obtain a commutative diagram in $\mathcal{D}^{\otimes}$ as follows.

$$\begin{tikzcd}
\mathrm{pr}_{\mathrm{Fun}}(A)(X) \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}(f)(X)"] \arrow[d, "\mathrm{pr}_{\mathrm{Fun}}(A)(\varphi)"'] & \mathrm{pr}_{\mathrm{Fun}}(B)(X) \arrow[d, "\mathrm{pr}_{\mathrm{Fun}}(B)(\varphi)"] \\
\mathrm{pr}_{\mathrm{Fun}}(A)(Y) \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}(f)(Y)"'] & \mathrm{pr}_{\mathrm{Fun}}(B)(Y)
\end{tikzcd} \tag{$*$}$$

As f is $\mathrm{pr}_{\mathcal{C}}$-cocartesian, the top and bottom horizontal morphisms are p-cocartesian by (1). As A lies in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$, the left vertical morphism is p-cocartesian as well. That the right vertical morphism is also p-cocartesian now follows from [HTT, 2.4.1.7].

Proof of (3): Let $f\colon A \to B$ be a morphism in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ and assume that for every object Y in $\mathcal{O}'$ the morphism $\mathrm{pr}_{\mathrm{Fun}}(f)(Y)$ in $\mathcal{D}^{\otimes}$ is p-cocartesian. Let $X \simeq X_1 \oplus \cdots \oplus X_n$ be an object in $\mathcal{O}'^{\otimes}_{\langle n\rangle}$, and denote by $\gamma_i\colon X \to X_i$ the inert morphism in $\mathcal{O}'^{\otimes}$ lying over ρ^i. We have to show that then also $\mathrm{pr}_{\mathrm{Fun}}(f)(X)$ is p-cocartesian.

Let $1 \le i \le n$. Consider the following diagram in $\mathcal{D}^{\otimes}$

$$\begin{tikzcd}
 & D \arrow[rr, "\Psi"] \arrow[d, dashed, "\Theta"] & & D_i \arrow[d, dashed, "\Theta_i"] \\
\mathrm{pr}_{\mathrm{Fun}}(A)(X) \arrow[ur, bend left, "\Phi"] \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}(f)(X)"] \arrow[dr, "\mathrm{pr}_{\mathrm{Fun}}(A)(\gamma_i)"'] & \mathrm{pr}_{\mathrm{Fun}}(B)(X) \arrow[rr, "\mathrm{pr}_{\mathrm{Fun}}(B)(\gamma_i)"] & & \mathrm{pr}_{\mathrm{Fun}}(B)(X_i) \\
 & \mathrm{pr}_{\mathrm{Fun}}(A)(X_i) \arrow[urr, "\mathrm{pr}_{\mathrm{Fun}}(f)(X_i)"'] & &
\end{tikzcd}$$

lying over the following commutative diagram in $\mathcal{O}^{\otimes} \times \mathcal{C}$

$$\begin{tikzcd}
 & (\alpha(X), \mathrm{pr}_{\mathcal{C}}(B)) \arrow[rr, "{(\alpha(\gamma_i),\mathrm{id})}"] \arrow[d, "\mathrm{id}"] & & (\alpha(X_i), \mathrm{pr}_{\mathcal{C}}(B)) \arrow[d, "\mathrm{id}"] \\
(\alpha(X), \mathrm{pr}_{\mathcal{C}}(A)) \arrow[ur, bend left, "{(\mathrm{id},\mathrm{pr}_{\mathcal{C}}(f))}"] \arrow[r, "{(\mathrm{id},\mathrm{pr}_{\mathcal{C}}(f))}"] \arrow[dr, "{(\alpha(\gamma_i),\mathrm{id})}"'] & (\alpha(X), \mathrm{pr}_{\mathcal{C}}(B)) \arrow[rr, "{(\alpha(\gamma_i),\mathrm{id})}"] & & (\alpha(X_i), \mathrm{pr}_{\mathcal{C}}(B)) \\
 & (\alpha(X_i), \mathrm{pr}_{\mathcal{C}}(A)) \arrow[urr, "{(\mathrm{id},\mathrm{pr}_{\mathcal{C}}(f))}"'] & &
\end{tikzcd}$$

and such that the morphisms Φ and Ψ in $\mathcal{D}^{\otimes}$ are p-cocartesian lifts of $(\mathrm{id}_{\alpha(X)}, \mathrm{pr}_{\mathcal{C}}(f))$ and $(\alpha(\gamma_i), \mathrm{id}_{\mathrm{pr}_{\mathcal{C}}(B)})$, respectively, and such that the dashed morphisms are the canonical fillers. As γ_i is inert and A in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ the morphism $\mathrm{pr}_{\mathrm{Fun}}(A)(\gamma_i)$ is p-cocartesian, and the morphism $\mathrm{pr}_{\mathrm{Fun}}(f)(X_i)$ is p-cocartesian by assumption, as X_i is an object of $\mathcal{O}'$. Considering the outer diagram it thus follows from [HTT, 2.4.1.7] that Θ_i is p-cocartesian, and thus by [HTT, 2.4.1.5] an equivalence.

We now want to conclude that also Θ must be an equivalence. For this, note that as $p_{\mathrm{pr}_{\mathcal{C}}(B)}$ is a cocartesian fibration of ∞-operads, the following functor induced by the inert morphisms $\alpha(\gamma_i)$ on fibers

$$\left(\mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(B)}\right)_{\alpha(X)} \xrightarrow{\prod_{1\le i\le n}(\alpha(\gamma_i))_!} \prod_{1\le i\le n} \left(\mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(B)}\right)_{\alpha(X_i)}$$

is an equivalence of ∞-categories. By Proposition C.1.1.1 we can identify this functor with the following functor.

$$\mathcal{D}^{\otimes}_{(\alpha(X),\mathrm{pr}_{\mathcal{C}}(B))} \xrightarrow{\prod_{1\le i\le n}(\alpha(\gamma_i),\mathrm{id})_!} \prod_{1\le i\le n} \mathcal{D}^{\otimes}_{(\alpha(X_i),\mathrm{pr}_{\mathcal{C}}(B))}$$

The morphism Θ lies in $\mathcal{D}^{\otimes}_{(\alpha(X),\mathrm{pr}_{\mathcal{C}}(B))}$, and by definition $\Theta_i \simeq (\alpha(\gamma_i), \mathrm{id})_!(\Theta)$. As we previously concluded that Θ_i is an equivalence for every $1 \le i \le n$ we can thus conclude that Θ is an equivalence, and hence p-cocartesian by [HTT, 2.4.1.5]. As Φ is p-cocartesian by definition we can then use [HTT, 2.4.1.7] to deduce that $\mathrm{pr}_{\mathrm{Fun}}(f)(X)$ is also p-cocartesian. □

Remark 3.1.2.18. Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. A morphism $g\colon C \to C'$ in $\mathcal{C}$ induces on fibers of p an $\mathcal{O}$-monoidal functor[6] $G\colon \mathcal{D}^{\otimes}_{C} \to \mathcal{D}^{\otimes}_{C'}$. Combining the identifications

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}^{\otimes}_{X}) \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}^{\otimes})_X$$

from Remark 3.1.2.4 (for $X = C$ as well as $X = C'$) with Proposition 3.1.2.17, in particular description Proposition 3.1.2.17 (2), we can conclude that we can identify the functor induced by g on fibers of $\mathrm{pr}_{\mathcal{C}}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$ with the functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(G)$. ♢

Definition 3.1.2.19. Let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads. Then we define

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}\colon \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty}) \to \mathfrak{Cat}_{\infty}$$

the be the functor that the cocartesian fibration

$$\mathrm{pr}_{\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}\left(\widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})\right) \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$$

is classified by. ♢

[6] This is clear from Proposition 3.1.1.1.

Remark 3.1.2.20. Let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads. The functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}$ sends by Remark 3.1.2.4 an $\mathcal{O}$-monoidal ∞-category $\mathcal{C}$ to the ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$, so that we can interpret $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}$ as encoding the full functoriality of the construction of ∞-categories of $\mathcal{O}'$-algebras in $\mathcal{O}$-monoidal ∞-categories.

Now let $\mathcal{C}$ be an ∞-category, $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories, and $F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$ the functor corresponding to p under the equivalence in Remark 3.1.1.3. Then it follows from Remark 3.1.2.11 and naturality of the Grothendieck construction (see [GHN17, A.32] and [Maz19]) that the cocartesian fibration

$$\mathrm{pr}_{\mathcal{C}}\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$$

is classified by the following composition.

$$\mathcal{C} \xrightarrow{F} \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty}) \xrightarrow{\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}} \mathfrak{Cat}_{\infty} \qquad \diamondsuit$$

Proposition 3.1.2.21. *Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ as well as $\beta\colon \mathcal{O}''^{\otimes} \to \mathcal{O}'^{\otimes}$ morphisms of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.*

Then the functor

$$\beta^*\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})$$

constructed in Construction 3.1.2.12 and Proposition 3.1.2.14, which by construction is a functor over $\mathcal{C}$, is a functor of cocartesian fibrations, i. e. sends $\mathrm{pr}_{\mathcal{C}}$-cocartesian morphisms to $\mathrm{pr}_{\mathcal{C}}$-cocartesian morphisms. $\heartsuit$

Proof. By definition of β^* there is a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}) \arrow[rr, "\beta^*"] & & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{D}^{\otimes}) \\
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[u, "\mathrm{pr}_{\mathrm{Fun}}"] \arrow[rr, "\beta^*"] \arrow[dr, "\mathrm{pr}_{\mathcal{C}}"'] & & \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}) \arrow[u, "\mathrm{pr}_{\mathrm{Fun}}"'] \arrow[dl, "\mathrm{pr}_{\mathcal{C}}"] \\
& \mathcal{C} &
\end{tikzcd}$$

As the top horizontal functor clearly preserves pointwise p-cocartesian morphisms, criterion Proposition 3.1.2.17 (2) implies that the middle horizontal functor preserves $\mathrm{pr}_{\mathcal{C}}$-cocartesian morphisms. $\square$

3.1.2.6 Algebras in cocartesian families and products

Let $\mathcal{C}$ and $\mathcal{C}'$ be two $\mathcal{O}$-monoidal ∞-categories. Then there is an induced $\mathcal{O}$-monoidal structure on $\mathcal{C} \times \mathcal{C}'$, and it is reasonable to expect that there should be an equivalence as follows.

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C} \times \mathcal{C}') \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \times \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}')$$

The next proposition shows that this is indeed the case.

Proposition 3.1.2.22. *Let $\mathcal{C}$ be an ∞-category, let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads, and let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories that has the product-fiber property from Definition 3.1.1.7. Then the cocartesian fibrations*

$$\mathrm{pr}_{\mathcal{C}}\colon \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$$

and

$$\mathrm{pr}_{\mathcal{C}}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$$

have fibers compatible with products in the sense of Definition C.2.0.1. ♡

Proof. Let I be a set, let X_i be an object in $\mathcal{C}$ for every element i of I, and let $X \coloneqq \prod_{i\in I} X_i$. We have to prove that the two functors induced on fibers

$$\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_X \xrightarrow{\prod_{i\in I}\mathrm{pr}_{i!}} \prod_{i\in I} \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_{X_i} \tag{$*$}$$

and

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_X \xrightarrow{\prod_{i\in I}\mathrm{pr}_{i!}} \prod_{i\in I} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_{X_i} \tag{$**$}$$

are equivalences.

We start by considering the following commutative triangle induced by the projections $\mathrm{pr}_i\colon X \to X_i$.

$$\begin{tikzcd} \mathcal{D}^{\otimes}_X \arrow[rr, "\prod_{i\in I}(pr_{i!})"] \arrow[dr, "p_X"'] & & \mathcal{O}^{\otimes} \times_{\prod_{i\in I}\mathcal{O}^{\otimes}} \prod_{i\in I}\mathcal{D}^{\otimes}_{X_i} \arrow[dl, "\mathrm{pr}_1"] \\ & \mathcal{O}^{\otimes} & \end{tikzcd} \tag{$***$}$$

Both p_X and pr_1 in this diagram are cocartesian fibrations, and the horizontal functor sends p_X-cocartesian morphisms to pr_1-cocartesian morphisms. The statement for p_X and pr_1 follows from p being a cocartesian fibration and applying Proposition C.1.1.1, and in the case of the functor on the right also using that products of cocartesian fibrations are again cocartesian fibrations by [HTT, 3.1.2.1]. This also gives a description of the respective cocartesian morphisms, and with that the statement about the horizontal functor boils down to a statement about p-cocartesian morphisms that holds by [HTT, 2.4.1.7]. By assumption p has the product-fiber property, which precisely means that the horizontal functor in the above diagram is a fiberwise (over $\mathcal{O}^{\otimes}$) equivalence. It now follows from [HTT, 2.4.4.4] that the horizontal functor is itself an equivalence.

We now consider the first of the two functors, $(*)$. Unpacking the definition (Definition 3.1.2.1) of $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ as a pullback and using Proposition C.1.1.1 we can identify the functor $(*)$ with

$$\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big)_{\mathrm{id}_{\mathcal{O}^{\otimes}} \times \mathrm{const}_X} \xrightarrow{\prod_{i\in I}(\mathrm{id}\times \mathrm{pr}_i)_!} \prod \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big)_{\mathrm{id}_{\mathcal{O}^{\otimes}} \times \mathrm{const}_{X_i}}$$

where the fibers are taken over $\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C})$, and $\mathrm{id} \times \mathrm{pr}_i$ is the natural transformation of functors $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ from $\mathrm{id}_{\mathcal{O}^{\otimes}} \times \mathrm{const}_X$ to $\mathrm{id}_{\mathcal{O}^{\otimes}} \times \mathrm{const}_{X_i}$ that is given by the identity in the $\mathcal{O}^{\otimes}$ factor and pr_i in the $\mathcal{C}$ factor.

Using that $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$ commutes with pullbacks together with [HTT, 3.1.2.1] we can further identify functor $(*)$ with the functor

$$\mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}_X\big) \xrightarrow{\prod_{i\in I}(\mathrm{pr}_{i!})_*} \prod \mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}_{X_i}\big)$$

and in another step, using composability of pullback diagrams, that the functor $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$ commutes with products, Proposition C.1.1.1 and [HTT, 3.1.2.1] some more, we can further identify this with the following functor.

$$\mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}_X\big) \xrightarrow{(\prod_{i\in I}(\mathrm{pr}_{i!}))_*} \prod \mathrm{Fun}_{\mathcal{O}^{\otimes}}\left(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times_{\prod_{i\in I}\mathcal{O}^{\otimes}} \prod_{i\in I} \mathcal{D}^{\otimes}_{X_i}\right)$$

This exactly $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, -)$ applied to the horizontal functor in $(***)$, so this is an equivalence.

Using Proposition 3.1.2.2 one can see that under these equivalences the functor $(**)$ (which is a restriction of $(*)$ on domain and codomain to full subcategories) corresponds to the application of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(-)$ to the horizontal functor in $(***)$, so this functor is also an equivalence. □

Corollary 3.1.2.23. *Let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads. Then the cocartesian fibration*

$$\mathrm{pr}_{\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}\Big(\widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)\Big) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$

has fibers compatible with products in the sense of Definition C.2.0.1. ♡

Proof. Combine Proposition 3.1.2.22 and Proposition 3.1.1.9. □

3.1.3 Functorial construction of ∞-categories of left modules

In Definition 3.1.2.19 we constructed a functor

$$\mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}\colon \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \to \mathcal{C}\mathrm{at}_\infty$$

that sends an (Assoc-)monoidal[7] ∞-category $\mathcal{C}$ to $\mathrm{Alg}(\mathcal{C}) := \mathrm{Alg}_{/\mathsf{Assoc}}(\mathcal{C})$, the ∞-category of Assoc-algebras in $\mathcal{C}$, and can thus be interpreted as encoding the functoriality of the construction $\mathcal{C} \mapsto \mathrm{Alg}(\mathcal{C})$, see Remark 3.1.2.20.

In this section we will similarly construct a functor LMod that can be interpreted as encoding the functoriality of the construction that maps a pair $(\mathcal{C}, A)$ with $\mathcal{C}$ a monoidal ∞-category and A an associative algebra in $\mathcal{C}$, to the ∞-category $\mathrm{LMod}_A(\mathcal{C})$ of left A modules[8]. For functoriality in $\mathcal{C}$, a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ should induce a functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_{FA}(\mathcal{D})$ when A is an associative algebra in $\mathcal{C}$. For functoriality in A, we should be able to form the base change along a morphisms of algebras $f\colon A \to B$ in $\mathcal{C}$, i. e. restricting the action, providing us with a functor $\mathrm{LMod}_B(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C})$.

We already have constructed an ∞-category whose objects can be described as pairs $(\mathcal{C}, A)$ with $\mathcal{C}$ a monoidal ∞-category and A an associative algebra in $\mathcal{C}$, namely

$$\mathcal{A}\mathrm{lg} := \mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}\left(\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\right)$$

see Remark 3.1.2.20. By taking algebras in $\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ with respect to two other ∞-operads, we will obtain a commutative diagram as follows.

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od} \arrow[r] \arrow[dr] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj} \arrow[r] \arrow[d] & \mathcal{A}\mathrm{lg} \arrow[dl] \\
& \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) &
\end{tikzcd} \tag{3.4}$$

Objects in $\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od}$ can be described as tuples $(\mathcal{C}, A, M)$, with $\mathcal{C}$ a monoidal ∞-category, A an associative algebra in $\mathcal{C}$, and M a left module in $\mathcal{C}$ over A. Objects in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj}$ can be described as tuples $(\mathcal{C}, A, X)$, with $\mathcal{C}$ and A as before, but X just an object of $\mathcal{C}$. The functors in diagram (3.4) are the obvious forgetful functors.

However, $\mathcal{A}\mathrm{lg}$ is not quite the ∞-category needed to encode the functoriality of LMod that we alluded to at the start of this sub-subsection: A morphism from $(\mathcal{C}, A) \to (\mathcal{D}, B)$ consists of a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ and a morphism $F(A) \to B$ of algebras in $\mathcal{D}$. So for our sought-after functoriality of LMod we would like the algebra-part of those morphisms to go in the opposite direction. Luckily, the horizontal functors in diagram (3.4) are morphisms of cocartesian fibrations over $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$, so we can apply the fiberwise $-^{\mathrm{op}}$-construction to fix this. We obtain a commuting triangle

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p} \arrow[rr] \arrow[dr] & & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p} \arrow[dl] \\
& \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} &
\end{tikzcd} \tag{3.5}$$

[7] We follow e. g. [HA, 4.1.1.10] and call Assoc-monoidal ∞-categories just *monoidal ∞-categories*.

[8] See [HA, 4.2] for this “pointwise” construction of ∞-categories of left modules.

that turns out to be a morphism of cocartesian fibrations over $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. Now $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ is the category we are looking for, but the fiber of the cocartesian fibration

$$\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$$

over $(\mathcal{C}, A)$ is $\mathrm{LMod}_A(\mathcal{C})^{\mathrm{op}}$. By passing to the opposite category fiberwise, and converting the morphism of cocartesian fibrations to a natural transformation of functors to $\mathcal{C}\mathrm{at}_\infty$, we obtain a natural transformation that evaluated at $(\mathcal{C}, A)$ is given by the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$.

Let us now give a brief overview of the subsections below. We will start in Section 3.1.3.1 with reviewing the relevant ∞-operads as well as some morphisms between them that we will need. In Section 3.1.3.2 we will then carry out the construction of LMod as a functor from $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ to $\mathcal{C}\mathrm{at}_\infty$ as outlined above. If $\mathcal{C}$ is a presentable monoidal ∞-category and A is an algebra in $\mathcal{C}$, then $\mathrm{LMod}_A(\mathcal{C})$ is also presentable by [HA, 4.2.3.7 (1)]. In Section 3.1.3.3 we will define a variant $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ whose objects can be thought of as as pairs $(\mathcal{C}, A)$ where $\mathcal{C}$ is a presentable monoidal ∞-category and A is an algebra in $\mathcal{C}$, and show that LMod lifts to a functor from $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ to $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

3.1.3.1 Review of the relevant operads

Diagram (3.4) is constructed by taking algebras in $\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ with respect to different ∞-operads, so we begin by discussing the relevant ∞-operads in this section.

Lurie defines in [HA, 4.2.1][9] an ∞-operad LM, which encodes the structure of a left module over an associative algebra: If $\mathcal{C}$ is a symmetric monoidal ∞-category, then we can interpret an LM-algebra in $\mathcal{C}$ as a pair (A, M), where A is an associative algebra in $\mathcal{C}$ and M is a left module over A. Indeed, if $\mathcal{C}$ is a 1-category, then this description holds literally, with the usual classical notions of associative algebras and left modules over them, see [HA, 4.2.1.4]. The underlying ∞-category of LM is a discrete 1-category with two objects, which we denote by $\mathfrak{a}$ and $\mathfrak{m}$ as in [HA, 4.2.1.1]. In the interpretation of an LM-algebra in $\mathcal{C}$ as a pair (A, M) as before, the underlying object of A is given by evaluation at $\mathfrak{a}$ and the underlying object of M is given by evaluation at $\mathfrak{m}$.

We next fix notation for some morphisms of ∞-operads defined in [HA, 4.2.1] that we will need.

Definition 3.1.3.1. We let

$$\iota_{\mathsf{Assoc}} \colon \mathsf{Assoc}^{\otimes} \to \mathsf{LM}^{\otimes}$$

[9]Note that our conventions are such that what we denote by $\mathsf{LM}^{\otimes}$ is what Lurie writes as $\mathbf{LM}^{\otimes}$ or $\mathcal{LM}^{\otimes}$ (as we do not notationally distinguish between 1-categories as objects of Cat and $\mathcal{C}\mathrm{at}_\infty$). We also use LM to both denote to $\mathsf{LM}^{\otimes}_{\langle 1 \rangle}$ as well as a shorthand to talk about the ∞-operad $\mathsf{LM}^{\otimes} \to \mathsf{Fin}_*$, which should not be confused with with the different type of object that Lurie denotes by $\mathbf{LM}$ (see [HA, 4.2.1.1]).

be the morphism of ∞-operads defined in [HA, 4.2.1.10] and

$$\nu_{\mathsf{Assoc}}\colon \mathsf{LM}^{\otimes} \to \mathsf{Assoc}^{\otimes}$$

the morphism of ∞-operads defined in [HA, 4.2.1.9]. $\diamondsuit$

Continuing with our discussion from before, these to morphisms of ∞-operads can be interpreted as follows: ι_{Assoc} induces a functor

$$\mathrm{Alg}_{\mathsf{LM}}(\mathcal{C}) \to \mathrm{Alg}_{\mathsf{Assoc}}(\mathcal{C})$$

that can be interpreted as mapping (A, M) to A (see [HA, 4.2.1.3]), and ν_{Assoc} induces a functor

$$\mathrm{Alg}_{\mathsf{Assoc}}(\mathcal{C}) \to \mathrm{Alg}_{\mathsf{LM}}(\mathcal{C})$$

that can be interpreted as mapping A to (A, A), with the second A in the pair being A considered as a left module over itself (see [HA, 4.2.1.5]).

We will also need to make use of the trivial ∞-operad Triv defined in [HA, 2.1.1.20], over which algebras are nothing more than objects of the underlying ∞-category. Specifically, the underlying ∞-category of Triv is discrete with a unique object, and for any ∞-operad $\mathcal{O}^{\otimes}$, the functor $\mathrm{Alg}_{\mathsf{Triv}}(\mathcal{O}) \to \mathcal{O}$ induced by evaluation at this object is an equivalence, see [HA, 2.1.3.5].

We can now define an additional morphism of ∞-categories that we will need.

Definition 3.1.3.2. We let

$$\iota_{\mathsf{Triv}}\colon \mathsf{Triv}^{\otimes} \to \mathsf{LM}^{\otimes}$$

be the morphism of ∞-operads that under the equivalence

$$\mathrm{Alg}_{\mathsf{Triv}}(\mathsf{LM}) \xrightarrow{\simeq} \mathsf{LM}^{\otimes}_{\langle 1 \rangle} = \{\mathfrak{a}, \mathfrak{m}\}$$

corresponds to the element $\mathfrak{m}$. $\diamondsuit$

The previous discussion implies that we can interpret the functor induced by ι_{Triv}

$$\mathrm{Alg}_{\mathsf{LM}}(\mathcal{C}) \to \mathrm{Alg}_{\mathsf{Triv}}(\mathcal{C})$$

as mapping (A, M) to the underlying object of M.

3.1.3.2 Construction of LMod

We write $\mathcal{O}^{\otimes} \boxplus \mathcal{O}'^{\otimes}$ for the coproduct of ∞-operads as discussed in [HA, 2.2.3]. We are now ready to construct diagram (3.4): The sequence of morphisms of ∞-operads

$$\begin{tikzcd} \mathsf{Assoc}^{\otimes} \arrow[r, "\iota_1"] \arrow[rrr, bend right, "\mathrm{id}_{\mathsf{Assoc}^{\otimes}}"'] & \mathsf{Assoc}^{\otimes} \boxplus \mathsf{Triv}^{\otimes} \arrow[r, "\iota_{\mathsf{Assoc}} \boxplus \iota_{\mathsf{Triv}}"] & \mathsf{LM}^{\otimes} \arrow[r, "\nu_{\mathsf{Assoc}}"] & \mathsf{Assoc}^{\otimes} \end{tikzcd} \tag{3.6}$$

induces as in Construction 3.1.2.12 and Proposition 3.1.2.14 on algebras in the universal family of Assoc-monoidal ∞-categories p^{Assoc} (see Definition 3.1.1.4) a commutative diagram as follows, where we shorten $\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ to $\widetilde{\mathrm{Mon}}$.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathsf{LM}/\mathsf{Assoc}}\Big(\widetilde{\mathrm{Mon}}\Big) \arrow[r] \arrow[dr] & \mathrm{Alg}_{\mathsf{Assoc}\boxplus\mathsf{Triv}/\mathsf{Assoc}}\Big(\widetilde{\mathrm{Mon}}\Big) \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}\Big(\widetilde{\mathrm{Mon}}\Big) \arrow[dl] \\
& \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) &
\end{tikzcd}$$

The functors to $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ are the respective functors that are all called $\mathrm{pr}_{\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)}$ in Definition 3.1.2.1, and which are cocartesian fibrations by Proposition 3.1.2.17. The above diagram precisely implements the description of diagram (3.4) given in the introduction to Section 3.1.3, as we will see below in Remark 3.1.3.4. This justifies making the following definition.

Definition 3.1.3.3. We define

$$\begin{aligned}
\mathcal{A}\mathrm{lg} &\coloneqq \mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}\Big(\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\Big)\\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj} &\coloneqq \mathrm{Alg}_{\mathsf{Assoc}\boxplus\mathsf{Triv}/\mathsf{Assoc}}\Big(\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\Big)\\
\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od} &\coloneqq \mathrm{Alg}_{\mathsf{LM}/\mathsf{Assoc}}\Big(\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\Big)
\end{aligned}$$

and denote the respective functors[10] $\mathrm{pr}_{\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)}$ to $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ by $q_{\mathcal{A}\mathrm{lg}}$, $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj}}$, and $q_{\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od}}$, respectively. Furthermore, we denote the functors induced by the morphisms of ∞-operads in (3.6)

$$\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj} \qquad \text{and} \qquad \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj} \to \mathcal{A}\mathrm{lg}$$

by $U^{\mathcal{L}\mathcal{M}\mathrm{od}}_{\mathcal{O}\mathrm{bj}}$ and $U^{\mathcal{O}\mathrm{bj}}$, respectively. ♢

Remark 3.1.3.4. We can summarize our previous discussions as follows.

Let $G\colon \mathcal{C} \to \mathcal{C}'$ be a monoidal functor of monoidal ∞-categories that we also consider as a morphism of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. By definition, we can identify the monoidal functor induced by G on fibers of the universal family of Assoc-monoidal ∞-categories p^{Assoc} (see Definition 3.1.1.4) with $G^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{C}'^{\otimes}$ itself.

As $U^{\mathcal{L}\mathcal{M}\mathrm{od}}_{\mathcal{O}\mathrm{bj}}$ and $U^{\mathcal{O}\mathrm{bj}}$ are morphisms of cocartesian fibrations over the ∞-category $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ by Proposition 3.1.2.21, we obtain an induced commutative diagram as follows.

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{C}} \arrow[r, "\left(U^{\mathcal{L}\mathcal{M}\mathrm{od}}_{\mathcal{O}\mathrm{bj}}\right)_{\mathcal{C}}"] \arrow[d, "G_!"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj}_{\mathcal{C}} \arrow[r, "\left(U^{\mathcal{O}\mathrm{bj}}\right)_{\mathcal{C}}"] \arrow[d, "G_!"'] & \mathcal{A}\mathrm{lg}_{\mathcal{C}} \arrow[d, "G_!"'] \\
\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{C}'} \arrow[r, "\left(U^{\mathcal{L}\mathcal{M}\mathrm{od}}_{\mathcal{O}\mathrm{bj}}\right)_{\mathcal{C}'}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj}_{\mathcal{C}'} \arrow[r, "\left(U^{\mathcal{O}\mathrm{bj}}\right)_{\mathcal{C}'}"'] & \mathcal{A}\mathrm{lg}_{\mathcal{C}'}
\end{tikzcd}$$

[10]See Definition 3.1.2.1.

Using Remark 3.1.2.4, Remark 3.1.2.15, and Remark 3.1.2.18 we can identify this diagram with the following commutative diagram induced by G and the morphisms of ∞-operads in (3.6).

$$\begin{tikzcd}
\mathrm{Alg}_{\mathsf{LM}/\mathsf{Assoc}}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{Alg}_{\mathsf{LM}/\mathsf{Assoc}}(G)"'] & \mathrm{Alg}_{\mathsf{Assoc}\boxplus\mathsf{Triv}/\mathsf{Assoc}}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{Alg}_{\mathsf{Assoc}\boxplus\mathsf{Triv}/\mathsf{Assoc}}(G)"'] & \mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}(G)"'] \\
\mathrm{Alg}_{\mathsf{LM}/\mathsf{Assoc}}(\mathcal{C}') \arrow[r] & \mathrm{Alg}_{\mathsf{Assoc}\boxplus\mathsf{Triv}/\mathsf{Assoc}}(\mathcal{C}') \arrow[r] & \mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}(\mathcal{C}')
\end{tikzcd}$$

The ∞-category of algebras over a coproduct of ∞-operads can be identified with the product of the ∞-categories of algebras by [HA, 2.2.3.6][11], and the ∞-category of algebras over Triv is by [HA, 2.1.3.6] equivalent to the underlying ∞-category. Considering also the definition of LMod [HA, 4.2.1.16] we can thus identify the above diagram with the following diagram

$$\begin{tikzcd}
\mathrm{LMod}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{LMod}(G)"'] & \mathrm{Alg}(\mathcal{C}) \times \mathcal{C} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{Alg}(G)\times G"'] & \mathrm{Alg}(\mathcal{C}) \arrow[d, "\mathrm{Alg}(G)"'] \\
\mathrm{LMod}(\mathcal{C}') \arrow[r] & \mathrm{Alg}(\mathcal{C}') \times \mathcal{C}' \arrow[r, "\mathrm{pr}_1"'] & \mathrm{Alg}(\mathcal{C}')
\end{tikzcd}$$

where the left horizontal functors are on the first factor the forgetful functors ι^*_{Assoc} from left modules to algebras from [HA, 4.2.1.13] that send a pair (A, M) with A an associative algebra and M a left module over it to A, and on the second factor the forgetful functors $\mathrm{ev}_{\mathfrak{m}}$ that send a pair (A, M) to M considered as just an object of $\mathcal{C}$ or $\mathcal{C}'$. ♢

Next we fix the variance of morphisms in the fibers.

Definition 3.1.3.5. By applying the functor

$$\begin{aligned}
&\mathrm{co}\mathcal{CF}\mathrm{ib}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)) \\
&\to \mathrm{Fun}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty), \mathcal{C}\mathrm{at}_\infty) \\
&\xrightarrow{(-^{\mathrm{op}})_*} \mathrm{Fun}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty), \mathcal{C}\mathrm{at}_\infty) \\
&\to \mathrm{co}\mathcal{CF}\mathrm{ib}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty))
\end{aligned}$$

to the morphisms $U^{\mathcal{LM}\mathrm{od}}_{\mathcal{O}\mathrm{bj}}$ and $U^{\mathcal{O}\mathrm{bj}}$ of cocartesian fibrations over the ∞-category $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ we obtain morphisms of cocartesian fibrations $V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ as depicted in the following diagram.

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p} \arrow[r, "V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}"] \arrow[dr, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p} \arrow[r, "V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}"] \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}"] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[dl, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
& \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) &
\end{tikzcd}$$

[11]To apply this in our situation, combine this with Proposition E.2.0.3 and the fact that pullbacks commute with products.

We define $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}$, $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}$, $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$, $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$, $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$, and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ as indicated in the diagram. We furthermore define $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$ to be the composition $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}} \circ V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$. ♢

Proposition 3.1.3.6. *$V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ from Definition 3.1.3.5 are cocartesian fibrations and $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ is a morphism of cocartesian fibrations over $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$.*

Furthermore, a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}$ is $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$-cocartesian precisely if it is the composition of a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$-cocartesian morphism with a $(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian morphism for $\mathcal{C}$ a monoidal ∞-category. The analogous statement holds for $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian morphisms. ♡

Proof. By [GHN15, 9.6][12,13], to show that $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ are cocartesian fibrations, it suffices to show the following.

(1) $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$, $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ are cocartesian fibrations.

(2) The functor $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$ maps $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$-cocartesian morphisms to morphisms that are $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian, and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ maps $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian morphisms to morphisms that are $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian.

(3) Let $\mathcal{C}$ be an object of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then the functor
$$\left(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}\right)_{\mathcal{C}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}_{\mathcal{C}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}}$$
induced by $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$ on fibers over $\mathcal{C}$ is a cocartesian fibration.

(3') Let $\mathcal{C}$ be an object of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then the functor
$$\left(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\right)_{\mathcal{C}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_{\mathcal{C}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}}$$
induced by $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ on fibers over $\mathcal{C}$ is a cocartesian fibration.

(4) Let
$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\beta"'] & N \arrow[d, "\gamma"] \\ M' \arrow[r, "\delta"'] & N' \end{tikzcd} \tag{3.7}$$
be a commuting diagram in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}$ lying over the following diagram in $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$.
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"'] & \mathcal{D} \arrow[d, "\mathrm{id}"] \\ \mathcal{C} \arrow[r, "\phi"'] & \mathcal{D} \end{tikzcd}$$

[12] The referenced proposition can be summarized as saying that a morphism of cocartesian fibrations over some ∞-category $\mathcal{C}$ is itself a cocartesian fibration if the restriction to fibers over any object of $\mathcal{C}$ is a cocartesian fibration, and the functor on fibers induced by a morphism in $\mathcal{C}$ preserves those cocartesian morphisms of the fibers.

[13] [GHN17] is the published version of [GHN15], but does not contain [GHN15, 9.6].

Assume that α and δ are $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$-cocartesian and that the morphism β is $(U^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian. Then γ is $(U^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{D}}$-cocartesian.

(4') Let

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\beta"'] & N \arrow[d, "\gamma"] \\ M' \arrow[r, "\delta"'] & N' \end{tikzcd}$$

be a commuting diagram in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}$ lying over the following diagram in $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"'] & \mathcal{D} \arrow[d, "\mathrm{id}"] \\ \mathcal{C} \arrow[r, "\phi"'] & \mathcal{D} \end{tikzcd}$$

Assume that α and δ are $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian and that the morphism β is $(U^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian. Then γ is $(U^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{D}}$-cocartesian.

From the proof of [GHN15, 9.6] it also follows that the $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian morphisms will be precisely the compositions of $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian morphisms with $(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian morphisms for $\mathcal{C}$ an object of the ∞-category $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. A similar statement holds for $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$. From this it follows that to show that $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ is a morphism of cocartesian fibrations from $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$ to $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ it will suffice to show the following.

(5) $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ sends $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$-cocartesian morphisms to $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian morphisms.

(6) Let $\mathcal{C}$ be an object of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then the functor $(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ maps morphisms that are $(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian to morphisms that are $(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian.

Proof of (1), (2) and (5): Hold by definition.

Proof of (3): Let $\mathcal{C}$ be an object of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. By Remark 3.1.3.4 we can identify the functor $(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ with the opposite of the following forgetful functor.

$$\iota^*_{\mathsf{Assoc}}\colon \mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$$

This forgetful functor is a cartesian fibration by [HA, 4.2.3.2], and thus $(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ is a cocartesian fibration. Furthermore, [HA, 4.2.3.2] also implies that a morphism in $\mathrm{LMod}(\mathcal{C})$ is $(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian if and only if ev_{m} of that morphism is an equivalence.

Proof of (3'): Just as above we can identify the functor $(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ using Remark 3.1.3.4 with the opposite of the left vertical functor in the following pullback diagram.

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{C}) \times \mathcal{C} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{C} \arrow[d] \\ \mathrm{Alg}(\mathcal{C}) \arrow[r] & * \end{tikzcd} \tag{3.8}$$

It follows by Proposition C.1.1.1 and [HTT, 2.4.1.5] that $(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ is a cocartesian fibration and that a morphism in $\mathrm{Alg}(\mathcal{C}) \times \mathcal{C}$ is $(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian if and only if pr_2 of that morphism is an equivalence.

Proof of (6): Follows immediately from the description of the respective cocartesian morphisms given above together with the description of the functor $(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ in Remark 3.1.3.4.

Proof of (4) and (4'): The two proofs are analogous, so we only prove (4).

We use the same notation as in the statement of (4), and by the description of $(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{D}}$-cocartesian morphisms in the proof of (3) we have to show that $\mathrm{ev}_{\mathfrak{m}}(\gamma)$ is an equivalence. Applying $\mathrm{ev}_{\mathfrak{m}}$ to diagram (3.7) we obtain

$$\begin{tikzcd} \mathrm{ev}_{\mathfrak{m}}(M) \arrow[r, "\mathrm{ev}_{\mathfrak{m}}(\alpha)"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}(\beta)"'] & \mathrm{ev}_{\mathfrak{m}}(N) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}(\gamma)"] \\ \mathrm{ev}_{\mathfrak{m}}(M') \arrow[r, "\mathrm{ev}_{\mathfrak{m}}(\delta)"'] & \mathrm{ev}_{\mathfrak{m}}(N') \end{tikzcd} \tag{3.9}$$

where by Proposition 3.1.2.17 the top and bottom horizontal morphisms are p^{Assoc}-cocartesian. Furthermore, the vertical morphism $\mathrm{ev}_{\mathfrak{m}}(\beta)$ is an equivalence, so by [HTT, 2.4.1.5 and 2.4.1.7] we can conclude that $\mathrm{ev}_{\mathfrak{m}}(\gamma)$ is also an equivalence. □

Remark 3.1.3.7. Let $\mathcal{C}$ be a monoidal ∞-category, and let us consider it as an object in $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then using Remark 3.1.3.4 we can identify the diagram

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}_{\mathcal{C}} \arrow[rr, "(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}"] \arrow[dr, "(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}"'] & & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_{\mathcal{C}} \arrow[dl, "(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}"] \\ & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}} & \end{tikzcd}$$

with the following diagram.

$$\begin{tikzcd} \mathrm{LMod}(\mathcal{C})^{\mathrm{op}} \arrow[rr, "(\iota^*_{\mathsf{Assoc}})^{\mathrm{op}} \times (\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}"] \arrow[dr, "(\iota^*_{\mathsf{Assoc}})^{\mathrm{op}}"'] & & \mathrm{Alg}(\mathcal{C})^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \arrow[dl, "\mathrm{pr}_1"] \\ & \mathrm{Alg}(\mathcal{C})^{\mathrm{op}} & \end{tikzcd}$$

Let A be an associative algebra in $\mathcal{C}$. Then it follows that we can identify the functor $(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_A = ((V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}})_A$, with the following functor.

$$(\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}\colon\ \mathrm{LMod}_A(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$$

Let us now turn to morphisms in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ and induced functors on fibers. As the functor $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ is a cocartesian fibration, every morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ is the composite of a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphism and a morphism in a fiber. Let $G\colon \mathcal{C} \to \mathcal{C}'$ be a monoidal functor of monoidal ∞-categories, considered as a morphism in $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then by Remark 3.1.3.4 the induced functor on fibers

$$G_!\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}'}$$

can be identified with the functor

$$\mathrm{Alg}(G)^{\mathrm{op}}\colon\ \mathrm{Alg}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Alg}(\mathcal{C}')^{\mathrm{op}}$$

which sends an object A of $\mathrm{Alg}(\mathcal{C})$ to an associative algebra $\mathrm{Alg}(G)(A)$ in $\mathcal{C}'$, that has underlying object $G(A)$, and so we will sometimes also write $G(A)$ for $\mathrm{Alg}(G)(A)$. Hence a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ from an object A in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}}$ to an object A' in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}'}$ consists of the composition of a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphism $A \to G(A)$ lying over a monoidal functor $G\colon \mathcal{C} \to \mathcal{C}'$ and a morphism of associative algebras $A' \to G(A)$.

Let us first consider a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphism $\widetilde{G}\colon A \to G(A)$ in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ lying over a monoidal functor $G\colon \mathcal{C} \to \mathcal{C}'$. By the description of cocartesian morphisms with respect to $V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ in Proposition 3.1.3.6, we know that the the functors induced by this morphism on fibers of the cocartesian fibrations $V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ are the restrictions of of the functors induced by G on fibers of the cocartesian fibrations $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}$ and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$. Thus using Remark 3.1.3.4 again we can identify the induced commutative diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}_A \arrow[r, "(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_A"] \arrow[d, "\widetilde{G}_!"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_A \arrow[d, "\widetilde{G}_!"] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}_{G(A)} \arrow[r, "(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{G(A)}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_{G(A)}
\end{tikzcd}$$

with the following commutative diagram.

$$\begin{tikzcd}
\mathrm{LMod}_A(\mathcal{C})^{\mathrm{op}} \arrow[r, "(\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}"] \arrow[d, "\mathrm{LMod}(G)^{\mathrm{op}}"'] & \mathcal{C}^{\mathrm{op}} \arrow[d, "G^{\mathrm{op}}"] \\
\mathrm{LMod}_{G(A)}(\mathcal{C}')^{\mathrm{op}} \arrow[r, "(\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}"'] & \mathcal{C}'^{\mathrm{op}}
\end{tikzcd}$$

Let us now consider a morphism $f\colon A' \to A$ of associative algebras in some monoidal ∞-category $\mathcal{C}$, considered as a morphism $\widetilde{f}\colon A \to A'$ in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}} \simeq \mathrm{Alg}(\mathcal{C})^{\mathrm{op}}$. Again using the description of cocartesian morphisms from Proposition 3.1.3.6 together with Remark 3.1.3.4 and [HA, 4.2.3.2] we can identify the commutative diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}_A \arrow[r, "\left(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\right)_A"] \arrow[d, "\widetilde{f}_!"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_A \arrow[d, "\widetilde{f}_!"] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}_{A'} \arrow[r, "\left(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\right)_{A'}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_{A'}
\end{tikzcd}$$

with the following commutative diagram.

$$\begin{tikzcd}
\mathrm{LMod}_A(\mathcal{C})^{\mathrm{op}} \arrow[r, "(\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}"] \arrow[d, "\mathrm{LMod}_f(\mathrm{id}_{\mathcal{C}})^{\mathrm{op}}"'] & \mathcal{C}^{\mathrm{op}} \arrow[d, "\mathrm{id}"] \\
\mathrm{LMod}_{A'}(\mathcal{C})^{\mathrm{op}} \arrow[r, "(\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}"'] & \mathcal{C}^{\mathrm{op}}
\end{tikzcd}$$

◊

Definition 3.1.3.8. By Proposition 3.1.3.6 we have a morphism of cocartesian fibrations over $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ as depicted in the following diagram.

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p} \arrow[rr, "V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}"] \arrow[dr, "V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}"'] & & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p} \arrow[dl, "V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}"] \\
& \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} &
\end{tikzcd}$$

Under the equivalence

$$\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \xrightarrow[\simeq]{\mathrm{Gr}^{-1}} \mathrm{Fun}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}, \mathcal{C}\mathrm{at}_\infty) \xrightarrow{(-^{\mathrm{op}})_*} \mathrm{Fun}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}, \mathcal{C}\mathrm{at}_\infty)$$

the cocartesian fibrations $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ correspond to functors

$$\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$$

that we will denote by LMod and pr, respectively. The morphism of cocartesian fibrations $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ corresponds to a natural transformation from LMod to pr that we will denote by $\mathrm{ev}_{\mathfrak{m}}$. ◊

Remark 3.1.3.9. Let $\mathcal{C}$ be a monoidal ∞-category and A an associative algebra in $\mathcal{C}$. Then Remark 3.1.3.7 shows that the natural transformation

$\mathrm{ev}_{\mathfrak{m}}$ as defined in Definition 3.1.3.8 evaluated at A (considered as an object of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}}$) can be identified with the usual forgetful functor[14]

$$\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$$

justifying the notation we chose for the two functors and the natural transformation. Furthermore, Remark 3.1.3.7 shows that LMod, pr, and $\mathrm{ev}_{\mathfrak{m}}$ are similarly compatible with usual notations on morphisms. ♢

3.1.3.3 LMod and colimits

In this section we put together some results from [HA] that imply that the functor LMod interacts well with the property of admitting and being compatible with colimits.

Definition 3.1.3.10 ([HA, 4.8.1.1 and 4.8.3.5] and [HTT, 5.5.3.1])**.** Let $\mathfrak{I}$ be a collection of small ∞-categories and $\mathcal{O}^{\otimes}$ an ∞-operad.

We define an ∞-category $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})$ together with a monomorphism to $\mathcal{C}\mathrm{at}_\infty$ as the monomorphism that under the construction of Remark B.6.0.1 corresponds to the replete subcategory of $\mathrm{Ho}\,\mathcal{C}\mathrm{at}_\infty$ whose objects are ∞-categories that admit $\mathfrak{I}$-indexed colimits[15] and whose morphisms are represented by those functors that preserve $\mathfrak{I}$-indexed colimits.

We similarly define an ∞-category $\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)$ together with a monomorphism to $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ as the monomorphism corresponding to the replete subcategory of $\mathrm{Ho}\,\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ whose objects are the $\mathcal{O}$-monoidal ∞-categories that are compatible with $\mathfrak{I}$-indexed colimits in the sense of [HA, 3.1.1.19 and 3.1.1.18], and whose morphisms are represented by $\mathcal{O}$-monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ such that for every object X of $\mathcal{O}$ the underlying functor of ∞-categories $\mathcal{C}_X \to \mathcal{D}_X$ preserves $\mathfrak{I}$-indexed colimits.

Now let $\mathfrak{J}$ be the collection of all small ∞-categories. We denote by $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ the full subcategory of $\mathcal{C}\mathrm{at}_\infty(\mathfrak{J})$ spanned by the presentable ∞-categories[16].

We furthermore define $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)$ to be the full subcategory of the ∞-category $\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{J}}(\mathcal{C}\mathrm{at}_\infty)$ spanned by $\mathcal{O}$-monoidal ∞-categories which are presentable in the sense of [HA, 3.4.4.1]. ♢

Definition 3.1.3.11. Let $\mathfrak{I}$ be a collection of small ∞-categories. We define $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}$ and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ via the following pullback diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}} \arrow[rr] \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}"'] & & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}_{\mathsf{Assoc}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty) \arrow[rr] & & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)
\end{tikzcd}$$

[14]Here $\mathrm{LMod}_A(\mathcal{C})$ refers to what is defined in [HA, 4.2.1.13].
[15]This means that they must admit $\mathcal{I}$-indexed colimits for every $\mathcal{I}$ in $\mathfrak{I}$.
[16]See [HTT, 5.5]

where the lower horizontal functor is the inclusion from Definition 3.1.3.10. We similarly define $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$ via the following pullback diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \arrow[r] \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)
\end{tikzcd}$$

where the lower horizontal functor is the inclusion from Definition 3.1.3.10. ♢

Proposition 3.1.3.12 ([HA, 4.2.3.5 and 4.2.3.7]). *Assume that $\mathfrak{I}$ is a collection of small ∞-categories. Then the restriction of the natural transformation $\mathrm{ev}_{\mathfrak{m}}$ to $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}$ factors through $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})$. Analogously, the restriction to $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ factors through $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The situation is depicted in the following diagram.*

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \arrow[r, dashed, bend left, "\mathrm{LMod}"] \arrow[r, dashed, bend right, "\mathrm{pr}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] \arrow[d] & \mathcal{P}\mathrm{r}^{\mathrm{L}} \arrow[d] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}} \arrow[r, dashed, bend left, "\mathrm{LMod}"] \arrow[r, dashed, bend right, "\mathrm{pr}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] \arrow[d] & \mathcal{C}\mathrm{at}_\infty(\mathfrak{I}) \arrow[d] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[r, bend left, "\mathrm{LMod}"] \arrow[r, bend right, "\mathrm{pr}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] & \mathcal{C}\mathrm{at}_\infty
\end{tikzcd} \tag{3.10}$$

As suggested by the diagram, will denote the induced functors and natural transformations by the same name again. ♡

Proof. Let $E\colon [1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ be the functor encoding the natural transformation $\mathrm{ev}_{\mathfrak{m}}$. By definition the right vertical functors in diagram (3.10) are monomorphisms, so by Proposition B.4.3.1 the composition

$$[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}} \longrightarrow [1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \xrightarrow{E} \mathcal{C}\mathrm{at}_\infty$$

can be lifted to $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})$ if and only if $\mathrm{Im}(E \circ (\mathrm{id}_{[1]} \times (\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p})))$ is contained in $\mathrm{Im}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}) \to \mathcal{C}\mathrm{at}_\infty)$, and similarly for the lift to $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

In light of Remark 3.1.3.7 and Remark 3.1.3.9, this boils down to the following statements for any ∞-category $\mathcal{I}$, monoidal ∞-category $\mathcal{C}$, associative algebra A in $\mathcal{C}$, monoidal functor $G\colon \mathcal{C} \to \mathcal{D}$, and morphism of associative algebras $g\colon B \to G(A)$.

(1) If the monoidal ∞-category $\mathcal{C}$ is compatible with $\mathcal{I}$-indexed colimits in the sense of [HA, 3.1.1.18], then $\mathrm{LMod}_A(\mathcal{C})$ admits $\mathcal{I}$-indexed colimits.

(2) If $\mathcal{C}$ is a presentable monoidal ∞-category in the sense of [HA, 3.4.4.1], then $\mathrm{LMod}_A(\mathcal{C})$ is presentable.

(3) If the monoidal ∞-category $\mathcal{C}$ is compatible with $\mathcal{I}$-indexed colimits, then the forgetful functor

$$\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$$

preserves $\mathcal{I}$-indexed colimits.

(4) If $\mathcal{C}$ admits and G preserves $\mathcal{I}$-indexed colimits, then the functor induced by G and g

$$\mathrm{LMod}_g(G) \colon \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_B(\mathcal{D})$$

also preserves $\mathcal{I}$-indexed colimits.

Proof of (1): This is [HA, 4.2.3.5 (1)].
Proof of (2): This is [HA, 4.2.3.7 (1)].
Proof of (3): This is [HA, 4.2.3.5 (2)].
Proof of (4): This is a slight generalization of [HA, 4.2.3.7 (2)]. From the natural transformation $\mathrm{ev}_{\mathfrak{m}}$ we obtain a commuting diagram

$$\begin{tikzcd}
\mathrm{LMod}_A(\mathcal{C}) \arrow[r, "\mathrm{LMod}_g(G)"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"'] & \mathrm{LMod}_B(\mathcal{D}) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\
\mathcal{C} \arrow[r, "G"'] & \mathcal{D}
\end{tikzcd}$$

where by assumption the lower horizontal functor preserves $\mathcal{I}$-indexed colimits. It then follows immediately from [HA, 4.2.3.5 (2)] that the top horizontal functor also does so. □

3.2 LMod and monoidality

In this section we will start in Section 3.2.1 by showing that the functor $\mathrm{LMod} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ preserves products and can thus be upgraded to a symmetric monoidal functor with respect to the respective cartesian symmetric monoidal structures. Furthermore, this induces a symmetric monoidal structure on the restriction $\mathrm{LMod} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see Proposition 3.1.3.12). This will be shown in Section 3.2.3, after we discuss the relevant symmetric monoidal ∞-categories in Section 3.2.2.

3.2.1 LMod and products

In this short section we show that $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ preserves products and can thus be upgraded to a symmetric monoidal functor with respect to the respective cartesian symmetric monoidal structures.

Proposition 3.2.1.1. *The cocartesian fibrations*

$$\begin{aligned} q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p} &\to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\\ q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p} &\to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\\ q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} &\to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \end{aligned}$$

have fibers compatible with products in the sense of Definition C.2.0.1. ♡

Proof. Proposition F.2.0.1 implies that the ∞-category $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ admits products. Combining Remark C.2.0.2 with the fact that the functor $(-)^{\mathrm{op}}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ is an equivalence and thus preserves products we are reduced to showing that $q_{\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od}}$, $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj}}$, and $q_{\mathcal{A}\mathrm{lg}}$ have fibers compatible with products. But this follows from combining Proposition 3.1.2.22 with Proposition 3.1.1.9. □

Proposition 3.2.1.2. *The cocartesian fibrations*[17] $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$ *and* $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ *from Definition 3.1.3.5 have fibers compatible with products in the sense of Definition C.2.0.1.* ♡

Proof. These cocartesian fibrations are by definition also morphisms of cocartesian fibrations over $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. As those cocartesian fibrations have fibers compatible with products by Proposition 3.2.1.1, the statement follows from Proposition C.2.0.4. □

Proposition 3.2.1.3. *The* ∞*-category* $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ *admits all products and the functors*

$$\mathrm{LMod}, \mathrm{pr}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$$

preserve products. ♡

Proof. Follows directly from Proposition 3.2.1.2, Remark C.2.0.2, and the fact that $(-)^{\mathrm{op}}$ is an equivalence and thus preserves products. □

Remark 3.2.1.4. Let $\mathcal{C}$ and $\mathcal{C}'$ be monoidal ∞-categories and A and A' associative algebras in $\mathcal{C}$ and $\mathcal{C}'$, respectively. Then Proposition C.2.0.3 and Proposition 3.2.1.1 imply that the pair (A, A') considered as an object in

$$(\mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}'))^{\mathrm{op}} \simeq \mathrm{Alg}(\mathcal{C}\times\mathcal{C}')^{\mathrm{op}} \simeq \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}\times\mathcal{C}'}$$

is a product in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ of A and A'.

That LMod preserves products by Proposition 3.2.1.3 means in particular that there is an equivalence as follows.

$$\mathrm{LMod}_{(A,A')}(\mathcal{C}\times\mathcal{C}') \simeq \mathrm{LMod}_A(\mathcal{C}) \times \mathrm{LMod}_{A'}(\mathcal{C}')$$ ♢

[17] That they are cocartesian fibrations was shown in Proposition 3.1.3.6.

3.2.2 $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ as a symmetric monoidal ∞-category

To be able to make sense of the claim that $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ should be upgradable to a symmetric monoidal functor, we first need to define symmetric monoidal structures on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and in particular on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$. This is what we will discuss in this section.

We will start in Section 3.2.2.1 by recalling the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, before discussing the symmetric monoidal structure on the ∞-category $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ in Section 3.2.2.2. While we will be able to define $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ directly, showing that this is indeed a symmetric monoidal structure on $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ will require a fair amount of work comparing it to $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\otimes}$, the induced symmetric monoidal structure on algebras in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The reason why we bother to do this rather than just using $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\otimes}$ is that $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ is a better fit when discussing the symmetric monoidal structure on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$, which we do in Section 3.2.2.3.

3.2.2.1 The symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$

In this section we recall the symmetric monoidal structures on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})$ for $\mathfrak{I}$ a collection of small ∞-categories, closely following [HA, 4.8.1].

Definition 3.2.2.1 ([HA, 4.8.1.2, 4.8.1.4 and 4.8.1.15])**.** Let $\mathfrak{I}$ be a collection of small ∞-categories. We define a monomorphism

$$\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes} \to \mathcal{C}\mathrm{at}_\infty^{\times}$$

corresponding as in Remark B.6.0.1 to a replete subcategory H of $\mathrm{Ho}(\mathcal{C}\mathrm{at}_\infty^{\times})$ that we describe next.

An object $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ of $(\mathcal{C}\mathrm{at}_\infty)^{\times}_{\langle n\rangle}$ with $\mathcal{C}_1, \ldots, \mathcal{C}_n$ ∞-categories is to be an object of H if and only if each $\mathcal{C}_i$ admits all $\mathfrak{I}$-indexed colimits. A morphism $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n \to \mathcal{C}'_1 \oplus \cdots \oplus \mathcal{C}'_m$ lying over $\varphi\colon \langle n\rangle \to \langle m\rangle$ is to be in H if and only if for each $1 \le j \le m$ the associated functor

$$\prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{C}'_j$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

Now let $\mathfrak{J}$ be the collection of all small ∞-categories. We define $\mathcal{P}\mathrm{r}^{\mathrm{L}^{\otimes}}$ to be the full subcategory of $\mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^{\otimes}$ spanned by those objects $\mathcal{C} \simeq \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ where each $\mathcal{C}_i$ is presentable. $\diamondsuit$

Remark 3.2.2.2. It is clear from the definitions that the functors

$$(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}))^{\otimes}_{\langle 1\rangle} \to (\mathcal{C}\mathrm{at}_\infty)^{\times}_{\langle 1\rangle} \qquad \text{and} \qquad (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\otimes}_{\langle 1\rangle} \to (\mathcal{C}\mathrm{at}_\infty)(\mathfrak{J})^{\times}_{\langle 1\rangle}$$

which are induced by the functors defined in Definition 3.2.2.1 can be identified with the functors

$$\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}) \to \mathcal{C}\mathrm{at}_\infty \qquad \text{and} \qquad \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{J})$$

from Definition 3.1.3.10. ♢

Proposition 3.2.2.3 ([HA, 4.8.1.4 and 4.8.1.15])**.** *Let $\mathfrak{J}$ be the collection of all small ∞-categories, let $\mathfrak{I}$ be a subcollection of $\mathfrak{J}$ and $\mathfrak{I}'$ a subcollection of $\mathfrak{I}$. Then the following statements hold.*

(0) The monomorphism $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$ from Definition 3.2.2.1 factors through the monomorphism $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}')^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$ from Definition 3.2.2.1. The lift obtained in this manner is also a monomorphism.

(1) The compositions
$$\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times \to \mathsf{Fin}_*$$
and
$$\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty^\times \to \mathsf{Fin}_*$$
where the first functor is the monomorphism from Definition 3.2.2.1 and the second functor is the canonical morphism of ∞-operads, are cocartesian fibrations of ∞-operads.

(2) The functors
$$\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^\otimes \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I}')^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$$
from Definition 3.2.2.1 and (0) are lax symmetric monoidal with respect to the symmetric monoidal structures from (1).

(3) A morphism in $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^\otimes$ or $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes}$ is inert if an only if its image in $\mathcal{C}\mathrm{at}_\infty^\times$ is inert.

(4) The functor
$$\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes$$
is symmetric monoidal with respect to the symmetric monoidal structure from (1).

(5) A morphism in $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes}$ is cocartesian with respect to the canonical morphism of ∞-operads $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathsf{Fin}_*$ if and only if its image in $\mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes$ is cocartesian with respect to the canonical morphism of ∞-operads $\mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes \to \mathsf{Fin}_*$. ♡

Proof. Proof of (0): Immediate from the definition together with Proposition B.4.3.1 and Proposition B.1.2.1.

Proof of (1) and (2) for the compositions to $\mathcal{C}\mathrm{at}_\infty^\times$: This is [HA, 4.8.1.4 and 4.8.1.15].

Proof of (4): This is [HA, 4.8.1.15].

Proof of(3) and (5): The functors $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$ and $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty^\times$ were already shown to be morphisms of ∞-operads, and $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes$

was already shown to be symmetric monoidal. As these functors are also monomorphisms[18] and hence conservative by Proposition B.4.1.2, we can apply Proposition E.1.2.1 to deduce the claims.

Proof of the rest of (2): Follows directly from (3). □

3.2.2.2 The symmetric monoidal structure on $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)$

In this section we construct the symmetric monoidal structure on the ∞-category $\mathrm{Mon}_{\mathsf{Assoc}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)$. While defining $\mathrm{Mon}_{\mathsf{Assoc}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ is relatively straightforward, showing that this defines a symmetric monoidal structure (which is Proposition 3.2.2.10) will require a bit more work, requiring a comparison result between $\mathrm{Mon}_{\mathsf{Assoc}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ and $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\otimes}$ that will be shown in Proposition 3.2.2.8.

Definition 3.2.2.4 ([HA, 4.8.5.14])**.** Let $\mathfrak{I}$ be a collection of small ∞-categories and $\mathcal{O}$ an ∞-operad. We define a monomorphism[19]

$$\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$

corresponding as in Remark B.6.0.1 to a replete subcategory H of the 1-category of $\mathrm{Ho}(\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times})$ that we describe next.

An object $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ of $\mathrm{Ho}(\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times})$ is to be in H if and only if for each $1 \leq i \leq n$ the $\mathcal{O}$-monoidal ∞-category $\mathcal{C}_i$ is compatible with $\mathfrak{I}$-indexed colimits in the sense of [HA, 3.1.1.19 and 3.1.1.18]. A morphism in $\mathrm{Ho}(\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times})$ between two objects of H is to be in H if and only if $\mathrm{Ho}((\mathrm{ev}_{\langle 1\rangle})^{\times})$ maps that morphism to a morphism in the replete image $\mathrm{Im}(\mathrm{Ho}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes}) \to \mathrm{Ho}(\mathcal{C}\mathrm{at}_\infty^{\times}))$[20].

Now let $\mathfrak{I}$ be the collection of all small ∞-categories. We then define $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ to be the full subcategory of $\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ spanned by those objects $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ for which for each $1 \leq i \leq n$ the $\mathcal{O}$-monoidal ∞-category $\mathcal{C}_i$ is presentable $\mathcal{O}$-monoidal in the sense of [HA, 3.4.4.1]. ♢

Remark 3.2.2.5. It is clear from the definitions that the functors

$$\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}_{\langle 1\rangle} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}_{\langle 1\rangle}$$

and

$$\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}_{\langle 1\rangle} \to \mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\times}_{\langle 1\rangle}$$

which are induced by the functors defined in Definition 3.2.2.4 can be identified with the functors

$$\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$

[18] That $\mathcal{P}\mathrm{r}^{\mathrm{L}^{\otimes}} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes}$ is a monomorphism follows from Proposition B.4.4.1 and that $\mathcal{P}\mathrm{r}^{\mathrm{L}^{\otimes}} \to \mathcal{C}\mathrm{at}_\infty^{\times}$ is a monomorphism then follows from Proposition B.1.2.1.

[19] For products in $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ see Proposition F.2.0.1.

[20] This condition boils down to associated underlying functors of the form $\prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{C}'_j$ preserving $\mathfrak{I}$-indexed colimits separately in each variable.

and

$$\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)$$

from Definition 3.1.3.10. ♢

Remark 3.2.2.6. It follows directly from the definitions in Definition 3.2.2.4 together with Proposition B.4.3.1 that for $\mathfrak{I}$ a collection of small ∞-categories and $\mathfrak{I}'$ a subcollection of $\mathfrak{I}$ the monomorphism

$$\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$

factors through the monomorphism

$$\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}'}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$

and the lift is by Proposition B.1.2.1 again a monomorphism. ♢

For easier reference we introduce some notation that we are going to use in some statements and proof below.

Notation 3.2.2.7. The following notation will be used only when specifically invoked, but not elsewhere. In the notation below, $\mathfrak{I}$ will be a collection of small ∞-categories, $\mathfrak{I}'$ a subcollection of $\mathfrak{I}$, and $\mathcal{O}$ an ∞-operad.

- Some of the below notations will use a superscript or subscript $\mathfrak{I}$. In the case $\mathfrak{I} = \emptyset$ we will allow ourselves to drop this superscript or subscript.
- We denote by
$$p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$$
the canonical morphism of ∞-operads.
- We let α be the bifunctor defined as the following composition.
$$\mathsf{Fin}_* \times \mathcal{O}^{\otimes} \xrightarrow{\mathrm{id}_{\mathsf{Fin}_*} \times p_{\mathcal{O}}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{-\wedge -} \mathsf{Fin}_*$$
- We denote by
$$p_{\mathfrak{I}}\colon \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes} \to \mathsf{Fin}_*$$
$$p_{\mathcal{P}\mathrm{r}}\colon \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \to \mathsf{Fin}_*$$
$$p_{\mathrm{Alg},\mathfrak{I}}\colon\ \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}))^{\otimes} \to \mathsf{Fin}_*$$
$$p_{\mathrm{Alg},\mathcal{P}\mathrm{r}}\colon\ \mathrm{Alg}_{\mathcal{O}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}\right)^{\otimes} \to \mathsf{Fin}_*$$
the canonical morphism of ∞-operads, where for $p_{\mathrm{Alg},\mathfrak{I}}$ and $p_{\mathrm{Alg},\mathcal{P}\mathrm{r}}$ this is with respect to the induced symmetric monoidal structures as in Proposition E.4.2.3 with respect to the bifunctor α.

- We will denote the lax symmetric monoidal functors from Proposition 3.2.2.3 (2) as indicated below.

$$\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \xrightarrow{(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}} \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})^{\otimes} \xrightarrow{(\Phi^{\mathfrak{I}}_{\mathfrak{I}'})^{\otimes}} \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I}')^{\otimes} \xrightarrow{\left(\Phi^{\mathfrak{I}'}\right)^{\otimes}} \mathcal{C}\mathrm{at}_{\infty}^{\times}$$

  We set $(\Phi^{\mathcal{P}\mathrm{r}})^{\otimes} \coloneqq (\Phi^{\mathfrak{I}})^{\otimes} \circ (\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}$.

- We will denote the monomorphisms from Definition 3.2.2.4 and Remark 3.2.2.6 as indicated below.

$$\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \xrightarrow{(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}} \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \xrightarrow{(\Psi^{\mathfrak{I}}_{\mathfrak{I}'})^{\otimes}} \mathrm{Mon}^{\mathfrak{I}'}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$$
$$\xrightarrow{\left(\Psi^{\mathfrak{I}'}\right)^{\otimes}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\times}$$

  We set $(\Psi^{\mathcal{P}\mathrm{r}})^{\otimes} \coloneqq (\Psi^{\mathfrak{I}})^{\otimes} \circ (\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}$.

- We denote by

$$p_{\mathrm{Mon}} \colon \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\times} \to \mathsf{Fin}_{*}$$

  the canonical morphism of ∞-operads, and define $p_{\mathrm{Mon},\mathfrak{I}}$ and $p_{\mathrm{Mon},\mathcal{P}\mathrm{r}}$ as the following compositions.

$$p_{\mathrm{Mon},\mathfrak{I}} \coloneqq p_{\mathrm{Mon}} \circ (\Psi^{\mathfrak{I}})^{\otimes}$$
$$p_{\mathrm{Mon},\mathcal{P}\mathrm{r}} \coloneqq p_{\mathrm{Mon}} \circ (\Psi^{\mathcal{P}\mathrm{r}})^{\otimes}$$

- The cartesian symmetric monoidal structure $\mathcal{C}\mathrm{at}_{\infty}^{\times}$ comes with a cartesian structure

$$\pi \colon \mathcal{C}\mathrm{at}_{\infty}^{\times} \to \mathcal{C}\mathrm{at}_{\infty}$$

  that we will denote by π, see [HA, 2.4.1.5]. Similarly, we denote the cartesian structure

$$\pi_{\mathrm{Mon}} \colon \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\times} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$$

  of $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\times}$ by π_{Mon}. ♢

Proposition 3.2.2.8 ([HA, 4.8.5.16 (1)]). *In this proposition we will make use of Notation 3.2.2.7.*

Let $\mathfrak{I}$ be a collection of small ∞-categories, $\mathfrak{I}'$ a subcollection of $\mathfrak{I}$, and $\mathcal{O}$ an ∞-operad. Then there is a commutative diagram as follows such that the *horizontal functors are equivalences*

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}\right)^{\otimes} \arrow[r, "\Theta^{\otimes}_{\mathcal{P}\mathrm{r}}", "\simeq"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}\left(\Phi^{\mathcal{P}\mathrm{r}}_{\mathcal{J}}\right)^{\otimes}"'] & \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[d, "\left(\Psi^{\mathcal{P}\mathrm{r}}_{\mathcal{J}}\right)^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{J}))^{\otimes} \arrow[r, "\Theta^{\otimes}_{\mathcal{J}}", "\simeq"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}\left(\Phi^{\mathcal{J}}_{\mathcal{J}'}\right)^{\otimes}"'] & \mathrm{Mon}^{\mathcal{J}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[d, "\left(\Psi^{\mathcal{J}}_{\mathcal{J}'}\right)^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{J}'))^{\otimes} \arrow[r, "\Theta^{\otimes}_{\mathcal{J}'}", "\simeq"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}\left(\Phi^{\mathcal{J}'}\right)^{\otimes}"'] & \mathrm{Mon}^{\mathcal{J}'}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[d, "\left(\Psi^{\mathcal{J}'}\right)^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[r, "\Theta^{\otimes}", "\simeq"'] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\times}
\end{tikzcd} \tag{3.11}$$

The functor $\Theta^{\otimes}$ can be chosen in such a way that for every object X in $\mathcal{O}$ there is a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[rr, "\Theta^{\otimes}", "\simeq"'] \arrow[dr, "\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}"] \arrow[ddr, bend right, "p_{\mathrm{Alg}}"'] & & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\times} \arrow[dl, "(\mathrm{ev}_X)^{\times}"'] \arrow[ddl, bend left, "p_{\mathrm{Mon}}"] \\
& \mathcal{C}\mathrm{at}^{\times}_{\infty} \arrow[d, dashed, "p"] & \\
& \mathsf{Fin}_* &
\end{tikzcd} \tag{3.12}$$

where the functors to $\mathcal{C}\mathrm{at}^{\times}_{\infty}$ are the symmetric monoidal forgetful functors[21]*.*
Furthermore, $\Theta^{\otimes}$ can be chosen such that the underlying equivalence

$$\Theta\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$$

is the equivalence from [HA, 2.4.2.5], i. e. there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[r, "\Theta"] \arrow[d] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[dd] \\
\mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}^{\times}_{\infty}\right) \arrow[d] & \\
\mathrm{Fun}\left(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}^{\times}_{\infty}\right) \arrow[r, "\pi_*"'] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd} \tag{3.13}$$

[21] See Proposition E.4.2.3 (5) for the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \to \mathcal{C}\mathrm{at}^{\times}_{\infty}$ that is given by evaluation at X and Proposition F.2.0.1 for the forgetful functor $\mathrm{ev}_X\colon \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathcal{C}\mathrm{at}_{\infty}$ preserving products and hence inducing a functor $(\mathrm{ev}_X)^{\times}\colon \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\times} \to \mathcal{C}\mathrm{at}^{\times}_{\infty}$.

where the vertical functors are the canonical projections or inclusions. ♡

Proof. We start by constructing $\Theta^{\otimes}$ together with diagram (3.12).

By Proposition F.3.0.2 there is a functor π_{Alg} making the following diagram commute

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[rr, dashed, "\pi_{\mathrm{Alg}}"] \arrow[d, "\iota_{\mathrm{Alg}}"'] & & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \arrow[dd] \\
\mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty^{\times}\big) \times_{\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \arrow[d, "\mathrm{pr}_1"'] & & \\
\mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty^{\times}\big) \arrow[rr, "\pi_*"'] & & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty)
\end{tikzcd} \tag{3.14}$$

where ι_{Alg} is as in Proposition E.4.2.3 and the unlabeled vertical functor on the right is the inclusion. Furthermore, Proposition F.3.0.2 also shows that π_{Alg} is a cartesian structure. Applying [HA, 2.4.1.7] we obtain a symmetric monoidal functor $\Theta^{\otimes}$ making the following diagram commute.

$$\begin{tikzcd}
 & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) & \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[ur, "\pi_{\mathrm{Alg}}"] \arrow[rr, "\Theta^{\otimes}"] \arrow[dr, "p_{\mathrm{Alg}}"'] & & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \arrow[ul, "\pi_{\mathrm{Mon}}"'] \arrow[dl, "p_{\mathrm{Mon}}"] \\
 & \mathsf{Fin}_* &
\end{tikzcd} \tag{3.15}$$

Of diagram (3.12) that we want to construct we have thus constructed $\Theta^{\otimes}$ as a functor over Fin_*. The two forgetful functors to $\mathcal{C}\mathrm{at}_\infty^{\times}$ are already given as functors over Fin_*, so it remains to construct a filler for the small triangle at the top, considered as a diagram over Fin_*.

So let X be an object of $\mathcal{O}$. As both the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}} \mathcal{C}\mathrm{at}_\infty^{\times}$$

as well as the composition

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\Theta^{\otimes}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \xrightarrow{(\mathrm{ev}_X)^{\times}} \mathcal{C}\mathrm{at}_\infty^{\times}$$

are symmetric monoidal, giving a homotopy between them as symmetric monoidal functors (and hence functors over Fin_*) is by [HA, 2.4.1.7] equivalent to giving a homotopy of weak cartesian structures between

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}} \mathcal{C}\mathrm{at}_\infty^{\times} \xrightarrow{\pi} \mathcal{C}\mathrm{at}_\infty$$

and the following composition.

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\Theta^{\otimes}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \xrightarrow{(\mathrm{ev}_X)^{\times}} \mathcal{C}\mathrm{at}_\infty^{\times} \xrightarrow{\pi} \mathcal{C}\mathrm{at}_\infty$$

Such a homotopy is encoded in the outer commutative diagram depicted below

$$\begin{tikzcd}
 & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \arrow[r, "(\mathrm{ev}_X)^{\times}"] \arrow[d, "\pi_{\mathrm{Mon}}"] & \mathcal{C}\mathrm{at}_\infty^{\times} \arrow[d, "\pi"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[ur, "\Theta^{\otimes}"] \arrow[r, "\pi_{\mathrm{Alg}}"'] \arrow[d, "\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}"'] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, "\mathrm{ev}_X"'] \arrow[d] & \mathcal{C}\mathrm{at}_\infty \arrow[d, equal] \\
\mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty^{\times}\big) \arrow[r, "\pi_*"'] \arrow[dr, "\mathrm{ev}_X"'] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[r, "\mathrm{ev}_X"'] & \mathcal{C}\mathrm{at}_\infty \\
 & \mathcal{C}\mathrm{at}_\infty^{\times} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where the upper left commutative triangle is the one from (3.15), the upper right commutative square arises from the functoriality of the construction $(-)^{\times}$, the middle left commutative square is the one from (3.14), the middle lower commutative square is one by definition, and the bottom commutative square arises from naturality of ev_X.

We have now constructed $\Theta^{\otimes}$ as a functor over Fin_* as well as diagram (3.12) for every object X of $\mathcal{O}$. Let us now consider diagram (3.13) concerning the underlying functor Θ. The composition of the inclusion of

$$\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}_{\langle 1 \rangle}$$

into $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$ with π_{Mon} is by definition homotopic to the identity, so we obtain from the commutative diagram (3.15) a homotopy between Θ and the the composition

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\pi_{\mathrm{Alg}}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$

The desired commutative diagram (3.13) can now be obtained by combining this with commutative diagram (3.14).

From this description of Θ it now follows from [HA, 2.4.2.5] that Θ is an equivalence. Using that $\Theta^{\otimes}$ is symmetric monoidal we can thus conclude from [HA, 2.1.3.8] that $\Theta^{\otimes}$ is an equivalence as well.

To construct diagram (3.11), we will show the following claims for each collection of small ∞-categories $\mathfrak{I}$.

(A) $(\Psi^{\mathfrak{I}})^{\otimes}$ is a monomorphism.

(B) $\mathrm{Alg}(\Phi^{\mathfrak{I}})^{\otimes}$ is a monomorphism.

(C) $\mathrm{Im}(\mathrm{Ho}(\Theta^{\otimes} \circ \mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathfrak{J}})^{\otimes}))$ is equal to $\mathrm{Im}(\mathrm{Ho}((\Psi^{\mathfrak{J}})^{\otimes}))$.

Let us assume claims (A), (B), and (C) for the moment and deduce the statements we have to prove.

Existence of an equivalences $\Theta^{\otimes}_{\mathfrak{J}}$ together with commutative squares of the form

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{J}))^{\otimes} \arrow[r, "\Theta^{\otimes}_{\mathfrak{J}}", "\simeq"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathfrak{J}})^{\otimes}"'] & \mathrm{Mon}^{\mathfrak{J}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[d, "(\Psi^{\mathfrak{J}})^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[r, "\Theta^{\otimes}", "\simeq"'] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}
\end{tikzcd}$$

then follows from Proposition B.4.3.1, see also Remark B.6.0.1. That there is a compatibility square between $\Theta^{\otimes}_{\mathfrak{J}'}$ and $\Theta^{\otimes}_{\mathfrak{J}}$ follows immediately from the uniqueness part of Proposition B.4.3.1 using that $(\Psi^{\mathfrak{J}'})^{\otimes}$ is a monomorphism.

Finally, we need to construct the dashed equivalence fitting into the square depicted at the top of the commutative diagram below, where $\mathfrak{J}$ is the collection of all small ∞-categories, and X is an object of $\mathcal{O}$.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}\right)^{\otimes} \arrow[rr, dashed, "\Theta^{\otimes}_{\mathcal{P}\mathrm{r}}", "\simeq"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}"'] & & \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[d, "(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{J}))^{\otimes} \arrow[rr, "\Theta^{\otimes}_{\mathfrak{J}}", "\simeq"'] \arrow[dr, "\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}} \circ \mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathfrak{J}})^{\otimes}"'] & & \mathrm{Mon}^{\mathfrak{J}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[dl, "(\mathrm{ev}_X)^{\times} \circ (\Psi^{\mathfrak{J}})^{\otimes}"] \\
& \mathcal{C}\mathrm{at}^{\times}_\infty &
\end{tikzcd} \tag{3.16}$$

The functor $(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ is by definition the inclusion of the fully faithful subcategory of $\mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^{\otimes}$ spanned by objects $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ such that the ∞-category $\Phi^{\mathfrak{J}}(\mathcal{C}_i)$ is presentable for each $1 \leq i \leq n$, see Definition 3.2.2.1. It follows from the definition of the induced functor $\mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ in Remark E.4.2.2 together with Proposition B.3.0.1, Proposition B.5.1.1, Remark B.5.1.2, and Proposition B.5.3.1, that $\mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ is again a fully faithful functor with essential image spanned by objects $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ such that the underlying ∞-category $(\mathrm{ev}_X \circ \mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathfrak{J}}))(\mathcal{C}_i)$ of $\mathcal{C}_i$ is presentable for each $1 \leq i \leq n$ and object X of $\mathcal{O}$[22].

[22]We are using here that only functors preserving inert morphisms are in the essential image of $\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}$ – this implies that we only need to check the presentability condition for objects X of $\mathcal{O}$ rather than all of $\mathcal{O}^{\otimes}$.

The functor $(\Psi^{\mathcal{P}r}_{\mathfrak{J}})^{\otimes}$ is by definition (see Definition 3.2.2.4) the inclusion of the fully faithful subcategory described in the same way[23], so as $\Theta^{\otimes}_{\mathfrak{J}}$ is compatible with the forgetful functors to $\mathcal{C}\mathrm{at}^{\times}_{\infty}$ we can use Proposition B.4.3.1 to complete diagram (3.16).

We now turn towards proving (A), (B), and (C). We will simplify notation and write $\Phi \coloneqq \Phi^{\mathfrak{J}}$ and $\Psi \coloneqq \Psi^{\mathfrak{J}}$.

Proof of (A): That $\Psi^{\otimes}$ is a monomorphism holds by definition, see Definition 3.2.2.4.

Proof of (B): By Remark E.4.2.2, there is a commutative diagram as follows

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J}))^{\otimes} \arrow[r, "\iota'_{\mathrm{Alg}}"] \arrow[d, dashed, "\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}"'] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J})^{\otimes}\Big) \times_{\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \arrow[d, "(\Phi^{\otimes})_* \times_{\mathrm{id}} \mathrm{id}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[r, "\iota_{\mathrm{Alg}}"'] & \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}^{\times}_{\infty}\big) \times_{\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_*
\end{tikzcd}$$

where ι_{Alg} and ι'_{Alg} are as in Proposition E.4.2.3. $\Phi^{\otimes}$ is by definition (see Definition 3.2.2.1) a monomorphism, so $(\Phi^{\otimes})_*$ is a monomorphism by Proposition B.5.1.1 and then it follows that $(\Phi^{\otimes})_* \times_{\mathrm{id}} \mathrm{id}$ is a monomorphism by Proposition B.5.3.1. As ι_{Alg} and ι'_{Alg} are fully faithful by definition and hence monomorphisms by Proposition B.4.4.1, it follows from Proposition B.1.2.1 that $\mathrm{Alg}(\Phi)^{\otimes}$ is a monomorphism.

Proof of (C): To describe $\mathrm{Im}(\mathrm{Ho}(\Theta^{\otimes} \circ \mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$ we will go through the same steps of (B) and identify the replete image of the respective functor at each step. We start with $\Phi^{\otimes}$, for which $\mathrm{Im}(\mathrm{Ho}(\Phi^{\otimes}))$ is described in Definition 3.2.2.1.

Combining this with Proposition B.5.1.1 we can describe $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_*))$ as follows.

(ObjI) A functor $A\colon \mathcal{O}^{\otimes} \to \mathcal{C}\mathrm{at}^{\times}_{\infty}$, considered as an object of the 1-category $\mathrm{Ho}(\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}^{\times}_{\infty}))$, lies in $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_*))$ if and only if the following hold.

(ObjI.1) For each object X of $\mathcal{O}^{\otimes}$, if $A(X) \simeq \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_k$ then for each $1 \leq i \leq k$ the ∞-category $\mathcal{C}_i$ admits all $\mathfrak{J}$-indexed colimits.

(ObjI.2) If β is a morphism in $\mathcal{O}^{\otimes}$, and

$$A(\beta)\colon \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_k \to \mathcal{C}'_1 \oplus \cdots \oplus \mathcal{C}'_l$$

lies over a morphism $\varphi\colon \langle k \rangle \to \langle l \rangle$ of Fin_*, then for each $1 \leq j \leq l$ the associated functor

$$\prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{C}'_j$$

[23]So spanned by objects $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ such that the underlying ∞-category of $\mathcal{C}_i$ is presentable for each $1 \leq i \leq n$.

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

(MorI) A natural transformation $f\colon A \to B$ of functors $\mathcal{O}^{\otimes} \to \mathfrak{Cat}_{\infty}^{\times}$, considered as a morphism of $\mathrm{Ho}(\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}^{\times}))$, is in $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_*))$ if and only if the following hold.

(MorI.1) A and B are in $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_*))$.

(MorI.2) For every object X of $\mathcal{O}^{\otimes}$ the morphism

$$f_X\colon \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_k \simeq A(X) \to B(X) \simeq \mathcal{C}'_1 \oplus \cdots \oplus \mathcal{C}'_l$$

lying over a morphism $\varphi\colon \langle k\rangle \to \langle l\rangle$ is such that for every $1 \leq j \leq l$ the associated functor

$$\prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{C}'_j$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

Describing $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_* \times_{\mathrm{id}} \mathrm{id}))$ needs little extra work, it follows from Proposition B.5.3.1 that an object or morphism of

$$\mathrm{Fun}\big(\mathcal{O}, \mathfrak{Cat}_{\infty}^{\times}\big) \times_{\mathrm{Fun}(\mathcal{O}, \mathsf{Fin}_*)} \mathsf{Fin}_*$$

is in $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_* \times_{\mathrm{id}} \mathrm{id}))$ if and only if its projection to the first factor is and object or morphism of $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_*))$.

The functor ι'_{Alg} is defined as the inclusion of the full subcategory of objects whose projection to the first factor is a functor $\mathcal{O}^{\otimes} \to \mathfrak{Cat}_{\infty}(\mathfrak{I})^{\otimes}$ that preserves inert morphisms, and ι_{Alg} is defined analogously. As by Proposition 3.2.2.3 (3) a morphism in $\mathfrak{Cat}_{\infty}(\mathfrak{I})^{\otimes}$ is inert if and only if $\Phi^{\otimes}$ maps that morphism to an inert morphism in $\mathfrak{Cat}_{\infty}^{\times}$, we can conclude that an object or morphism of $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})^{\otimes})$ is in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$ if and only if $\mathrm{Ho}(\iota_{\mathrm{Alg}})$ maps it into $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_* \times_{\mathrm{id}} \mathrm{id}))$. This leads to the following description of $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$.

We will notationally identify $\langle n\rangle \wedge \langle m\rangle$ with $(\langle n\rangle^{\circ} \times \langle m\rangle)_*$ and thus write non-basepoint elements of $\langle n\rangle \wedge \langle m\rangle$ as pairs (i,j) with $1 \leq i \leq n$ and $1 \leq j \leq m$.

(ObjII) An object A of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})^{\otimes}_{\langle n\rangle}$, considered as an object of the 1-category $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})^{\otimes})$, is in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$ if and only if the following hold.

(ObjII.1) For each $k \geq 0$ and object X in $\mathcal{O}^{\otimes}_{\langle k\rangle}$, if

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(X) \simeq \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)}$$

then for each $1 \leq i_1 \leq n$ and $1 \leq i_2 \leq k$ the ∞-category $\mathcal{C}_{(i_1,i_2)}$ admits all $\mathfrak{I}$-indexed colimits.

(ObjII.2) If $\varphi\colon \langle k\rangle \to \langle l\rangle$ is a morphism in Fin_* and $f\colon X \to Y$ a morphism in $\mathcal{O}^{\otimes}$ lying over φ, and

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(f)\colon \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)} \to \mathcal{C}'_{(1,1)} \oplus \cdots \oplus \mathcal{C}'_{(n,l)}$$

then for each $1 \leq j_1 \leq n$ and $1 \leq j_2 \leq l$ the associated functor

$$\prod_{\varphi(i)=j_2} \mathcal{C}_{(j_1,i)} \to \mathcal{C}'_{(j_1,j_2)}$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

(MorII) A morphism $f\colon A \to B$ of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes}$, lying over a morphism $\varphi\colon \langle n\rangle \to \langle m\rangle$ in Fin_* and considered as a morphism of the 1-category $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes})$, is in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$ if and only if the following hold.

(MorII.1) A and B are in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$.

(MorII.2) For every $k \geq 0$ and object X in $\mathcal{O}^{\otimes}_{\langle k\rangle}$ the morphism

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(f)_X : \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)} \to \mathcal{C}'_{(1,1)} \oplus \cdots \oplus \mathcal{C}'_{(m,k)}$$

is such that for every $1 \leq j_1 \leq m$ and $1 \leq j_2 \leq k$ the associated functor

$$\prod_{\varphi(i)=j_1} \mathcal{C}_{(i,j_2)} \to \mathcal{C}'_{(j_1,j_2)}$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

We will now replace these conditions with equivalent descriptions that are more amenable to describing what happens under the equivalence $\Theta^{\otimes}$.

Let $A \simeq A_1 \oplus \cdots \oplus A_n$ be an object of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes}_{\langle n\rangle}$, let $k \geq 0$, let $X \simeq X_1 \oplus \cdots \oplus X_k$ be an object of $\mathcal{O}^{\otimes}_{\langle k\rangle}$, and let

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(X) \simeq \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)}$$

be the usual decomposition. Let $1 \leq i \leq n$ and let $g_i\colon A \to A_i$ be an inert morphism lying over ρ^i. It follows from Proposition E.4.2.3 (2) that the morphism $(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(g_i)(X)$ can be identified with the inert morphism

$$\mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)} \to \mathcal{C}_{(i,1)} \oplus \cdots \oplus \mathcal{C}_{(i,k)}$$

in $\mathfrak{Cat}^{\times}_\infty$ over $\rho^i \wedge \mathrm{id}_{\langle k\rangle}$. Furthermore, as A_i lies in

$$\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes}_{\langle 1\rangle} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$$

(see Proposition E.4.2.3 (0)) and thus preserves inert morphisms, we also obtain an equivalence as follows.

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A_i)(X) \simeq \bigoplus_{1\leq j\leq k} (\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A_i)(X_j)$$

It follows that condition (ObjII.1) is equivalent to the following condition.

(ObjIII.1) For each $1 \le i \le n$ and object X of $\mathcal{O}$, the underlying ∞-category[24] $\mathrm{ev}_X(A_i)$ in $\mathfrak{Cat}_\infty$ is an ∞-category that admits all $\mathfrak{I}$-indexed colimits.

Similarly one obtains that if $f\colon X \to Y$ is a morphism in $\mathcal{O}^\otimes$ lying over $\varphi\colon \langle k\rangle \to \langle l\rangle$ then

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(f)\colon \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)} \to \mathcal{C}'_{(1,1)} \oplus \cdots \oplus \mathcal{C}'_{(n,l)}$$

can be identified with a sum $A_1(f) \oplus \cdots \oplus A_n(f)$ in $\mathfrak{Cat}_\infty^\times$, and for $1 \le j_1 \le n$ and $1 \le j_2 \le l$ the functor

$$\prod_{\varphi(i)=j_2} \mathcal{C}_{(j_1,i)} \to \mathcal{C}'_{(j_1,j_2)}$$

associated to $(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(f)$ can be identified with the analogous functor associated to

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A_{j_1})(f)\colon \mathcal{C}_{(j_1,1)} \oplus \cdots \oplus \mathcal{C}_{(j_1,k)} \to \mathcal{C}'_{(j_1,1)} \oplus \cdots \oplus \mathcal{C}'_{(j_1,l)}$$

at index (j_1, j_2). It follows that condition (ObjII.2) is equivalent to the following condition.

(ObjIII.2) For each $1 \le i \le n$, the $\mathcal{O}$-monoidal ∞-category A_i (which we consider as an object of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\otimes_{\langle 1\rangle} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$) is such that for every morphism $f\colon X_1 \oplus \cdots \oplus X_k \to Y$ in $\mathcal{O}$ lying over $\varphi\colon \langle k\rangle \to \langle 1\rangle$ the associated functor

$$\prod_{1\le j\le k} \mathrm{ev}_{X_j} A_i \to \mathrm{ev}_Y A_i$$

is compatible with $\mathfrak{I}$-indexed colimits separately in each variable.

Reformulations (ObjIII.1) and (ObjIII.2) allow us to rephrase (ObjI) as follows, by using the definitions of Θ (given by postcomposing with π) and the monomorphism $\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathfrak{Cat}_\infty) \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$ from Definition 3.1.3.10, which we can identify with Ψ by Remark 3.2.2.5.

(ObjIII) Let $A \simeq A_1 \oplus \cdots \oplus A_n$ be an object of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\otimes_{\langle n\rangle}$, and consider A as an object of $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\otimes)$. Then A is in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^\otimes))$ if and only if for each $1 \le i \le n$, the equivalence

$$\Theta\colon\ \mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty) \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$$

maps A_i to an object in $\mathrm{Im}(\mathrm{Ho}(\Psi))$.

[24]Here we consider A_i as an object of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\otimes_{\langle 1\rangle} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$

Using that $\Theta^{\otimes}$ maps $A_1 \oplus \cdots \oplus A_n$ to $\Theta(A_1) \oplus \cdots \oplus \Theta(A_n)$ by virtue of being lax monoidal, as well as the definition of $\Psi^{\otimes}$ in Definition 3.2.2.4, we finally obtain the following reformulation.

(ObjIV) Let A be an object $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes})$. Then A lies in the subcategory $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$ if and only if $\mathrm{Ho}(\Theta^{\otimes})(A)$ is in $\mathrm{Im}(\mathrm{Ho}(\Psi^{\otimes}))$.

This shows (C) for objects. Let us now turn towards reformulating (MorII). Let $f\colon A \to B$ be a morphism in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$, lying over a morphism $\varphi\colon \langle n\rangle \to \langle m\rangle$ in Fin_*, and let $X \simeq X_1 \oplus \cdots \oplus X_k$ be an object of $\mathcal{O}^{\otimes}_{\langle k\rangle}$. As $(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)$ and $(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(B)$ preserve inert morphisms, we can for $1 \leq j_2 \leq k$ identify the commutative diagram

$$\begin{tikzcd}
(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(X) \arrow[r, "(\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(f)_X"] \arrow[d, "(\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(A)(\rho^{j_2})"'] & (\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(B)(X) \arrow[d, "(\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(B)(\rho^{j_2})"] \\
(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(X_{j_2}) \arrow[r, "(\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(f)_{X_{j_2}}"'] & (\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(B)(X_{j_2})
\end{tikzcd}$$

lying over

$$\begin{tikzcd}
\langle n\rangle \wedge \langle k\rangle \arrow[r, "\varphi\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\rho^{j_2}"'] & \langle m\rangle \wedge \langle k\rangle \arrow[d, "\mathrm{id}\wedge\rho^{j_2}"] \\
\langle n\rangle \wedge \langle 1\rangle \arrow[r, "\varphi\wedge\mathrm{id}"'] & \langle m\rangle \wedge \langle 1\rangle
\end{tikzcd}$$

with a diagram as indicated below

$$\begin{tikzcd}
\mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)} \arrow[r] \arrow[d] & \mathcal{C}'_{(1,1)} \oplus \cdots \oplus \mathcal{C}'_{(m,k)} \arrow[d] \\
\mathcal{C}_{(1,j_2)} \oplus \cdots \oplus \mathcal{C}_{(n,j_2)} \arrow[r] & \mathcal{C}'_{(1,j_2)} \oplus \cdots \oplus \mathcal{C}'_{(m,j_2)}
\end{tikzcd}$$

and the functor

$$\prod_{\varphi(i)=j_1} \mathcal{C}_{(i,j_2)} \to \mathcal{C}'_{(j_1,j_2)}$$

associated to the top horizontal morphism at index (j_1, j_2) with $1 \leq j_1 \leq m$ can be identified with the functor associated to the bottom horizontal morphism at the same index. This implies that (MorII.2) is equivalent to the following condition.

(MorIII.2) For every object X of $\mathcal{O}$, if f is such that

$$(\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(f)\colon \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,1)} \to \mathcal{C}'_{(1,1)} \oplus \cdots \oplus \mathcal{C}'_{(m,1)}$$

then for every $1 \leq j \leq m$ the associated functor

$$\prod_{\varphi(i)=j} \mathcal{C}_{(i,1)} \to \mathcal{C}'_{(j,1)}$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

For X an object of $\mathcal{O}$, the composition $\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}$ is by commutativity of diagram (3.12) homotopic to the composition $(\mathrm{ev}_X)^\times \circ \Theta^\otimes$. Combining this with the definition of Ψ (see Definition 3.2.2.4) and the reformulation of (MorII.1) made possible by (ObjIV) we finally obtain the following.

(MorIV) Let $f\colon A \to B$ be a morphism in $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^\otimes)$. Then f is a morphism in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^\otimes))$ if and only if $\mathrm{Ho}(\Theta^\otimes)(f)$ is in $\mathrm{Im}(\mathrm{Ho}(\Psi^\otimes))$.

This shows (C) and thereby ends the proof. $\square$

Remark 3.2.2.9. In this remark we will make use of Notation 3.2.2.7.

Let $\mathfrak{I}$ be a collection of small ∞-categories, $\mathcal{O}$ an ∞-operad, and X and object of the underlying category $\mathcal{O}$. Diagram (3.11) constructed in Equation (3.11) can be extended to a commutative diagram as follows

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}\right)^{\otimes} \arrow[rr, "\Theta^{\otimes}_{\mathcal{P}\mathrm{r}}", "\simeq"'] \arrow[dd, "\mathrm{Alg}\left(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}\right)^{\otimes}"'] \arrow[dr, "E"'] & & \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[dl, dashed] \arrow[dd, "\left(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}\right)^{\otimes}"] \\
& \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \arrow[dd, "\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}" near start] & \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}))^{\otimes} \arrow[rr, crossing over, "\Theta^{\otimes}_{\mathfrak{I}}", "\simeq"'] \arrow[dd, "\mathrm{Alg}\left(\Phi^{\mathfrak{I}}\right)^{\otimes}"'] \arrow[dr, "E"'] & & \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[dl, dashed] \arrow[dd, "\left(\Psi^{\mathfrak{I}}\right)^{\otimes}"] \\
& \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes} \arrow[dd, "\Psi^{\mathfrak{I}}" near start] & \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[rr, crossing over, "\Theta^{\otimes}", "\simeq"'] \arrow[dr, "E"'] & & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \arrow[dl, "\left(\mathrm{ev}_{\langle 1 \rangle}\right)^{\times}"] \\
& \mathcal{C}\mathrm{at}_\infty^{\times} &
\end{tikzcd}$$

where we write E as an abbreviation for $\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}$.

We will refer to the functors

$$\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes}$$

and

$$\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}$$

as the forgetful functors and denote them by $(\mathrm{ev}_X)^{\otimes}$. $\diamondsuit$

Proposition 3.2.2.10 ([HA, 4.8.5.16 (1)]). *In this proposition we make use of Notation 3.2.2.7.*

Let $\mathfrak{I}$ be a collection of small ∞-categories, let $\mathfrak{I}'$ be a subcollection of $\mathfrak{I}$, and let $\mathfrak{J}$ be the collection of all small ∞-categories. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. Then the following statements hold.

(1) The functors $p_{\mathrm{Mon},\mathfrak{I}}$ and $p_{\mathrm{Mon},\mathcal{P}\mathrm{r}}$ are cocartesian fibrations of ∞-operads and thus exhibit $\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ and $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ as symmetric monoidal ∞-categories.

(2) The functors

$$\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\left(\Psi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}\right)^{\otimes}} \mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\left(\Psi_{\mathfrak{I}'}^{\mathfrak{I}}\right)^{\otimes}} \mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}'}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$$
$$\xrightarrow{\left(\Psi^{\mathfrak{I}'}\right)^{\otimes}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$

are lax symmetric monoidal with respect to the symmetric monoidal structures from (1).

(3) A morphism in $\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ or $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ is inert if and only if its image under $\left(\Psi^{\mathfrak{I}}\right)^{\otimes}$ or $\left(\Psi^{\mathcal{P}\mathrm{r}}\right)^{\otimes}$ in $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$ is inert.

(4) The functor

$$\left(\Psi_{\mathfrak{J}}^{\mathcal{P}\mathrm{r}}\right)^{\otimes}\colon \mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathrm{Mon}_{\mathcal{O}}^{\mathfrak{J}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$

is symmetric monoidal with respect to the symmetric monoidal structure from (1).

(5) A morphism f in $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ is $p_{\mathrm{Mon},\mathcal{P}\mathrm{r}}$-cocartesian if and only if $\left(\Psi_{\mathfrak{J}}^{\mathcal{P}\mathrm{r}}\right)^{\otimes}(f)$ is $p_{\mathrm{Mon},\mathfrak{J}}$-cocartesian.

(6) Let X be an object in $\mathcal{O}$. The forgetful functors

$$(\mathrm{ev}_X)^{\otimes}\colon \mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes}$$

and

$$(\mathrm{ev}_X)^{\otimes}\colon \mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}$$

from Remark 3.2.2.9 are symmetric monoidal. ♡

Proof. All of the statements will be shown by translating them to statements regarding $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}))^{\otimes}$ and $\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}\right)^{\otimes}$ using Proposition 3.2.2.8. The individual statements then all follow by combining parts of Proposition E.4.2.3 with parts of Proposition 3.2.2.3, as indicated in the table below.

Claim	Combine Proposition E.4.2.3	with Proposition 3.2.2.3
(1)	(3)	(1)
(2)	(7)	(2)
(3)	(2) and (9)	(3)
(4)	(8)	(4)
(5)	(4) and (9)	(5)
(6)	(5)	

□

3.2.2.3 The symmetric monoidal structure on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$

By Proposition 3.2.1.1 and Proposition C.2.0.3 the cocartesian fibration

$$q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$$

preserves products (see also Remark 3.2.1.4). By [HA, 2.4.1.8] we thus obtain an induced symmetric monoidal functor

$$q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$

between the respective cartesian symmetric monoidal structures.

In this section we upgrade $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$ to symmetric monoidal functors in a compatible way.

Definition 3.2.2.11 ([HA, 4.8.5.14])**.** Let $\mathfrak{I}$ be a collection of small ∞-categories. We define functors

$$q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}} \to \mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$$

and

$$q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \to \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$$

as pullbacks, as indicated in the following pullback diagrams

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}} \arrow[r, "(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}"] \arrow[d, "q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[d, "q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[r, "(\Psi^{\mathfrak{I}})^{\otimes}"'] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\times}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \arrow[r, "(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}})^{\otimes}"] \arrow[d, "q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[d, "q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[r, "(\Psi^{\mathcal{P}\mathrm{r}})^{\otimes}"'] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\times}
\end{tikzcd}$$

where the lower horizontal functors are the ones defined in Notation 3.2.2.7.
♢

Remark 3.2.2.12. Passing to fibers over $\langle 1\rangle$ we obtain a pullback diagram

$$\begin{tikzcd}
(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}})^{\otimes}_{\langle 1\rangle} \arrow[r] \arrow[d, "(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}})^{\otimes}_{\langle 1\rangle}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times}_{\langle 1\rangle} \arrow[d, "(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})^{\times}_{\langle 1\rangle}"] \\
\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}_{\langle 1\rangle} \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\times}_{\langle 1\rangle}
\end{tikzcd}$$

that can be identified using Remark 3.2.2.5 with the pullback diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}} \arrow[r] \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd}$$

from Definition 3.1.3.11. A similar statement holds for $(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}})^{\otimes}_{\langle 1\rangle}$. $\diamondsuit$

Proposition 3.2.2.13 ([HA, 4.8.5.16 (1)])**.** *In this proposition we use notation from Notation 3.2.2.7.*

Let $\mathfrak{I}$ be a collection of small ∞-categories, $\mathfrak{I}'$ a subcollection of $\mathfrak{I}$, and $\mathfrak{J}$ the collection of all small ∞-categories.

(0) The functors $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}$ and $(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}})^{\otimes}$ from Definition 3.2.2.11 are monomorphisms in $\mathcal{C}\mathrm{at}_{\infty}$, and $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}$ factors as a composition of a monomorphisms $(\widetilde{\Psi}^{\mathfrak{I}}_{\mathfrak{I}'})^{\otimes}$ with $(\widetilde{\Psi}^{\mathfrak{I}'})^{\otimes}$. Similarly, $(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}})^{\otimes}$ factors as a composition of a monomorphism $(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ with $(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}$.

(1) The functors $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ and $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$ as defined in Definition 3.2.2.11 are cocartesian fibrations of ∞-operads.

(2) The compositions $p_{\mathrm{Mon},\mathfrak{I}}\circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ and $p_{\mathrm{Mon},\mathcal{P}\mathrm{r}}\circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$ exhibit $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$ and $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$ as symmetric monoidal ∞-categories.

(3) The morphisms of ∞-operads $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ and $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$ are symmetric monoidal.

(4) Let f be a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$. Then f is $p_{\mathrm{Mon},\mathfrak{I}}\circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian if and only if $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}(f)$ is $p_{\mathrm{Mon},\mathfrak{I}}$-cocartesian and $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}(f)$ is $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian. An analogous statement holds for morphisms in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$.

(5) The functors $(\widetilde{\Psi}^{\mathfrak{I}}_{\mathfrak{I}'})^{\otimes}$ and $(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ of Definition 3.2.2.11 are lax symmetric monoidal.

(6) Let f be a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$. Then f is inert if and only if $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}(f)$ is inert. An analogous statement holds for morphisms in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$.

(7) *The functor*

$$(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$$

of (0) is symmetric monoidal.

(8) *Let f be a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$. Then f is $p_{\mathrm{Mon},\mathcal{P}\mathrm{r}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$-cocartesian if and only if $(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}(f)$ is $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian.* ♡

Proof. Proof of (0): That the functors factor as indicated follows from composability of pullback diagrams [HTT, 4.4.2.1] together with Remark 3.2.2.6. By Proposition B.5.2.1, pullbacks of monomorphisms are again monomorphisms, so that the functors in question are monomorphisms follows from Definition 3.2.2.4 and Remark 3.2.2.6.

Proof of (1): The functor $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ is a pullback of $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$, which is a cocartesian fibration of ∞-operads by Proposition 3.2.1.1 and Proposition C.2.0.6. As cocartesian fibrations of ∞-operads are stable under taking pullbacks along morphisms of ∞-operads[25] and $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\times}$ is a morphism of ∞-operads by Proposition 3.2.2.10 (2), we can conclude that $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ is also a cocartesian fibration of ∞-operads, and thus in particular a morphism of ∞-operads by [HA, 2.1.2.14].

Proof of (2): As the ∞-operad $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$ is in fact a symmetric monoidal ∞-category[26] by Proposition 3.2.2.10 (1), it follows[27] with (1) that $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$ is a symmetric monoidal ∞-category as well.

Proof of (3): Follows immediately from Proposition C.1.3.1.

Proof of (4): We do the case of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$, as the proof for $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$ is completely analogous. Let f be a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$. Because $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ maps $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian morphisms to $p_{\mathrm{Mon},\mathfrak{I}}$-cocartesian morphisms by (3), it follows from [HTT, 2.4.1.3 (3)] that f is $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian if and only if $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}(f)$ is $p_{\mathrm{Mon},\mathfrak{I}}$-cocartesian and f is $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian. The claim now follows from Proposition C.1.1.1.

Proof of (6): We again only discuss the case of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$, as the proof for $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$ is completely analogous. In light of (4) it suffices to show that if f is a morphism of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$ lying over an inert morphism in Fin_*, then $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}(f)$ is $p_{\mathrm{Mon}} \circ q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian if and only if $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}(f)$ and $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}(f)$ are inert.

Combining that $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ is a morphism of ∞-operads with [HTT, 2.4.1.3 (3)] we obtain that $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}(f)$ being $p_{\mathrm{Mon}} \circ q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian is equivalent to

[25] This is a special case of the functoriality of cocartesian families of monoidal ∞-categories discussed in Remark 3.1.1.3 – in this case we consider [0]-families, which are just cocartesian fibrations of ∞-operads.

[26] I. e. the canonical morphism of ∞-operads $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \to \mathsf{Fin}_*$ is a cocartesian fibration.

[27] Cocartesian fibrations are closed under composition by [HTT, 2.4.2.3 (3)].

$(\widetilde{\Psi}^{\mathcal{I}})^{\otimes}(f)$ as well as

$$\left(q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \circ \left(\widetilde{\Psi}^{\mathcal{I}}\right)^{\otimes}\right)(f) \simeq \left(\left(\Psi^{\mathcal{I}}\right)^{\otimes} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{I}}}\right)(f)$$

being inert. The claim now follows by applying Proposition 3.2.2.10 (3).

Proof of (5): Immediate consequence of (6).

Proof of (8): Analogous to the proof of (6), using that $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ is even symmetric monoidal and Proposition 3.2.2.10 (5).

Proof of (7): Immediate consequence of (8). □

3.2.3 LMod as a symmetric monoidal functor

In Section 3.1 we constructed a natural transformation $\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod} \to \mathrm{pr}$ of functors $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_{\infty}$, see Definition 3.1.3.8. It was shown in Proposition 3.2.1.3 that $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ admits products and that LMod and pr preserve products. This makes $\mathrm{ev}_{\mathfrak{m}}$ into a morphism in $\mathrm{Fun}^{\times}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}, \mathcal{C}\mathrm{at}_{\infty})$, the full subcategory of $\mathrm{Fun}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}, \mathcal{C}\mathrm{at}_{\infty})$ spanned by the product-preserving functors. [HA, 2.4.1.8] then implies that $\mathrm{ev}_{\mathfrak{m}}$ can be upgraded to a natural transformation ${\mathrm{ev}_{\mathfrak{m}}}^{\times} \colon \mathrm{LMod}^{\times} \to \mathrm{pr}^{\times}$ of symmetric monoidal functors $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \to \mathcal{C}\mathrm{at}_{\infty}^{\times}$.

We also investigated the behavior of $\mathrm{ev}_{\mathfrak{m}}$ with respect to algebras in *presentable* symmetric monoidal ∞-categories, showing in Proposition 3.1.3.12 that $\mathrm{ev}_{\mathfrak{m}}$ lifts to a natural transformation of functors $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Finally, in Section 3.2.2 we constructed symmetric monoidal structures on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and upgraded the inclusion functors to $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ and $\mathcal{C}\mathrm{at}_{\infty}$ to lax symmetric monoidal functors (see Proposition 3.2.2.3 and Proposition 3.2.2.13).

The situation is depicted in the non-dashed part of the following diagram. Squares that contain parallel arrows on opposing sides are to be interpreted as encoding two commutative diagrams, one considering only the arrows at the top, and one only considering the arrows at the bottom, as well as a compatible homotopy between the two natural transformations from the source corner to the target corner that one obtains by pre-composing and post-composing.

$$
\begin{tikzcd}
 & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \arrow[rr, dashed, bend left] \arrow[rr, dashed, bend right] \arrow[rr, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] \arrow[dl] & & \mathcal{P}\mathrm{r}^{\mathrm{L}^{\otimes}} \arrow[dl] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow \mathrm{ev}^{\times}_{\mathfrak{m}}"] & & \mathcal{C}\mathrm{at}^{\times}_{\infty} & \\
 & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \arrow[uu] \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] \arrow[dl] & & \mathcal{P}\mathrm{r}^{\mathrm{L}} \arrow[uu] \arrow[dl] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[uu] \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] & & \mathcal{C}\mathrm{at}_{\infty} \arrow[uu] &
\end{tikzcd}
\tag{3.17}
$$

The vertical functors are all inclusions of the fiber over $\langle 1 \rangle$, the bottom square was constructed in Proposition 3.1.3.12, and the front square can be obtained from [HA, 2.4.1.8]. To be more precise about how the above cube is to be interpreted with regards to parallel arrows, we could also depict the cube (3.17) in the form shown below (as just a standard commuting cube in $\mathcal{C}\mathrm{at}_{\infty}$), using that natural transformations are equivalently encoded as functors out of a product with [1].

$$
\begin{tikzcd}
 & {[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}} \arrow[rr, dashed, "\mathrm{ev}^{\otimes}_{\mathfrak{m}}"] \arrow[dl] & & \mathcal{P}\mathrm{r}^{\mathrm{L}^{\otimes}} \arrow[dl] \\
{[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times}} \arrow[rr, "\mathrm{ev}^{\times}_{\mathfrak{m}}"] & & \mathcal{C}\mathrm{at}^{\times}_{\infty} & \\
 & {[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}} \arrow[uu] \arrow[rr, "\mathrm{ev}_{\mathfrak{m}}"] \arrow[dl] & & \mathcal{P}\mathrm{r}^{\mathrm{L}} \arrow[uu] \arrow[dl] \\
{[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \arrow[uu] \arrow[rr, "\mathrm{ev}_{\mathfrak{m}}"] & & \mathcal{C}\mathrm{at}_{\infty} \arrow[uu] &
\end{tikzcd}
\tag{3.18}
$$

The goal of this section is to complete the cube as indicated by the dashed arrows, and in such a way that $\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod} \to \mathrm{pr}$ in its incarnation as a natural transformation of functors $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is upgraded to a natural transformation of symmetric monoidal functors.

Proposition 3.2.3.1 ([HA, 4.8.5.16 (3) and (4)]). *Let $\mathfrak{I}$ be a collection of small ∞-categories that includes $\mathbf{\Delta}^{\mathrm{op}}$. Then the restriction to $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$ of the* *natural transformation* $\mathrm{ev}_{\mathfrak{m}}^{\times}$ *of symmetric monoidal functors* $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \to \mathcal{C}\mathrm{at}_{\infty}^{\times}$ *factors through* $\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})^{\otimes}$*. Analogously, the restriction to* $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}^{\otimes}$ *factors through* $\mathcal{P}\mathrm{r}^{\mathrm{L}^{\otimes}}$*. The situation is depicted in the following commutative diagram.*

$$
\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}^{\otimes} \arrow[rr, dashed, bend left, "\mathrm{LMod}^{\otimes}"] \arrow[rr, dashed, bend right, "\mathrm{pr}^{\otimes}"'] \arrow[rr, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \arrow[d, "\left(\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}\right)^{\otimes}"'] & & \mathcal{P}\mathrm{r}^{\mathrm{L}^{\otimes}} \arrow[d, "\left(\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}\right)^{\otimes}"] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}^{\otimes} \arrow[rr, dashed, bend left, "\mathrm{LMod}^{\otimes}"] \arrow[rr, dashed, bend right, "\mathrm{pr}^{\otimes}"'] \arrow[rr, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \arrow[d, "\left(\widetilde{\Psi}^{\mathfrak{I}}\right)^{\otimes}"'] & & \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})^{\otimes} \arrow[d, "\left(\Phi^{\mathfrak{I}}\right)^{\otimes}"] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[rr, bend left, "\mathrm{LMod}^{\times}"] \arrow[rr, bend right, "\mathrm{pr}^{\times}"'] \arrow[rr, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}^{\times}"] & & \mathcal{C}\mathrm{at}_{\infty}^{\times}
\end{tikzcd}
\tag{3.19}
$$

Furthermore, the two natural transformations $\mathrm{ev}_{\mathfrak{m}}^{\otimes}$ *that we obtain in this manner are natural transformations of symmetric monoidal functors, and the underlying diagram of underlying* ∞*-categories of diagram* (3.19) *can be identified with diagram* (3.10) *from Proposition 3.1.3.12.* ♡

Proof. In this proof we will use Notation 3.2.2.7 as well as the notation from Definition 3.2.2.11 and Proposition 3.2.2.13.

Reformulation of the lifting problem: We first note that by combining Proposition 3.2.2.3 (0) with Definition 3.2.2.1 and with Proposition B.4.4.1 and Proposition B.1.2.1 the right vertical functors $\left(\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}\right)^{\otimes}$ and $\left(\Phi^{\mathfrak{I}}\right)^{\otimes}$ in diagram (3.19) are monomorphisms. In this situation Proposition B.4.3.1 implies that the dashed lifts in the following diagram are essentially unique if they exist.

$$
\begin{tikzcd}
{[1]} \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}^{\otimes} \arrow[r, dashed, "\mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \arrow[d, "\mathrm{id} \times \left(\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}\right)^{\otimes}"'] & \mathcal{P}\mathrm{r}^{\mathrm{L}^{\otimes}} \arrow[d, "\left(\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}\right)^{\otimes}"] \\
{[1]} \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}^{\otimes} \arrow[r, dashed, "\mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \arrow[d, "\mathrm{id} \times \left(\widetilde{\Psi}^{\mathfrak{I}}\right)^{\otimes}"'] & \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})^{\otimes} \arrow[d, "\left(\Phi^{\mathfrak{I}}\right)^{\otimes}"] \\
{[1]} \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[r, "\mathrm{ev}_{\mathfrak{m}}^{\times}"] & \mathcal{C}\mathrm{at}_{\infty}^{\times}
\end{tikzcd}
\tag{$*$}
$$

Furthermore, Proposition B.4.3.1 also implies that these lifts exists if and only if the following two inclusions of replete subcategories of $\mathrm{Ho}(\mathcal{C}\mathrm{at}_\infty^\times)$ hold.

$$\begin{aligned}\mathrm{Im}\Big(\mathrm{Ho}\Big(\mathrm{ev}_{\mathfrak{m}}^\times \circ \Big(\mathrm{id} \times \Big(\widetilde{\Psi}^{\mathfrak{I}}\Big)^{\otimes}\Big)\Big)\Big) &\subseteq \mathrm{Im}\Big(\mathrm{Ho}\Big((\Phi^{\mathfrak{I}})^{\otimes}\Big)\Big) \\ \mathrm{Im}\Big(\mathrm{Ho}\Big(\mathrm{ev}_{\mathfrak{m}}^\times \circ \Big(\mathrm{id} \times (\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}\Big) \circ \Big(\mathrm{id} \times (\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}})^{\otimes}\Big)\Big)\Big) &\subseteq \mathrm{Im}\big(\mathrm{Ho}\big((\Phi^{\mathfrak{I}})^{\otimes} \circ (\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}})^{\otimes}\big)\big)\end{aligned} \tag{A}$$

Verification of the inclusion of replete images for fibers over Fin_*: We start by checking those inclusions for objects and morphisms lying in a fiber over $\langle n\rangle$ for some $n \geq 0$. Because $(\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}})^{\otimes}$, $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}$, $(\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}})^{\otimes}$, and $(\Phi^{\mathfrak{I}})^{\otimes}$ are all morphisms of ∞-operads (see Proposition 3.2.2.3 (2) and Proposition 3.2.2.13 (5)), we can identify the diagram induced by $(*)$ on fibers over $\langle n\rangle$ with the following diagram.

$$\begin{tikzcd}
{[1]} \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}^{\times n} \arrow[r, dashed] \arrow[d, "\mathrm{id} \times (\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}})^{\otimes}"'] & \mathcal{P}\mathrm{r}^{\mathrm{L}\,\times n} \arrow[d, "(\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}})^{\otimes}"] \\
{[1]} \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}^{\times n} \arrow[r, dashed] \arrow[d, "\mathrm{id} \times (\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}"'] & \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\times n} \arrow[d, "(\Phi^{\mathfrak{I}})^{\otimes}"] \\
{[1]} \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times n} \arrow[r, "\mathrm{ev}_{\mathfrak{m}}^{\times n}"] & \mathcal{C}\mathrm{at}_\infty^{\times n}
\end{tikzcd} \tag{$**$}$$

By Remark 3.2.2.12 and Remark 3.2.2.2 this diagram can be identified with the n-fold product of the lifting problem solved in Proposition 3.1.3.12, so we deduce that the inclusions (A) hold for objects as well as for morphisms lying over an identity morphism in Fin_*.

Reduction of the presentable case to the other cases: Suppose for the moment that we have shown the first inclusion of (A) for all families of small ∞-categories. Given that we already know the second inclusion on objects, the second inclusion will follow if $(\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}})^{\otimes}$ and $(\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}})^{\otimes}$ are fully faithful for $\mathfrak{I}$ the family of all small ∞-categories. That $\big(\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}\big)^{\otimes}$ is fully faithful is the case by Definition 3.2.2.1, and $\Big(\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}\Big)^{\otimes}$ is fully faithful combining Proposition B.5.2.1 with Definition 3.2.2.11 and Definition 3.2.2.4.

Verification of the inclusion of replete images for morphisms: Let

$$\Gamma\colon A_1 \oplus \cdots \oplus A_n \to B_1 \oplus \cdots \oplus B_m$$

be a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\times$ lying over a morphism

$$G\colon \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n \to \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_m$$

in $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^\times$ lying over a morphism

$$\gamma\colon \langle n\rangle \to \langle m\rangle$$

in Fin_*. Note that by Remark 3.1.3.7 we can interpret A_i as an object of $\mathrm{Alg}(\mathcal{C}_i)$ and similarly for B_j. Assume that Γ lies in the replete image of $(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}$. By applying Proposition B.5.2.1, the definition of $(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}$ in Definition 3.2.2.11, as well as Definition 3.2.2.4 we can unpack this to see that this implies in particular that the underlying ∞-categories of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and $\mathcal{D}_1, \ldots, \mathcal{D}_m$ admit $\mathfrak{J}$-indexed colimits, that the tensor product functors on $\mathcal{C}_1, \ldots, \mathcal{C}_n, \mathcal{D}_1, \ldots, \mathcal{D}_m$ are compatible with $\mathfrak{J}$-indexed colimits, and that for every $1 \leq j \leq m$ the functor

$$\prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{D}_j$$

associated to G preserves $\mathfrak{J}$-indexed colimits in each variable separately. Applying $\mathrm{ev}^{\times}_{\mathsf{m}}$ to Γ we obtain a commutative diagram as follows in $\mathcal{C}\mathrm{at}^{\times}_{\infty}$ (see Remark 3.1.3.9).

$$\begin{tikzcd}
\mathrm{LMod}_{A_1}(\mathcal{C}_1) \oplus \cdots \oplus \mathrm{LMod}_{A_n}(\mathcal{C}_n) \arrow[r, "\mathrm{LMod}^{\times}(\Gamma)"] \arrow[d, "\mathrm{ev}^{\times}_{\mathsf{m}}(A_1 \oplus \cdots \oplus A_n)"'] & \mathrm{LMod}_{B_1}(\mathcal{D}_1) \oplus \cdots \oplus \mathrm{LMod}_{B_m}(\mathcal{D}_m) \arrow[d, "\mathrm{ev}^{\times}_{\mathsf{m}}(B_1 \oplus \cdots \oplus B_m)"] \\
\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n \arrow[r, "\mathrm{pr}^{\times}(\Gamma)"'] & \mathcal{C}'_1 \oplus \cdots \oplus \mathcal{D}_m
\end{tikzcd}$$

What we have to show is that this diagram is in the replete image of $(\Phi^{\mathfrak{J}})^{\otimes}$. What we have already shown when considering objects and morphisms in fibers over Fin_* already implies that the four objects as well as the vertical morphisms are in the replete image of $(\Phi^{\mathfrak{J}})^{\otimes}$, so it only remains to show this for the horizontal morphisms. By definition (see Definition 3.2.2.1) this means that we have to show that for every $1 \leq j \leq m$ the two horizontal functors in the following commutative diagram associated to the diagram above preserve $\mathfrak{J}$-indexed colimits separately in each variable (see Remark 3.1.3.9 for the identifications made here – in particular the functors called ev_{m} are the actual evaluation functors).

$$\begin{tikzcd}
\prod_{\varphi(i)=j} \mathrm{LMod}_{A_i}(\mathcal{C}_i) \arrow[r] \arrow[d, "\prod_{\varphi(i)=j} \mathrm{ev}_{\mathsf{m}}"'] & \mathrm{LMod}_{B_j}(\mathcal{D}_j) \arrow[d, "\mathrm{ev}_{\mathsf{m}}"] \\
\prod_{\varphi(i)=j} \mathcal{C}_i \arrow[r] & \mathcal{D}_j
\end{tikzcd}$$

The bottom horizontal functor is the same one as the functor associated to G that we already mentioned preserving $\mathfrak{J}$-indexed colimits separately in each variable. We also already know that the left vertical functor is a product of functors that preserve $\mathfrak{J}$-indexed colimit, so it follows that the compositions from the top left to the bottom right preserve $\mathfrak{J}$-indexed colimits separately in each variable. As the tensor product in the monoidal ∞-category $\mathcal{D}_j$ is compatible with $\mathfrak{J}$-indexed colimits, we can now apply [HA, 4.2.3.5] to deduce that the top horizontal functor also preserves $\mathfrak{J}$-indexed colimits separately in each variable.

On showing that the induced functors are symmetric monoidal: We have now constructed a commutative diagram (3.19). We next need to prove that the induced functors $\mathrm{LMod}^{\otimes}$ and $\mathrm{pr}^{\otimes}$ are symmetric monoidal[28], i. e. that they preserve morphisms that are cocartesian with respect to the canonical morphism of ∞-operads to Fin_* (see [HA, 2.1.3.7]).

Proof that the induced functors are lax monoidal: As all solid arrows in diagram (3.19) are lax monoidal (so preserve inert morphisms)[29], and the right vertical morphisms of that diagram reflect inert morphisms by Proposition 3.2.2.3 (3), we can already conclude that the functors called $\mathrm{LMod}^{\otimes}$ and $\mathrm{pr}^{\otimes}$ preserve inert morphisms, i. e. are lax monoidal.

Reduction of what needs to be checked for symmetric monoidality: Let $\mathfrak{J}$ be the collection of all small ∞-categories. Note that in the commutative diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \arrow[rr, bend left, "\mathrm{LMod}^{\otimes}"] \arrow[rr, bend right, "\mathrm{pr}^{\otimes}"'] \arrow[rr, phantom, "\Downarrow \mathrm{ev}^{\otimes}_{\mathfrak{m}}"] \arrow[d, "(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}"'] & & \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \arrow[d, "(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}"] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{J}} \arrow[rr, bend left, "\mathrm{LMod}^{\otimes}"] \arrow[rr, bend right, "\mathrm{pr}^{\otimes}"'] \arrow[rr, phantom, "\Downarrow \mathrm{ev}^{\otimes}_{\mathfrak{m}}"] & & \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J})^{\otimes}
\end{tikzcd} \tag{3.20}$$

the left vertical functor is symmetric monoidal by Proposition 3.2.2.13 (7) and the right vertical functor reflects cocartesian morphisms with respect to the canonical morphisms of ∞-operads to Fin_* by Proposition 3.2.2.3 (5). If we show that the two bottom horizontal morphisms of ∞-operads are symmetric monoidal it will thus follow that the same is true for the two top horizontal ones.

Taking into account Proposition E.1.1.1 it thus remains to show that the functors

$$\mathrm{LMod}^{\otimes}, \mathrm{pr}^{\otimes} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{J}} \to \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J})^{\otimes}$$

map $p_{\mathrm{Mon},\mathfrak{J}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}}$-cocartesian lifts of μ and ϵ (see Proposition E.1.1.1 for the definitions) to $p_{\mathfrak{J}}$-cocartesian morphisms.

[28] The ∞-category of symmetric monoidal functors from one symmetric monoidal ∞-category to another one is a full subcategory of the ∞-category of functors over Fin_* (see [HA, 2.1.3.7]), so there is no extra condition that we need to check for $\mathrm{ev}_{\mathfrak{m}}$.

[29] See Proposition 3.2.2.13 (5) for the left vertical functors and Proposition 3.2.2.3 (2) for the right vertical functors. The bottom horizontal functor is symmetric monoidal by construction.

Cocartesian lifts of ϵ: Denote by $\emptyset$ the unique object in $(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}})^{\otimes}_{\langle 0\rangle}$, and let

$$\widetilde{E}' \colon \emptyset \to A$$

be a $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian lift of ϵ lying over a $p_{\mathrm{Mon},\mathfrak{I}}$-cocartesian morphism[30]

$$E' \colon \emptyset \to \mathcal{C}$$

in $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$.

That E' is $p_{\mathrm{Mon},\mathfrak{I}}$-cocartesian implies that the functor

$$E \colon \mathbb{1}_{\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})} \to \mathcal{C}$$

associated to E' is an equivalence, so that we can identify $\mathcal{C}$ with the unit[31] $\mathbb{1}_{\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})}$ in $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$.

By Proposition 3.2.2.13 (4) the morphism $(\Psi^{\mathfrak{I}})^{\otimes}(\widetilde{E}')$ is $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian. The commutative diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[r, "\pi_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \arrow[d, "q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \arrow[r, "\pi_{\mathrm{Mon}}"'] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)
\end{tikzcd}$$

where the horizontal functors are the cartesian structures is a pullback diagram by Proposition 3.2.1.1 and Proposition F.1.0.2. Applying Proposition C.1.1.1 we conclude that the functor

$$\mathbb{1}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \to A$$

associated to $(\Psi^{\mathfrak{I}})^{\otimes}(\widetilde{E}')$ (where $\mathbb{1}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ is the final object in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$, so the unit object in the cartesian symmetric monoidal structure) is a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian lift of the monoidal functor[32]

$$e \colon [0] \to \mathcal{C}$$

associated to $(\Phi^{\mathcal{I}})^{\otimes}(E')$. The final object $\mathbb{1}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ can then using Remark 3.1.3.7, Proposition 3.2.1.1, and Proposition C.2.0.3 be identified with the final object in

$$\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{[0]} \simeq \mathrm{Alg}([0])^{\mathrm{op}}$$

---

[30] $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ is symmetric monoidal by Proposition 3.2.2.13 (3).

[31] By Proposition 3.2.2.10 (6) the forgetful functor

$$(\mathrm{ev}_{\langle 1\rangle})^{\otimes} \colon \mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes}$$

is symmetric monoidal, so the underlying ∞-category of the monoidal unit $\mathbb{1}_{\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)}$ of $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ is given by the monoidal unit of $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})$.

[32] The final object of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ (which is also the monoidal unit with respect to the cartesian symmetric monoidal structure) is by Proposition 3.2.2.10 (6) given by the essentially unique monoidal structure on the ∞-category that is final in $\mathcal{C}\mathrm{at}_\infty$, the discrete category [0] that has a single object and only the identity morphism.

which is the unit object $\mathbb{1}_{[0]}$ in $[0]$[33]. That the morphism $\mathbb{1}_{[0]} \to A$ is $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian then implies using Remark 3.1.3.7 that A can be identified as an object of

$$\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}} \simeq \mathrm{Alg}(\mathcal{C})^{\mathrm{op}}$$

with $e(\mathbb{1}_{[0]}) \simeq \mathbb{1}_{\mathcal{C}}$.

Getting back to showing that $\mathrm{LMod}^{\otimes}$ and $\mathrm{pr}^{\otimes}$ map $\widetilde{E}'$ to a $p_{\mathfrak{I}}$-cocartesian morphism, we obtain the following commutative diagram in $\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})$ by applying $\mathrm{ev}_{\mathsf{m}}^{\otimes}$ to $\widetilde{E}'$.

$$\begin{tikzcd}
\emptyset \arrow[r, "\mathrm{LMod}^{\otimes}(\widetilde{E}')"] \arrow[d, "\mathrm{ev}_{\mathsf{m}}"'] & \mathrm{LMod}_A(\mathcal{C}) \arrow[d, "\mathrm{ev}_{\mathsf{m}}"] \\
\emptyset \arrow[r, "\mathrm{pr}^{\otimes}(\widetilde{E}')"'] & \mathcal{C}
\end{tikzcd}$$

It suffices to show that the associated horizontal functors as depicted in the diagram below are equivalences.

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathrm{LMod}_A(\mathcal{C}) \arrow[d, "\mathrm{ev}_{\mathsf{m}}"] \\
\mathbb{1}_{\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})} \arrow[r, "E"'] & \mathcal{C}
\end{tikzcd}$$

That E is an equivalence was already noted, and the right vertical functor ev_{m} is an equivalence by [HA, 4.2.4.9], as A is the unit object in $\mathcal{C}$.

Cocartesian lifts of μ: Let $\mathcal{C}$ and $\mathcal{D}$ be two objects in $\mathrm{Mon}_{\mathsf{Assoc}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_{\infty})$, let A be an algebra in $\mathcal{C}$, and let B be an algebra in $\mathcal{D}$. We can use an analysis completely analogous to the ϵ-case to describe a $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian lift $\widetilde{M}' \colon A \oplus B \to A \otimes_{\mathfrak{I}} B$. Let us just note that from the lax symmetric monoidal functor $\mathrm{Mon}_{\mathsf{Assoc}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$ we obtain a monoidal functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_{\mathfrak{I}} \mathcal{D}$, and the induced functor on algebras sends the pair (A, B) to an object $A \otimes_{\mathfrak{I}} B$ of $\mathrm{Alg}(\mathcal{C} \otimes_{\mathfrak{I}} \mathcal{D})$, and it is this algebra considered as an object in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ that is the target of $\widetilde{M}'$.

$\mathrm{ev}_{\mathsf{m}}^{\otimes}$ applied to $\widetilde{M}'$ yields a commutative diagram (after passing to the associated functors, as before)

$$\begin{tikzcd}
\mathrm{LMod}_A(\mathcal{C}) \times \mathrm{LMod}_B(\mathcal{D}) \arrow[r, "\simeq"] \arrow[d, "- \otimes_{\mathfrak{I}} -"'] & \mathrm{LMod}_{(A,B)}(\mathcal{C} \times \mathcal{D}) \arrow[d, "\mathrm{LMod}(- \otimes_{\mathfrak{I}} -)"] \\
\mathrm{LMod}_A(\mathcal{C}) \otimes_{\mathfrak{I}} \mathrm{LMod}_B(\mathcal{D}) \arrow[r] \arrow[d, "\mathrm{ev}_{\mathsf{m}} \otimes_{\mathfrak{I}} \mathrm{ev}_{\mathsf{m}}"'] & \mathrm{LMod}_{A \otimes_{\mathfrak{I}} B}(\mathcal{C} \otimes_{\mathfrak{I}} \mathcal{D}) \arrow[d, "\mathrm{ev}_{\mathsf{m}}"] \\
\mathcal{C} \otimes_{\mathfrak{I}} \mathcal{D} \arrow[r, "\mathrm{id}"'] & \mathcal{C} \otimes_{\mathfrak{I}} \mathcal{D}
\end{tikzcd}$$

[33] In this case this is completely clear because there is only an essentially unique algebra in [0], but we could also invoke [HA, 3.2.1.8].

and we have to show that the bottom and middle horizontal functors are equivalences. This can be done by applying [HA, 4.7.3.16], and the verification of the necessary hypotheses is carried out in [HA, Proof of 4.8.5.16 (4)]. While our settings are slightly different, for example our functor was constructed on an ∞-category where morphisms of algebras have the opposite variance compared to Lurie's ∞-category, these differences are not relevant in the proof, the most that would need to be changed for our setting is replacing RMod by LMod.

Note that this is the step that requires the assumption that $\mathbf{\Delta}^{\mathrm{op}}$ is contained in $\mathfrak{I}$.

Compatibility of the constructed diagram with diagram (3.10) *from Proposition 3.1.3.12*: Finally, it only remains to show that the underlying diagram of (3.19) on underlying ∞-categories can be identified with diagram (3.10) from Proposition 3.1.3.12. But this follows from $\Phi^{\mathfrak{I}}$ and $\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}$ being monomorphisms together with the uniqueness part of Proposition B.4.3.1. □

3.3 Bialgebras

Let C be a symmetric monoidal category. An (associative) *algebra* A in C consists of a multiplication $A \otimes A \to A$ and a unit $\mathbb{1}_{\mathsf{C}} \to A$ such that diagrams encoding associativity and unitality commute. The notion of (coassociative) *coalgebras* A in C is dual to this; instead of a multiplication we require a *comultiplication* $A \to A \otimes A$, and instead of a unit we require a *counit* $A \to \mathbb{1}_{\mathsf{C}}$, satisfying diagrams encoding coassociativity and counitality. Instead of defining coalgebras from scratch like this we can also define them in terms of algebras: A coalgebra in C is the same thing as an algebra in C^{op}.

We are often not only interested in individual algebras A in C, but the category of all (associative) algebras in C, which we denote by $\mathrm{Alg}_{\mathsf{Assoc}}(\mathsf{C})$. The data of a morphism of algebras $A \to B$ just consists of a morphism in C from the underlying object of A to the underlying object of B, but we require that this morphism is compatible with the respective multiplication and unit morphisms. If we want morphisms of coalgebras to similarly be given by morphisms of underlying objects that are compatible with comultiplication and counit, then we need to fix having passed to the opposite category by doing it a second time, leading to the definition of the category of (coassociative) coalgebras as

$$\mathrm{coAlg}_{\mathsf{Assoc}}(\mathsf{C}) \coloneqq \mathrm{Alg}_{\mathsf{Assoc}}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$$

This is the perspective that is most suitable to extend the definition to the ∞-categorical setting.

Definition 3.3.0.1. Let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphisms of ∞-operads and $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ an $\mathcal{O}$-monoidal ∞-category. Then we set

$$\mathrm{coAlg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \coloneqq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$$

where $\mathcal{C}^{\mathrm{op}}$ carries the $\mathcal{O}$-monoidal structure described in [HA, 2.4.2.7][34]. $\diamondsuit$

Notation 3.3.0.2. We will use similar notational shortcuts for coAlg as for Alg. In particular, in the situation of Definition 3.3.0.1:

- If α is the identity, then we will shorten $\mathrm{coAlg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C})$ to $\mathrm{coAlg}_{/\mathcal{O}}(\mathcal{C})$.
- If $\mathcal{O}^{\otimes} = \mathsf{Fin}_*$, then we write $\mathrm{coAlg}_{\mathcal{O}'}(\mathcal{C})$ instead of $\mathrm{coAlg}_{\mathcal{O}'/\mathsf{Comm}}(\mathcal{C})$.
- We write $\mathrm{coAlg}(\mathcal{C})$ for $\mathrm{coAlg}_{/\mathsf{Assoc}}(\mathcal{C})$ or $\mathrm{coAlg}_{\mathsf{Assoc}}(\mathcal{C})$.
- We write $\mathrm{coCAlg}(\mathcal{C})$ for $\mathrm{coAlg}_{\mathsf{Comm}}(\mathcal{C})$. $\diamondsuit$

The category $\mathrm{Alg}_{\mathsf{Assoc}}(\mathsf{C})$ inherits a symmetric monoidal structure from C, so that we can form the category

$$\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Assoc}}(\mathsf{C}) \coloneqq \mathrm{coAlg}_{\mathsf{Assoc}}(\mathrm{Alg}_{\mathsf{Assoc}}(\mathsf{C}))$$

of *bialgebras* in C. Unpacking the definition, a bialgebra in C consists of an object A in C together with a multiplication, unit, comultiplication, and counit, satisfying associativity, coassociativity, unitality, and counitality, and such that comultiplication and counit are morphisms of algebras. In this classical setting it is very easy to see that comultiplication and counit are morphisms of algebras if and only if multiplication and unit are morphisms of coalgebras, so that there is a canonical isomorphism

$$\mathrm{coAlg}_{\mathsf{Assoc}}(\mathrm{Alg}_{\mathsf{Assoc}}(\mathsf{C})) \cong \mathrm{Alg}_{\mathsf{Assoc}}(\mathrm{coAlg}_{\mathsf{Assoc}}(\mathsf{C}))$$

or ordinary categories, and we could have taken either side as a definition for the category of bialgebras $\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Assoc}}(\mathsf{C})$.

Unfortunately, the situation is not quite as easy in the setting of ∞-categories. For the case of commutative and cocommutative bialgebras in a symmetric monoidal ∞-category it is shown in [Lur18, 3.3.4] that the two possible definitions coincide. The case of either commutative or cocommutative bialgebras is handled in [Rak20, 2.1.2]. In all these cases, the crucial input to the proof is the fact that tensor products of commutative algebras happen to be coproducts in the ∞-category of commutative algebras [HA, 3.2.4.7], so the proof strategies do not generalize easily to bialgebras which are neither commutative nor cocommutative. Luckily we will not need to use that the two possible definitions are equivalent in this text. Instead, for us *bialgebra* will always mean *coalgebra in algebras*.

Definition 3.3.0.3. Let $\alpha\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads, and $\mathcal{C}$ an $\mathcal{O}''$-monoidal ∞-category. Then we define

$$\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C}) \coloneqq \mathrm{coAlg}_{/\mathcal{O}}\Big(\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})\Big)$$

where $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})$ carries the $\mathcal{O}$-monoidal structure of Proposition E.4.2.3, and call $\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C})$ the ∞-category of $\mathcal{O}', \mathcal{O}$-bialgebras in $\mathcal{C}$. $\diamondsuit$

[34] So if the cocartesian fibration $p_{\mathcal{C}}$ is classified by a functor $F\colon \mathcal{O}^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}$, then the cocartesian fibration $(\mathcal{C}^{\mathrm{op}})^{\otimes} \to \mathcal{O}^{\otimes}$ is classified by the composite $(-)^{\mathrm{op}} \circ F$.

Warning 3.3.0.4. In the notation $\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C})$, the ∞-operad stated *first*, $\mathcal{O}'$, is employed in the *algebra* direction, and $\mathrm{Alg}_{\mathcal{O}'}$ is also what is applied first (i. e. innermost) to $\mathcal{C}$ in our definition. $\diamondsuit$

Remark 3.3.0.5. Let $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ and $p_{\mathcal{O}'}\colon \mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$ be ∞-operads and $\mathcal{C}$ a symmetric monoidal ∞-category.

There is a canonical bifunctor of ∞-operads

$$\alpha\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \xrightarrow{(-\wedge-)\circ(p_{\mathcal{O}}\times p_{\mathcal{O}'})} \mathsf{Fin}_*$$

with respect to which we can form the ∞-category of $\mathcal{O}',\mathcal{O}$-bialgebras as in Definition 3.3.0.3.

Note that if we let β be the canonical bifunctor of ∞-operads

$$\beta\colon \mathsf{Fin}_* \times \mathcal{O}'^{\otimes} \xrightarrow{(-\wedge-)\circ(\mathrm{id}\times p_{\mathcal{O}'})} \mathsf{Fin}_*$$

then α is the composition $\alpha = \beta\circ(p_{\mathcal{O}}\times\mathrm{id})$. Let $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})'^{\otimes}$ be the $\mathcal{O}$-monoidal category from Proposition E.4.2.3 with respect to α and let $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ be the symmetric monoidal ∞-category from Proposition E.4.2.3 with respect to β. It then follows from Remark E.4.2.4 that there is a pullback diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})'^{\otimes} \arrow[r] \arrow[d, "\mathrm{pr}_2\circ\iota_{\mathrm{Alg}}"'] & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \arrow[d, "\mathrm{pr}_2\circ\iota_{\mathrm{Alg}}"] \\
\mathcal{O}^{\otimes} \arrow[r, "p_{\mathcal{O}}"'] & \mathsf{Fin}_*
\end{tikzcd}$$

in $\mathfrak{Cat}_{\infty}$, and all morphisms in the square are morphisms of ∞-operads, while the vertical morphisms are even cocartesian fibrations of ∞-operads by Proposition E.4.2.3 (3).

Passing to fiberwise opposites, applying Remark E.2.0.4, and passing to opposites again we then obtain an induced equivalence

$$\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C}) = \mathrm{coAlg}_{/\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})') \xrightarrow{\simeq} \mathrm{coAlg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})) \qquad \diamondsuit$$

3.3.1 Bialgebras in (co)cartesian symmetric monoidal ∞-categories

Let $\mathcal{C}$ be a cocartesian symmetric monoidal ∞-category[35]. Then if $\mathcal{O}$ is a reduced[36] ∞-operad, then [HA, 2.4.3.9] shows that the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ is an equivalence. In other words, every object of $\mathcal{C}$ carries an essentially unique $\mathcal{O}$-algebra structure. This implies analogous results for bialgebras of cocartesian or cartesian symmetric monoidal ∞-categories, as the next two propositions show.

[35]See [HA, 2.4.0.1] for a definition and [HA, 2.4.3] for further discussion.
[36]See [HA, 2.3.4.1].

The first of the two, Proposition 3.3.1.1 can be summarized as saying that every coalgebra in a cocartesian symmetric monoidal ∞-category can be upgraded to a bialgebra in an essentially unique way. The second, Proposition 3.3.1.2, instead says that any algebra in a cartesian symmetric monoidal ∞-category can be upgraded to a bialgebra in an essentially unique way.

Proposition 3.3.1.1. *Let $\mathcal{C}$ be a cocartesian symmetric monoidal ∞-category, let $\mathcal{O}$ be an ∞-operad, let $\mathcal{O}'$ be a reduced ∞-operad, and let $\mathfrak{o}$ be the essentially unique underlying object of $\mathcal{O}'$.*

Then the following composite functor is an equivalence

$$\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C}) \simeq \mathrm{coAlg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})) \xrightarrow{\mathrm{coAlg}_{\mathcal{O}}(\mathrm{ev}_{\mathfrak{o}})} \mathrm{coAlg}_{\mathcal{O}}(\mathcal{C})$$

where the first functor is the equivalence discussed in Remark 3.3.0.5 and the second functor is induced on coalgebras by the symmetric monoidal functor $\mathrm{ev}_{\mathfrak{o}}$ *from Proposition E.4.2.3 (5).* ♡

Proof. As the functor

$$\mathrm{ev}_{\mathfrak{o}}^{\otimes} \colon \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$$

is symmetric monoidal, with the underlying functor being an equivalence by [HA, 2.4.3.9] (as $\mathcal{C}$ is cocartesian symmetric monoidal), it follows from [HA, 2.1.3.8] that $\mathrm{ev}_{\mathfrak{o}}^{\otimes}$ is an equivalence of symmetric monoidal ∞-categories. It follows that the induced functor on $\mathcal{O}$-coalgebras is an equivalence. ☐

Proposition 3.3.1.2. *Let $\mathcal{C}$ be a cartesian symmetric monoidal ∞-category, let $\mathcal{O}$ be a reduced ∞-operad with essentially unique underlying object $\mathfrak{o}$, and let $\mathcal{O}'$ be an ∞-operad.*

Then the forgetful functor

$$\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\mathrm{ev}_{\mathfrak{o}}^{\mathrm{op}}} (\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$$

is an equivalence, where the first equivalence is the one from Remark 3.3.0.5. ♡

Proof. By Proposition F.3.0.2, the symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ is cartesian, so the symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\mathrm{op}}$ is cocartesian, so that the statement follows from [HA, 2.4.3.9]. ☐

3.4 Modules over bialgebras

In Section 3.2 we upgraded LMod to a symmetric monoidal functor from $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ to $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. In this section we will try to better understand the functor induced on ∞-categories of $\mathcal{O}$-algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ when $\mathcal{O}$ is an ∞-operad. By Proposition 3.2.2.8 there is an equivalence

$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \simeq \mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, so that this functor can be interpreted as producing presentable monoidal ∞-categories out of $\mathcal{O}$-algebras in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ in a functorial way.

In Section 3.4.1 we will give a description of the domain of this functor. The result can be roughly summarized as follows: An $\mathcal{O}$-algebra in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ is given by a pair $(\mathcal{O}^{\otimes}, A)$ where $\mathcal{C}$ is an $\mathcal{O} \otimes \mathsf{Assoc}$-monoidal ∞-category and A is an $\mathsf{Assoc}, \mathcal{O}$-bialgebra in $\mathcal{C}$.

In Section 3.4.2 we will then discuss LMod as a functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}) \to \mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_{\infty})$$

and describe the $\mathcal{O}$-monoidal structure on an $\mathsf{Assoc}, \mathcal{O}$-bialgebra in more concrete terms. We will thus see that this construction really implements the idea described in the introduction to Chapter 3.

3.4.1 Algebras in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$

The goal of this section is to give a description of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}})$. It will turn out that the presentability condition plays little role in the discussion, so to illustrate the results we will start by unpacking a bit what objects in $\mathrm{Mon}_{\mathsf{Fin}_*}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p})$ are. Specifically, let us try to understand the multiplication functor induced by the active morphism $\mu\colon \langle 2\rangle \to \langle 1\rangle$.

So let $\mathcal{C}$ be a monoidal ∞-category and let A be an Assoc-algebra in $\mathcal{C}$. By Remark 3.1.3.7 this specifies an object of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ lying over $\mathcal{C}$ that we denote by $(\mathcal{C}, A)$.

Suppose $(\mathcal{C}, A)$ is the underlying object of a commutative monoid in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. We want to describe the multiplication

$$(\mathcal{C}, A) \times (\mathcal{C}, A) \to (\mathcal{C}, A)$$

where the product is taken in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. Propositions 3.2.1.1 and C.2.0.3 imply that the product is given by $(\mathcal{C} \times \mathcal{C}, (A, A))$. So the multiplication map is given by a morphism

$$(\mathcal{C} \times \mathcal{C}, (A, A)) \to (\mathcal{C}, A)$$

in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. We can factor this morphism as indicated in the commutative triangle below

$$\begin{tikzcd}
 & (\mathcal{C}, F((A, A))) \arrow[d, "(\mathrm{id}_{\mathcal{C}}, f)"] \\
(\mathcal{C} \times \mathcal{C}, (A, A)) \arrow[ur, "\widetilde{F}"] \arrow[r] & (\mathcal{C}, A)
\end{tikzcd}$$

where $\widetilde{F}$ is a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphism lifting a monoidal functor

$$F^{\otimes} \colon (\mathcal{C} \times \mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$$

and f is a morphism of algebras $A \to F((A, A))$ (see also Remark 3.1.3.7). The monoidal functor $F^{\otimes}$ grants us a second tensor product functor on $\mathcal{C}$, which by the Eckmann-Hilton argument can be identified with the original one. Thus f can be identified with a morphism of algebras $\Delta\colon A \to A \otimes A$, and this provides the comultiplication of a bialgebra structure on A.

To approach such a description more rigorously, we use that the cocartesian fibration of ∞-operads $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \to \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ (see Proposition 3.2.2.13 (1)) induces a cocartesian fibration

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}) \to \mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\Big)$$

for every ∞-operad $\mathcal{O}$, see Definition 3.4.1.2 and Proposition 3.4.1.3 below.

We start this section by discussing in Construction 3.4.1.1 how we can identify the codomain of this cocartesian fibration $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty))$ with $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$, the ∞-category of presentable $\mathcal{O} \otimes \mathsf{Assoc}$-monoidal ∞-categories.

Most of the remainder of this section will then be occupied by determining the fiber of $\mathrm{Alg}_{\mathcal{O}}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}})$ over a presentable $\mathcal{O}\otimes\mathsf{Assoc}$-monoidal ∞-category $\mathcal{C}$, and in Proposition 3.4.1.15 we will show that the fiber over $\mathcal{C}$ can be identified with $\mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathcal{C})^{\mathrm{op}}$.

Construction 3.4.1.1. Let $\mathcal{O}$, $\mathcal{O}'$, as well as $\mathcal{O}''$ be ∞-operads, and let $\alpha\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads exhibiting $\mathcal{O}''$ as a tensor product of $\mathcal{O}$ and $\mathcal{O}'$, and let $\mathfrak{I}$ be a collection of small ∞-categories. Then there is a commutative diagram as follows, explained below. To save space we abbreviate expressions such as $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}\mathrm{at}_\infty)$ by $\mathrm{Mon}_{\mathcal{O}'}$, i. e. we omit the $\mathcal{C}\mathrm{at}_\infty$ in parentheses.

$$\begin{tikzcd}
 & & \mathrm{Mon}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}) \arrow[d, "\simeq"] \\
\mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}'}\Big) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}'}\Big) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}) \arrow[d, "\simeq"] \\
\mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Alg}_{\mathcal{O}'}\Big(\mathcal{P}\mathrm{r}^{\mathrm{L}}\Big)\Big) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}))) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}) \arrow[d, "\simeq"] \\
\mathrm{Alg}_{\mathcal{O}''}\Big(\mathcal{P}\mathrm{r}^{\mathrm{L}}\Big) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Alg}_{\mathcal{O}''} \arrow[d, "\simeq"] \\
\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}''} \arrow[r] & \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}''} \arrow[r] & \mathrm{Mon}_{\mathcal{O}''}
\end{tikzcd}$$

The equivalence at the top right is the one from [HA, 2.4.2.5], i. e. is the one induced by $\pi_{\mathrm{Mon}*}$. The top two squares are induced on $\mathcal{O}$-algebras by the commutative diagram constructed in Proposition 3.2.2.8, which is a commutative diagram of ∞-operads by Proposition 3.2.2.10. The middle two squares are obtained from naturality of the equivalences constructed in Proposition E.5.0.2

and Proposition E.5.0.1 as discussed in Remark F.3.0.4, applied to the morphisms of ∞-operads

$$\mathcal{P}\mathrm{r}^{\mathrm{L}^{\otimes}} \to \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}^{\times}$$

from Proposition 3.2.2.3 (2). Finally, the commutative diagram constructed in Proposition 3.2.2.8 induces a commutative diagram on underlying ∞-categories that yields the bottom two commutative squares. ♢

Definition 3.4.1.2. Let $\mathcal{O}$ be an ∞-operad and $\mathfrak{I}$ a collection of small ∞-categories. We define the following ∞-categories and morphisms of ∞-categories by applying $\mathrm{Alg}_{\mathcal{O}}$ to the morphisms of ∞-operads (see Proposition 3.2.2.13 (1)) $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ and $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$. The equivalences used are the ones from Construction 3.4.1.1.

$$\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathfrak{I}} \coloneqq \mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}})$$
$$\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}} \coloneqq \mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}})$$

$$\begin{tikzcd}
\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathfrak{I}} \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}})"] \arrow[dr, dashed, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathfrak{I}}}"'] & \mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})\Big) \arrow[d, "\simeq"] \\
& \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}} \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}})"] \arrow[dr, dashed, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}}"'] & \mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})\Big) \arrow[d, "\simeq"] \\
& \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd}$$

We will also write $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$ for $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\emptyset}}$ and $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}$ for $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\emptyset}$. ♢

Proposition 3.4.1.3. *In the situation of Definition 3.4.1.2, the functors* $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathfrak{I}}}$ *and* $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}}$ *are cocartesian fibrations.* ♡

Proof. Combine Proposition 3.2.2.13 (1) with Proposition E.3.2.1. □

We start the process of identifying the fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathfrak{I}}}$ and $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}}$ by reducing the problem to $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$.

Proposition 3.4.1.4. *We use Notation 3.2.2.7 in this proposition. Let $\mathfrak{I}$ be a collection of small ∞-categories and let $\mathcal{O}$ be an ∞-operad. Then there is* *a pullback diagram in* $\mathcal{C}\mathrm{at}_\infty$ *as follows.*

$$\begin{tikzcd} \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathfrak{I}} \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(\widetilde{\Psi}^{\mathfrak{I}})"] \arrow[d, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathfrak{I}}}"'] & \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}} \arrow[d, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}"] \\ \mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, "(\Psi^{\mathfrak{I}})"'] & \mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \end{tikzcd}$$

In particular, if $\mathcal{C}$ *is an object in* $\mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)$*, then we can identify the fiber* $(\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathfrak{I}})_{\mathcal{C}}$ *with* $(\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}})_{(\Psi^{\mathfrak{I}})^{\otimes}(\mathcal{C})}$*, and if* $F\colon \mathcal{C}\to\mathcal{D}$ *is a morphism in the* ∞*-category* $\mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)$ *we can identify the induced functor on fibers of* $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathfrak{I}}}$ *with the functor induced by* $\left(\Psi^{\mathfrak{I}}\right)^{\otimes}(F)$ *on fibers of* $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$*.*

Analogous statements hold for $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}}$*.* ♡

Proof. We only prove the case of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathfrak{I}}}$, the case of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}}$ is completely analogous.

By Definition 3.2.2.11 we have a pullback diagram

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}^{\otimes} \arrow[r, "(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}"] \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}^{\otimes}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}^{\times}"] \\ \mathrm{Mon}_{\mathsf{Assoc}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[r, "(\Psi^{\mathfrak{I}})^{\otimes}"'] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \end{tikzcd}$$

where $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}^{\times}$ is a cocartesian fibration of ∞-operads (see Proposition 3.2.2.13 (1)) and $(\Psi^{\mathfrak{I}})^{\otimes}$ is a morphism of ∞-operads (see Proposition 3.2.2.10 (2)). Combining Proposition E.1.3.1 and Proposition E.3.1.1 we conclude that the the top square in the following commutative diagram is a pullback square[37]

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(\widetilde{\Psi}^{\mathfrak{I}})"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}})"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})"] \\ \mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Mon}_{\mathsf{Assoc}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)\Big) \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(\Psi^{\mathfrak{I}})"] \arrow[d, "\simeq"'] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)) \arrow[d, "\simeq"] \\ \mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, "\Psi^{\mathfrak{I}}"'] & \mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \end{tikzcd}$$

[37] The two $\Psi^{\mathfrak{I}}$ in the diagram are different functors, the same notation only arises here because the operad does not occur in the notation.

where the lower commuting square is the one from Construction 3.4.1.1. This proves the claim, as the the left and right vertical compositions are by definition $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{J}}_{\mathcal{O}}}$ and $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$. ☐

Before starting to analyze the fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$, it will be helpful to describe the equivalences from Construction 3.4.1.1 more concretely as done in the following proposition.

Proposition 3.4.1.5. *Let $\mathcal{O}$, $\mathcal{O}'$, as well as $\mathcal{O}''$ be ∞-operads, and let $\alpha\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads exhibiting $\mathcal{O}''$ as a tensor product of $\mathcal{O}$ and $\mathcal{O}'$.*

Then there is a commutative diagram as follows

$$\begin{tikzcd}
\mathrm{Mon}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}\mathrm{at}_{\infty})) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}_{\infty})) \\
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}\mathrm{at}_{\infty})) \arrow[d, "\simeq"'] & \\
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}\mathrm{at}_{\infty})) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}^{\times}_{\infty}\big)\big) \arrow[uu, "(\pi_*)_*"'] \\
\mathrm{BiFunc}(\mathcal{O}, \mathcal{O}'; \mathcal{C}\mathrm{at}_{\infty}) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}^{\times}_{\infty}\big) \arrow[u, "\widehat{(-)}"'] \\
\mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{C}\mathrm{at}^{\times}_{\infty}\big) \arrow[u, "\alpha^*"'] \arrow[d, "\pi_*"] \\
\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[r] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}) \arrow[uuuuu, bend right=90, "\widehat{(-\circ\alpha)}"']
\end{tikzcd}$$

where vertical functors on the left are the ones from Construction 3.4.1.1 (where we split up the equivalence in the middle in its two steps from Proposition E.5.0.2 and Proposition E.5.0.1) and the horizontal functors are the the compositions of the canonical inclusions and projections. ♡

Proof. The top square is obtained from the construction of the equivalence $\Theta^{\otimes}$ by combining the commutative diagrams (3.15) and (3.14) occurring in the proof of Proposition 3.2.2.8. The two middle squares are from Proposition F.3.0.3. The bottom square is diagram (3.13) from Proposition 3.2.2.8. Finally, the commutative rectangle on the right is obtained from naturality of $\widehat{(-)}$. ☐

The cocartesian fibration $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$ is constructed in multiple steps from the universal cocartesian family of Assoc-monoidal ∞-categories, but ends up with $\mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$ as a codomain. The next proposition relates the universal cocartesian family of Assoc-monoidal ∞-categories with the universal cocartesian family of $\mathsf{Assoc} \otimes \mathcal{O}$-monoidal ∞-categories.

Proposition 3.4.1.6. *Let $\mathcal{O}$, $\mathcal{O}'$, as well as $\mathcal{O}''$ be ∞-operads and let $\alpha\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads exhibiting $\mathcal{O}''$ as the tensor product of $\mathcal{O}$ and $\mathcal{O}'$. Then there is a commutative diagram as follows such that both squares are pullback diagrams, and where other parts of the diagram will be explained further below.*

$$
\begin{tikzcd}
\widetilde{\mathrm{Mon}}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[d, "p^{\mathcal{O}''}"'] & \widetilde{\mathrm{Mon}}_{\alpha}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[l] \arrow[r] \arrow[d, "p^{\alpha}"] & \widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[d, "p^{\mathcal{O}}"] \\
\mathcal{O}''^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[l] \arrow[r] & \mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)
\end{tikzcd}
\tag{3.21}
$$

The left and right vertical functors are the universal cocartesian families of monoidal ∞-categories defined in Definition 3.1.1.4, whereas the middle vertical functor is a functor we newly define here as the pullback of either the left or right square. The bottom left horizontal functor is $\alpha \times \mathrm{id}$, and the bottom right vertical functor is the the product of $\mathrm{id}_{\mathcal{O}^{\otimes}}$ with the following composition

$$
\begin{aligned}
\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) &\to \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)) \\
&\to \mathcal{O}'^{\otimes} \times \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)\big) \xrightarrow{\mathrm{ev}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)
\end{aligned}
\tag{3.22}
$$

where the first functor uses the equivalence from Proposition 3.4.1.5 interpreting $\mathcal{O}''$ as the tensor product $\mathcal{O}' \otimes \mathcal{O}$ via $\alpha \circ \tau$, where τ is the symmetry equivalence $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \simeq \mathcal{O}'^{\otimes} \times \mathcal{O}^{\otimes}$, and the second functor is the product of the identity and the canonical inclusion. ♡

Proof. Both $p^{\mathcal{O}''}$ and $p^{\mathcal{O}}$ are by definition cocartesian fibrations, with $p^{\mathcal{O}}$ classified by[38] the composition

$$
\mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \to \mathcal{O}^{\otimes} \times \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty\big) \xrightarrow{\mathrm{ev}} \mathcal{C}\mathrm{at}_\infty
$$

where the first functor is the product of the identity functor and the canonical inclusion, and similarly for $p^{\mathcal{O}''}$. So by naturality of the Grothendieck construction[39] it suffices to show that the composition of the left bottom horizontal functor in diagram (3.21) with the functor the left vertical cocartesian fibration is classified by is homotopic to the composition of the right bottom horizontal functor with the functor the right vertical cocartesian fibration is classified by. For this consider the following three commutative diagrams, where we will denote the various canonical inclusions by ι and abbreviate

[38] See Definition 3.1.1.4.
[39] See [GHN17, A.32] and [Maz19].

$\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ by $\mathrm{Mon}_{\mathcal{O}}$ and analogously for $\mathcal{O}'$ and $\mathcal{O}''$.

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''} \arrow[r, "\mathrm{id}\times\mathrm{id}\times\iota"] \arrow[d] & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathrm{id}\times\mathrm{id}\times\alpha^*"] \\
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Mon}_{\mathcal{O}}) \arrow[d, "\mathrm{id}\times\mathrm{id}\times\iota"'] & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathrm{id}\times\mathrm{id}\times\widehat{-\circ\tau}"] \\
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathrm{Mon}_{\mathcal{O}}) \arrow[r, "\mathrm{id}\times\mathrm{id}\times\iota_*"] \arrow[d, "\mathrm{id}\times\mathrm{ev}"'] & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty)) \arrow[d, "\mathrm{id}\times\mathrm{ev}"] \\
\mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}} \arrow[r, "\mathrm{id}\times\iota"] & \mathcal{O}^{\otimes} \times \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathrm{ev}"] \\
 & \mathcal{C}\mathrm{at}_\infty
\end{tikzcd} \tag{$*$}$$

In the above diagram, the top square arises from Proposition 3.4.1.5 and the bottom square uses naturality of evaluation. The next two commutative diagrams only use various naturalities and functorialities.

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathrm{id}\times\mathrm{id}\times\widehat{-\circ\tau}"] \arrow[ddr, bend left, "\mathrm{ev}"] & \\
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty)) \arrow[d, "\mathrm{id}\times\mathrm{ev}"] & \\
\mathcal{O}^{\otimes} \times \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[r, "\mathrm{ev}"] & \mathcal{C}\mathrm{at}_\infty
\end{tikzcd} \tag{$**$}$$

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, "\mathrm{id}\times\mathrm{id}\times\iota"] \arrow[d, "\alpha\times\mathrm{id}"'] & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathrm{id}\times\mathrm{id}\times\alpha^*"] \arrow[ddl, "\alpha\times\mathrm{id}"'] \\
\mathcal{O}''^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathrm{id}\times\iota"'] & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathrm{ev}"] \\
\mathcal{O}''^{\otimes} \times \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[r, "\mathrm{ev}"] & \mathcal{C}\mathrm{at}_\infty
\end{tikzcd} \tag{$***$}$$

The composite of the lower left (right) horizontal functor in diagram (3.21) with the functor the left (right) vertical cocartesian fibration is classified by is precisely the composite via the bottom left corner from the top left to the bottom right corner in diagram $(***)$ (in diagram $(*)$). Diagrams $(*)$, $(**)$, and $(***)$ show that these two composites are homotopic, which proves the claim. $\square$

We next go through the steps used to construct $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ from p^{Assoc} and show how we can identify $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ with a functor obtained from p^α as in Proposition 3.4.1.6. We will use the right pullback square in (3.21) to compare constructions obtained from p^α with the intermediate steps on the way to $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$, while using the left pullback square to be able to describe those constructions in a way helpful to ultimately describe fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ as ∞-categories of bialgebras.

Definition 3.4.1.7. Let $\mathcal{O}'$ as well as $\mathcal{O}''$ be two ∞-operads and let furthermore $\alpha\colon \mathsf{Assoc}^\otimes \times \mathcal{O}'^\otimes \to \mathcal{O}''^\otimes$ be a bifunctor of ∞-operads that exhibits $\mathcal{O}''$ as the tensor product of Assoc and $\mathcal{O}$.

Using that the right square in (3.21) is a pullback diagram we can interpret p^α from Proposition 3.4.1.6 as a cocartesian $\mathcal{O}'^\otimes \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)$-family of Assoc-monoidal ∞-categories. Passing to Assoc-algebras we obtain by Proposition 3.1.2.10 a pullback, where we will denote the ∞-category on the top left and functor on the left as indicated, and the functor on the right is the one from Definition 3.1.3.3.

$$\begin{tikzcd}
\mathcal{A}^\otimes = \mathrm{Alg}_{/\mathsf{Assoc}}\left(\widetilde{\mathrm{Mon}}_\alpha(\mathcal{C}\mathrm{at}_\infty)^\otimes\right) \arrow[r] \arrow[d, "q_{\mathcal{A}}"'] & \mathcal{A}\mathrm{lg} \arrow[d, "q_{\mathcal{A}\mathrm{lg}}"] \\
\mathcal{O}'^\otimes \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)
\end{tikzcd}$$

$\diamondsuit$

Remark 3.4.1.8. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{O}$ be an ∞-operad, and let $p\colon \mathcal{D}^\otimes \to \mathcal{O}^\otimes \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.

Note that the projection $\mathrm{pr}_2\colon \mathcal{O}^\otimes \times \mathcal{C} \to \mathcal{C}$ is a cocartesian fibration[40], and pr_2-cocartesian morphisms are those that are (equivalent to) an identity morphism in the first factor.

By [HTT, 2.4.2.3 (3)] and Proposition C.1.3.1 we obtain a morphism of cocartesian fibrations over $\mathcal{C}$ as follows.

$$\begin{tikzcd}
\mathcal{D}^\otimes \arrow[rr, "p"] \arrow[dr, "\mathrm{pr}_2 \circ p"'] & & \mathcal{O}^\otimes \times \mathcal{C} \arrow[dl, "\mathrm{pr}_2"] \\
& \mathcal{C} &
\end{tikzcd}$$

[40] This is for example easy to see by using that it is the pullback of the functor $\mathcal{O}^\otimes \to *$ along $\mathcal{C} \to *$.

If $f\colon X \to Y$ is a morphism in $\mathcal{C}$, then we obtain an induced commutative square on fibers as follows.

$$\begin{tikzcd} \mathcal{D}_X^{\otimes} \arrow[r, "f_!"] \arrow[d, "p_X"'] & \mathcal{D}_Y^{\otimes} \arrow[d, "p_Y"] \\ \mathcal{O}^{\otimes} \arrow[r, "\mathrm{id}"'] & \mathcal{O}^{\otimes} \end{tikzcd}$$

By the description of pr_2-cocartesian morphisms given above the induced functor on fibers of pr_2 is the identity, and by assumption on p the two vertical functors are cocartesian fibrations of ∞-operads. We thus obtain a commuting triangle

$$\begin{tikzcd} \mathcal{D}_X^{\otimes} \arrow[rr, "p_X"] \arrow[dr, "\mathrm{pr}_2 \circ p"'] & & \mathcal{D}_Y^{\otimes} \arrow[dl, "p_Y"] \\ & \mathcal{O}^{\otimes} & \end{tikzcd}$$

that by Proposition 3.1.1.1 is an $\mathcal{O}$-monoidal functor. It is this $\mathcal{O}$-monoidal functor that we will refer to as the induced $\mathcal{O}$-monoidal functor on fibers over f. ♢

Proposition 3.4.1.9. *Assume we are in the situation of Definition 3.4.1.7, and let $\mathcal{C}$ be an $\mathcal{O}''$-monoidal ∞-category. Then the fiber of $q_{\mathcal{A}}$ over $\mathcal{C}$ (considered as an object of $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)$) can be identified with the $\mathcal{O}'$-monoidal ∞-category of Assoc-algebras*[41] *$\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes}$ from Proposition E.4.2.3.*

Furthermore, if $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{O}''$-monoidal functor, then the induced $\mathcal{O}'$-monoidal functor on fibers of $q_{\mathcal{A}}$ fits into a commutative diagram as follows

$$\begin{tikzcd} \mathcal{A}_{\mathcal{C}}^{\otimes} \arrow[r, "F_!"] \arrow[d, "\simeq"'] & \mathcal{A}_{\mathcal{D}}^{\otimes} \arrow[d, "\simeq"] \\ \mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(F)^{\otimes}"'] & \mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{D})^{\otimes} \end{tikzcd}$$

where $\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(F)^{\otimes}$ is the induced functor from Proposition E.4.2.3 and the vertical equivalences are the ones from the first claim of this proposition. ♡

Proof. Consider the following commutative diagram, where the top pullback square is the one from Definition 3.1.2.1, and the bottom square is the induced pullback square by applying $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, -)$ to the left pullback square in diagram (3.21) of Proposition 3.4.1.6. We abbreviate $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)$ by $\mathrm{Mon}_{\mathcal{O}''}$

[41] With respect to the bifunctor of ∞-operads $\alpha \circ \tau$.

to save space.

$$\begin{tikzcd}
\mathcal{A}^{\otimes} \arrow[d] \arrow[drr, "q_{\mathcal{A}}"] & & \\
\widetilde{\mathcal{A}}^{\otimes} := \widetilde{\mathrm{Alg}}_{/\mathsf{Assoc}}\Big(\widetilde{\mathrm{Mon}}_{\alpha}(\mathcal{C}\mathrm{at}_{\infty})\Big) \arrow[rr, "\mathrm{pr}"] \arrow[d, "\mathrm{pr}_{\mathrm{Fun}}"'] & & \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''} \arrow[d, "\widehat{\mathrm{id}}_{\mathsf{Assoc}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}}"] & (*) \\
\mathrm{Fun}\Big(\mathsf{Assoc}^{\otimes}, \widetilde{\mathrm{Mon}}_{\alpha}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}\Big) \arrow[rr, "p^{\alpha}_{*}"] \arrow[d] & & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathsf{Assoc}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}\big) \arrow[d, "(\alpha \times \mathrm{id})_{*}"] \\
\mathrm{Fun}\Big(\mathsf{Assoc}^{\otimes}, \widetilde{\mathrm{Mon}}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}\Big) \arrow[rr, "p^{\mathcal{O}''}_{*}"] & & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{O}''^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}\big)
\end{tikzcd}$$

$\mathcal{A}^{\otimes}$ is by definition[42] the full subcategory of $\widetilde{\mathcal{A}}^{\otimes}$ spanned by those objects that are mapped by $\mathrm{pr}_{\mathrm{Fun}}$ to functors $\mathsf{Assoc}^{\otimes} \to \widetilde{\mathrm{Mon}}_{\alpha}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$ that send inert morphisms to p^{α}-cocartesian ones. By the description of p^{α}-cocartesian morphisms afforded by the left pullback square in diagram (3.21) of Proposition 3.4.1.6 in combination with Proposition C.1.1.1 we can thus identify $\mathcal{A}^{\otimes}$ with the full subcategory of $\widetilde{\mathcal{A}}^{\otimes}$ spanned by those objects that map to functors $\mathsf{Assoc}^{\otimes} \to \widetilde{\mathrm{Mon}}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$ which send inert morphisms to $p^{\mathcal{O}''}$-cocartesian ones. Similarly, we obtain from Proposition 3.1.2.17 that a morphism in $\mathcal{A}^{\otimes}$ is $q_{\mathcal{A}}$-cocartesian if and only if it maps to a natural transformation of functors $\mathsf{Assoc}^{\otimes} \to \widetilde{\mathrm{Mon}}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$ that is pointwise $p^{\mathcal{O}''}$-cocartesian.

Now let $\mathcal{C}$ be an $\mathcal{O}''$-monoidal ∞-category. Then there is a commutative cube as follows[43].

$$\begin{tikzcd}
& \widetilde{\mathrm{Alg}}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[rr] \arrow[dl, dashed] \arrow[dd] & & \mathcal{O}'^{\otimes} \times \{\mathcal{C}\} \arrow[dl] \arrow[dd] \\
\widetilde{\mathcal{A}}^{\otimes} \arrow[rr] \arrow[dd] & & \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''} \arrow[dd] & \\
& \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[rr] \arrow[dl] & & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{O}''^{\otimes} \times \{\mathcal{C}\}\big) \arrow[dl] \\
\mathrm{Fun}\Big(\mathsf{Assoc}^{\otimes}, \widetilde{\mathrm{Mon}}_{\mathcal{O}''}\Big) \arrow[rr] & & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{O}''^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}\big) &
\end{tikzcd}$$

[42]See Definition 3.1.2.3.

[43]We abbreviate $\widetilde{\mathrm{Mon}}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$ and $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$ as $\widetilde{\mathrm{Mon}}_{\mathcal{O}''}$ and $\mathrm{Mon}_{\mathcal{O}''}$.

The front square is the composite pullback diagram from $(*)$. The bottom square is the pullback square obtained by applying $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, -)$ to the pullback diagram of the identification of $\mathcal{C}^{\otimes}$ as the fiber of $p^{\mathcal{O}''}$ over $\mathcal{C}$, see Remark 3.4.1.8. The back one is the pullback diagram from Proposition E.4.2.3. That there is a commutative square as indicated on the right, where the top functor is the product of the identity with the inclusion of $\{\mathcal{C}\}$, can be checked by unpacking the definitions and using naturality. We obtain the induced top and left square and filler for the cube (using that the front square is a pullback square), and it follows from [HTT, 4.4.2.1] that the top square is also a pullback diagram.

The description of $\mathcal{A}^{\otimes}$ as a full subcategory of $\widetilde{\mathcal{A}}^{\otimes}$ we gave above together with the definition of $\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes}$ as a full subcategory of $\widetilde{\mathrm{Alg}}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes}$ in Remark E.4.2.1 and an argument very similar to the one in the proof of Proposition 3.1.2.2 show that the dashed functor in the above diagram induces an equivalence

$$\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathcal{A}^{\otimes}$$

on full subcategories.

The description of the functor induced on fibers by a morphism $F\colon \mathcal{C} \to \mathcal{D}$ of $\mathcal{O}''$-monoidal ∞-categories follows from the description given above for $q_{\mathcal{A}}$-cocartesian morphisms together with the fact that the $\mathcal{O}''$-monoidal functor induced by F (considered as a morphism in $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$) on fibers of $p^{\mathcal{O}''}$ can by construction (see Definition 3.1.1.4) be identified with F. $\square$

Proposition 3.4.1.10. *Assume we are in the situation of Definition 3.4.1.7. Then $q_{\mathcal{A}}$ is a cocartesian $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$-family of $\mathcal{O}'$-monoidal ∞-categories.* $\heartsuit$

Proof. Follows from the definition[44] together with Proposition 3.4.1.9 and Proposition E.4.2.3 (3). $\square$

Definition 3.4.1.11. Assume we are in the situation of Definition 3.4.1.7. We let

$$q_{\mathcal{A}'}\colon \mathcal{A}'^{\otimes} \to \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$$

be the cocartesian fibration obtained by applying the functor

$$\begin{aligned} &\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}\big(\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})\big)\\ &\to \mathrm{Fun}\big(\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty}), \mathcal{C}\mathrm{at}_{\infty}\big)\\ &\xrightarrow{(-^{\mathrm{op}})_*} \mathrm{Fun}\big(\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty}), \mathcal{C}\mathrm{at}_{\infty}\big)\\ &\to \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}\big(\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})\big)\end{aligned}$$

to $q_{\mathcal{A}}\colon \mathcal{A}^{\otimes} \to \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$. $\diamondsuit$

[44] Definition 3.1.1.2 with variant Proposition 3.1.1.1 (2).

Proposition 3.4.1.12. *Assume we are in the situation of Definition 3.4.1.7. Then there is a pullback diagram as follows*

$$\begin{tikzcd} \mathcal{A}'^{\otimes} \arrow[r] \arrow[d, "q_{\mathcal{A}'}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\ \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \end{tikzcd}$$

where the bottom functor is the composition (3.22)*.* ♡

Proof. Follows immediately from Definition 3.4.1.11 and Definition 3.1.3.5 together with Proposition 3.4.1.6 and naturality of the Grothendieck construction. □

Proposition 3.4.1.13. *Assume we are in the situation of Definition 3.4.1.7. Then the following hold.*

(1) $q_{\mathcal{A}'}$ *from Definition 3.4.1.11 is again a cocartesian* $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)$*-family of* $\mathcal{O}'$*-monoidal* ∞*-categories.*

(2) Let $\mathcal{C}$ *be a* $\mathcal{O}''$*-monoidal* ∞*-category. Then the fiber of* $q_{\mathcal{A}'}$ *over* $\mathcal{C}$ *is, as an* $\mathcal{O}'$*-monoidal* ∞*-category, equivalent to* $(\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\mathrm{op}})^{\otimes}$*, the opposite* $\mathcal{O}'$*-monoidal* ∞*-category of* $\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes}$*.*

(3) Let $F\colon \mathcal{C} \to \mathcal{D}$ *be a* $\mathcal{O}''$*-monoidal functor. Then there is a commutative square*

$$\begin{tikzcd} \mathcal{A}'^{\otimes}_{\mathcal{C}} \arrow[r, "F_!"] \arrow[d, "\simeq"'] & \mathcal{A}'^{\otimes}_{\mathcal{D}} \arrow[d, "\simeq"] \\ (\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\mathrm{op}})^{\otimes} \arrow[r, "(\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(F)^{\mathrm{op}})^{\otimes}"'] & (\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{D})^{\mathrm{op}})^{\otimes} \end{tikzcd}$$

where the top functor is the one induced on fibers of $q_{\mathcal{A}'}$*, and the vertical functors are the equivalences from (2).* ♡

Proof. Follows directly from $q_{\mathcal{A}}$ being a cocartesian family of $\mathcal{O}'$-monoidal ∞-categories by Proposition 3.4.1.10 and the description of its fibers in Proposition 3.4.1.9. □

Proposition 3.4.1.14. *Let* $\mathcal{O}'$ *as well as* $\mathcal{O}''$ *be* ∞*-operads and let furthermore* $\alpha\colon \mathsf{Assoc}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ *be a bifunctor of* ∞*-operads that exhibits* $\mathcal{O}''$ *as the tensor product of* Assoc *and* $\mathcal{O}$*.*

Then there is a commutative triangle as follows such that the horizontal functor is an equivalence

$$\begin{tikzcd} \mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}') \arrow[rr, "\simeq", no head] \arrow[dr, "\mathrm{pr}_{\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)}"'] & & \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}'} \arrow[dl, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}'}}"] \\ & \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) & \end{tikzcd}$$

where the functor on the left is as in Definition 3.1.2.3 and Definition 3.1.2.1, applied to the cocartesian family of $\mathcal{O}'$-monoidal ∞-categories $q_{\mathcal{A}'}$ from Definition 3.4.1.11 and Proposition 3.4.1.13. ♡

Proof. By naturality of the construction $-^\times$ and [HA, 2.4.2.5] there is a commutative diagram as follows

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}'}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})"'] & \mathrm{Mon}_{\mathcal{O}'}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[d, "\mathrm{Mon}_{\mathcal{O}'}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})"] \\ \mathrm{Alg}_{\mathcal{O}'}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)) \arrow[r, "\simeq"'] & \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)) \end{tikzcd}$$

such that the two horizontal functors are equivalences. It follows from Definition 3.4.1.2 and Construction 3.4.1.1 that there is a commutative square

$$\begin{tikzcd} \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}'} \arrow[r, "\simeq", no head] \arrow[d, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}'}}"'] & \mathrm{Mon}_{\mathcal{O}'}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[d, "\mathrm{Mon}_{\mathcal{O}'}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})"] \\ \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, "\simeq"', no head] & \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)) \end{tikzcd}$$

such that the bottom horizontal functor is the equivalence from Construction 3.4.1.1.

Thus it suffices to show that there is a commutative square as follows

$$\begin{tikzcd} \mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}') \arrow[r, "\simeq", no head] \arrow[d, "\mathrm{pr}_{\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)}"'] & \mathrm{Mon}_{\mathcal{O}'}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[d, "\mathrm{Mon}_{\mathcal{O}'}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})"] \\ \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, "\simeq"', no head] & \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)) \end{tikzcd}$$

such that the bottom horizontal functor is the equivalence from Construction 3.4.1.1.

Now we consider the following diagram[45] that will be explained in detail

[45] We abbreviate $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)$ and $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ as $\mathrm{Mon}_{\mathcal{O}''}$ and $\mathrm{Mon}_{\mathsf{Assoc}}$.

below.

$$\begin{tikzcd}
 & \mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}') \arrow[d] \arrow[rr, dashed] & & \mathrm{Mon}_{\mathcal{O}'}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[ddl, "\iota"'] \arrow[d, dashed, "\varphi"] \arrow[ddd, bend left] \\
 & \widetilde{\mathrm{Alg}}_{/\mathcal{O}'}(\mathcal{A}') \arrow[dl, "\mathrm{pr}_{\mathrm{Fun}}"'] \arrow[rr, dashed, "\vartheta"] \arrow[dd] & & \mathcal{P} \arrow[dl, "\psi"] \arrow[dd] \\
\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{A}'^{\otimes}\big) \arrow[rr] \arrow[dd] & & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\big) \arrow[dd] & \\
 & \mathrm{Mon}_{\mathcal{O}''} \arrow[rr] \arrow[dl] & & \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Mon}_{\mathsf{Assoc}}) \arrow[dl] \\
\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}\big) \arrow[rr] & & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathrm{Mon}_{\mathsf{Assoc}}\big) &
\end{tikzcd}$$

The front square is $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$ applied to the pullback square from Proposition 3.4.1.12. In particular, the front square is again a pullback square. The bottom square arises from naturality of $\widehat{-}$ and the fact that $\widehat{\mathrm{ev}} = \mathrm{id}$. The bottom back horizontal equivalence is the one from Construction 3.4.1.1 and Proposition 3.4.1.5. The left square is the pullback square defining $\widetilde{\mathrm{Alg}}_{/\mathcal{O}'}(\mathcal{A}')$, see Definition 3.1.2.1. We define the right square to be a pullback square.

As the left and right squares in the cube are pullback diagrams, we obtain an induced functor ϑ together with fillers for the top and back square and the cube.

The right big square arises from applying the natural transformation

$$\mathrm{Mon}_{\mathcal{O}'}(-) \to \mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$$

to $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$. We obtain the induced functor φ and the two commutative triangles on the right. By definition, ι and the bottom functor from the back to the front on the right side are fully faithful. As the small square is a pullback square and taking pullbacks preserves fully faithful functors by Proposition B.5.2.1, ψ is fully faithful as well. By considering the top triangle on the right side we then deduce that φ is also fully faithful[46].

What we have to show is that there is a dashed top back horizontal functor making the back big rectangle commute and which is an equivalence. As the front, left, and right squares are pullback squares it follows from [HTT, 4.4.2.1] that the back lower square is a pullback square as well. As the lower back horizontal functor is an equivalence, it follows that ϑ is an equivalence

[46] It follows immediately from Definition B.2.0.1 that functors being fully faithful satisfies the two-out-of-three-property.

too. It thus suffices to show that an object A of $\widetilde{\mathrm{Alg}}_{/\mathcal{O}'}(\mathcal{A}')$ is in the essential image of the functor from $\mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}')$ if and only if $\vartheta(A)$ is in the essential image of φ (see Proposition B.4.3.1).

We first consider the essential image of φ, which consists of precisely those objects that are mapped by ψ to an object that is in the essential image of ι i. e. is an $\mathcal{O}'$-monoid. By definition [HA, 2.4.2.1], a functor $F\colon \mathcal{O}'^{\otimes} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ is an $\mathcal{O}'$-monoid if and only if for every $n \geq 0$, objects X_i in $\mathcal{O}'$ for every $1 \leq i \leq n$, and inert morphisms $r_i\colon X_1 \oplus \cdots \oplus X_n \to X_i$ lying over ρ^i, the morphisms $F(r_i)$ exhibit $F(X_1 \oplus \cdots \oplus X_n)$ as the product of $(F(X_i))_{1\leq i\leq n}$. By the description of products in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ from Proposition 3.2.1.1 and Proposition C.2.0.3 this is equivalent to the morphisms $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}(F(r_i))$ exhibiting $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}(F(X_1 \oplus \cdots \oplus X_n))$ as the product of $(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}(F(X_i)))_{1\leq i\leq n}$ and $F(r_i)$ being $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian for every $1 \leq i \leq n$. Thus F is in the essential image of ι if and only if $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \circ F$ is an $\mathcal{O}'$-monoid and F maps inert morphisms to $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphisms. By Proposition B.5.2.1, a functor $F\colon \mathcal{O}'^{\otimes} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ lies in the essential image of ψ if and only if $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \circ F$ is an $\mathcal{O}'$-monoid. It follows that an object A of $\mathcal{P}$ is in the essential image of φ if and only if $\psi(A)$ maps inert morphisms to $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphisms.

By definition[47], an object A of $\widetilde{\mathrm{Alg}}_{/\mathcal{O}'}(\mathcal{A}')$ is in the essential image of the inclusion from $\mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}')$ if and only if $\mathrm{pr}_{\mathrm{Fun}}(A)$ maps inert morphisms to $q_{\mathcal{A}'}$-cocartesian morphisms. By Proposition 3.4.1.12 and Proposition C.1.1.1 this is the case if and only if $\psi(\vartheta(A))$ maps inert morphisms to $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphisms. Thus an object A of $\widetilde{\mathrm{Alg}}_{/\mathcal{O}'}(\mathcal{A}')$ is in the essential image of the functor from $\mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}')$ if and only if $\vartheta(A)$ is in the essential image of φ, which finishes the proof. □

With Proposition 3.4.1.14 we can now finally discuss the fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$.

Proposition 3.4.1.15. *Let $\mathfrak{I}$ be a collection of small ∞-categories, let $\mathcal{O}$ be an ∞-operad. Then the following hold.*

(1) *Let $\mathcal{C}$ be an $\mathsf{Assoc} \otimes \mathcal{O}$-monoidal ∞-category that is compatible with $\mathfrak{I}$-indexed colimits, and that we also consider as an object of the ∞-category $\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then the fiber of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}}}$ over $\mathcal{C}$ can be identified with $\mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathcal{C})^{\mathrm{op}}$.*

(2) *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then there is a commutative diagram*

$$\begin{tikzcd}
(\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}})_{\mathcal{C}} \arrow[r, "F_!"] \arrow[d, "\simeq"'] & (\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}})_{\mathcal{D}} \arrow[d, "\simeq"] \\
\mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathcal{C})^{\mathrm{op}} \arrow[r, "\mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathcal{F})^{\mathrm{op}}"'] & \mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathcal{D})^{\mathrm{op}}
\end{tikzcd}$$

[47] Definition 3.1.2.3

where the top horizontal functor is the one induced by F on fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}}}$ and the vertical equivalences are those from (1).

Analogous statements holds for $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}}$. ♡

Proof. By Proposition 3.4.1.4 and Proposition 3.4.1.14 we can consider fibers of

$$\mathrm{pr}_{\mathrm{Mon}_{\mathsf{Assoc}\otimes\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)}\colon \mathrm{Alg}_{/\mathcal{O}}(\mathcal{A}) \to \mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$$

instead. For this we can combine Proposition 3.4.1.13 with Remark 3.1.2.18 and then need only compare with the definition of BiAlg in Definition 3.3.0.3. □

3.4.2 LMod as a functor from $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$

In this short section we discuss LMod as a functor from $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}$ to $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$.

Definition 3.4.2.1. Let $\mathfrak{I}$ be a collection of small ∞-categories that includes $\boldsymbol{\Delta}^{\mathrm{op}}$ and $\mathcal{O}$ an ∞-operad.

Applying $\mathrm{Alg}_{\mathcal{O}}(-)$ to the natural transformation of symmetric monoidal functors denoted by $\mathrm{ev}^{\otimes}_{\mathfrak{m}}\colon \mathrm{LMod}^{\otimes} \to \mathrm{pr}^{\otimes}$ of Proposition 3.2.3.1 and postcomposing with the underlying equivalences of Proposition 3.2.2.8[48] we obtain natural transformations that we will again denote by $\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{LMod} \to \mathrm{pr}$, as depicted in the commutative diagram below

$$\begin{tikzcd}
\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}"'] & \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}} \arrow[l, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}}"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})"'] \arrow[r, bend left, "\mathrm{LMod}", ""{name=A1, below}] \arrow[r, bend right, "\mathrm{pr}"', ""{name=B1, above}] & \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}"] \\
\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "\Psi^{\mathfrak{I}}"'] & \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}} \arrow[l, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}}}"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\widetilde{\Psi}^{\mathfrak{I}})"'] \arrow[r, bend left, "\mathrm{LMod}", ""{name=A2, below}] \arrow[r, bend right, "\mathrm{pr}"', ""{name=B2, above}] & \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "\Psi^{\mathfrak{I}}"] \\
\mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) & \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}} \arrow[l, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}"'] \arrow[r, bend left, "\mathrm{LMod}", ""{name=A3, below}] \arrow[r, bend right, "\mathrm{pr}"', ""{name=B3, above}] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)
\arrow[Rightarrow, from=A1, to=B1, "\mathrm{ev}_{\mathfrak{m}}"]
\arrow[Rightarrow, from=A2, to=B2, "\mathrm{ev}_{\mathfrak{m}}"]
\arrow[Rightarrow, from=A3, to=B3, "\mathrm{ev}_{\mathfrak{m}}"]
\end{tikzcd}$$

where the functors Ψ and $\widetilde{\Psi}$ are as in Notation 3.2.2.7 and Definition 3.2.2.11, and the left part of the diagram is induced by the pullback squares of Def-

[48]So $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \simeq \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ etc.

inition 3.2.2.11, which are commutative squares of ∞-operads by Proposition 3.2.2.13. $\diamondsuit$

Remark 3.4.2.2. By Proposition E.4.2.3 (8) the functor induced on $\mathcal{O}$-algebras by a symmetric monoidal functor can again be upgraded to a symmetric monoidal functor with respect to the induced symmetric monoidal structures. It follows that the natural transformations $\mathrm{ev}_{\mathfrak{m}}$ defined in Definition 3.4.2.1 acquire the structure of natural transformations of symmetric monoidal functors between symmetric monoidal ∞-categories. $\diamondsuit$

Remark 3.4.2.3. Let $\mathcal{O}$ be an ∞-operad. Using [HA, 2.4.2.5] and the definition of the equivalence

$$\Theta\colon \operatorname{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \to \operatorname{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$

as in diagram (3.13) of Proposition 3.2.2.8, we can identify the functor

$$\operatorname{LMod}\colon \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}} \to \operatorname{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$

with the functor induced by the product-preserving functor

$$\operatorname{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$$

on $\mathcal{O}$-monoids.

Let $\mathcal{C}$ be a symmetric monoidal ∞-category, A an associative algebra in $\mathcal{C}$, and consider $(\mathcal{C}, A)$ as an object of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. In the introduction to Section 3.4.1 we discussed how the multiplication morphism induced by the active morphism $\mu\colon \langle 2\rangle \to \langle 1\rangle$ looks like for a commutative monoid structure on $(\mathcal{C}, A)$. Concretely, the multiplication morphism factors as a composition

$$(\mathcal{C}\times\mathcal{C}, (A, A)) \xrightarrow{\widetilde{-\otimes-}} (\mathcal{C}, A\otimes A) \xrightarrow{(\mathrm{id},\Delta)} (\mathcal{C}, A)$$

where $\widetilde{-\otimes-}$ is a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian lift of the tensor product functor

$$-\otimes-\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$$

and (id,Δ) is a morphism in the fiber of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ over $\mathcal{C}$ – so in $\operatorname{Alg}(\mathcal{C})^{\mathrm{op}}$ – given by a morphism of algebras $\Delta\colon A \to A\otimes A$, encoding the comultiplication.

Let us now discuss the induced multiplication on $\operatorname{LMod}_A(\mathcal{C})$, using Remark 3.1.3.7. The multiplication functor can be identified with the composition

$$\begin{gathered}\operatorname{LMod}_A(\mathcal{C})\times\operatorname{LMod}_A(\mathcal{C}) \xrightarrow{\simeq} \operatorname{LMod}_{(A,A)}(\mathcal{C}\times\mathcal{C})\\ \xrightarrow{\operatorname{LMod}_{(A,A)}(-\otimes-)} \operatorname{LMod}_{A\otimes A}(\mathcal{C}) \xrightarrow{\operatorname{LMod}_\Delta(\mathcal{C})} \operatorname{LMod}_A(\mathcal{C})\end{gathered}$$

where the first functor arises from compatibility of LMod with products, the second is induced by $\widetilde{-\otimes-}$, and the last functor is given by restriction of the action along Δ.

Let now X and Y be two objects in $\mathrm{LMod}_A(\mathcal{C})$. Then $\mathrm{LMod}_{(A,A)}(-\otimes -)$ sends (X,Y) to the left $A\otimes A$-module in $\mathcal{C}$ whose underlying object in $\mathcal{C}$ is $X\otimes Y$ and where the action by $A\otimes A$ is the tensor-factor-wise one, i. e.

$$(A\otimes A)\otimes(X\otimes Y)\simeq(A\otimes X)\otimes(A\otimes Y)\to X\otimes Y \tag{3.23}$$

where the first morphism uses the symmetric monoidal structure on $\mathcal{C}$ and the second is the tensorwise action of A on X and Y, respectively. Finally, $\mathrm{LMod}_\Delta(\mathcal{C})$ restricts this action along Δ.

The unit morphism, as well as the case of ∞-operads other than the commutative one can be unpacked analogously, and hence the functor

$$\mathrm{LMod}\colon \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}\to\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$

really implements the construction sketched at the very beginning of Chapter 3. $\diamondsuit$

We end this section by considering the case of 1-categories, for which the constructions discussed so far reduce to the classical ones.

Remark 3.4.2.4. Let C be a 1-category. The data of a symmetric monoidal structure on C in the classical sense is equivalent to the the data of a symmetric monoidal structure on C considered as an ∞-category, so there is no ambiguity when talking about symmetric monoidal structures on C[49].

So assume now that C is a symmetric monoidal 1-category. By [HA, 4.1.1.2 and 2.1.3.3] the ∞-categories $\mathrm{Alg}(\mathsf{C})$ and $\mathrm{CAlg}(\mathsf{C})$ of associative and commutative algebras in C are 1-categories and can be identified with the usual classical 1-categories of associative and commutative algebras in C. Let $\mathcal{O}$ be either the ∞-operad Assoc or Comm. Then we can also conclude that the ∞-category $\mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathsf{C})$ can be identified with the classical 1-category of $\mathsf{Assoc},\mathcal{O}$-bialgebras in C.

Similarly, if A is an associative algebra in C, then by [HA, 4.2.1.3] the ∞-category $\mathrm{LMod}_A(\mathsf{C})$ is a 1-category that can be identified with the usual classical 1-category of left modules over A. The discussion in Remark 3.4.2.3 furthermore implies that if A is an $\mathsf{Assoc},\mathsf{Comm}$-bialgebra in C, then we can also identify the symmetric monoidal structure on $\mathrm{LMod}_A(\mathsf{C})$ with the classical one that was sketched in the introduction to Chapter 3. $\diamondsuit$

[49] The discussion in [HA, after 2.0.0.6 and condition (M2)] can be summarized as follows: The data of a symmetric monoidal structure on C in the classical sense (up to symmetric monoidal equivalence) is equivalent to the data of a cocartesian fibration of ∞-operads $p\colon \mathsf{C}^\otimes\to\mathsf{Fin}_*$ (up to symmetric monoidal equivalence) such that $\mathsf{C}^\otimes$ is a 1-category.

But if $p\colon \mathsf{C}^\otimes\to\mathsf{Fin}_*$ is any cocartesian fibration of ∞-operads with $\mathsf{C}^\otimes_{\langle 1\rangle}\simeq\mathsf{C}$, then $\mathsf{C}^\otimes$ is automatically a 1-category. Indeed, using that Fin_* is a 1-category it suffices to show that for every pair of objects X and Y of $\mathsf{C}^\otimes$ and morphism $f\colon p(X)\to p(Y)$ in Fin_* the fiber of the map $\mathrm{Map}_{\mathsf{C}^\otimes}(X,Y)\to\mathrm{Map}_{\mathsf{Fin}_*}(p(X),p(Y))$ over f is discrete. But by [HTT, 2.4.4.2], this fiber is equivalent to $\mathrm{Map}_{\mathsf{C}^\otimes_{p(Y)}}(f_!X,Y)$, which is discrete as $\mathsf{C}^\otimes_{p(Y)}\simeq\mathsf{C}^{\times n}$ is a 1-category (here n is such that $p(Y)\cong\langle n\rangle$).

Chapter 4

Mixed complexes

Let A be an associative k-algebra. As will be discussed in Chapter 6, the Hochschild homology functor $\mathrm{HH}_{\mathbb{T}}$ produces out of A an object of $\mathcal{D}(k)$ with action by the circle group $\mathbb{T}$, so an object of $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$. It will be useful to have a strict model for $\mathrm{HH}_{\mathbb{T}}(A)$, by which we mean an object representing $\mathrm{HH}_{\mathbb{T}}(A)$ in a model category whose underlying ∞-category comes with an equivalence to $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$. This can indeed by done; there is a result of Hoyois [Hoy18], which we will discuss in more detail in Section 6.3.4.1, that provides us with a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{LMod}_k(\mathsf{Ab})) \arrow[rr, "\text{Standard Hochschild complex}"] \arrow[d] & & \mathsf{Mixed} \arrow[d] \\
\mathrm{Alg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"'] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[r, dash, "\simeq"'] & \mathcal{M}\mathrm{ixed}
\end{tikzcd}$$

The standard Hochschild complex functor appearing in this diagram has as codomain the model category Mixed of *strict mixed complexes*, which are chain complexes of k-modules together with some extra structure that encodes the circle action. The ∞-category $\mathcal{M}\mathrm{ixed}$ of *mixed complexes* is (equivalent to) the underlying ∞-category of Mixed, and also equivalent to $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$, as we will see in Chapter 5.

In order to be able to make sense of this, this chapter will introduce and discuss Mixed and $\mathcal{M}\mathrm{ixed}$. We begin in Section 4.1 with reviewing chain complexes, primarily to fix notation. In Section 4.2 we will then discuss Mixed, including the closed symmetric monoidal structure that can be defined on it as well as the model structure. We then turn to the corresponding ∞-categories. We will collect the properties we need from $\mathcal{D}(k)$ in Section 4.3. Finally, we discuss the underlying ∞-categories of the model categories Mixed and $\mathrm{Alg}(\mathsf{Mixed})$ in Section 4.4.

4.1 Chain complexes

In this section we briefly review the 1-category of chain complexes of modules over the commutative ring k, to fix notation and sign conventions. We refer to books like [Wei94] for a thorough introduction to homological algebra. The book [Lod98], which we will use as our main reference for classical Hochschild homology, also reviews chain complexes in more detail than we do.

4.1.1 $\mathsf{Ch}(k)$ as a 1-category

To fix notation we briefly review the 1-category of chain complexes of k-modules.

Definition 4.1.1.1. We denote by $\mathsf{Ch}(k)$ the 1-category of chain complexes of k-modules. We use homological grading, so an object X of $\mathsf{Ch}(k)$ consists of k-modules X_n for every integer n together with boundary operators $\partial_n^X \colon X_n \to X_{n-1}$ (we will often omit the sub- and superscript when they are clear from context) satisfying $\partial \circ \partial = 0$.

If x is an element of X_n for some integer n, then we define $\deg_{\mathsf{Ch}}(x) := n$ and call n the *(chain) degree* of x.

If n is an integer, then we denote by $\mathsf{Ch}(k)_{\geq n} = \mathsf{Ch}(k)_{n\leq}$ the full subcategory of $\mathsf{Ch}(k)$ that is spanned by those objects X for which $X_m \cong 0$ if $m < n$. The full subcategories $\mathsf{Ch}(k)_{\leq n}$ and $\mathsf{Ch}(k)_{n_1\leq,\leq n_2}$ are defined analogously. ♢

Definition 4.1.1.2. Let X be an object of $\mathsf{Ch}(k)$ and n an integer. Then we denote by $X[n]$ the *n-fold shift* of X, which is also an object of $\mathsf{Ch}(k)$ that is defined as follows.

$$(X[n])_m := X_{m-n} \qquad \partial_m^{X[n]} := (-1)^n \cdot \partial_{m-n}^X$$

We can extend the construction $X \mapsto X[n]$ to an endofunctor of $\mathsf{Ch}(k)$ by setting $(f[n])_m := f_{m-n}$ for morphisms f.

Note that some authors denote what we call $X[n]$ by $X[-n]$, see for example [Wei94, Translation 1.2.8]. The convention we use is chosen to be consistent with [HA, 1.1.2.7]. ♢

Definition 4.1.1.3. If X is a k-module, then we will often consider X as a chain complex of k-modules concentrated in degree 0 without comment. This is the chain complex X' defined as follows.

$$X'_n := \begin{cases} X & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

If we want to make clear we are considering X as a chain complex rather than a module we will use $X[0]$. ♢

4.1.2 $\mathsf{Ch}(k)$ as a closed symmetric monoidal 1-category

In this short section we recall the closed symmetric monoidal structure on $\mathsf{Ch}(k)$, in particular to fix signs.

Definition 4.1.2.1. We equip $\mathsf{Ch}(k)$ with the usual symmetric monoidal structure, described as follows. For X and Y two objects of $\mathsf{Ch}(k)$ and f and g two morphisms in $\mathsf{Ch}(k)$ their tensor product is given by the following formulas[1].

$$\begin{aligned}
(X \otimes Y)_n &:= \bigoplus_{i+j=n} X_i \otimes Y_j \\
\partial_n^{X \otimes Y}(x \otimes y) &:= \partial^X(x) \otimes y + (-1)^{\deg_{\mathsf{Ch}}(x)} x \otimes \partial^Y(y) \\
(f \otimes g)(x \otimes y) &:= f(x) \otimes g(y)
\end{aligned}$$

The monoidal unit is $k[0]$, and the symmetry isomorphism is given by the isomorphism $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ that sends $x \otimes y$ to $(-1)^{\deg_{\mathsf{Ch}}(x) \deg_{\mathsf{Ch}}(y)} y \otimes x$.

$\mathsf{Ch}(k)$ can be upgraded to a closed symmetric monoidal category, with internal homomorphism objects given by the following formulas.

$$\begin{aligned}
\mathrm{HOM}_{\mathsf{Ch}(k)}(X, Y)_n &= \prod_{i \in \mathbb{Z}} \mathrm{HOM}_{\mathrm{LMod}_k(\mathsf{Ab})}(X_i, Y_{i+n}) \\
\left(\partial^{\mathrm{HOM}_{\mathsf{Ch}(k)}(X,Y)}(f)\right) &= \partial^Y \circ f - (-1)^{\deg_{\mathsf{Ch}}(f)} f \circ \partial^X
\end{aligned}$$ ♢

Remark 4.1.2.2. The tensor product is compatible with the shift functors defined in Definition 4.1.1.2; For every integer n there are isomorphisms natural in X and Y as follows

$$(X[n]) \otimes Y \xrightarrow{\cong} (X \otimes Y)[n] \xleftarrow{\cong} X \otimes (Y[n]) \tag{4.1}$$

where the first isomorphisms maps $x \otimes y$ to $x \otimes y$, but the second isomorphism introduces a sign by mapping $x \otimes y$ to $(-1)^{n \deg_{\mathsf{Ch}}(x)} x \otimes y$. That one of the two isomorphisms must introduce signs is related to the following compatibility: The first isomorphism in (4.1) is equal to the composition

$$(X[n]) \otimes Y \cong Y \otimes (X[n]) \cong (Y \otimes X)[n] \cong (X \otimes Y)[n]$$

where the first and third isomorphism is (induced by) the symmetry isomorphism τ and the middle isomorphism is the second one from (4.1).

The sign is easier to remember if one thinks of $Y[n]$ as $(-)[n]$ applied to Y. Then the shift construction is commuted past X, and hence introduces a sign if the degree of the element of x as well as n are both odd. ♢

4.1.3 $\mathsf{Ch}(k)$ as a model category

We recall the main properties of the projective model structure on $\mathsf{Ch}(k)$ for later use.

[1] When we write $X_i \otimes Y_j$ this refers to the tensor product in $\mathrm{LMod}_k(\mathsf{Ab})$, i. e. the relative tensor product over k.

Fact 4.1.3.1. $\mathsf{Ch}(k)$ *can be given the* projective *model structure where the weak equivalences are the quasiisomorphisms and the fibrations are the levelwise surjective morphism, see [HA, 7.1.2.8] and [Hov99, 2.3.11]. This model structure is left proper and combinatorial [HA, 7.1.2.8]. Furthermore, with respect to the closed symmetric monoidal structure discussed in Section 4.1.2, this model structure is a symmetric monoidal model structure [HA, 7.1.2.11] with cofibrant unit*[2] *and satisfies the monoid axiom [HA, 7.1.4.3].* ♧

When we refer to the model structure on $\mathsf{Ch}(k)$, we will always mean the projective model structure from Fact 4.1.3.1 – while there are other model structures on $\mathsf{Ch}(k)$, the projective one is the only one we will use in this text.

4.1.4 Homotopies in $\mathsf{Ch}(k)$

In this section we record that the notion of homotopy between morphisms from a cofibrant to a fibrant chain complex coincides with the usual notion of chain homotopy.

Proposition 4.1.4.1 ([Hov99, Between 2.3.11 and 2.3.12])**.** *Let Y be a chain complex. Then the operator of degree -1 on the graded k-module $P \coloneqq Y \times Y \times Y[-1]$ defined as*

$$\partial((x, y, z)) \coloneqq (\partial x, \partial y, -\partial(z) + x - y)$$

upgrades P into a chain complex. Furthermore the assignments $x \mapsto (x, x, 0)$ and $(x, y, z) \mapsto (x, y)$ define morphisms of chain complexes

$$Y \xrightarrow{i} P \xrightarrow{p} Y \times Y$$

which exhibit P as a path object for Y. ♡

Proof. The calculation

$$\begin{aligned} \partial(\partial((x, y, z))) &= \partial((\partial x, \partial y, -\partial(z) + x - y)) \\ &= (\partial(\partial x), \partial(\partial y), -\partial(-\partial(z) + x - y) + \partial x - \partial y) \\ &= (0, 0, 0) \end{aligned}$$

shows that P is a chain complex, and similarly simple calculations show that i and p are morphisms of chain complexes.

It is clear that p is levelwise surjective, so p is a fibration. It thus remains to show that i is a quasiisomorphism. For this consider $r\colon P \to Y$ defined by $(x, y, z) \mapsto x$. This is also a chain map, and $r \circ i = \mathrm{id}_Y$. It thus suffices

[2] The definition of a (symmetric) monoidal model category in [HA, 4.1.7] differs slightly from the definition in [Hov99, 4.2.6]: Lurie requires that the unit object is cofibrant, while Hovey replaces this condition with a weaker condition. See Section 4.2.2.2 for a more detailed discussion.

to show that $i \circ r$ is chain homotopic to the identity. For this consider the chain homotopy h from P to P that is defined by $(x, y, z) \mapsto (0, z, 0)$. Then we obtain

$$\begin{aligned}
&\partial(h((x,y,z))) + h(\partial((x,y,z))) \\
&= \partial((0,z,0)) + h((\partial x, \partial y, -\partial(z) + x - y)) \\
&= (0, \partial z, -z) + (0, -\partial(z) + x - y, 0) \\
&= (0, x - y, -z) \\
&= (x, x, 0) - (x, y, z) \\
&= (i \circ r - \mathrm{id}_P)((x,y,z))
\end{aligned}$$

and thus h is a chain homotopy from $i \circ r$ to id_P. □

Proposition 4.1.4.2 ([Hov99, Between 2.3.11 and 2.3.12])**.** *Let X be a cofibrant chain complex, Y a fibrant chain complex, and f and g two morphisms $X \to Y$ in $\mathsf{Ch}(k)$. Then f and g are homotopic (in the sense of model categories) if and only if there exists a chain homotopy from f to g, i. e. there exists a morphism h of graded k-modules that increases degree by 1 from X to Y satisfying the following relation.*

$$\partial \circ h + h \circ \partial = f - g \qquad \heartsuit$$

Proof. By [Hov99, 1.2.6], as X is cofibrant and Y is fibrant, left and right homotopy define the same equivalence relations on morphisms from X to Y. Furthermore, to check for right homotopies, we can use any path object for Y. Thus f and g are homotopic if and only if there exists a morphism of chain complexes $H\colon X \to P$ such that $p \circ H = f \times g$, where P and p are as in Proposition 4.1.4.1. As a graded k-module, P is given by $Y \times Y \times Y[-1]$, so we can write H as $H = h_0 \times h_1 \times h$, where h_0, h_1, and h are morphisms of graded k-modules from X to Y, where h increases degree by 1. The condition $p \circ H$ amounts to $h_0 = f$ and $h_1 = g$. The remaining data of h is then only constrained by the requirement that H be a morphism of chain complexes. This amounts to the equation

$$\partial \circ H = H \circ \partial$$

needing to hold. The left hand side is given by

$$\partial \circ H = \partial \circ (f \times g \times h) = ((\partial \circ f) \times (\partial \circ g) \times (-\partial \circ h + f - g))$$

and the right hand side is given by

$$H \circ \partial = (f \times g \times h) \circ \partial = ((f \circ \partial) \times (g \circ \partial) \times (h \circ \partial))$$

so, as equality in the first two factors follows automatically from f and g being morphisms of chain complexes, this boils down to

$$-\partial \circ h + f - g = h \circ \partial$$

which is equivalent to the equation from the statement. □

4.1.5 Extension of scalars

While we will usually keep the commutative ring k fixed, it will sometimes be useful to consider functoriality in k. For this we record the following statement.

Fact 4.1.5.1 ([Hov99, Page 48 and before 4.2.17 on page 114]). *Let $\varphi\colon k \to k'$ be a morphism of commutative rings.*

Then extension and restriction of scalars along φ induces a Quillen adjunction as follows.

$$\begin{tikzcd} \mathsf{Ch}(k) \arrow[r, shift left, "k' \otimes_k -"] \arrow[r, phantom, "\perp"] & \mathsf{Ch}(k') \arrow[l, shift left, "\varphi^*"] \end{tikzcd}$$

Furthermore, $k' \otimes_k -$ preserves fibrations and can be upgraded to a symmetric monoidal functor, making the adjunction into a symmetric monoidal Quillen adjunction in the sense of [Hov99, 4.2.16]. The right adjoint φ^* then obtains the structure of a lax symmetric monoidal functor, but is in general not symmetric monoidal. ♧

4.2 Strict mixed complexes

In this section we discuss strict mixed complexes. Strict mixed complexes where introduced by Kassel in [Kas87], where they are called *mixed complexes*. We will use the additional adjective *strict* to distinguish between the model category of strict mixed complexes Mixed and its underlying ∞-category of mixed complexes $\mathcal{M}\mathrm{ixed}$. A strict mixed complex roughly consists of a chain complex X together with a homomorphism $\mathrm{d}_n\colon X_n \to X_{n+1}$ increasing degree by 1 for every integer n, and satisfying $\mathrm{d} \circ \mathrm{d} = 0$ and $\partial \mathrm{d} + \mathrm{d}\partial = 0$, see Remark 4.2.1.4. The main examples of strict mixed complexes arise in the setting of Hochschild homology: The standard Hochschild complex of an associative ring carries the natural structure of a mixed complex, as will be discussed in Section 6.3.1. This was already alluded to in the introduction of Chapter 4, and in that context the operator d is the extra structure that encodes the circle action.

In Section 4.2.1, we will start by discussing Mixed as a closed symmetric monoidal 1-category. We will then discuss model structures on Mixed as well as $\mathrm{Alg}(\mathsf{Mixed})$ in Section 4.2.2 and discuss their properties and how they relate to each other, for example along the various forgetful functors. Finally, in Section 4.2.3, we will discuss the notion of strongly homotopy linear morphisms of strict mixed complexes, which are a form of weak morphisms between strict mixed complexes that only commute with d up to coherent homotopy.

4.2.1 Mixed as a closed symmetric monoidal 1-category

In this section we define the 1-category of strict mixed complexes Mixed and discuss its closed symmetric monoidal structure as well as algebra objects in Mixed. As Mixed will be defined as the category of left modules over a cocommutative bialgebra D in $\mathsf{Ch}(k)$, we start in Section 4.2.1.1 by defining the D, which then allows us to define Mixed as a symmetric monoidal category in Section 4.2.1.2 by using the results from Section 3.4. We will unpack the symmetric monoidal structure in Section 4.2.1.4 and discuss algebras in Mixed in Section 4.2.1.5. The symmetric monoidal structure will then be upgraded to a closed symmetric monoidal structure in Section 4.2.1.6. Finally, when discussing examples in Chapter 10 it will be helpful to depict mixed complexes diagrammatically, so we introduce the conventions we will use for this in Section 4.2.1.3. Examples of such diagrams will also appear as Example 4.2.1.11 in Section 4.2.1.4.

4.2.1.1 The bialgebra D

Construction 4.2.1.1. Define D to be the chain complex of k-modules $k \cdot \{1\} \oplus k \cdot \{\mathrm{d}\}$ with 1 of degree 0 and d of degree 1. In other words, D is the chain complex with zero differentials and a copy of k generated by 1 in degree 0, and a copy of k generated by an element we call d in degree 1.

Then D can be given a unique structure of a commutative algebra in $\mathsf{Ch}(k)$ such that the element 1 in degree 0 is the unit[3].

Furthermore, there is a unique way to extend this structure to a commutative and cocommutative bialgebra in $\mathsf{Ch}(k)$. Indeed, if $\epsilon\colon \mathrm{D} \to k$ is the counit of such a bialgebra structure, then $\epsilon(1) = 1$ is determined by the requirement that ϵ is a morphism of algebras, and $\epsilon(d) = 0$ is clear for degree reasons. If $\Delta\colon \mathrm{D} \to \mathrm{D} \otimes \mathrm{D}$ is the comultiplication of such a bialgebra structure, then again as Δ is an algebra morphism we must have $\Delta(1) = 1 \otimes 1$. We can write $\Delta(\mathrm{d})$ as $a \cdot (1 \otimes \mathrm{d}) + b \cdot (\mathrm{d} \otimes 1)$ for some elements a and b of k. But from counitality we can conclude that a and b must both be 1. Hence we must have $\Delta(\mathrm{d}) = \mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}$. That ϵ and Δ defined like this really define a commutative and cocommutative bialgebra can easily be checked.

While we will usually just write D, we will also denote this commutative and cocommutative bialgebra by D_k if we want to make the base ring explicit. It follows immediately from the construction that if $\varphi\colon k \to k'$ is a morphism of commutative rings, then the symmetric monoidal functor[4]

$$k' \otimes_k -\colon \mathsf{Ch}(k) \to \mathsf{Ch}(k')$$

maps D_k to $\mathrm{D}_{k'}$, as a commutative and cocommutative bialgebra. ♢

[3] 1 being the unit already pins down products $x \cdot y$ if one of x and y is in degree 0, and if x and y are both in degree 1 then the product is 0 for degree reasons.

[4] See Fact 4.1.5.1.

4.2.1.2 Definition of Mixed

We can now define the symmetric monoidal category of strict mixed complexes.

Definition 4.2.1.2. We denote by Mixed the symmetric monoidal category

$$\mathsf{Mixed} \coloneqq \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$$

where the symmetric monoidal structure we consider here is the one from Definition 3.4.2.1, see also Remark 3.4.2.4. We will call Mixed the category of *strict mixed complexes*.

We will sometimes have reason to use strict mixed complexes whose underlying chain complex is cofibrant with respect to the projective model structure (see Fact 4.1.3.1). We will thus use the notation

$$\mathsf{Mixed}_{\mathrm{cof}} \coloneqq \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$$

for the full symmetric monoidal subcategory of Mixed spanned by those strict mixed complexes whose underlying chain complex is cofibrant.

If we want to make the base ring explicit we will also use the notation Mixed_k and $\mathsf{Mixed}_{k,\mathrm{cof}}$. ◊

Remark 4.2.1.3. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. The symmetric monoidal functor

$$k' \otimes_k -\colon \mathsf{Ch}(k) \to \mathsf{Ch}(k') \tag{4.2}$$

from Fact 4.1.5.1 induces by Definition 3.4.2.1 and Remark 3.4.2.4 a symmetric monoidal functor as indicated at the top of the following commutative diagram.

$$\begin{tikzcd} \mathsf{Mixed}_k \arrow[r, "k'\otimes_k -"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"'] & \mathsf{Mixed}_{k'} \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\ \mathsf{Ch}(k) \arrow[r, "k'\otimes_k -"'] & \mathsf{Ch}(k') \end{tikzcd} \tag{4.3}$$

As (4.2) preserves cofibrant objects by Fact 4.1.5.1, the top horizontal functor restricts to a symmetric monoidal functor from $\mathsf{Mixed}_{k,\mathrm{cof}}$ to $\mathsf{Mixed}_{k',\mathrm{cof}}$.

Furthermore, as the forgetful functors $\mathrm{ev}_{\mathfrak{m}}$ detect colimits by [HA, 4.2.3.5 (2)] and the bottom horizontal functor $k' \otimes_k -$ in (4.3) preserves colimits by Fact 4.1.5.1, the top horizontal functor in (4.3) preserves colimits as well. ◊

Remark 4.2.1.4. Let us unpack what an object of Mixed is. A D-module consists of an underlying chain complex X together with a morphism

$$\mu\colon \mathrm{D} \otimes X \to X$$

of chain complexes, the action of D on X, satisfying associativity and unitality.

Unpacking the definition of the tensor product in $\mathsf{Ch}(k)$ and the definition of D we see that the data of μ corresponds to the data of morphisms of abelian groups

$$\mu(1 \otimes -)_n \colon X_n \to X_n \qquad \text{and} \qquad \mu(\mathrm{d} \otimes -)_n \colon X_n \to X_{n+1}$$

for every integer n. Those morphisms have to satisfy a condition corresponding to μ being a morphism of chain complexes.

Let us first note that unitality of the action is equivalent to $\mu(1 \otimes -)_n$ being the identity for every n, so this piece of data is redundant. If x is an element of X_n for some n, let us write $\mathrm{d}(x)$ for $\mu(\mathrm{d} \otimes x)$. Then μ being a morphism of chain complexes is equivalent to $\partial \mathrm{d} + \mathrm{d}\partial = 0$. Finally, associativity of the action is equivalent to $\mathrm{d} \circ \mathrm{d} = 0$.

A morphism of D-modules $f \colon X \to Y$ can similarly be unpacked to be a morphism of underlying chain complexes (which we also denote by f) such that $f \circ \mathrm{d}^X = \mathrm{d}^Y \circ f$.

The upshot of the above discussion is that the category of strict mixed complexes is isomorphic to the category of chain complexes with an extra operator d that increases degree by 1, and that satisfies the two equations $\partial \mathrm{d} + \mathrm{d}\partial = 0$ and $\mathrm{d}^2 = 0$. In the rest of the text we will often switch back and forth between these two perspectives. ♢

As an example, we define a very basic family of strict mixed complexes.

Definition 4.2.1.5. Let n be an integer. Then we denote by D_n the strict mixed complex with underlying chain complex $\mathbb{Z} \cdot \{1\} \oplus \mathbb{Z} \cdot \{\delta\}[1]$ (so the same underlying chain complex as D itself), and with d defined by $\mathrm{d}(1) = n \cdot \delta$. ♢

Remark 4.2.1.6. As a D-module, D is isomorphic to D_1. Also note that D_n is isomorphic to D_{-n}. ♢

4.2.1.3 Diagrams depicting strict mixed complexes

Convention 4.2.1.7. It will sometimes be helpful to diagrammatically depict strict mixed complexes for which the underlying graded abelian group is free on some basis $(b_i)_{i \in I}$ for a set I. In that case we will use the following conventions.

- Basis elements are represented by vertices of the diagram.
- A non-squiggly black arrow from b_i to b_j is used to represent the b_j-coefficient of $\partial(b_i)$. More concretely, if we write $\partial(b_i)$ as a linear combination $\sum_{l \in I} a_l \cdot b_l$ of basis elements, with a_l elements of k, then the label of such a non-squiggly black arrow will be a_j. If $a_j = 0$, then we will omit the arrow.

- d is represented completely analogously with red squiggly arrows.
- If an arrow has no label without further comment, then the the missing label is to be interpreted as 1.
- Sometimes we will drop the signs of the labels, or the labels altogether. In these cases we will point this out in the text. ♢

Example 4.2.1.8. The strict mixed complex $\mathrm{D}_n \oplus \mathrm{D}_m[1]$ for n and m integers can be depicted as follows, where we use $1'$ and δ' for the basis elements of D_m.

$$\begin{tikzcd}
 & \delta' \\
\delta & 1' \arrow[u, squiggly, red, "-m"'] \\
1 \arrow[u, squiggly, red, "n"'] &
\end{tikzcd}$$

The sign arises from the isomorphism $\mathrm{D} \otimes (\mathrm{D}_m[1]) \cong (\mathrm{D} \otimes \mathrm{D}_m)[1]$, see Remark 4.1.2.2. ♢

Example 4.2.1.9. Let n be an integer. The following is an example of an acyclic strict mixed complex.

$$\begin{tikzcd}
\delta' \arrow[dr] & \\
1' \arrow[u, squiggly, red, "-n"] \arrow[dr] & \delta \\
 & 1 \arrow[u, squiggly, red, "n"']
\end{tikzcd}$$

♢

4.2.1.4 The symmetric monoidal structure on Mixed

Remark 4.2.1.10. Let us unpack the symmetric monoidal structure on Mixed. By Definition 3.4.2.1 the forgetful functor $\mathsf{Mixed} \to \mathsf{Ch}(k)$ is symmetric monoidal, so if X and Y are two strict mixed complexes, then the underlying chain complex of $X \otimes Y$ must be the tensor product of underlying chain complexes, and it remains to figure out how d acts. Using Remark 3.4.2.4, this action arises from the composition

$$\mathrm{D} \otimes X \otimes Y \xrightarrow{\Delta \otimes \mathrm{id}_X \otimes \mathrm{id}_Y} \mathrm{D} \otimes \mathrm{D} \otimes X \otimes Y \xrightarrow{\mathrm{id}_{\mathrm{D}} \otimes \tau_{\mathrm{D},X} \otimes \mathrm{id}_Y} \mathrm{D} \otimes X \otimes \mathrm{D} \otimes Y$$
$$\xrightarrow{\mu^X \otimes \mu^Y} X \otimes Y$$

where Δ is the comultiplication of D as defined in Construction 4.2.1.1, τ is the symmetry isomorphism reviewed in Definition 4.1.2.1, and μ^X and μ^Y are the action morphisms on X and Y, respectively.

By unpacking the definitions we obtain the following.

$$
\begin{aligned}
&\mathrm{d}^{X\otimes Y}(x\otimes y)\\
&= \left(\mu^X\otimes\mu^Y\right)\circ\left(\mathrm{id}_{\mathrm{D}}\otimes\tau_{\mathrm{D},X}\otimes\mathrm{id}_Y\right)\circ\left(\Delta\otimes\mathrm{id}_X\otimes\mathrm{id}_Y\right)(\mathrm{d}\otimes x\otimes y)\\
&= \left(\mu^X\otimes\mu^Y\right)\circ\left(\mathrm{id}_{\mathrm{D}}\otimes\tau_{\mathrm{D},X}\otimes\mathrm{id}_Y\right)\circ(\mathrm{d}\otimes 1\otimes x\otimes y+1\otimes\mathrm{d}\otimes x\otimes y)\\
&= \left(\mu^X\otimes\mu^Y\right)\circ\Big(\mathrm{d}\otimes x\otimes 1\otimes y+(-1)^{\deg_{\mathsf{Ch}}(x)}1\otimes x\otimes\mathrm{d}\otimes y\Big)\\
&= \mathrm{d}^X(x)\otimes y+(-1)^{\deg_{\mathsf{Ch}}(x)}x\otimes\mathrm{d}^Y(y)
\end{aligned}
$$

The monoidal unit of Mixed is the unique strict mixed complex with underlying chain complex $k[0]$. ♢

Example 4.2.1.11. As an example we discuss the tensor product $\mathrm{D}_n\otimes\mathrm{D}_m$ for n and m positive integers.

The strict mixed complex $\mathrm{D}_n\otimes\mathrm{D}_m$ can be depicted as follows.

$$
\begin{tikzcd}
& \delta\otimes\delta & \\
\delta\otimes 1 \arrow[ur, squiggly, "-m"] & & 1\otimes\delta \arrow[ul, squiggly, "n"'] \\
& 1\otimes 1 \arrow[ul, squiggly, "n"] \arrow[ur, squiggly, "m"'] &
\end{tikzcd}
$$

Let i, j be integers such that $\gcd(n,m)=in+jm$. Then another basis for the free abelian group generated by $\delta\otimes 1$ and $1\otimes\delta$ is given by the two elements

$$
\frac{n}{\gcd(n,m)}\cdot\delta\otimes 1+\frac{m}{\gcd(n,m)}\cdot 1\otimes\delta \qquad\text{and}\qquad j\cdot\delta\otimes 1-i\cdot 1\otimes\delta.
$$

Thus we can also depict $\mathrm{D}_n\otimes\mathrm{D}_m$ as follows.

$$
\begin{tikzcd}
& \delta\otimes\delta & \\
\frac{n}{\gcd(n,m)}\cdot\delta\otimes 1+\frac{m}{\gcd(n,m)}\cdot 1\otimes\delta & & j\cdot\delta\otimes 1-i\cdot 1\otimes\delta \arrow[ul, squiggly, "-\gcd(n,m)"'] \\
& 1\otimes 1 \arrow[ul, squiggly, "\gcd(n,m)"] &
\end{tikzcd}
$$

Thus $\mathrm{D}_n\otimes\mathrm{D}_m$ is isomorphic in Mixed to $\mathrm{D}_{\gcd(n,m)}\oplus\mathrm{D}_{\gcd(n,m)}[1]$. ♢

4.2.1.5 Algebras in Mixed

As we will later also consider algebras in Mixed, we unpack the definition in the following remark.

Remark 4.2.1.12. As the forgetful functor from Mixed to $\mathsf{Ch}(k)$ is symmetric monoidal, every algebra in strict mixed complexes has an underlying differential graded algebra (i. e. an algebra in $\mathsf{Ch}(k)$). An algebra in Mixed then consists of a differential graded algebra together with a strict mixed complex structure on the underlying chain complex A, such that the unit morphism $k \to A$ and the multiplication morphism $A \otimes A \to A$ are morphisms of strict mixed complexes.

Making use of Remark 4.2.1.10 we can rephrase this as the requirement that $\mathrm{d}(1) = 0$ and that the Leibniz rule

$$\mathrm{d}(x \cdot y) = \mathrm{d}(x) \cdot y + (-1)^{\deg_{\mathsf{Ch}}(x)} x \cdot \mathrm{d}(y)$$

is satisfied for every element x and y of A.

Note that the Leibniz rule for $x = y = 1$ implies $\mathrm{d}(1) = 2\,\mathrm{d}(1)$ and hence $\mathrm{d}(1) = 0$, so if the Leibniz rule holds, then this condition is redundant.

Commutative algebras in Mixed have the analogous description, they consist of a commutative differential graded algebra together with a strict mixed complex structure on the underlying chain complex satisfying the Leibniz rule. ♢

4.2.1.6 The closed symmetric monoidal structure on Mixed

Construction 4.2.1.13. Let X and Y be two strict mixed complexes. We can define an operator d increasing degree by one on $\mathrm{HOM}_{\mathsf{Ch}(k)}(X, Y)$ by letting d act on f by the following formula.

$$\mathrm{d}(f) = \mathrm{d}^Y \circ f - (-1)^{\deg_{\mathsf{Ch}}(f)} f \circ \mathrm{d}^X$$

By unwrapping the definitions it is straightforward to check that this definition satisfies $\mathrm{d} \circ \mathrm{d} = 0$ and $\mathrm{d} \circ \partial + \partial \circ \mathrm{d} = 0$ and thus defines a strict mixed complex, which we will denote by $\mathrm{HOM}_{\mathsf{Mixed}}(X, Y)$. ♢

Proposition 4.2.1.14. *Let*

$$\begin{aligned} \varphi\colon\ \mathrm{Mor}_{\mathsf{Ch}(k)}(-_1 \otimes -_2, -_3) &\xrightarrow{\cong} \mathrm{Mor}_{\mathsf{Ch}(k)}\big(-_1, \mathrm{HOM}_{\mathsf{Ch}(k)}(-_2, -_3)\big) \\ f &\mapsto (x \mapsto (y \mapsto f(x \otimes y))) \end{aligned}$$

be the natural isomorphism that is part of the closed symmetric monoidal structure on $\mathsf{Ch}(k)$*. Then* φ *restricts to a natural isomorphism as follows.*

$$\mathrm{Mor}_{\mathsf{Mixed}}(-_1 \otimes -_2, -_3) \xrightarrow{\cong} \mathrm{Mor}_{\mathsf{Mixed}}(-_1, \mathrm{HOM}_{\mathsf{Mixed}}(-_2, -_3))$$

In particular, this makes Mixed *into a* closed *symmetric monoidal category.* ♡

Proof. Let X, Y, and Z be strict mixed complexes and $f : X \otimes Y \to Z$ a morphism of chain complexes. The statement then follows from the following chain of equivalences.

$$\begin{aligned}
&f \text{ is a morphism of strict mixed complexes}\\
\iff &\forall x \in X \colon \forall y \in Y \colon\\
&\mathrm{d}^Z(f(x \otimes y)) = f\Big(\mathrm{d}^X(x) \otimes y\Big) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x \otimes \mathrm{d}^Y(y))\\
\iff &\forall x \in X \colon \forall y \in Y \colon\\
&\mathrm{d}^Z(\varphi(f)(x)(y)) = \varphi(f)\Big(\mathrm{d}^X(x)\Big)(y) + (-1)^{\deg_{\mathsf{Ch}}(x)} \varphi(f)(x)\Big(\mathrm{d}^Y(y)\Big)\\
\iff &\forall x \in X \colon \forall y \in Y \colon\\
&\mathrm{d}^Z(\varphi(f)(x)(y)) - (-1)^{\deg_{\mathsf{Ch}}(x)} \varphi(f)(x)\Big(\mathrm{d}^Y(y)\Big) = \varphi(f)\Big(\mathrm{d}^X(x)\Big)(y)\\
\iff &\forall x \in X \colon\ \mathrm{d}(\varphi(f)(x)) = \varphi(f)\Big(\mathrm{d}^X(x)\Big)\\
\iff &\varphi(f) \text{ is a morphism of strict mixed complexes}
\end{aligned}$$

□

4.2.2 Mixed and $\mathrm{Alg}(\mathsf{Mixed})$ as model categories

In this section we construct model structures on Mixed and $\mathrm{Alg}(\mathsf{Mixed})$ and discuss various properties that they have. We will start in Section 4.2.2.1 by reviewing a general result by Schwede and Shipley concerning when one can lift a model structure from a closed symmetric monoidal category with compatible model structure to a model structure on categories of algebras or modules over an algebra. We then apply this in Section 4.2.2.2 to $\mathsf{Ch}(k)$ in order to obtain a model structure on $\mathsf{Mixed} = \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$. We will also show that this model structure is again suitably compatible with the closed symmetric monoidal structure on Mixed, so that we can further lift the model structure from Mixed to $\mathrm{Alg}(\mathsf{Mixed})$, which we do in Section 4.2.2.3. As discussed in Section 4.2.1.5, an algebra in Mixed consists of a chain complex that has both an algebra structure as well as a strict mixed complex structure, satisfying that the Leibniz rule. We thus obtain two forgetful functors on $\mathrm{Alg}(\mathsf{Mixed})$: One forgetting the strict mixed complex structure and mapping to $\mathrm{Alg}(\mathsf{Ch}(k))$, and one forgetting the algebra structure and mapping to Mixed. Together with the forgetful functors from $\mathrm{Alg}(\mathsf{Ch}(k))$ and Mixed to $\mathsf{Ch}(k)$ they fit into a commutative diagram, and the main result of Section 4.2.2.3 is Proposition 4.2.2.12, in which various properties of those forgetful functors are shown. Finally, it will in practice be helpful to have a concrete description of homotopies in the model categories Mixed as well as $\mathrm{Alg}(\mathsf{Ch}(k))$ and $\mathrm{Alg}(\mathsf{Mixed})$, so we discuss them in Sections 4.2.2.4, 4.2.2.5 and 4.2.2.6.

4.2.2.1 Model categories of algebras and modules

In order to construct model structures on $\mathsf{Mixed} = \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$ and $\mathrm{Alg}(\mathsf{Mixed})$ we will make use of a general theorem by Schwede and Shipley that allows one to lift model structures to categories of modules and algebras. We recall their result as Theorem 4.2.2.1 below.

Theorem 4.2.2.1 ([SS00, Theorem 4.1])**.** *Let C be a combinatorial model category with a closed symmetric monoidal structure such that the tensor product functor is a Quillen bifunctor (i. e. the pushout product axiom is satisfied) and satisfying the monoid axiom (see [SS00, 3.3]).*

Then there is a combinatorial model structure on $\mathrm{Alg}(\mathsf{C})$ such that the following statements hold.

(1) The adjunction

$$\mathrm{Free}^{\mathrm{Alg}} \colon \mathsf{C} \rightleftarrows \mathrm{Alg}(\mathsf{C}) : \mathrm{ev}_{\mathfrak{a}}$$

where $\mathrm{Free}^{\mathrm{Alg}}$ is the free algebra functor and $\mathrm{ev}_{\mathfrak{a}}$ is the forgetful functor, is a Quillen adjunction.

(2) $\mathrm{Alg}(\mathsf{C})$ is cofibrantly generated with the set of generating (acyclic) cofibrations given by application of $\mathrm{Free}^{\mathrm{Alg}}$ to the set of generating (acyclic) cofibrations of C.

(3) $\mathrm{ev}_{\mathfrak{a}}$ preserves and reflects weak equivalences and fibrations.

(4) If the unit of C is cofibrant, then $\mathrm{ev}_{\mathfrak{a}}$ preserves cofibrant objects and cofibrations between cofibrant objects.

Let A be an algebra in C. Then there is a combinatorial model structure on $\mathrm{LMod}_A(\mathsf{C})$ such that the following statements hold.

(5) The adjunction

$$\mathrm{Free}^{\mathrm{LMod}_A} \colon \mathsf{C} \rightleftarrows \mathrm{LMod}_A(\mathsf{C}) : \mathrm{ev}_{\mathfrak{m}}$$

where $\mathrm{Free}^{\mathrm{LMod}_A}$ is the functor sending an object X to the free A-module $A \otimes X$ and $\mathrm{ev}_{\mathfrak{m}}$ is the forgetful functor, is a Quillen adjunction.

(6) $\mathrm{LMod}_A(\mathsf{C})$ is cofibrantly generated with set of generating (acyclic) cofibrations given by application of $\mathrm{Free}^{\mathrm{LMod}_A}$ to the set of generating (acyclic) cofibrations of C.

(7) $\mathrm{ev}_{\mathfrak{m}}$ preserves and reflects weak equivalences and fibrations.

(8) If the underlying object of A is cofibrant in C, then $\mathrm{ev}_{\mathfrak{m}}$ preserves cofibrations. ♡

Proof.

Construction of the model structures: By definition (see [HTT, A.2.6.1]), a combinatorial model category has presentable underlying category, so in particular every object is small (see [HTT, A.1.1.2]). Furthermore, combinatorial model categories are by definition also cofibrantly generated, so all the conditions to applying [SS00, 4.1] are satisfied. We thus obtain the existence of cofibrantly generated model structures on $\mathrm{Alg}(\mathsf{C})$ and $\mathrm{LMod}_A(\mathsf{C})$. Let us now turn to the various properties of these model structures that we claimed.

Proof of claims (1), (2), (3), (5), (6), and (7): See the proof of [SS00, 4.1] as well as [SS00, 2.3 and the description right before 2.3].

Proof of (4): Part of the statement of [SS00, 4.1 (3)].

Proof that the model structures are combinatorial: It remains to show that $\mathrm{Alg}(\mathsf{C})$ and $\mathrm{LMod}_A(\mathsf{C})$ are presentable. We refer to [HTT, A.1.1.2] for a definition of presentable categories. That the two categories are cocomplete is already part of them being model categories, and as the forgetful functors to C are faithful it is also clear that the morphisms sets are small. It thus suffices to show that the two categories are accessible[5]; condition [HTT, A.1.1.2 (2)] then follows directly from definition and [HTT, A.1.1.2 (3)] follows from [AR94, 2.2 (3) and 1.16]. See also [HTT, 5.5.1.1 and 5.5.0.1].

But both $\mathrm{Alg}(\mathsf{C})$ and $\mathrm{LMod}_A(\mathsf{C})$ are categories of algebras over an accessible monad on C[6], so they are again accessible by [AR94, 2.78].

Proof of claim (8): ev_{m} preserves colimits[7], so to show that ev_{m} preserves cofibrations it suffices to show that ev_{m} preserves generating cofibrations. So let $i\colon X \to Y$ be a cofibration in C. We claim that $\mathrm{ev}_{\mathsf{m}}(\mathrm{Free}^{\mathrm{LMod}_A}(i)) = \mathrm{id}_A \otimes i$ is again a cofibration. But this follows from $-\otimes-$ being a Quillen bifunctor[8]. □

4.2.2.2 The model structure on Mixed

The general result Theorem 4.2.2.1 allows us to define a combinatorial model structure on Mixed that is lifted from the projective model structure on $\mathsf{Ch}(k)$ – all prerequisites to apply Theorem 4.2.2.1 are covered by Fact 4.1.3.1.

Definition 4.2.2.2. We equip $\mathsf{Mixed} = \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$ with the combinatorial model structure from Theorem 4.2.2.1 that is lifted from the projective model structure on $\mathsf{Ch}(k)$. ♢

[5] See [AR94, 2.2 (1)] for a definition. An object is called κ-presentable (presentable) in [AR94, 1.13] precisely if it is called κ-compact (small) in [HTT, A.1.1.1]. Thus (keeping in mind we already know that the categories in question are cocomplete), [AR94, 2.2 (1)] asks for existence of a regular cardinal κ and a small set of κ-compact objects such that every object can be obtained as a κ-filtered colimit of objects from that set.

[6] The proof of [SS00, 4.1] uses this fact, so see there for more details.

[7] Because we assume that the symmetric monoidal structure on C is closed, the tensor product preserves colimits separately in each variable, so we can apply [HA, 4.2.3.5].

[8] See [Hov99, 4.2.1] for a definition. We apply the property to the cofibrations $0 \to A$ and i, and use that the morphism $(0 \to A) \,\square\, i$ can be identified with $\mathrm{id}_A \otimes i$.

Proposition 4.2.2.3. *Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the extension of scalars functor*

$$k' \otimes_k -\colon \mathsf{Mixed}_k \to \mathsf{Mixed}_{k'}$$

from Remark 4.2.1.3 preserves cofibrations as well as weak equivalences between objects with cofibrant underlying chain complex. ♡

Proof. We first show that the functor preserves cofibrations. As it preserves colimits by Remark 4.2.1.3, it suffices to show that the functor preserves generating cofibrations. But this follows immediately from compatibility with the free module functors by Proposition E.7.4.1 in combination with

$$k' \otimes_k -\colon \mathsf{Ch}(k) \to \mathsf{Ch}(k') \tag{$*$}$$

preserving cofibrations by Fact 4.1.5.1.

That the functor preserves weak equivalences between objects with cofibrant underlying chain complex follows directly from the forgetful functors $\mathrm{ev}_{\mathfrak{m}}$ detecting weak equivalences, the diagram (4.3) in Remark 4.2.1.3 commuting, and $(*)$ preserving weak equivalences between cofibrant objects by Fact 4.1.5.1. □

Proposition 4.2.2.4. *The underlying chain complex of* D *is cofibrant.* ♡

Proof. Follows from [Hov99, 2.3.6]. □

So we have now obtained a model structure on Mixed. We have also already previously discussed a closed symmetric monoidal structure on Mixed, see Proposition 4.2.1.14. We would like to show that these to structures are in fact compatible and make Mixed into a symmetric monoidal model structure. However, there are slightly different definitions of what properties a monoidal model structure needs to satisfy, and not all are true in this case. What all definitions require is that the tensor product is a Quillen bifunctor. As explained in [SS00, 3.2] and [Hov99, below 4.2.6], this does not quite suffice to obtain an induced monoidal structure on the homotopy category, a condition on the unit object is also necessary. This is because the derived tensor product is formed by tensoring cofibrant replacements of the two objects one wants to tensor. If the unit object is not cofibrant, there is no guarantee that the derived tensor product with the unit object is weakly equivalent to the original object. One condition to guarantee that this is nevertheless the case is given in [Hov99, 4.2.6] as part of Hovey's definition of monoidal model structures. This condition is always satisfied when the unit is in fact cofibrant, and Lurie requires this more restrictive condition for monoidal model categories [HA, Start of 4.1.7].

The unit object in Mixed is $\mathbb{Z}$ (see Remark 4.2.1.10), which is unfortunately not cofibrant (see Proposition 4.2.2.5 directly below), so we can not directly apply some of the result concerning monoidal model categories proven in [HA], like the result on rectification of algebras [HA, 4.1.8.4]. However, Hovey's condition is satisfied, and we will be able to work around the obstacles to deducing the analogous result to [HA, 4.1.8] in Proposition 4.4.2.3 in Section 4.4.2.

Proposition 4.2.2.5. *The unit object* $\mathbb{Z}$ *of* Mixed *(see Remark 4.2.1.10) is not cofibrant with respect to the model structure from Definition 4.2.2.2.* ♡

Proof. Consider the counit $\epsilon\colon \mathrm{D} \to \mathbb{Z}$. This is a morphism of mixed complexes, and also a fibration in Mixed as it is levelwise surjective and $\mathrm{ev}_{\mathfrak{m}}$ detects fibrations by Theorem 4.2.2.1 (7). If $\mathbb{Z}$ were cofibrant in Mixed, then there would have to exist a section of ϵ as strict mixed complexes. However, the unique section in $\mathsf{Ch}(k)$ is not a morphism of strict mixed complexes, as $\mathrm{d}(1) = \mathrm{d} \neq 0$ in D. □

Proposition 4.2.2.6. *The model structure on* Mixed *from Definition 4.2.2.2 is a symmetric monoidal model structure (in the sense of [Hov99, 4.2.6]) with respect to the closed symmetric monoidal structure from Definition 4.2.1.2 and Proposition 4.2.1.14.* ♡

Proof. Proof that $-\otimes-$ *is a Quillen bifunctor*: Let $f\colon W \to X$ be a cofibration and $p\colon Y \to Z$ a fibration in Mixed. By [Hov99, 4.2.2] if suffices to show that the induced morphism

$$\mathrm{HOM}_{\mathsf{Mixed}}(X,Y) \to \mathrm{HOM}_{\mathsf{Mixed}}(X,Z) \times_{\mathrm{HOM}_{\mathsf{Mixed}}(W,Z)} \mathrm{HOM}_{\mathsf{Mixed}}(W,Y)$$

is a fibration in Mixed, and acyclic if f or p is acyclic. But this follows immediately from $\mathsf{Ch}(k)$ having the corresponding property by Fact 4.1.3.1 and [Hov99, 4.2.2], in combination with $\mathrm{ev}_{\mathfrak{m}}$ preserving and detecting fibrations and weak equivalences by Theorem 4.2.2.1 (7), preserving cofibrations by Theorem 4.2.2.1 (8) and Proposition 4.2.2.4, and mapping $\mathrm{HOM}_{\mathsf{Mixed}}$ to $\mathrm{HOM}_{\mathsf{Ch}(k)}$ by Construction 4.2.1.13.

Proof of [Hov99, 4.2.6 (2)]: We have to show that if $0 \to \mathbb{Z}^{\mathrm{cof}} \xrightarrow{f} \mathbb{Z}$ is a factorization in Mixed of $0 \to \mathbb{Z}$ into a cofibration followed by an acyclic fibration, then tensoring f with the identity of any cofibrant object on either side yields a weak equivalence. By Proposition 4.2.2.4 and Theorem 4.2.2.1 (7) and (8), the forgetful functor $\mathrm{ev}_{\mathfrak{m}}\colon \mathsf{Mixed} \to \mathsf{Ch}(k)$ preserves weak equivalences as well as cofibrations, and also detects weak equivalences. Furthermore, $\mathrm{ev}_{\mathfrak{m}}$ is also symmetric monoidal.

Hence it suffices to show that for a cofibrant chain complex X it holds that

$$\mathrm{ev}_{\mathfrak{m}}(\mathbb{Z}^{\mathrm{cof}}) \otimes X \xrightarrow{\mathrm{ev}_{\mathfrak{m}}(f)\otimes \mathrm{id}_X} \mathrm{ev}_{\mathfrak{m}}(\mathbb{Z}) \otimes X$$

is a weak equivalence in $\mathsf{Ch}(k)$. But note that while $\mathbb{Z}$ is not cofibrant an an object in Mixed, it *is* cofibrant as a chain complex. Hence

$$\mathrm{ev}_{\mathfrak{m}}(\mathbb{Z}^{\mathrm{cof}}) \xrightarrow{\mathrm{ev}_{\mathfrak{m}}(f)} \mathrm{ev}_{\mathfrak{m}}(\mathbb{Z}) = \mathbb{Z}$$

is a weak equivalence between cofibrant objects. As $\mathsf{Ch}(k)$ is a symmetric monoidal model category, $-\otimes X$ preserves acyclic cofibrations, and hence sends weak equivalences between cofibrant objects to weak equivalences (see [Hov99, 1.1.12]), so the claim follows. $\square$

We next show that Mixed satisfies the monoid axiom. Definitions of the monoid axiom can be found in [SS00, 3.3] and [HA, 4.1.8.1], however these two definitions are stated in a slightly different way, so we briefly discuss them first in the next remark.

Remark 4.2.2.7. Let C be a combinatorial model category that is equipped with a symmetric monoidal structure.

Let U be the subclass of morphisms of C that are of the form $\mathrm{id}_X \otimes i$, with X an object in C and i an acyclic cofibration. Let $\overline{U}$ be the weakly saturated class of morphisms generated by U[9]. Let $\widetilde{U}$ be the subclass of morphisms of C that can be obtained as a transfinite composition of pushouts of morphisms in U. Finally, let $\widetilde{U}'$ be the subclass of morphisms of C that are retracts of morphisms in $\widetilde{U}$.

Then [SS00, 3.3] asks that all morphisms in $\widetilde{U}$ are weak equivalences, and [HA, 4.1.8.1] asks that all morphisms in $\overline{U}$ are weak equivalences.

From the definitions it is clear that $\widetilde{U}'$ is contained in $\overline{U}$. On the other hand, [HTT, A.1.2.8] implies that $\overline{U}$ is contained in $\widetilde{U}'$. As weak equivalences are closed under retracts, $\widetilde{U}$ is contained in the class of weak equivalences if and only if $\widetilde{U}' = \overline{U}$ is, so definitions [SS00, 3.3] and [HA, 4.1.8.1] are equivalent. $\diamondsuit$

Proposition 4.2.2.8. *The symmetric monoidal model category*[10] Mixed *satisfies the monoid axiom.* $\heartsuit$

Proof. In this proof we use the following notation. If S is a class of morphisms in some monoidal category C, then we denote by $\mathsf{C}\otimes S$ the class of all morphisms of the form $\mathrm{id}_X \otimes s$ where X is an object of C and s is an element of S. We denote by $\overline{S}$ the weakly saturated class of morphisms generated by S in the sense of [HTT, A.1.2.2].

Denote by W the class of weak equivalences of $\mathsf{Ch}(k)$, and by I a set of generating acyclic cofibrations of $\mathsf{Ch}(k)$. We also define $\mathrm{Free}^{\mathsf{Mixed}}$ to be $\mathrm{Free}^{\mathrm{LMod}_{\mathrm{D}}}$, the left adjoint to the forgetful functor $\mathrm{ev}_{\mathsf{m}}\colon \mathsf{Mixed} \to \mathsf{Ch}(k)$.

What we have to show is that the class of morphisms

$$\overline{\mathsf{Mixed} \otimes \{\text{acyclic cofibrations in } \mathsf{Mixed}\}}$$

[9] See [HTT, A.1.2.2] for a definition. This is smallest subclass of morphisms of C containing U that is closed under taking pushouts along morphisms of C, transfinite compositions, and retracts.

[10] In the sense of [Hov99, 4.2.6].

is contained in the class of weak equivalences of Mixed, which by Theorem 4.2.2.1 (8) is equivalent to showing the following.

$$\mathrm{ev}_{\mathfrak{m}}\left(\overline{\mathsf{Mixed} \otimes \{\text{acyclic cofibrations in } \mathsf{Mixed}\}}\right) \subseteq W$$

This will follow from the following easy claims.

(1) $\overline{\mathsf{Mixed} \otimes \{\text{acyclic cofibrations in } \mathsf{Mixed}\}} = \overline{\mathsf{Mixed} \otimes \mathrm{Free}^{\mathsf{Mixed}}(I)}$

(2) $\mathrm{ev}_{\mathfrak{m}}\left(\overline{\mathsf{Mixed} \otimes \mathrm{Free}^{\mathsf{Mixed}}(I)}\right) \subseteq \overline{\mathrm{ev}_{\mathfrak{m}}\left(\mathsf{Mixed} \otimes \mathrm{Free}^{\mathsf{Mixed}}(I)\right)}$

(3) $\mathrm{ev}_{\mathfrak{m}}\left(\mathsf{Mixed} \otimes \mathrm{Free}^{\mathsf{Mixed}}(I)\right) \subseteq \mathsf{Ch}(k) \otimes \{\text{acyclic cofibrations in } \mathsf{Ch}(k)\}$

(4) $\overline{\mathsf{Ch}(k) \otimes \{\text{acyclic cofibrations in } \mathsf{Ch}(k)\}} \subseteq W$

Proof of claim (1): The class of acyclic cofibrations in Mixed is by Theorem 4.2.2.1 (6) equal to $\overline{\mathrm{Free}^{\mathsf{Mixed}}(I)}$. As the tensor product functor on Mixed preserves colimits in each variable the claim follows.

Proof of claim (2): Follows from $\mathrm{ev}_{\mathfrak{m}}$ preserving colimits.

Proof of claim (3): Let i be a generating acyclic cofibration of $\mathsf{Ch}(k)$ and X a strict mixed complex. Then we have

$$\mathrm{ev}_{\mathfrak{m}}\left(\mathrm{id}_X \otimes \mathrm{Free}^{\mathsf{Mixed}}(i)\right) \cong \mathrm{id}_{X \otimes \mathrm{D}} \otimes i$$

where we use that $\mathrm{ev}_{\mathfrak{m}}$ is symmetric monoidal, so the claim follows.

Proof of claim (4): Follows from $\mathsf{Ch}(k)$ satisfying the monoid axiom, see Fact 4.1.3.1. □

4.2.2.3 The model structure on Alg(Mixed)

We can now put together the various results regarding the model structure on Mixed and apply Theorem 4.2.2.1 in order to obtain a combinatorial model structure on Alg(Mixed).

Proposition 4.2.2.9. *There are combinatorial model structures on the* 1-*categories* Alg(Mixed) *and* Alg($\mathsf{Ch}(k)$) *with the properties listed in Theorem 4.2.2.1.* ♡

Proof. By Definition 4.2.2.2 the model structure on Mixed is combinatorial, by Proposition 4.2.1.14 there is a closed symmetric monoidal structure on Mixed, by Proposition 4.2.2.6 the model structure satisfies the pushout product axiom, and by Proposition 4.2.2.8 the monoid axiom is satisfied. $\mathsf{Ch}(k)$ has all these properties as well by Fact 4.1.3.1. We can thus apply Theorem 4.2.2.1. □

We end this section by discussing the various forgetful functors, and show some properties that they have that will be useful later.

Notation 4.2.2.10. There is a commutative diagram of forgetful functors as follows.

$$\begin{tikzcd} & \mathrm{Alg}(\mathsf{Mixed}) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}"'] \arrow[dr, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})"] & \\ \mathsf{Mixed} \arrow[dr, "\mathrm{ev}_{\mathfrak{m}}"'] & & \mathrm{Alg}(\mathsf{Ch}(k)) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}"] \\ & \mathsf{Ch}(k) & \end{tikzcd} \tag{4.4}$$

To be able to distinguish the two forgetful functors from categories of algebras to their underlying categories, we give the forgetful functor from $\mathrm{Alg}(\mathsf{Mixed})$ to Mixed an extra superscript Mixed.

The functors $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$, $\mathrm{ev}_{\mathfrak{m}}$, and $\mathrm{ev}_{\mathfrak{a}}$ all have left adjoints according to Theorem 4.2.2.1. We denote

- the left adjoint to $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ by $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$.
- the left adjoint to $\mathrm{ev}_{\mathfrak{m}}$ by $\mathrm{Free}^{\mathsf{Mixed}}$.
- the left adjoint to $\mathrm{ev}_{\mathfrak{a}}$ by $\mathrm{Free}^{\mathrm{Alg}}$. ♢

Proposition 4.2.2.11. *The commutative square*

$$\begin{tikzcd} \mathrm{Alg}(\mathsf{Mixed}) \arrow[r, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}"] \arrow[d, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})"'] & \mathsf{Mixed} \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\ \mathrm{Alg}(\mathsf{Ch}(k)) \arrow[r, "\mathrm{ev}_{\mathfrak{a}}"'] & \mathsf{Ch}(k) \end{tikzcd}$$

from Notation 4.2.2.10 is left adjointable[11]*, i. e. the push-pull transformation*

$$\mathrm{Free}^{\mathrm{Alg}} \circ \mathrm{ev}_{\mathfrak{m}} \to \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}}) \circ \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$$

is a natural isomorphism. ♡

Proof. As the symmetric monoidal structures on Mixed and $\mathsf{Ch}(k)$ are compatible with colimits[12], and $\mathrm{ev}_{\mathfrak{m}}$ is symmetric monoidal and preserves colimits[13], this is a special case of Proposition E.7.2.2 (2). □

We can now collect some properties of the various forgetful functors.

[11] See [HTT, 7.3.1.1] for a definition.

[12] As both symmetric monoidal categories are closed symmetric monoidal, see Definition 4.1.2.1 and Proposition 4.2.1.14.

[13] See for example [HA, 4.2.3.5].

Proposition 4.2.2.12. *The following table summarizes what kind of morphisms or constructions the various forgetful functors from Notation 4.2.2.10 preserve (marked with a P) or detect (marked with a D).*

Functor	*isomorphisms*	*weak equivalences*	*fibrations*	*cofibrations*	*cofib'*[14]	*lim*	*sifted colim*	*colim*
$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$	*D*	*D*	*D*			*D*	*D*	
$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$	*D*	*D*	*D*	*P*	*P*	*D*	*D*	*D*
$\mathrm{ev}_{\mathfrak{m}}$	*D*	*D*	*D*	*P*	*P*	*D*	*D*	*D*
$\mathrm{ev}_{\mathfrak{a}}$	*D*	*D*	*D*		*P*	*D*	*D*	

All properties that make use of a model structure are to be understood with respect to the model structures from Fact 4.1.3.1, Definition 4.2.2.2, and Proposition 4.2.2.9. ♡

Proof. Weak equivalences and fibrations: That $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$, $\mathrm{ev}_{\mathfrak{m}}$, and $\mathrm{ev}_{\mathfrak{a}}$ detect weak equivalences and fibrations is Theorem 4.2.2.1 (3) and (7). From commutativity of the diagram (4.4) we obtain the same for $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$.

Limits and sifted colimits: That limits and colimits in module categories[15] are calculated on underlying objects is a standard categorical fact, see for example [HA, 4.2.3.3 and 4.2.3.5]. Similarly, it is standard that limits and sifted colimits[16] of algebras are calculated on underlying objects, see for example [HA, 3.2.2.5] and [HA, 3.2.3.1]. Again, as the three other functors detect limits and sifted colimits, this also follows for $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$.

Isomorphisms: That $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$, $\mathrm{ev}_{\mathfrak{m}}$, and $\mathrm{ev}_{\mathfrak{a}}$ are conservative, i. e. detect isomorphisms, is standard, and then it again follows that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ is conservative as well. However, we could also deduce this from all four functors detecting sifted colimits, as detecting isomorphisms is equivalent to detecting $[0]$-colimits.

Colimits: That $\mathrm{ev}_{\mathfrak{m}}$ detects colimits was already mentioned above. As $\mathrm{ev}_{\mathfrak{m}}$ is also symmetric monoidal, it then follows from Proposition E.7.3.1 that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ preserves colimits as well. As $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ is conservative, this implies that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ even detects colimits.

Cofibrations and cofibrations between cofibrant objects: It follows from Theorem 4.2.2.1 (8) in combination with D being cofibrant in $\mathsf{Ch}(k)$ by Proposition 4.2.2.4 that $\mathrm{ev}_{\mathfrak{m}}$ preserves cofibrations. Furthermore it follows from Theorem 4.2.2.1 (4) in combination with the monoidal unit of $\mathsf{Ch}(k)$ being cofibrant by Fact 4.1.3.1 that $\mathrm{ev}_{\mathfrak{a}}$ preserves cofibrant objects and cofibrations between cofibrant objects.

[14] Cofibrant objects and cofibrations between cofibrant objects.

[15] This is true for categories of modules in a monoidal category whose tensor product functor preserves colimits in each variable separately, which is the case for $\mathsf{Ch}(k)$, as it is a closed symmetric monoidal category.

[16] This again requires the assumption that the tensor product preserves sifted colimits in each variable separately, which is the case for both $\mathsf{Ch}(k)$ and Mixed.

It remains to show that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ preserves cofibrations. As we already showed that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ preserves colimits, it suffices for this to show that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ maps generating cofibrations to cofibrations. Generating cofibrations of $\mathrm{Alg}(\mathsf{Mixed})$ are by Theorem 4.2.2.1 (2) and (6) morphisms of the form $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}(\mathrm{Free}^{\mathsf{Mixed}}(i))$ with i a (generating) cofibration in $\mathsf{Ch}(k)$. By Proposition 4.2.2.11 there is a natural isomorphism as follows.

$$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}}) \circ \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})} \circ \mathrm{Free}^{\mathsf{Mixed}} \cong \mathrm{Free}^{\mathrm{Alg}} \circ \mathrm{ev}_{\mathfrak{m}} \circ \mathrm{Free}^{\mathsf{Mixed}}$$

As $\mathrm{Free}^{\mathrm{Alg}}$ and $\mathrm{Free}^{\mathsf{Mixed}}$ preserve cofibrations as left Quillen functors[17] and $\mathrm{ev}_{\mathfrak{m}}$ was already shown to preserve cofibrations, the claim follows. □

Proposition 4.2.2.13. *Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the extension of scalars functor*

$$k' \otimes_k -\colon\ \mathrm{Alg}(\mathsf{Mixed}_k) \to \mathrm{Alg}(\mathsf{Mixed}_{k'})$$

that is induced on algebras by the symmetric monoidal functor

$$k' \otimes_k -\colon \mathsf{Mixed}_k \to \mathsf{Mixed}_{k'}$$

from Remark 4.2.1.3 preserves colimits and cofibrations. ♡

Proof. The extension of scalars functor

$$k' \otimes_k -\colon \mathsf{Mixed}_k \to \mathsf{Mixed}_{k'}$$

is by Remark 4.2.1.3 symmetric monoidal and preserves colimits. As the tensor product functors of Mixed_k and $\mathsf{Mixed}_{k'}$ also preserve colimits in each variable separately by Proposition 4.2.2.6 we can apply Proposition E.7.3.1 to conclude that the induced functor

$$k' \otimes_k -\colon\ \mathrm{Alg}(\mathsf{Mixed}_k) \to \mathrm{Alg}(\mathsf{Mixed}_{k'})$$

preserves colimits.

To show that this functor also preserves cofibrations it now suffices to show that it maps generating cofibrations to cofibrations. So let $i\colon X \to Y$ be a cofibration in Mixed_k. We have to show that

$$k' \otimes_k \mathrm{Free}_{\mathsf{Mixed}_k}^{\mathrm{Alg}(\mathsf{Mixed}_k)}(i)$$

is a cofibration in $\mathrm{Alg}(\mathsf{Mixed}_{k'})$. But by Proposition E.7.2.2 we can identify this morphism with

$$\mathrm{Free}_{\mathsf{Mixed}_{k'}}^{\mathrm{Alg}(\mathsf{Mixed}_{k'})}(k' \otimes_k i)$$

which is a cofibration as $\mathrm{Free}_{\mathsf{Mixed}_{k'}}^{\mathrm{Alg}(\mathsf{Mixed}_{k'})}$ is a left Quillen functor by Theorem 4.2.2.1 (5) and

$$k' \otimes_k -\colon \mathsf{Mixed}_k \to \mathsf{Mixed}_{k'}$$

preserves cofibrations by Proposition 4.2.2.3. □

[17] See Theorem 4.2.2.1 (1) and (5).

4.2.2.4 Homotopies in Mixed

In this section we describe homotopies in Mixed, continuing from and proceeding analogously to Section 4.1.4.

Proposition 4.2.2.14. *Let Y be a strict mixed complex. Then defining an operator* d *that increases degree by one on P from Proposition 4.1.4.1 as*

$$\mathrm{d}((x,y,z)) \coloneqq (\mathrm{d}\,x, \mathrm{d}\,y, -\mathrm{d}\,z)$$

upgrades P to a strict mixed complex. Furthermore, the morphisms i and p that were defined in Proposition 4.1.4.1 are compatible with this strict mixed structure, exhibiting P as a path object for Y in Mixed*.* ♡

Proof. It is clear that d as defined in the statement is k-linear and increases degree by 1. Let (x,y,z) be an element in P. Then the short calculation

$$\mathrm{d}(\mathrm{d}((x,y,z))) = \mathrm{d}((\mathrm{d}\,x, \mathrm{d}\,y, -\mathrm{d}\,z)) = (\mathrm{d}(\mathrm{d}\,x), \mathrm{d}(\mathrm{d}\,y), \mathrm{d}(\mathrm{d}\,z)) = (0,0,0)$$

shows that d squares to zero, and the following calculation shows that it also holds that $\mathrm{d}\circ\partial+\partial\circ\mathrm{d} = 0$, so that P indeed becomes a strict mixed complex.

$$\begin{aligned}
&(\mathrm{d}\circ\partial+\partial\circ\mathrm{d})((x,y,z)) \\
&= \mathrm{d}((\partial x, \partial y, -\partial(z)+x-y)) + \partial((\mathrm{d}\,x, \mathrm{d}\,y, -\mathrm{d}\,z)) \\
&= (\mathrm{d}(\partial(x)), \mathrm{d}(\partial(y)), -\mathrm{d}(-\partial(z)+x-y)) \\
&\quad + (\partial(\mathrm{d}(x)), \partial(\mathrm{d}(y)), -\partial(-\mathrm{d}(z)) + \mathrm{d}\,x - \mathrm{d}\,y) \\
&= \big(\mathrm{d}(\partial(x)) + \partial(\mathrm{d}(x)), \mathrm{d}(\partial(y)) + \partial(\mathrm{d}(y)), \\
&\qquad \mathrm{d}(\partial(z)) - \mathrm{d}(x) + \mathrm{d}(y) + \partial(\mathrm{d}(z)) + \mathrm{d}(x) - \mathrm{d}(y)\big) \\
&= (0,0,0)
\end{aligned}$$

It is clear that i and p are compatible with d, making them into morphisms in Mixed. As the forgetful functor $\mathrm{ev}_{\mathsf{m}}\colon \mathsf{Mixed} \to \mathsf{Ch}(k)$ detects weak equivalences and fibrations by Proposition 4.2.2.12, it now follows from Proposition 4.1.4.1 that i and p exhibit P as a path object for Y. □

Proposition 4.2.2.15. *Let X be a cofibrant and Y a fibrant object in* Mixed*, with respect to the model structure of Definition 4.2.2.2, and f and g two morphisms $X \to Y$ in* Mixed*. Then f and g are homotopic if and only if there exists a* chain homotopy of strict mixed complexes *h from f to g, by which we mean a chain homotopy h from f to g in the sense of Proposition 4.1.4.2 satisfying additionally*[18]

$$h(\mathrm{d}(x)) = -\mathrm{d}(h(x)) \tag{4.5}$$

for all elements x of X. ♡

[18] To remember the sign, note that both d and h have odd degree, so commuting them should be expected to introduce a sign.

Proof. Note that by [Hov99, 1.2.6], as X is cofibrant and Y is fibrant, the left and right homotopy relations coincide, and the right homotopy relation can be tested using any path object for Y. For this we use the path object P from Proposition 4.2.2.14.

Arguing analogously to the proof of Proposition 4.1.4.2, we see that f and g are homotopic as morphisms of strict mixed complexes if and only if there exists a morphism of strict mixed complexes $H = f \times g \times h\colon X \to P$. That H is a morphism of chain complexes amounts, just like in Proposition 4.1.4.2, to

$$\partial \circ h + h \circ \partial = f - g$$

but this time H needs to additionally commute with d, so for x an element of X the following equality must hold.

$$(f(\mathrm{d}(x)), g(\mathrm{d}(x)), h(\mathrm{d}(x))) = \mathrm{d}((f(x), g(x), h(x))) \tag{$*$}$$

The right hand side is given by

$$\mathrm{d}((f(x), g(x), h(x))) = (\mathrm{d}(f(x)), \mathrm{d}(g(x)), -\mathrm{d}(h(x)))$$

so as f and g are morphisms of strict mixed complexes we can conclude that equality $(*)$ is equivalent to the following equation.

$$h(\mathrm{d}(x)) = -\mathrm{d}(h(x))$$

□

4.2.2.5 Homotopies in $\mathrm{Alg}(\mathsf{Ch}(k))$

In this section we describe homotopies in $\mathrm{Alg}(\mathsf{Ch}(k))$. The statements of the first two propositions, concerning an appropriate path object and a concrete description of the resulting homotopies, are completely analogous to the propositions in Sections 4.1.4 and 4.2.2.4. However, this section has an additional helpful result that reduces the amount of data that needs to be specified and the amount of properties that need to be checked to construct homotopies out of differential graded algebras whose underlying $\mathbb{Z}$-graded k-algebra is free.

Proposition 4.2.2.16. *Let Y be a differential graded k-algebra. Then defining a multiplication on the chain complex P that was defined in Proposition 4.1.4.1 as*

$$(x, y, z) \cdot (x', y', z') \coloneqq \left(xx', yy', zy' + (-1)^{\deg_{\mathsf{Ch}}(x)} xz'\right)$$

upgrades P to a differential graded k-algebra with unit $(1, 1, 0)$. Furthermore, the morphisms i and p that were defined in Proposition 4.1.4.1 are compatible with this multiplicative structure, exhibiting P as a path object for Y in $\mathrm{Alg}(\mathsf{Ch}(k))$. ♡

Proof. It is clear that $(1,1,0)$ is a unit for the multiplication that was defined in the statement, and that multiplication is k-linear in both factors. For associativity we carry out the following calculations.

$$\begin{aligned}
&((x,y,z)\cdot(x',y',z'))\cdot(x'',y'',z'')\\
&=\Big(xx',yy',zy'+(-1)^{\deg_{\mathsf{Ch}}(x)}xz'\Big)\cdot(x'',y'',z'')\\
&=\Big(xx'x'',yy'y'',zy'y''+(-1)^{\deg_{\mathsf{Ch}}(x)}xz'y''+(-1)^{\deg_{\mathsf{Ch}}(x)+\deg_{\mathsf{Ch}}(x')}xx'z''\Big)
\end{aligned}$$

$$\begin{aligned}
&(x,y,z)\cdot((x',y',z')\cdot(x'',y'',z''))\\
&=(x,y,z)\cdot\Big(x'x'',y'y'',z'y''+(-1)^{\deg_{\mathsf{Ch}}(x')}x'z''\Big)\\
&=\Big(xx'x'',yy'y'',zy'y''+(-1)^{\deg_{\mathsf{Ch}}(x)}xz'y''+(-1)^{\deg_{\mathsf{Ch}}(x)+\deg_{\mathsf{Ch}}(x')}xx'z''\Big)
\end{aligned}$$

The next calculations show that the Leibniz rule is also satisfied, making P into a differential graded algebra.

$$\begin{aligned}
&\partial((x,y,z)\cdot(x',y',z'))\\
&=\partial\Big(\Big(xx',yy',zy'+(-1)^{\deg_{\mathsf{Ch}}(x)}xz'\Big)\Big)\\
&=\Big(\partial(xx'),\partial(yy'),-\partial\Big(zy'+(-1)^{\deg_{\mathsf{Ch}}(x)}xz'\Big)+xx'-yy'\Big)\\
&=\Big(\partial(x)x'+(-1)^{\deg_{\mathsf{Ch}}(x)}x\partial(x'),\partial(y)y'+(-1)^{\deg_{\mathsf{Ch}}(x)}y\partial(y'),\\
&\quad-\partial(z)y'-(-1)^{\deg_{\mathsf{Ch}}(x)+1}z\partial(y')\\
&\quad-(-1)^{\deg_{\mathsf{Ch}}(x)}\partial(x)z'-x\partial(z')+xx'-yy'\Big)
\end{aligned}$$

$$\begin{aligned}
&\partial((x,y,z))\cdot(x',y',z')+(-1)^{\deg_{\mathsf{Ch}}(x)}(x,y,z)\cdot\partial((x',y',z'))\\
&=(\partial(x),\partial(y),-\partial(z)+x-y)\cdot(x',y',z')\\
&\quad+(-1)^{\deg_{\mathsf{Ch}}(x)}(x,y,z)\cdot(\partial(x'),\partial(y'),-\partial(z')+x'-y')\\
&=\Big(\partial(x)x',\partial(y)y',-\partial(z)y'+xy'-yy'-(-1)^{\deg_{\mathsf{Ch}}(x)}\partial(x)z'\Big)\\
&\quad+(-1)^{\deg_{\mathsf{Ch}}(x)}\,(x\partial(x'),y\partial(y'),\\
&\quad z\partial(y')-(-1)^{\deg_{\mathsf{Ch}}(x)}x\partial(z')+(-1)^{\deg_{\mathsf{Ch}}(x)}xx'-(-1)^{\deg_{\mathsf{Ch}}(x)}xy'\Big)\\
&=\Big(\partial(x)x'+(-1)^{\deg_{\mathsf{Ch}}(x)}x\partial(x'),\partial(y)y'+(-1)^{\deg_{\mathsf{Ch}}(x)}y\partial(y'),\\
&\quad-\partial(z)y'+xy'-yy'-(-1)^{\deg_{\mathsf{Ch}}(x)}\partial(x)z'\\
&\quad+(-1)^{\deg_{\mathsf{Ch}}(x)}z\partial(y')-x\partial(z')+xx'-xy'\Big)
\end{aligned}$$

$$= \Big(\partial(x)x' + (-1)^{\deg_{\mathsf{Ch}}(x)} x\partial(x'), \partial(y)y' + (-1)^{\deg_{\mathsf{Ch}}(x)} y\partial(y'),$$
$$- \partial(z)y' + (-1)^{\deg_{\mathsf{Ch}}(x)} z\partial(y')$$
$$-(-1)^{\deg_{\mathsf{Ch}}(x)}\partial(x)z' - x\partial(z') + xx' - yy'\Big)$$

It is immediate from the formula for multiplication on P that the the morphisms of chain complexes $i\colon Y \to P$ and $p\colon P \to Y \times Y$ from Proposition 4.1.4.1 become morphisms of differential graded algebras. As weak equivalences and fibrations in $\mathrm{Alg}(\mathsf{Ch}(k))$ are detected by the forgetful functor to $\mathsf{Ch}(k)$ by Proposition 4.2.2.12, it now follows from Proposition 4.1.4.1 that i and p exhibit P as a path object for Y. We remark that a more conceptual approach to constructing this path object is described in [SS00, Section *Chain complexes* on pages 503 and 504], though there are some differences in signs. □

Proposition 4.2.2.17. *Let X be a cofibrant and Y a fibrant object in* $\mathrm{Alg}(\mathsf{Ch}(k))$*, with respect to the model structure of Proposition 4.2.2.9, and f and g two morphisms $X \to Y$ in* $\mathrm{Alg}(\mathsf{Ch}(k))$. *Then f and g are homotopic if and only if there exists a* chain homotopy of differential graded k-algebras h *from f to g, by which we mean a chain homotopy h from f to g in the sense of Proposition 4.1.4.2 satisfying additionally*

$$h(x \cdot x') = h(x)g(x') + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)h(x') \tag{4.6}$$

for all elements x and x' of X. ♡

Proof. Note that by [Hov99, 1.2.6], as X is cofibrant and Y is fibrant, the left and right homotopy relations coincide, and the right homotopy relation can be tested using any path object for Y. For this we use the path object P from Proposition 4.2.2.16.

Arguing completely analogously to the proof of Proposition 4.1.4.2, we see that f and g are homotopic as morphisms of differential graded algebras if and only if there exists a morphism of differential graded algebras $H = f \times g \times h\colon X \to P$. That H is a morphism of chain complexes amounts, just like in Proposition 4.1.4.2, to

$$\partial \circ h + h \circ \partial = f - g$$

but this time H needs to additionally preserve the unit, which is equivalent to $h(1) = 0$, and the multiplication, so for x and x' elements of X the following equality must hold.

$$(f(x \cdot x'), g(x \cdot x'), h(x \cdot x')) = (f(x), g(x), h(x)) \cdot (f(x'), g(x'), h(x')) \tag{$*$}$$

The right hand side is given by

$$(f(x), g(x), h(x)) \cdot (f(x'), g(x'), h(x'))$$
$$= \Big(f(x) \cdot f(x'), g(x) \cdot g(x'), h(x)g(x') + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)h(x')\Big)$$

so as f and g are multiplicative we conclude that equality $(*)$ is equivalent to the following equation.

$$h(x \cdot x') = h(x)g(x') + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)h(x')$$

Finally, applying this equation for $x = x' = 1$ we obtain that $h(1) = 2h(1)$ and hence $h(1) = 0$. $\square$

The following proposition will sometimes be helpful in defining homotopies of differential graded k-algebras.

Proposition 4.2.2.18. *Let X and Y be objects in $\mathrm{Alg}(\mathsf{Ch}(k))$, and assume that the underlying $\mathbb{Z}$-graded k-algebra of X is free on a $\mathbb{Z}$-graded subset Z of X.*

Let f and g be morphisms of differential graded algebras from X to Y and h a map from Z to Y that increases degree by 1. Then there is a unique extension of h to a morphism of $\mathbb{Z}$-graded k-modules of degree 1 from X to Y such that

$$h(x \cdot x') = h(x)g(x') + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)h(x') \tag{4.7}$$

holds for all elements x and x' of X. That unique extension is given by defining h on the basis given by words in Z by

$$h(z_1 \cdots z_l) := \sum_{1 \le i \le l} (-1)^{\sum_{1 \le j \le i-1} \deg_{\mathsf{Ch}}(z_j)} \cdot f(z_1 \cdots z_{i-1}) \cdot h(z_i) \cdot g(z_{i+1} \cdots z_l) \tag{4.8}$$

for $l \ge 0$ and $z_1, \ldots, z_l \in Z$, and then extending k-linearly.

Furthermore, such an extension h satisfies

$$\partial \circ h + h \circ \partial = f - g \tag{4.9}$$

if and only if this holds on elements of Z. ♡

Proof. We first show uniqueness of the extension. As h must be k-linear, it suffices to show that the h is already uniquely given on words in Z. This we do by induction on the word length. By the Leibniz rule (4.7), h must map 1 to 0 (use $x = x' = 1$), so h is uniquely determined on words in Z of length 0. It is also uniquely determined on elements of Z themselves, as we prescribe the value on those elements. The induction step then follows directly from (4.7).

Now define h as in (4.8). It is clear from the definition that this definition extends the prescribed valued on Z. To verify that (4.7) holds we first note that both sides of the equation are k-linear in both x and x', so that it suffices to check this on a k-basis of X. So let $w = z_1 \cdots z_l$ and $w' = z'_1 \cdots z'_{l'}$ be words in Z. Then the following calculation shows that (4.7) is satisfied.

$$\begin{aligned}
&h(w \cdot w') \\
&= \sum_{1 \le i \le l} (-1)^{\sum_{1 \le j \le i-1} \deg_{\mathsf{Ch}}(z_j)} \cdot f(z_1 \cdots z_{i-1}) \cdot h(z_i) \cdot g(z_{i+1} \cdots z_l \cdot w') \\
&\quad + \sum_{1 \le i \le l'} \Big((-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (-1)^{\sum_{1 \le j \le i-1} \deg_{\mathsf{Ch}}(z'_j)} \\
&\qquad\qquad \cdot f\big(w \cdot z'_1 \cdots z'_{i-1}\big) \cdot h(z'_i) \cdot g\big(z'_{i+1} \cdots z'_{l'}\big) \Big) \\
&= \left(\sum_{1 \le i \le l} (-1)^{\sum_{1 \le j \le i-1} \deg_{\mathsf{Ch}}(z_j)} \cdot f(z_1 \cdots z_{i-1}) \cdot h(z_i) \cdot g(z_{i+1} \cdots z_l) \right) \cdot g(w') \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot \\
&\qquad \left(\sum_{1 \le i \le l'} (-1)^{\sum_{1 \le j \le i-1} \deg_{\mathsf{Ch}}(z'_j)} \cdot f\big(z'_1 \cdots z'_{i-1}\big) \cdot h(z'_i) \cdot g\big(z'_{i+1} \cdots z'_{l'}\big) \right) \\
&= h(w) \cdot g(w') + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot h(w')
\end{aligned}$$

It remains to show the assertion concerning (4.9). That if equality holds in general, then it also holds on Z is clear. So assume that (4.9) holds on Z. As both sides of the equation are k-linear it again suffices to show (4.9) on the k-basis given by words in Z. We show this by induction on the word length. For the element 1 (i. e. the unique word of length 0) we obtain $h(1) = 0$ and $\partial(1) = 0$ from the respective Leibniz rules, and the right hand side of (4.9) is zero as well as $f(1) = 1 = g(1)$. On words of length 1, i. e. elements of Z, the equation (4.9) holds by assumption. So now let w be an element of X on which (4.9) holds, and z an element of Z. Then the following calculation shows that (4.9) also holds for $w \cdot z$, thereby finishing the proof.

$$\partial(h(w \cdot z)) + h(\partial(w \cdot z))$$

We first apply the Leibniz rule twice, for both h and ∂.

$$\begin{aligned}
&= \partial\Big(h(w) \cdot g(z) + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot h(z) \Big) \\
&\quad + h\Big(\partial(w) \cdot z + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot w \cdot \partial(z) \Big) \\
&= \partial(h(w)) \cdot g(z) + (-1)^{\deg_{\mathsf{Ch}}(w)+1} h(w) \cdot \partial(g(z)) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot \partial(f(w)) \cdot h(z) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot \partial(h(z)) \\
&\quad + h(\partial(w)) \cdot g(z) + (-1)^{\deg_{\mathsf{Ch}}(w)-1} \cdot f(\partial(w)) \cdot h(z) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot h(w) \cdot g(\partial(z)) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot h(\partial(z))
\end{aligned}$$

Next we reorder the summands.

$$
\begin{aligned}
&= \partial(h(w)) \cdot g(z) + h(\partial(w)) \cdot g(z) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)+1} h(w) \cdot \partial(g(z)) + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot h(w) \cdot g(\partial(z)) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot \partial(f(w)) \cdot h(z) + (-1)^{\deg_{\mathsf{Ch}}(w)-1} \cdot f(\partial(w)) \cdot h(z) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot \partial(h(z)) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot h\partial((z)) \\
&= (\partial(h(w)) + h(\partial(w))) \cdot g(z) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} h(w) \cdot (-\partial(g(z)) + g(\partial(z))) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (\partial(f(w)) - f(\partial(w))) \\
&\quad + f(w) \cdot (\partial(h(z)) + h\partial((z)))
\end{aligned}
$$

Now we can apply the induction hypothesis, and that f and g are morphisms of chain complexes.

$$
\begin{aligned}
&= (f(w) - g(w)) \cdot g(z) + f(w) \cdot (f(z) - g(z)) \\
&= f(w) \cdot g(z) - g(w) \cdot g(z) + f(w) \cdot f(z) - f(w) \cdot g(z) \\
&= f(w \cdot z) - g(w \cdot z)
\end{aligned}
$$

□

4.2.2.6 Homotopies in $\mathrm{Alg}(\mathsf{Mixed})$

Now we turn to homotopies of algebras in strict mixed complexes. This results in this section are analogous to those in the preceding Section 4.2.2.5, and obtained by combining those results with those from Section 4.2.2.4.

Proposition 4.2.2.19. *Let Y be an object in $\mathrm{Alg}(\mathsf{Mixed})$. Then the strict mixed structure defined in Proposition 4.2.2.14 on the chain complex P from Proposition 4.1.4.1 satisfies the Leibniz rule with respect to the multiplication from Proposition 4.2.2.16, upgrading P to an object in $\mathrm{Alg}(\mathsf{Mixed})$. Furthermore, the morphisms i and p exhibit P as a path object for Y in $\mathrm{Alg}(\mathsf{Mixed})$.* ♡

Proof. Let (x, y, z) and (x', y', z') be two elements of P. Then the following calculation shows that d satisfies the Leibniz rule.

$$
\begin{aligned}
&\mathrm{d}((x, y, z) \cdot (x', y', z')) \\
&= \mathrm{d}\Big(\Big(x \cdot x', y \cdot y', z \cdot y' + (-1)^{\deg_{\mathsf{Ch}}(x)} \cdot x \cdot z'\Big)\Big) \\
&= \Big(\mathrm{d}(x \cdot x'), \mathrm{d}(y \cdot y'), -\mathrm{d}\Big(z \cdot y' + (-1)^{\deg_{\mathsf{Ch}}(x)} \cdot x \cdot z'\Big)\Big) \\
&= \Big(\mathrm{d}(x) \cdot x' + (-1)^{\deg_{\mathsf{Ch}}(x)} \cdot x \cdot \mathrm{d}(x'), \mathrm{d}(y) \cdot y' + (-1)^{\deg_{\mathsf{Ch}}(y)} \cdot y \cdot \mathrm{d}(y'), \\
&\qquad - \mathrm{d}(z) \cdot y' - (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot z \cdot \mathrm{d}(y') \\
&\qquad\qquad - (-1)^{\deg_{\mathsf{Ch}}(x)} \cdot \mathrm{d}(x) \cdot z' - (-1)^{\deg_{\mathsf{Ch}}(x)} \cdot (-1)^{\deg_{\mathsf{Ch}}(x)} \cdot x \cdot \mathrm{d}(z')\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\mathrm{d}(x)\cdot x', \mathrm{d}(y)\cdot y', -\mathrm{d}(z)\cdot y' - (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot \mathrm{d}(x)\cdot z'\Big)\\
&\quad + \Big((-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot \mathrm{d}(x'), (-1)^{\deg_{\mathsf{Ch}}(y)}\cdot y\cdot \mathrm{d}(y'),\\
&\quad\ - (-1)^{\deg_{\mathsf{Ch}}(z)}\cdot z\cdot \mathrm{d}(y') - (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot(-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot \mathrm{d}(z')\Big)\\
&= \Big(\mathrm{d}(x)\cdot x', \mathrm{d}(y)\cdot y', (-\mathrm{d}(z))\cdot y' + (-1)^{\deg_{\mathsf{Ch}}(\mathrm{d}\,x)}\cdot \mathrm{d}(x)\cdot z'\Big)\\
&\quad + \Big((-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot \mathrm{d}(x'), (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot y\cdot \mathrm{d}(y'),\\
&\quad\ + (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot z\cdot \mathrm{d}(y') + (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot(-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot(-\mathrm{d}(z'))\Big)\\
&= (\mathrm{d}(x), \mathrm{d}(y), -\mathrm{d}(z))\cdot(x', y', z')\\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot(x, y, z)\cdot(\mathrm{d}(x'), \mathrm{d}(y'), -\mathrm{d}(z'))\\
&= \mathrm{d}((x, y, z))\cdot(x', y', z') + (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot(x, y, z)\cdot \mathrm{d}((x', y', z'))
\end{aligned}
$$

This upgrades P to an object in $\mathrm{Alg}(\mathsf{Mixed})$. As i and p are compatible with both the strict mixed structure by Proposition 4.2.2.14 and the multiplicative structure by Proposition 4.2.2.16 we can conclude that i and p also lift to morphisms in $\mathrm{Alg}(\mathsf{Mixed})$. As weak equivalences and fibrations in $\mathrm{Alg}(\mathsf{Mixed})$ are detected by the forgetful functor to $\mathsf{Ch}(k)$ by Proposition 4.2.2.12, it now follows from Proposition 4.1.4.1 that i and p exhibit P as a path object for Y. □

Proposition 4.2.2.20. *Let X be a cofibrant and Y a fibrant object in* $\mathrm{Alg}(\mathsf{Mixed})$*, with respect to the model structure of Proposition 4.2.2.9, and f and g two morphisms $X \to Y$ in* $\mathrm{Alg}(\mathsf{Mixed})$*. Then f and g are homotopic if and only if there exists a* chain homotopy of algebras of strict mixed complexes *h from f to g, by which we mean a chain homotopy h from f to g in the sense of Proposition 4.1.4.2 that is simultaneously a chain homotopy of differential graded algebras from f to g in the sense of Proposition 4.2.2.17 and a chain homotopy of strict mixed complexes from f to g in the sense of Proposition 4.2.2.15.* ♡

Proof. Note that by [Hov99, 1.2.6], as X is cofibrant and Y is fibrant, the left and right homotopy relations coincide, and the right homotopy relation can be tested using any path object for Y. For this we use the path object P from Proposition 4.2.2.19.

Arguing completely analogously to the proof of Proposition 4.1.4.2, we see that f and g are homotopic as morphisms of algebras in strict mixed complexes if and only if there exists a morphism of algebras in strict mixed complexes $H = f \times g \times h\colon X \to P$. While an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ is more than a chain complex that is equipped with both a strict mixed and an algebra structure, as d needs to additionally satisfy the Leibniz rule, morphisms of algebras in strict mixed complexes are just morphisms of chain complexes that are compatible with both multiplication and the strict mixed structure. Thus the claim now follows directly by combining the proofs of Propositions 4.2.2.15 and 4.2.2.17. □

The following proposition is an analogue of Proposition 4.2.2.18 and will sometimes be helpful when trying to define a chain homotopy of algebras in strict mixed complexes.

Proposition 4.2.2.21. *Let X and Y be objects in* $\mathrm{Alg}(\mathsf{Mixed})$*, and let Z be a $\mathbb{Z}$-graded subset of X. Assume that Z is disjoint from $\mathrm{d}\,Z$ and that the underlying $\mathbb{Z}$-graded k-algebra of X is free on $Z \cup \mathrm{d}\,Z$.*

Let f and g be morphisms of algebras of strict mixed complexes from X to Y, and h a map from Z to Y that increases degree by 1. Then there is a unique extension of h to a morphism of $\mathbb{Z}$-graded k-modules of degree 1 from X to Y such that

$$h(x \cdot x') = h(x)g(x') + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)h(x') \tag{4.10}$$

and

$$h(\mathrm{d}(x)) = -\,\mathrm{d}(h(x)) \tag{4.11}$$

holds for all elements x and x' of X. That unique extension is given by first extending h to $Z \cup \mathrm{d}\,Z$ via

$$h(\mathrm{d}\,z) := -\,\mathrm{d}(h(z)) \tag{4.12}$$

for z an element of Z, and then defining h on the basis given by words in Z and $\mathrm{d}\,Z$ by

$$h(z_1 \cdots z_l) := \sum_{1 \le i \le l} (-1)^{\sum_{1\le j \le i-1} \deg_{\mathsf{Ch}}(z_j)} \cdot f(z_1 \cdots z_{i-1}) \cdot h(z_i) \cdot g(z_{i+1} \cdots z_l) \tag{4.13}$$

for $l \geq 0$ and $z_1, \ldots, z_l \in Z \cup \mathrm{d}\,Z$, and then extending k-linearly.

Furthermore, such an extension h satisfies

$$\partial \circ h + h \circ \partial = f - g \tag{4.14}$$

if and only if this holds on elements of Z. ♡

Proof. We first show uniqueness of the extension. By (4.11) the extension to $Z \cup \mathrm{d}\,Z$ as in (4.12) is uniquely determined, and then uniqueness of the extension from $Z \cup \mathrm{d}\,Z$ to X follows from Proposition 4.2.2.18.

Now define h as in (4.12) and (4.13). Then h is extended from $Z \cup \mathrm{d}\,Z$ as in Proposition 4.2.2.18, so Proposition 4.2.2.18 show that (4.10) holds. To show that (4.11) holds, we start by noting that (4.11) holds on elements of $Z \cup \mathrm{d}\,Z$. For elements of Z this is by construction, and for $\mathrm{d}\,Z$ this is shown by the following small calculation, where $z \in Z$.

$$h(\mathrm{d}(\mathrm{d}\,z)) = h(0) = 0 = \mathrm{d}(\mathrm{d}(h(z))) = -\,\mathrm{d}(h(\mathrm{d}(z)))$$

As both sides of (4.11) are k-linear, it suffices to show (4.11) on the k-basis given by words in $Z \cup \mathrm{d}\, Z$. By what we just argued (4.11) holds on words of length 1, and as $\mathrm{d}(1) = 0$ and $h(1) = 0$ by the respective Leibniz rules we also have that (4.11) holds for words of length 0. We now show that (4.11) holds for words of length greater than 1 by induction. So let z and z' be elements of Z such that (4.11) holds on them. Then we have to show that (4.11) also holds for $z \cdot z'$, which we do with the following calculation, using the Leibniz rule for d as well as the Leibniz rule for h (i. e. (4.10)), which we already showed.

$$\begin{aligned}
&h(\mathrm{d}(z \cdot z')) \\
&= h\Big(\mathrm{d}(z) \cdot z' + (-1)^{\deg_{\mathsf{Ch}}(z)} z \cdot \mathrm{d}(z')\Big) \\
&= h(\mathrm{d}(z)) \cdot g(z') + (-1)^{\deg_{\mathsf{Ch}}(\mathrm{d}(z))} \cdot f(\mathrm{d}(z)) \cdot h(z') \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot h(z) \cdot g(\mathrm{d}(z')) + (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot f(z) \cdot h(\mathrm{d}(z')) \\
&= -\mathrm{d}(h(z)) \cdot g(z') - (-1)^{\deg_{\mathsf{Ch}}(z)}\, \mathrm{d}(f(z)) \cdot h(z') \\
&\quad - (-1)^{\deg_{\mathsf{Ch}}(h(z))} h(z) \cdot \mathrm{d}(g(z')) - (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot (-1)^{\deg_{\mathsf{Ch}}(z)} f(z) \cdot \mathrm{d}(h(z')) \\
&= -\mathrm{d}(h(z)) \cdot g(z') - (-1)^{\deg_{\mathsf{Ch}}(h(z))} h(z) \cdot \mathrm{d}(g(z')) \\
&\quad - (-1)^{\deg_{\mathsf{Ch}}(z)}\, \mathrm{d}(f(z)) \cdot h(z') - (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot (-1)^{\deg_{\mathsf{Ch}}(z)} f(z) \cdot \mathrm{d}(h(z')) \\
&= -\mathrm{d}(h(z) \cdot g(z')) - (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot \mathrm{d}(f(z) \cdot h(z')) \\
&= -\mathrm{d}\Big(h(z) \cdot g(z') + (-1)^{\deg_{\mathsf{Ch}}(z)} f(z) \cdot h(z')\Big) \\
&= -\mathrm{d}(h(z \cdot z'))
\end{aligned}$$

It remains to show the assertion concerning (4.14). So assume that (4.14) holds on elements of Z. Then we first show that (4.14) also holds on elements of $\mathrm{d}\, Z$. Indeed, the following calculation verifies (4.14) for $\mathrm{d}\, z$ if z is an element of Z, where we use the compatibility of all the involved morphisms and operators with d.

$$\begin{aligned}
\partial(h(\mathrm{d}\, z)) + h(\partial(\mathrm{d}\, z)) &= -\partial(\mathrm{d}(h(z))) - h(\mathrm{d}(\partial(z))) = \mathrm{d}(\partial(h(z))) + \mathrm{d}(h(\partial(z))) \\
&= \mathrm{d}((\partial \circ h + h \circ \partial)(z)) = \mathrm{d}(f(z) - g(z)) \\
&= f(\mathrm{d}(z)) - g(\mathrm{d}(z))
\end{aligned}$$

Now that we know that (4.14) is satisfied on all of $Z \cup \mathrm{d}\, Z$ it immediately follows from Proposition 4.2.2.18 that (4.14) already holds on all of Z. □

4.2.3 Strongly homotopy linear morphisms of strict mixed complexes

Let X and Y be strict mixed complexes and $f\colon X \to Y$ a morphism of the underlying chain complexes. We might then want to lift f to a morphism of strict mixed complexes, which is possible if and only if f commutes with the differential d, or equivalently if $f \circ d - d \circ f$ is zero. In practice it may however happen that f only commutes with d up to homotopy rather than strictly. In this case $f \circ d - d \circ f$ is nullhomotopic, but not zero, and we could record this by letting $f^{(1)}$ be a nullhomotopy[19] of $f \circ d - d \circ f$. We can now ask whether this additional data $f^{(1)}$ commutes with d. Again, this may only be the case up to a homotopy $f^{(2)}$. If we keep going in this manner we arrive at the notion of a *strongly homotopy linear morphism* of strict mixed complexes. We will give a full definition in Section 4.2.3.1.

To relate the notion of strongly homotopy linear morphisms with the homotopy theory of strict mixed complexes as developed in Section 4.2.2, we are then going to show in Section 4.2.3.2 that a strongly homotopy linear morphism $f\colon X \to Y$ corresponds to a (strict) morphism $f^{\mathrm{strict}}\colon X \to Y^{\mathrm{shl}}$ of strict mixed complexes, where Y^{shl} is a thickened version of Y coming with a quasiisomorphism of strict mixed complexes $Y \to Y^{\mathrm{shl}}$. We can thus interpret the strongly homotopy linear morphism f as encoding a zigzag as depicted below.

$$\begin{tikzcd} X \arrow[r, "f^{\mathrm{strict}}"'] \arrow[rr, bend left, dashed, "f"] & Y^{\mathrm{shl}} & Y \arrow[l, "\simeq"'] \end{tikzcd}$$

4.2.3.1 Definition of strongly homotopy linear morphisms

Below we record the definition of strongly homotopy linear morphisms that was sketched in the introduction to Section 4.2.3.

Definition 4.2.3.1 ([Kas87, 2.2] and [Lod98, 2.5.14])**.** Let X and Y be strict mixed complexes. A strongly homotopy linear morphism from X to Y consists of morphisms of graded k-modules $f^{(i)}\colon X \to Y$ of degree $2i$ for all $i \geq 0$, satisfying

$$\partial \circ f^{(i)} - f^{(i)} \circ \partial = f^{(i-1)} \circ \mathrm{d} - \mathrm{d} \circ f^{(i-1)} \tag{4.15}$$

where we set $f^{(-1)} = 0$. Note that the condition for $i = 0$ implies that $\partial \circ f^{(0)} = f^{(0)} \circ \partial$, so that $f^{(0)}$ is a morphism of chain complexes. ♢

Remark 4.2.3.2. We can compose strongly homotopy linear morphisms with (strict) morphisms of strict mixed complexes. To be more concrete, let X and Y be strict mixed complexes, $g^{(\bullet)}\colon X \to Y$ a strongly homotopy linear morphism, and $f\colon X' \to X$ and $h\colon Y \to Y'$ morphisms of strict mixed complexes. Then we make the following definition.

$$(hgf)^{(i)} := h \circ g^{(i)} \circ f \qquad \text{for } i \geq 0$$

[19] As $f \circ d - d \circ f$ is a morphism of odd degree, this would take the form $\partial f^{(1)} - f^{(1)} \partial = fd - df$, compare with Definition 4.1.2.1.

This defines a strongly homotopy linear morphism hgf from X' to Y', whose underlying morphism of chain complexes is the composition of underlying morphisms of chain complexes. That hgf really is a strongly homotopy linear morphism can be easily checked using that f and h commute with both ∂ and d, as seen below.

$$\begin{aligned}\partial(hgf)^{(i)} - (hgf)^{(i)}\partial &= \partial h g^{(i)} f - h g^{(i)} f \partial \\ &= h\Big(\partial g^{(i)} - g^{(i)}\partial\Big) f \\ &= h\Big(g^{(i-1)}\mathrm{d} - \mathrm{d}g^{(i-1)}\Big) f \\ &= hg^{(i-1)} f\mathrm{d} - \mathrm{d}hg^{(i-1)} f \\ &= (hgf)^{(i-1)}\mathrm{d} - \mathrm{d}(hgf)^{(i-1)} \qquad \diamondsuit\end{aligned}$$

4.2.3.2 Strongly homotopy linear morphisms as zigzags

We begin this section with the construction of the strict mixed complex Y^{shl} that was mentioned in the introduction to Section 4.2.3, before explaining how to reinterpret a strongly homotopy linear morphism $f\colon X \to Y$ as a morphism of strict mixed complexes $f^{\mathrm{strict}}\colon X \to Y^{\mathrm{shl}}$.

Definition 4.2.3.3. Let Y be a strict mixed complex. Then define Y^{shl} to be the $\mathbb{Z}$-graded k-module

$$Y^{\mathrm{shl}}_n \coloneqq \prod_{m\geq 0} Y[-m]$$

so that $Y^{\mathrm{shl}}_n = \prod_{m\geq n} Y_m$ for any integer n. We furthermore define operators ∂ and d of degrees -1 and 1 on Y^{shl} as follows, where $(y_n, y_{n+1}, \ldots)$ is an element of Y^{shl}_n and e.g. $\partial(y_n, y_{n+1}, \ldots)_m$ refers to the Y_m-component of Y^{shl}_{n-1}.

$$\partial(y_n, y_{n+1}, \ldots)_{n-1+i} \coloneqq \begin{cases} \partial(y_n) & \text{if } i = 0 \\ -\partial(y_{n+i}) & \text{if } i > 0 \text{ is odd} \\ \partial(y_{n+i}) - y_{n-1+i} & \text{if } i > 0 \text{ is even}\end{cases}$$

$$\mathrm{d}(y_n, y_{n+1}, \ldots)_{n+1+i} \coloneqq \begin{cases} -\,\mathrm{d}(y_{n+i}) & \text{if } i \geq 0 \text{ is odd} \\ \mathrm{d}(y_{n+i}) + y_{n+1+i} & \text{if } i \geq 0 \text{ is even}\end{cases}$$

The special case for $i = 0$ in the formula for ∂ can be avoided by declaring y_{n-1} to be 0.

Finally, we let $\iota^{\mathrm{shl}}_Y\colon Y \to Y^{\mathrm{shl}}$ be the morphism of $\mathbb{Z}$-graded k-modules that is given by $\iota^{\mathrm{shl}}_Y(y) \coloneqq (y, 0, 0, \ldots)$ for every element y of Y. $\diamondsuit$

Remark 4.2.3.4. The following diagram[20] depicts how one can think of Y^{shl}. The picture only shows part of Y^{shl}, which continues towards the right, top, and bottom, but not towards the left.

$$\begin{tikzcd}
Y_{n+1} \arrow[d, shift right, "\partial"'] & Y_{n+2} \arrow[d, shift left, "-\partial"] \arrow[dr, "-\mathrm{id}"'] & Y_{n+3} \arrow[d, shift right, "\partial"'] & Y_{n+4} \arrow[d, shift left, "-\partial"] \arrow[dr, "-\mathrm{id}"'] & Y_{n+5} \arrow[d, shift right, "\partial"'] \\
Y_n \arrow[u, squiggly, shift right, "\mathrm{d}"'] \arrow[d, shift right, "\partial"'] & Y_{n+1} \arrow[u, squiggly, shift left, "-\mathrm{d}"] \arrow[ul, squiggly, "\mathrm{id}"'] \arrow[d, shift left, "-\partial"] \arrow[dr, "-\mathrm{id}"'] & Y_{n+2} \arrow[u, squiggly, shift right, "\mathrm{d}"'] \arrow[d, shift right, "\partial"'] & Y_{n+3} \arrow[u, squiggly, shift left, "-\mathrm{d}"] \arrow[ul, squiggly, "\mathrm{id}"'] \arrow[d, shift left, "-\partial"] \arrow[dr, "-\mathrm{id}"'] & Y_{n+4} \arrow[u, squiggly, shift right, "\mathrm{d}"'] \arrow[d, shift right, "\partial"'] \\
Y_{n-1} \arrow[u, squiggly, shift right, "\mathrm{d}"'] & Y_n \arrow[u, squiggly, shift left, "-\mathrm{d}"] \arrow[ul, squiggly, "\mathrm{id}"'] & Y_{n+1} \arrow[u, squiggly, shift right, "\mathrm{d}"'] & Y_{n+2} \arrow[u, squiggly, shift left, "-\mathrm{d}"] \arrow[ul, squiggly, "\mathrm{id}"'] & Y_{n+3} \arrow[u, squiggly, shift right, "\mathrm{d}"']
\end{tikzcd}$$

♢

Proposition 4.2.3.5. *Let Y be a strict mixed complex and Y^{shl} as in Definition 4.2.3.3. Then ∂ and d as defined in Definition 4.2.3.3 define a strict mixed complex structure on Y^{shl} which makes $\iota_Y^{\mathrm{shl}}\colon Y \to Y^{\mathrm{shl}}$ into a quasiisomorphism of strict mixed complexes.* ♡

Proof. We begin by showing that ∂ and d upgrade Y^{shl} to a strict mixed complex. It is easiest to convince oneself of this by considering the diagram in Remark 4.2.3.4, but we also provide a proof by unpacking the formulas. So let $(y_n, y_{n+1}, \ldots)$ be an element of Y_n^{shl}. Then we obtain the following calculations, first for odd i and then for even i[21], showing that ∂ squares to zero.

$$\begin{aligned}
&\partial(\partial((y_n, y_{n+1}, \ldots)))_{n-2+i} && \text{(assuming } i \text{ is odd)}\\
&= -\partial\big(\partial((y_n, y_{n+1}, \ldots))_{n-1+i}\big)\\
&= -\partial(-\partial(y_{n+i}))\\
&= \partial(\partial(y_{n+i}))\\
&= 0
\end{aligned}$$

$$\begin{aligned}
&\partial(\partial((y_n, y_{n+1}, \ldots)))_{n-2+i} && \text{(assuming } i \text{ is even)}\\
&= \partial\big(\partial((y_n, y_{n+1}, \ldots))_{n-1+i}\big) - \partial((y_n, y_{n+1}, \ldots))_{n-2+i}\\
&= \partial(\partial(y_{n+i}) - y_{n-1+i}) + \partial(y_{n-1+i})\\
&= 0 - \partial(y_{n-1+i}) + \partial(y_{n-1+i})\\
&= 0
\end{aligned}$$

[20] This diagram uses some of the pictorial elements from Convention 4.2.1.7, but is only meant to help with intuition rather than as a precise depiction of an isomorphism class of strict mixed complexes. For example Y^{shl} is the product of the rows, whereas interpreting the picture while following Convention 4.2.1.7 too closely would suggest taking the sum.

[21] In the case of $i = 0$ we set $y_{n-1} = 0$ so that we can use the same formulas as for even $i > 0$.

The proof that d squares to 0 is completely analogous. Similarly, the following calculation shows $\partial\mathrm{d} + \mathrm{d}\partial = 0$.

$$\begin{aligned}
&\big((\partial\mathrm{d} + \mathrm{d}\partial)((y_n, y_{n+1}, \ldots))\big)_{n+i} \qquad\qquad \text{(assuming } i \text{ is odd)}\\
&= -\partial\big(\mathrm{d}((y_n, y_{n+1}, \ldots))_{n+1+i}\big) - \mathrm{d}\big(\partial((y_n, y_{n+1}, \ldots))_{n-1+i}\big)\\
&= \partial(\mathrm{d}(y_{n+i})) + \mathrm{d}(\partial(y_{n+i}))\\
&= (\partial\mathrm{d} + \mathrm{d}\partial)(y_{n+i})\\
&= 0
\end{aligned}$$

$$\begin{aligned}
&\big((\partial\mathrm{d} + \mathrm{d}\partial)((y_n, y_{n+1}, \ldots))\big)_{n+i} \qquad\qquad \text{(assuming } i \text{ is even)}\\
&= \partial\big(\mathrm{d}((y_n, y_{n+1}, \ldots))_{n+1+i}\big) - \mathrm{d}((y_n, y_{n+1}, \ldots))_{n+i}\\
&\quad + \mathrm{d}\big(\partial((y_n, y_{n+1}, \ldots))_{n-1+i}\big) + \partial((y_n, y_{n+1}, \ldots))_{n+i}\\
&= \partial(\mathrm{d}(y_{n+i}) + y_{n+1+i}) + \mathrm{d}(y_{n-1+i})\\
&\quad + \mathrm{d}(\partial(y_{n+i}) - y_{n-1+i}) - \partial(y_{n+1+i})\\
&= (\partial\mathrm{d} + \mathrm{d}\partial)(y_{n+i}) + \partial(y_{n+1+i}) + \mathrm{d}(y_{n-1+i})\\
&\quad - \mathrm{d}(y_{n-1+i}) - \partial(y_{n+1+i})\\
&= 0
\end{aligned}$$

It remains to show that $\iota_Y^{\mathrm{shl}}\colon Y \to Y^{\mathrm{shl}}$ is a morphism of strict mixed complexes as well as a quasiisomorphism. That ι_Y^{shl} is compatible with the boundary operator and differential is clear from the formulas. It thus remains to show that it is a quasiisomorphism. For this, let $Y^{\mathrm{shl},i}$ for $i \geq 1$ be the sub-$\mathbb{Z}$-graded k-module of Y^{shl} given by the factor $Y[-(2i-1)] \times Y[-2i]$. If we let $Y^{\mathrm{shl},0}$ be the first factor of Y^{shl}, i. e. $Y^{\mathrm{shl},0} = Y$, then we obtain a product decomposition

$$Y^{\mathrm{shl}} \cong \prod_{i \geq 0} Y^{\mathrm{shl},i}$$

as $\mathbb{Z}$-graded k-modules. It is immediate from the formulas for the boundary operator that each $Y^{\mathrm{shl},i}$ is closed under ∂, making this also product decomposition considered as chain complexes. As ι_Y^{shl} is the inclusion of the first factor it thus remains to show that for each $i \geq 1$ the chain complex $Y^{\mathrm{shl},i}$ is acyclic. To do so, we define a contracting homotopy as follows.

$$\begin{aligned}
h\colon Y_n^{\mathrm{shl},i} = Y_{n+2i-1} \oplus Y_{n+2i} &\to Y_{n+1}^{\mathrm{shl},i} = Y_{n+2i} \oplus Y_{n+2i+1}\\
(y_{n+2i-1}, y_{n+2i}) &\mapsto (-y_{n+2i}, 0)
\end{aligned}$$

The following calculations shows that h is a contracting homotopy of $Y^{\mathrm{shl},i}$, where (y_{n+2i-1}, y_{n+2i}) is an element of $Y_n^{\mathrm{shl},i}$.

$$\begin{aligned}
&(\partial h + h\partial)((y_{n+2i-1}, y_{n+2i}))\\
&= \partial((-y_{n+2i}, 0)) + h((-\partial(y_{n+2i-1}), \partial(y_{n+2i}) - y_{n+2i-1}))\\
&= (-\partial(-y_{n+2i}), 0 - (-y_{n+2i})) + (-\partial(y_{n+2i}) + y_{n+2i-1}, 0)
\end{aligned}$$

$$
\begin{aligned}
&= (\partial(y_{n+2i}) - \partial(y_{n+2i}) + y_{n+2i-1}, y_{n+2i}) \\
&= (y_{n+2i-1}, y_{n+2i})
\end{aligned}
$$

The following diagram depicts the situation for $i = 1$ diagrammatically as in Remark 4.2.3.4, with the contracting homotopy h indicated with the dashed blue arrow.

$$
\begin{tikzcd}
Y_{n+2} \arrow[d, "-\partial"] \arrow[dr, "-\mathrm{id}"'] & Y_{n+3} \arrow[d, "\partial"'] \\
Y_{n+1} \arrow[d, "-\partial"] \arrow[dr, "-\mathrm{id}"'] & Y_{n+2} \arrow[ul, dashed, "-\mathrm{id}"'] \\
Y_n & Y_{n+1} \arrow[ul, dashed, "-\mathrm{id}"']
\end{tikzcd}
$$

□

The proof of Proposition 4.2.3.5 shows that ι_Y^{shl} has a retraction given by the projection to the first factor, but only as chain complexes. While the projection to the first factor is not compatible with the differential, it can however be upgraded to a strongly homotopy linear morphism, as we will explain next.

Proposition 4.2.3.6. *Let Y be a strict mixed complex. Define $(p_Y^{\mathrm{shl}})^{(i)}$ for each $i \geq 0$ to be the morphisms of $\mathbb{Z}$-graded k-modules from Y^{shl} to Y of degree $2i$ that is the projection to the $2i$-th factor, i. e. is defined as follows.*

$$
(p_Y^{\mathrm{shl}})_n^{(i)} \colon Y_n^{\mathrm{shl}} \to Y_n, \qquad (y_n, y_{n+1}, y_{n+2}, \ldots) \mapsto y_{n+2i}
$$

Then this makes p_Y^{shl} into a strongly homotopy linear morphism from Y^{shl} to Y. Furthermore, the underlying morphism of chain complexes of p_Y^{shl} is a quasiisomorphism. ♡

Proof. That $(p_Y^{\mathrm{shl}})^{(0)}$ is a morphism of chain complexes is clear. As $(p_Y^{\mathrm{shl}})^{(0)}$ is a left inverse of ι_Y^{shl}, it also follows immediately from ι_Y^{shl} being a quasi-isomorphism by Proposition 4.2.3.5 that $(p_Y^{\mathrm{shl}})^{(0)}$ is a quasiisomorphism as well.

It remains to show that the compatibility relations required of $(p_Y^{\mathrm{shl}})^{(i)}$ for $i \geq 0$ in order to make p_Y^{shl} into a strongly homotopy linear morphism are satisfied. So let $i \geq 1$ be an integer and $(y_n, y_{n+1}, \ldots)$ an element of Y_n^{shl}. Then the following calculations show the claim.

$$
\begin{aligned}
&\Big(\partial \circ (p_Y^{\mathrm{shl}})^{(i)} - (p_Y^{\mathrm{shl}})^{(i)} \circ \partial\Big)((y_n, y_{n+1}, y_{n+2}, \ldots)) \\
&= \partial(y_{n+2i}) - \partial((y_n, y_{n+1}, y_{n+2}, \ldots))_{n-1+2i} \\
&= \partial(y_{n+2i}) - (\partial(y_{n+2i}) - y_{n-1+2i})
\end{aligned}
$$

$$= y_{n-1+2i}$$

$$\begin{aligned}
&\Big((p_Y^{\mathrm{shl}})^{(i-1)} \circ \mathrm{d} - \mathrm{d} \circ (p_Y^{\mathrm{shl}})^{(i-1)}\Big)((y_n, y_{n+1}, y_{n+2}, \ldots))\\
&= \mathrm{d}((y_n, y_{n+1}, y_{n+2}, \ldots))_{n+1+2i-2} - \mathrm{d}(y_{n+2i-2})\\
&= (\mathrm{d}(y_{n+2i-2}) + y_{n+1+2i-2}) - \mathrm{d}(y_{n+2i-2})\\
&= y_{n+1+2i-2} = y_{n-1+2i}
\end{aligned}$$

□

The relevance of Y^{shl} and p_Y^{shl} stems from the fact that p_Y^{shl} is the *universal* strongly homotopy linear morphism to Y; we show next that any other strongly homotopy morphism with codomain Y factors uniquely as the composition of a (strict) morphism of strict mixed complexes to Y^{shl} with p_Y^{shl}.

Proposition 4.2.3.7. *Let X and Y be strict mixed complexes and $f\colon Y \to Y$ a strongly homotopy linear morphism. Then there is a unique morphism of strict mixed complexes $g\colon X \to Y^{\mathrm{shl}}$ such that $f = p_Y^{\mathrm{shl}} \circ g$[22].* ♡

Proof. We first show existence. Define a morphism of $\mathbb{Z}$-graded k-modules g as

$$g\colon X \to Y^{\mathrm{shl}} = \prod_{m\geq 0} Y[-m]$$

$$g(x)_{n+2i} = f^{(i)}(x)$$

$$g(x)_{n+2i+1} = \Big(f^{(i)}\mathrm{d} - \mathrm{d}f^{(i)}\Big)(x) = \Big(\partial f^{(i+1)} - f^{(i+1)}\partial\Big)(x)$$

for $i \geq 0$ and x elements of X_n, and where $g(x)_{n+m}$ refers to the component in Y_{n+m}. As $\big(p_Y^{\mathrm{shl}}\big)^{(i)}$ is projection to the $2i$-th factor, it is clear that f is the composition $p_Y^{\mathrm{shl}} \circ g$, so it only remains to show that g is a morphism of strict mixed complexes. This is proven by the following calculations, where $i \geq 0$ and x is an element of X_n.

$$\begin{aligned}
&(\partial g - g\partial)(x)_{n-1+2i}\\
&= \partial(g(x))_{n-1+2i} - f^{(i)}(\partial(x))\\
&= \partial(g(x)_{n+2i}) - g(x)_{n-1+2i} - f^{(i)}(\partial(x))\\
&= \partial\Big(f^{(i)}(x)\Big) - \Big(\partial f^{(i)} - f^{(i)}\partial\Big)(x) - f^{(i)}(\partial(x))\\
&= 0
\end{aligned}$$

This shows what is needed for g to be a morphism of chain complexes for only the even components, now we check the odd components.

$$(\partial g - g\partial)(x)_{n+2i}$$

[22] See Remark 4.2.3.2 for the composition of a strongly homotopy linear morphism with a morphism of strict mixed complexes.

$$
\begin{aligned}
&= -\partial(g(x)_{n+2i+1}) - \Big(\partial f^{(i+1)} - f^{(i+1)}\partial\Big)(\partial(x))\\
&= -\partial\Big(\Big(\partial f^{(i+1)} - f^{(i+1)}\partial\Big)(x)\Big) - \Big(\partial f^{(i+1)} - f^{(i+1)}\partial\Big)(\partial(x))\\
&= \Big(-\partial\partial f^{(i+1)} + \partial f^{(i+1)}\partial - \partial f^{(i+1)}\partial + f^{(i+1)}\partial\partial\Big)(x)\\
&= 0
\end{aligned}
$$

Next we verify that g commutes with d, beginning with the even components.

$$
\begin{aligned}
&(\mathrm{d}g - g\mathrm{d})(x)_{n+1+2i}\\
&= \mathrm{d}(g(x)_{n+2i}) + g(x)_{n+1+2i} - f^{(i)}(\mathrm{d}(x))\\
&= \mathrm{d}\Big(f^{(i)}(x)\Big) + \Big(f^{(i)}\mathrm{d} - \mathrm{d}f^{(i)}\Big)(x) - f^{(i)}(\mathrm{d}(x))\\
&= 0
\end{aligned}
$$

Finally, we check compatibility with d on odd components.

$$
\begin{aligned}
&(\mathrm{d}g - g\mathrm{d})(x)_{n+2+2i}\\
&= -\mathrm{d}(g(x)_{n+1+2i}) - \Big(f^{(i)}\mathrm{d} - \mathrm{d}f^{(i)}\Big)(\mathrm{d}(x))\\
&= -\mathrm{d}\Big(\Big(f^{(i)}\mathrm{d} - \mathrm{d}f^{(i)}\Big)(x)\Big) - \Big(f^{(i)}\mathrm{d} - \mathrm{d}f^{(i)}\Big)(\mathrm{d}(x))\\
&= \Big(-\mathrm{d}f^{(i)}\mathrm{d} + \mathrm{d}\mathrm{d}f^{(i)} - f^{(i)}\mathrm{d}\mathrm{d} + \mathrm{d}f^{(i)}\mathrm{d}\Big)(x)\\
&= 0
\end{aligned}
$$

This shows existence. It remains to show that such a lift g is already uniquely determined by f. So let $g\colon X \to Y^{\mathrm{shl}}$ be any morphism of strict mixed complexes such that $f = p_Y^{\mathrm{shl}} \circ g$. We can immediately read off that the even components must be given by

$$g(x)_{n+2i} = f^{(i)}(x) \qquad \text{for } n \in \mathbb{Z}, i \geq 0 \text{ and } x \in X_n.$$

So now let x be an element of X_n and $i \geq 0$. Then the following calculation, using that g is a morphism of chain complexes, shows that $g(x)_{n+2i+1}$ is also already determined by f.

$$
\begin{aligned}
&g(x)_{n+2i+1}\\
&= \partial\big(g(x)_{n+2i+2}\big) - \big(\partial\big(g(x)_{n+2i+2}\big) - g(x)_{n+2i+1}\big)\\
&= \partial\big(g(x)_{n+2i+2}\big) - \partial(g(x))_{n+2i+1}\\
&= \partial\big(g(x)_{n+2i+2}\big) - g(\partial(x))_{n-1+2i+2}\\
&= \partial\Big(f^{(i+1)}(x)\Big) - f^{(i+1)}(\partial(x))
\end{aligned}
$$

□

Definition 4.2.3.8. Let X and Y be strict mixed complexes and $f\colon X \to Y$ a strongly homotopy linear morphism. Then we denote by f^{strict} the unique morphism of strict mixed complexes $X \to Y^{\mathrm{shl}}$ lifting f as in Proposition 4.2.3.7. The assignment $f \mapsto f^{\mathrm{strict}}$ defines a bijection from the set of strongly homotopy linear morphisms $X \to Y$ to the set of morphisms of strict mixed complexes $X \to Y^{\mathrm{shl}}$. $\diamondsuit$

4.3 The derived category of k

The derived category of k is an ∞-category $\mathcal{D}(k)$ that can be constructed by inverting the quasiisomorphisms in the category $\mathsf{Ch}(k)$ of chain complexes of (ordinary) k-modules. In this section we discuss $\mathcal{D}(k)$ and record the main properties that we will need later – most of them are proven in various places in [HA].

We begin in Section 4.3.1 by proving some useful statements concerning semiadditive ∞-categories, which we will need in Section 4.3.2, where we will collect the main properties of $\mathcal{D}(k)$. We finish this section with Section 4.3.4, where we state some properties of the truncation functors on $\mathcal{D}(k)$ that we will need in Chapter 5.

4.3.1 Semiadditive ∞-categories

In this section we prove some small helpful results regarding semiadditive ∞-categories that we will need in Section 4.3.2.

Proposition 4.3.1.1. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category such that the underlying ∞-category $\mathcal{C}$ is semiadditive ∞-category*[23]*. Then $\mathcal{C}^{\otimes}$ is cartesian if and only if it is cocartesian.* $\heartsuit$

Proof. The property of symmetric monoidal structures being (co)cartesian is defined in [HA, 2.4.0.1]. The symmetric monoidal structure $\mathcal{C}^{\otimes}$ is cartesian if the unit object $\mathbb{1}_{\mathcal{C}}$ is final and if for every pair of objects X and Y of $\mathcal{C}$ the morphisms

$$X \simeq X \otimes \mathbb{1}_{\mathcal{C}} \leftarrow X \otimes Y \to \mathbb{1}_{\mathcal{C}} \otimes Y \simeq Y$$

induced by the essentially unique morphisms $X \to \mathbb{1}_{\mathcal{C}}$ and $Y \to \mathbb{1}_{\mathcal{C}}$ exhibit $X \otimes Y$ as a product of X and Y.

Analogously, for $\mathcal{C}^{\otimes}$ being cocartesian the unit object must be initial, and the analogously defined morphisms

$$X \simeq X \otimes \mathbb{1}_{\mathcal{C}} \to X \otimes Y \leftarrow \mathbb{1}_{\mathcal{C}} \otimes Y \simeq Y$$

must exhibit $X \otimes Y$ as a coproduct of X and Y.

[23] By this we mean that $\mathcal{C}$ admits finite products and finite coproducts and has the following two properties. Firstly, the (essentially unique) morphism from an initial object to a final object must be an equivalence (i. e. $\mathcal{C}$ has *zero objects*). Secondly, for any two objects X and Y of $\mathcal{C}$ the morphism

$$X \amalg Y \xrightarrow{\left[\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right]} X \times Y$$

must be an equivalence (i. e. $\mathcal{C}$ has *biproducts*).

As $\mathcal{C}$ is assumed to be semiadditive, every initial object is automatically final as well, and every final object is automatically initial, which shows equivalence of the first part of the respective definitions. For the second part, let X and Y be two objects of $\mathcal{C}$. Note that the compositions

$$X \otimes \mathbb{1}_{\mathcal{C}} \to X \otimes Y \to X \otimes \mathbb{1}_{\mathcal{C}}$$

and

$$\mathbb{1}_{\mathcal{C}} \otimes Y \to X \otimes Y \to \mathbb{1}_{\mathcal{C}} \otimes Y$$

are, by functoriality of the tensor product, homotopic to the identity. Functoriality also implies that the following square commutes

$$\begin{tikzcd}
X \otimes \mathbb{1}_{\mathcal{C}} \arrow[r] \arrow[d] & X \otimes Y \arrow[d] \\
\mathbb{1}_{\mathcal{C}} \otimes \mathbb{1}_{\mathcal{C}} \arrow[r] & \mathbb{1}_{\mathcal{C}} \otimes Y
\end{tikzcd}$$

which shows that the composition

$$X \otimes \mathbb{1}_{\mathcal{C}} \to X \otimes Y \to \mathbb{1}_{\mathcal{C}} \otimes Y$$

and analogously

$$\mathbb{1}_{\mathcal{C}} \otimes Y \to X \otimes Y \to X \otimes \mathbb{1}_{\mathcal{C}}$$

are zero morphisms. We can conclude that the following triangle commutes.

$$\begin{tikzcd}
X \amalg Y \arrow[rr, "\begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix}"] \arrow[dr] & & X \times Y \\
& X \otimes Y \arrow[ur] &
\end{tikzcd}$$

The second condition for $\mathcal{C}^{\otimes}$ being (co)cartesian is that the morphism on the right (left) is an equivalence for every X and Y. As the horizontal morphism is an equivalence by virtue of $\mathcal{C}$ being semiadditive, it follows that those two conditions are equivalent. $\square$

Proposition 4.3.1.2. *Let $\mathcal{C}$ be a semiadditive ∞-category, let $\mathcal{D}$ be an ∞-category admitting finite products, and let F_1 and F_2 be two functors*

$$F_1, F_2 \colon \mathcal{C} \to \mathrm{CMon}(\mathcal{D})$$

such that F_1 preserves products.

Denote the forgetful functor $\mathrm{CMon}(\mathcal{D}) \to \mathcal{D}$ by V and assume that $V \circ F_1$ is naturally equivalent to $V \circ F_2$. Then there is also a natural equivalence between F_1 and F_2. $\heartsuit$

Proof. As $\mathcal{D}$ has finite products we can upgrade $\mathcal{D}$ to a symmetric monoidal ∞-category with respect to the cartesian symmetric monoidal structure $\mathcal{D}^{\times}$ (see [HA, 2.4.1.5]). Applying [HA, 2.4.1.5 (5) and 2.4.2.5] we obtain an equivalence of ∞-categories

$$\mathrm{CMon}(\mathcal{D}) \simeq \mathrm{CAlg}(\mathcal{D})$$

which is compatible with the respective forgetful functors to $\mathcal{D}$. Denote the composite of F_i with this equivalence by F_i'. It suffices to show that F_1' is naturally equivalent to F_2'.

Note that as V detects products by Proposition F.2.0.1 the equivalence $V \circ F_1 \simeq V \circ F_2$ and F_1 preserving products implies that F_2 preserves products as well. Hence both F_1' and F_2' preserve products too, so they induce symmetric monoidal functors as follows (see [HA, 2.4.1.8]).

$${F_i'}^{\times} \colon \mathcal{C}^{\times} \to \mathrm{CAlg}(\mathcal{D})^{\times}$$

We obtain the following commutative diagram for $i = 1$ and $i = 2$

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{C}) \arrow[r, "\mathrm{CAlg}(F_i')"] \arrow[d, "U_{\mathcal{C}}"'] & \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{D})) \arrow[d, "U_{\mathrm{CAlg}(\mathcal{D})}"] \\
\mathcal{C} \arrow[r, "F_i'"'] & \mathrm{CAlg}(\mathcal{D})
\end{tikzcd}$$

where the vertical functors are the forgetful functors forgetting the "outer" algebra structure. By Proposition 4.3.1.1, the cartesian symmetric monoidal structure $\mathcal{C}^{\times}$ is also cocartesian, so it follows from [HA, 2.4.3.9] that $U_{\mathcal{C}}$ is an equivalence. It thus suffices to show that $U_{\mathrm{CAlg}(\mathcal{D})} \circ \mathrm{CAlg}(F_1')$ is homotopic to $U_{\mathrm{CAlg}(\mathcal{D})} \circ \mathrm{CAlg}(F_2')$.

The symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{D})$ used in forming the ∞-category $\mathrm{CAlg}(\mathrm{CAlg}(\mathcal{D}))$ in the above diagram is the cartesian one $\mathrm{CAlg}(\mathcal{D})^{\times}$. There is also a symmetric monoidal structure induced by $\mathcal{D}^{\times}$ on $\mathrm{CAlg}(\mathcal{D})$, which we denote by $\mathrm{CAlg}(\mathcal{D})^{\otimes}$, see Propositions E.4.2.3 and E.6.0.1. By Proposition F.3.0.2 in combination with [HA, 2.4.1.7] and [HA, 2.4.2.5], there is a symmetric monoidal equivalence $\mathrm{CAlg}(\mathcal{D})^{\otimes} \simeq \mathrm{CAlg}(\mathcal{D})^{\times}$ whose underlying functor of ∞-categories is the identity. We can thus replace $\mathrm{CAlg}(\mathcal{D})^{\times}$ implicitly used in $\mathrm{CAlg}(\mathrm{CAlg}(\mathcal{D}))$ with $\mathrm{CAlg}(\mathcal{D})^{\otimes}$.

By Proposition E.6.0.1 there is then a natural equivalence between $U_{\mathrm{CAlg}(\mathcal{D})}$ and $\mathrm{CAlg}(U_{\mathcal{D}})$, where $U_{\mathcal{D}} \colon \mathrm{CAlg}(\mathcal{D}) \to \mathcal{D}$ is the forgetful functor. We obtain

$$\begin{aligned}
U_{\mathrm{CAlg}(\mathcal{D})} \circ \mathrm{CAlg}(F_i') &\simeq \mathrm{CAlg}(U_{\mathcal{D}}) \circ \mathrm{CAlg}(F_i') \\
&\simeq \mathrm{CAlg}(U_{\mathcal{D}} \circ F_i') \\
&\simeq \mathrm{CAlg}(V \circ F_i)
\end{aligned}$$

so as $V \circ F_1 \simeq V \circ F_2$ by assumption we conclude

$$U_{\mathrm{CAlg}(\mathcal{D})} \circ \mathrm{CAlg}(F_2') \simeq U_{\mathrm{CAlg}(\mathcal{D})} \circ \mathrm{CAlg}(F_2')$$

which is what we needed to show. □

4.3.2 Properties of $\mathcal{D}(k)$

Proposition 4.3.2.1. *The following hold.*

(1) *$\mathcal{D}(k)$ is*[24] *the presentable symmetric monoidal ∞-category underlying the combinatorial and symmetric monoidal model category $\mathsf{Ch}(k)$ carrying the projective model structure from Fact 4.1.3.1.*

We will denote the symmetric monoidal functor $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ by γ. We will also sometimes denote the composition of γ with the cofibrant replacement functor $\mathsf{Ch}(k) \to \mathsf{Ch}(k)^{\mathrm{cof}}$ by γ again[25].

(2) *$\mathcal{D}(k)$ is stable.*

(3) *$\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ preserves coproducts.*

(4) *There are natural equivalences for every integer n as follows*[26].

$$\gamma(-)[n] \simeq \gamma(-[n])$$

From now on we will write k for $\gamma(k)$.

(5) *There is a natural isomorphism of functors $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ab}$ as follows.*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}(k[n], \gamma(-)) \cong \mathrm{H}_n(-)$$

(6) *Let $\mathsf{Ch}(k)'_{\geq 0}$ and $\mathsf{Ch}(k)'_{\leq 0}$ be the full subcategories of $\mathsf{Ch}(k)$ spanned by the chain complexes whose homology is concentrated in non-negative and non-positive, respectively, degree. Let $\mathcal{D}(k)_{\geq 0}$ be the essential image of the restriction of γ to $(\mathsf{Ch}(k))'_{\geq 0}$, and analogously for $\mathcal{D}(k)_{\leq 0}$. Then the pair $(\mathcal{D}(k)_{\geq 0}, \mathcal{D}(k)_{\leq 0})$ determines a t-structure on $\mathcal{D}(k)$.*

   Furthermore, $\mathcal{D}(k)_{\geq 0}$ is also the essential image of $\mathsf{Ch}(k)_{\geq 0}$ from Definition 4.1.1.1 and $\mathcal{D}(k)_{\leq 0}$ is the essential image of $\mathsf{Ch}(k)_{\leq 0}$.

(7) *There is a symmetric monoidal equivalence preserving the respective t-structures between $\mathcal{D}(k)$ and the ∞-category of k-modules in spectra $\mathrm{LMod}_k(\mathcal{S}\mathrm{p})$ (where the tensor product is the tensor product over k, see [HA, 4.5], and the t-structure is defined in [HA, 7.1.1.10 and 7.1.1.13]).*

(8) *The t-structure on $\mathcal{D}(k)$ is compatible with the symmetric monoidal structure in the sense of [HA, 2.2.1.3].*

[24] We will take this as the definition for $\mathcal{D}(k)$, but will also also point out in the proof below why other possible definitions used in [HA] are equivalent.

[25] Note that the restriction of this functor to $\mathsf{Ch}(k)^{\mathrm{cof}}$ is homotopic to the original functor γ.

[26] See Definition 4.1.1.2 for a definition of the shift in $\mathsf{Ch}(k)$ and [HA, 1.1.2.7] for a definition of the shift in the stable ∞-category $\mathcal{D}(k)$.

(9) There is a commutative diagram

$$\begin{tikzcd}
\mathrm{LMod}_k(\mathsf{Ab}) \arrow[r, "(-)[0]"] \arrow[d, dashed] & \mathsf{Ch}(k) \arrow[d, "\gamma"] \\
\mathcal{D}(k)^{\heartsuit} \arrow[r] & \mathcal{D}(k)
\end{tikzcd}$$

of ∞-categories, where $\mathcal{D}(k)^{\heartsuit} = \mathcal{D}(k)_{\geq 0} \cap \mathcal{D}(k)_{\leq 0}$ is the heart of $\mathcal{D}(k)$, see [HA, 1.2.1.11], and the lower horizontal functor the inclusion.

Furthermore, the dashed functor is an equivalence. We can thus identify the heart of $\mathcal{D}(k)$ with $\mathrm{LMod}_k(\mathsf{Ab})$. $\heartsuit$

Proof. Proof of Claim (1): The projective model structure on chain complexes with the required properties was discussed in Fact 4.1.3.1. For the construction of $\mathcal{D}(k)$ as the symmetric monoidal ∞-category underlying $\mathsf{Ch}(k)$ see [HA, 7.1.2.12]. The proof that $\mathcal{D}(k)$ is *presentable* symmetric monoidal can be found in the proof of [HA, 7.1.2.13].

Finally, let us note that different ways of constructing $\mathcal{D}(k)$ are used in [HA]. They are however all equivalent by [HA, 7.1.2.9] and [HA, 1.3.5.15][27], so there is no problem in using results concerning $\mathcal{D}(k)$ from different places in [HA].

Proof of Claim (2): This is [HA, 1.3.5.9].

Proof of Claim (3): By (1) and [HA, 1.3.4.25 and 1.3.4.24] this follows from the fact that coproducts of cofibrant chain complexes are already homotopy coproducts[28].

Proof of Claim (4): We start by proving that $\gamma(k[n]) \cong \gamma(k)[n]$. First note that as k is projective as a k-module, the chain complexes $k[n]$ are cofibrant in the projective model structure by [Hov99, 2.3.6]. Now consider the following pushout diagram of cofibrant objects in $\mathsf{Ch}(k)$

$$\begin{tikzcd}
k[n] \arrow[r] \arrow[d] & D^{n+1}(k) \arrow[d] \\
0 \arrow[r] & k[n+1]
\end{tikzcd}$$

where $D^{n+1}(k)$ is the chain complex with $D^{n+1}(k)_m = k$ if $m = n$ or $m = n+1$ and $D^{n+1}(k)_m = 0$ otherwise, and with differential from degree $n+1$ to degree n the identity, and where the morphisms $k[n] \to D^{n+1}(k)$ and $D^{n+1}(k) \to k[n+1]$ are the obvious inclusion and projection. The morphism

[27] The construction of $\mathcal{D}(\mathcal{A})$ considered in [HA, 1.3.5] applies to the case $\mathcal{A} = \mathrm{LMod}_k(\mathsf{Ab})$ (the category of ordinary k-modules), as $\mathrm{LMod}_k(\mathsf{Ab})$ is a Grothendieck abelian category in the sense of [HA, 1.3.5.1].

[28] Unpacking the projective model structure (see [HTT, A.2.8.2]) on $\mathrm{Fun}(\mathsf{J}, \mathsf{Ch}(k))$ for a discrete category J one can easily see that such a functor is cofibrant if and only if it is pointwise cofibrant.

$k[n] \to D^{n+1}(k)$ is a cofibration[29], so it follows from [HTT, A.2.4.4, variant (i)] that this diagram is a homotopy pushout diagram in $\mathsf{Ch}(k)$. Applying [HA, 1.3.4.24] and using that $D^{n+1}(k)$ is acyclic we can conclude that for every integer n there is a pushout diagram in $\mathcal{D}(k)$ of the following form.

$$\begin{tikzcd}
\gamma(k[n]) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \gamma(k[n+1])
\end{tikzcd}$$

Using that $\gamma(k)[0] = \gamma(k) = \gamma(k[0])$ it now follows that $\gamma(k[n]) \cong \gamma(k)[n]$ by inducting up and down[30] from 0.

The general statement now follows by combining that by Remark 4.1.2.2 there is a natural isomorphism

$$-[n] \cong (k \otimes -)[n] \cong k[n] \otimes -$$

of endofunctors of $\mathsf{Ch}(k)^{\mathrm{cof}}$ and that as the tensor product functor of $\mathcal{D}(k)$ preserves colimits in each variable separately, there is also such a natural equivalence of endofunctors of $\mathcal{D}(k)$, with the fact that γ is symmetric monoidal.

Proof of Claim (5): We start by showing that the compositions of the two functors with the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ are naturally equivalent. Applying Proposition A.1.0.1 to $\mathsf{Ch}(k)$, we obtain a natural isomorphism as follows.

$$\mathrm{Mor}_{\mathrm{Ho}\,\mathcal{D}(k)}(\gamma(-), \gamma(-)) \cong \mathrm{Mor}_{\mathrm{Ho}\,\mathsf{Ch}(k)}(-, -)$$

A standard calculation using left homotopies (see [Hov99, 1.2.4 in combination with 1.2.6 and 1.2.10]) shows that[31]

$$\mathrm{Mor}_{\mathrm{Ho}\,\mathsf{Ch}(k)}(k[n], -) \cong \mathrm{H}_n(-) \tag{4.16}$$

so that we have obtained a natural equivalence between the respective compositions with the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$. This forgetful functor factors as the composition of the forgetful functors $\mathsf{Ab} \to \mathrm{CMon}(\mathsf{Set})$ and $\mathrm{CMon}(\mathsf{Set}) \to \mathsf{Set}$. As $\mathsf{Ab} \to \mathrm{CMon}(\mathsf{Set})$ is the inclusion of a full subcategory, it suffices to show that the two functors in question are naturally equivalent as functors to $\mathrm{CMon}(\mathsf{Set})$.

For this we apply Proposition 4.3.1.2. The category $\mathsf{Ch}(k)^{\mathrm{cof}}$ is semiadditive (coproducts of cofibrant objects are again cofibrant by [Hov99, 1.1.11]) and Set admits finite products, so it remains to show that $\mathrm{H}_n(-)$ as a functor

[29] It is even one of the generating cofibrations discussed in [Hov99, 2.3.3 and 2.3.11].

[30] The downwards induction uses that $\mathcal{D}(k)$ is stable.

[31] The main point is that a cylinder object for $k[n]$ is given by $k[n] \amalg k[n] \xrightarrow{i_0 \amalg i_1} C \xrightarrow{s} k[n]$ where on underlying graded abelian groups $i_0 \amalg i_1$ is the inclusion into $k[n] \oplus k[n] \oplus k[n+1]$, and where ∂^C_{n+1} sends 1 to $(1,0)-(0,1)$. One also needs to use that every object of $\mathsf{Ch}(k)$ is fibrant, and then the rest is unpacking the definition.

$\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathrm{CMon}(\mathsf{Set})$ preserve products. The forgetful functor from commutative monoids to sets detects products (see Proposition F.2.0.1), so it suffices to show that $\mathrm{H}_n(-)$ preserves products as a functor into Set. But this is clear, as direct sums in $\mathsf{Ch}(k)$ are formed levelwise, and direct sums are both limits as well as colimits, so are compatible with forming kernels and cokernels.

Proof of Claim (6): The first part is [HA, 1.3.5.16 and 1.3.5.21]. The second part follows immediately from the observation that every chain complex with homology concentrated in nonnegative or nonpositive degrees is quasiisomorphic to a chain complex itself concentrated in those degrees, by truncating.

Proof of Claim (7): By [HA, 7.1.2.13] there is an equivalence

$$\theta\colon \mathcal{D}(k) \to \mathrm{LMod}_k(\mathcal{S}\mathrm{p})$$

of symmetric monoidal ∞-categories. It remains to show that θ is compatible with the respective t-structures.

As a monoidal equivalence, θ preserves monoidal units, so $\theta(k) \simeq k$, which implies that there is a sequence of natural isomorphism for $n \geq 2$ of functors $\mathcal{D}(k) \to \mathsf{Set}$ as follows.

$$\mathrm{H}_n(-)$$

Using Claim (5).

$$\cong \mathrm{Mor}_{\mathrm{Ho}\,\mathcal{D}(k)}(k[n], -)$$

Applying $\mathrm{Ho}\,\theta$.

$$\cong \mathrm{Mor}_{\mathrm{Ho}\,\mathrm{LMod}_k(\mathcal{S}\mathrm{p})}(k[n], \theta(-))$$

Using that the functor $\mathrm{Free}\colon \mathcal{S}\mathrm{p} \to \mathrm{LMod}_k(\mathcal{S}\mathrm{p})$ is left adjoint to the forgetful functor. See [HA, 4.2.4.8] and [HTT, 5.2.2.9].

$$\cong \mathrm{Mor}_{\mathrm{Ho}\,\mathcal{S}\mathrm{p}}(\mathbb{S}[n], \theta(-)) \cong \pi_0\big(\mathrm{Map}_{\mathcal{S}\mathrm{p}}(\mathbb{S}[n], \theta(-))\big)$$

Using that $n \geq 0$.

$$\cong \pi_n\big(\mathrm{Map}_{\mathcal{S}\mathrm{p}}(\mathbb{S}, \theta(-))\big)$$

Using the adjunction $\Sigma^\infty \dashv \Omega^\infty_*$.

$$\cong \pi_n\big(\mathrm{Map}_{\mathcal{S}_*}\big(S^0, \Omega^\infty_*\theta(-)\big)\big) \cong \pi_n(\Omega^\infty_*\theta(-))$$

Using [HA, 1.4.3.8].

$$\cong \pi_n(\theta(-))$$

By using that H_* and π_* are both compatible with shifts[32], we can conclude[33] that $\mathrm{H}_n(-) \cong \pi_n(\theta(-))$ for every integer n, which implies that θ is compatible

[32] For π_* this is by definition, see [HA, 1.2.1.11], for H_* this follows from Claim (5) and (4)

[33] A priori this is only a natural bijection – which is also all we need, as an abelian group is isomorphic to 0 if and only if its underlying set consists of a single element – but one can also apply Proposition 4.3.1.2 to deduce that this bijection in fact preserves the group structure.

with the respective t-structures on $\mathcal{D}(k)$ and $\mathrm{LMod}_k(\mathcal{S}\mathrm{p})$ as follows directly from their respective definitions.

Proof of Claim (8): The t-structure on $\mathrm{LMod}_k(\mathcal{S}\mathrm{p})$ is compatible with the symmetric monoidal structure by [HA, 7.1.3.10], so this also holds for $\mathcal{D}(k)$ by Claim (7).

Proof of Claim (9): Every chain complex concentrated in degree 0 has obviously vanishing homology outside of degree 0, so $\gamma\circ(-)[0]$ factors through the full subcategory $\mathcal{D}(k)^{\heartsuit}$ of $\mathcal{D}(k)$.

The induced functor is essentially surjective by the second part of (6). If two morphisms f and g in $\mathrm{LMod}_k(\mathsf{Ab})$ map to homotopic morphisms, then they induce the same morphisms on $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}(k[0],-)$, so by (5) $\mathrm{H}_0(f[0]) = \mathrm{H}_0(g[0])$, and hence $f = g$. Thus $\mathrm{Ho}(\mathrm{LMod}_k(\mathsf{Ab})) \to \mathrm{Ho}(\mathcal{D}(k)^{\heartsuit})$ is faithful. Finally, let X and Y be k-modules and $f\colon \gamma(X[0]) \to \gamma(Y[0])$ a morphism in $\mathrm{Ho}(\mathcal{D}(k))$. There is a zigzag of quasiisomorphisms

$$X[0] \cong (\tau_{\leq 0}\circ\tau_{\geq 0})(X^{\mathrm{cof}}) \leftarrow \tau_{\geq 0}(X[0]^{\mathrm{cof}}) \to X[0]^{\mathrm{cof}}$$

in $\mathsf{Ch}(k)$. As $Y[0]$ is fibrant we can by Proposition A.1.0.1 and [Hov99, 1.2.10 (iii)] find a morphism $\overline{f}\colon X^{\mathrm{cof}} \to Y[0]$ representing f, i. e. the dashed composite

$$\begin{tikzcd}
\gamma(X[0]) \arrow[r, dash, "\simeq"] \arrow[rrrd, dashed] & \gamma\big((\tau_{\leq 0}\circ\tau_{\geq 0})\big(X^{\mathrm{cof}}\big)\big) & \gamma\big(\tau_{\geq 0}\big(X[0]^{\mathrm{cof}}\big)\big) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \gamma\big(X[0]^{\mathrm{cof}}\big) \arrow[d, "\gamma(\overline{f})"] \\
 & & & \gamma(Y[0])
\end{tikzcd}$$

where the top line is obtained by applying γ to the above zigzag, is homotopic to a representative of f. But it is easy to see that $\overline{f}$ can be strictly lifted to a morphism from $X[0]$, as $Y[0]$ is concentrated in degree 0. This shows that the functor $\mathrm{Ho}(\mathrm{LMod}_k(\mathsf{Ab})) \to \mathrm{Ho}(\mathcal{D}(k)^{\heartsuit})$ is full.

As the ∞-category $\mathcal{D}(k)^{\heartsuit}$ is a 1-category by [HA, 1.2.1.12], this shows that the functor $\mathrm{LMod}_k(\mathsf{Ab}) \to \mathcal{D}(k)^{\heartsuit}$ is an equivalence. ☐

Remark 4.3.2.2. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the symmetric monoidal functor

$$k' \otimes_k -\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ch}(k')^{\mathrm{cof}}$$

from Fact 4.1.5.1 preserves weak equivalences and so induces by [HA, 4.1.7.4] a commutative diagram of symmetric monoidal functors as follows.

$$\begin{tikzcd}
\mathsf{Ch}(k)^{\mathrm{cof}} \arrow[r, "k'\otimes_k -"] \arrow[d, "\gamma"'] & \mathsf{Ch}(k')^{\mathrm{cof}} \arrow[d, "\gamma"] \\
\mathcal{D}(k) \arrow[r, "k'\otimes_k -"'] & \mathcal{D}(k')
\end{tikzcd}$$

Furthermore, it follows from Fact 4.1.5.1 using [HA, 1.3.4.27] that the functor

$$k' \otimes_k - \colon \mathcal{D}(k) \to \mathcal{D}(k') \tag{4.17}$$

is left adjoint to the functor

$$\varphi^* \colon \mathcal{D}(k') \to \mathcal{D}(k)$$

that is induced by the composition

$$\mathsf{Ch}(k')^{\mathrm{cof}} \xrightarrow{\varphi^*} \mathsf{Ch}(k) \xrightarrow{(-)^{\mathrm{cof}}} \mathsf{Ch}(k)^{\mathrm{cof}}$$

where the second functor is the cofibrant replacement functor. In particular, functor (4.17) preserves small colimits.

As $k' \otimes_k -$ is a symmetric monoidal functor, we can use [HA, 7.3.2.7] to upgrade the adjunction $k' \otimes_k - \dashv \varphi^*$ to an adjunction[34]

$$\begin{tikzcd}
\mathcal{D}(k)^{\otimes} \arrow[rr, shift left, "(k' \otimes_k -)^{\otimes}"] \arrow[rr, phantom, "\perp"] \arrow[dr] & & \mathcal{D}(k')^{\otimes} \arrow[ll, shift left, "(\varphi^*)^{\otimes}"] \arrow[dl] \\
& \mathsf{Fin}_* &
\end{tikzcd}$$

relative to Fin_* in the sense of [HA, 7.3.2.3], and such that $(\varphi^*)^{\otimes}$ is lax symmetric monoidal. ♢

4.3.3 Homology

Homology is a very important invariant of chain complexes, and for $\mathcal{D}(k)$ as well. In this section we will discuss how the different definitions are compatible, as well as some properties that we will need.

Definition 4.3.3.1 ([HA, 1.2.1.11])**.** Let n be an integer. We define a functor

$$\mathrm{H}_n \colon \mathcal{D}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$$

to be the composition

$$\mathcal{D}(k) \xrightarrow{(-)[-n]} \mathcal{D}(k) \xrightarrow{\tau_{\geq 0} \circ \tau_{\leq 0}} \mathcal{D}(k)^{\heartsuit} \simeq \mathrm{LMod}_k(\mathsf{Ab})$$

where the equivalence is the one from Proposition 4.3.2.1 (9). ♢

[34]The functors to Fin_* are to be the canonical cocartesian fibrations of ∞-operads.

Proposition 4.3.3.2. *Let n be an integer. Then there is a commutative diagram*

$$\begin{tikzcd} \mathsf{Ch}(k) \arrow[d, "\gamma"'] \arrow[dr, "\mathrm{H}_n"] & \\ & \mathrm{LMod}_k(\mathsf{Ab}) \\ \mathcal{D}(k) \arrow[ur, "\mathrm{H}_n"'] & \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$. ♡

Proof. We need to show that $\mathrm{H}_n \circ \gamma$ and H_n are naturally isomorphic.

Denote by φ the equivalence $\mathrm{LMod}_k(\mathsf{Ab}) \to \mathcal{D}(k)^{\heartsuit}$ from Proposition 4.3.2.1 (9) and assume we have already shown the claim for $n = 0$. Then we can deduce the claim for general n using Proposition 4.3.2.1 (4), as we obtain equivalences of functors $\mathsf{Ch}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$ as follows.

$$\begin{aligned} & \mathrm{H}_n \circ \gamma \\ & = \varphi^{-1} \circ \tau_{\geq 0} \circ \tau_{\leq 0} \circ (-)[-n] \circ \gamma \\ & \cong \varphi^{-1} \circ \tau_{\geq 0} \circ \tau_{\leq 0} \circ \gamma \circ (-)[-n] \\ & \cong \mathrm{H}_0 \circ \gamma \circ (-)[-n] \\ & \cong \mathrm{H}_0 \circ (-)[-n] \\ & \cong \mathrm{H}_n \end{aligned}$$

We now turn to the case $n = 0$. Consider the natural transformations of endofunctors of $\mathsf{Ch}(k)$

$$\mathrm{id}_{\mathsf{Ch}(k)} \to \tau_{\leq 0} \leftarrow \tau_{\geq 0} \circ \tau_{\leq 0} \tag{4.18}$$

where $\tau_{\leq 0}$ and $\tau_{\geq 0}$ refer to the truncation functors for chain complexes. The endofunctor $\tau_{\geq 0} \circ \tau_{\leq 0}$ factors over the inclusion of chain complexes that are concentrated in degree 0, so it suffices to show the following.

(1) The precompositions of $\mathrm{H}_0 \colon \mathsf{Ch}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$ with the two natural transformations in (4.18) are natural isomorphisms.

(2) The precompositions of $\mathrm{H}_0 \circ \gamma \colon \mathsf{Ch}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$ with the two natural transformations in (4.18) are natural isomorphisms.

(3) The precompositions of H_0 and $\mathrm{H}_0 \circ \gamma$ with the inclusion of chain complexes concentrated in degree 0 are naturally isomorphic.

Proof of (1): Clear.

Proof of (2): We only consider the first natural transformation, the other case is similar. We need to show that the natural transformation

$$\tau_{\geq 0} \circ \tau_{\leq 0} \circ \gamma \circ \mathrm{id}_{\mathsf{Ch}(k)} \to \tau_{\geq 0} \circ \tau_{\leq 0} \circ \gamma \circ \tau_{\leq 0}$$

is a natural equivalence. Let X be a chain complex, and let f be the natural morphism $X \to \tau_{\leq 0}X$. Then f is an isomorphism in homology in non-positive degrees, while $\tau_{\leq 0}X$ has homology concentrated in non-positive degrees, so the homotopy fiber $\mathrm{hofib}(f)$ has homology concentrated in positive degrees. We obtain a pullback diagram

$$\begin{tikzcd} \gamma(\mathrm{hofib}(f)) \arrow[r] \arrow[d] & \gamma(X) \arrow[d] \\ 0 \arrow[r] & \gamma(\tau_{\leq 0}X) \end{tikzcd}$$

in $\mathcal{D}(k)$, with $\gamma(\mathrm{hofib}(f))$ lying in $\mathcal{D}(k)_{\geq 1}$. Applying $\tau_{\leq 0}\colon \mathcal{D}(k) \to \mathcal{D}(k)$ we obtain a pullback diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \tau_{\leq 0}(\gamma(X)) \arrow[d] \\ 0 \arrow[r] & \tau_{\leq 0}(\gamma(\tau_{\leq 0}X)) \end{tikzcd}$$

in $\mathcal{D}(k)$, which, as $\mathcal{D}(k)$ is stable, is also a pushout diagram, from which it follows that

$$\tau_{\leq 0}(\gamma(X)) \to \tau_{\leq 0}(\gamma(\tau_{\leq 0}X))$$

is an equivalence. The claim follows.

Proof of (3): What we need to show is that $\mathrm{H}_0 \circ (-)[0]$ and $\mathrm{H}_0 \circ \gamma \circ (-)[0]$ are naturally isomorphic as functors from $\mathrm{LMod}_k(\mathsf{Ab})$ to $\mathrm{LMod}_k(\mathsf{Ab})$.

$\mathrm{H}_0 \circ (-)[0]$ is naturally isomorphic to the identity functor right from the definition. For $\mathrm{H}_0 \circ \gamma \circ (-)[0]$ we can apply Proposition 4.3.2.1 (9) to obtain equivalences as follows.

$$\begin{aligned} & \mathrm{H}_0 \circ \gamma \circ (-)[0] \\ \simeq\; & \left(\varphi^{-1} \circ \tau_{\geq 0} \circ \tau_{\leq 0}\right) \circ \left(\left(\mathcal{D}(k)^{\heartsuit} \to \mathcal{D}(k)\right) \circ \varphi\right) \\ \simeq\; & \varphi^{-1} \circ \mathrm{id}_{\mathcal{D}(k)^{\heartsuit}} \circ \varphi \\ \simeq\; & \varphi^{-1} \circ \varphi \\ \simeq\; & \mathrm{id}_{\mathrm{LMod}_k(\mathsf{Ab})} \end{aligned}$$

□

Proposition 4.3.3.3. *Let n be an integer. Then there is a commutative diagram*

$$\begin{tikzcd} & \mathrm{LMod}_k(\mathsf{Ab}) \arrow[dd, "\mathrm{ev}_{\mathrm{m}}"] \\ \mathcal{D}(k) \arrow[ur, "\mathrm{H}_n"] \arrow[dr, "\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}(k[n],-)"'] & \\ & \mathsf{Ab} \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$. ♡

Proof. By [HA, 1.3.4.1] it suffices to show that there is a homotopy

$$\mathrm{ev}_{\mathfrak{m}} \circ \mathrm{H}_n \circ \gamma \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}(k[n], \gamma(-))$$

of functors $\mathcal{D}(k) \to \mathsf{Ab}$. The former functor is by Proposition 4.3.3.2 homotopic to the composition

$$\mathsf{Ch}(k) \xrightarrow{\mathrm{H}_n} \mathrm{LMod}_k(\mathsf{Ab}) \xrightarrow{\mathrm{ev}_{\mathfrak{m}}} \mathsf{Ab} \qquad (*)$$

and the latter functor is by Proposition 4.3.2.1 (5) homotopic to the functor

$$\mathsf{Ch}(k) \xrightarrow{\mathrm{H}_n} \mathsf{Ab}$$

which is by definition the same as the composition $(*)$. □

Notation 4.3.3.4. Let n be an integer. In light of Proposition 4.3.3.3 we will also denote the functor

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}(k[n], -) \colon \mathcal{D}(k) \to \mathsf{Ab}$$

by H_n. However, if it is not clear from context that we mean this functor, then usage of the notation H_n should be understood to refer to the functor with image in $\mathrm{LMod}_k(\mathsf{Ab})$. ♢

Proposition 4.3.3.5. *Let n be an integer. The functor*

$$\mathrm{H}_n \colon \mathcal{D}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$$

preserves products and coproducts. ♡

Proof. As the forgetful functor $\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod}_k(\mathsf{Ab}) \to \mathsf{Ab}$ detects limits and colimits, it suffices to show that the functor

$$\mathrm{H}_n \colon \mathcal{D}(k) \to \mathsf{Ab}$$

preserves products and coproducts.

We start by showing that it preserves products. As the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ preserves products, it suffices to show that the functor $\mathcal{D}(k) \to \mathsf{Set}$

$$\mathrm{Mor}_{\mathrm{Ho}(\mathcal{D}(k))}(k[n], -) \cong \pi_0\Big(\mathrm{Map}_{\mathcal{D}(k)}(k[n], -)\Big) \colon \mathcal{D}(k) \to \mathsf{Set}$$

preserves products, but this is clear as both $\mathrm{Map}_{\mathcal{D}(k)}(k[n], -)$ and π_0 preserve products.

For coproducts we use the commutative diagram constructed in Proposition 4.3.2.1 (5) that is depicted below.

$$\begin{tikzcd}
\mathsf{Ch}(k)^{\mathrm{cof}} \arrow[d, "\gamma"'] \arrow[dr, "\mathrm{H}_n"] & \\
\mathcal{D}(k) \arrow[r, "\mathrm{H}_n"'] & \mathsf{Ab}
\end{tikzcd}$$

As every object of $\mathcal{D}(k)$ is represented by a cofibrant chain complex (by definition) and γ preserves coproducts[35] it suffices to show that the functor H_n on chain complexes preserves coproducts, which is a classical exercise in homological algebra[36]. $\square$

Remark 4.3.3.6. The functor

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}(k, -)\colon \mathrm{Ho}(\mathcal{D}(k)) \to \mathsf{Ab}$$

is by [Nee01, 1.1.10] homological in the sense of [Nee01, 1.1.7]. As the forgetful functor from $\mathrm{LMod}_k(\mathsf{Ab})$ to Ab detects exact sequences, it follows from Proposition 4.3.3.3 that the functor

$$\mathrm{H}_0\colon \mathrm{Ho}(\mathcal{D}(k)) \to \mathrm{LMod}_k(\mathsf{Ab})$$

is an homological functor as well.

Any cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{D}(k)$ thus induces a long exact sequence

$$\cdots \xrightarrow{\mathrm{H}_0(-h[-1])} \mathrm{H}_0(X) \xrightarrow{\mathrm{H}_0(f)} \mathrm{H}_0(Y) \xrightarrow{\mathrm{H}_0(g)} \mathrm{H}_0(Z) \xrightarrow{\mathrm{H}_0(h)} \mathrm{H}_0(X[1]) \xrightarrow{\mathrm{H}_0(-f[1])} \cdots$$

in $\mathrm{LMod}_k(\mathsf{Ab})$ that we can identify with a long exact sequence

$$\cdots \to \mathrm{H}_1(Z) \to \mathrm{H}_0(X) \xrightarrow{\mathrm{H}_0(f)} \mathrm{H}_0(Y) \xrightarrow{\mathrm{H}_0(g)} \mathrm{H}_0(Z) \xrightarrow{\mathrm{H}_0(h)} \mathrm{H}_{-1}(X) \to \cdots \quad \diamondsuit$$

Proposition 4.3.3.7. *Let X be an object of $\mathcal{D}(k)$ so that $\mathrm{H}_n(X)$ is a free k-module with basis[37] $\{b_i\colon k[n] \to X\}_{i\in I_n}$ for every integer n.*

Then the morphism

$$\coprod_{n\in\mathbb{Z}, i\in I_n} k[n] \xrightarrow{\coprod_{n\in\mathbb{Z}, i\in I_n} b_i} X$$

is an equivalence in $\mathcal{D}(k)$. $\heartsuit$

[35] Coproducts of cofibrant objects are homotopy coproducts, then use [HA, 1.3.4.25 and 1.3.4.24].

[36] See for example [Rot08, Exercise 6.9]. One way to show this is as follows. One first considers finite coproducts, which are biproducts, so one can for example use additivity. Arbitrary coproducts can be written as filtered colimits of their finite subcoproducts (this is true also for ∞-categories by [HTT, Special case of the proof of 4.2.3.11] but can of course also be shown in a more elementary way for our application), so it then suffices to show that filtered colimits in $\mathrm{LMod}_k(\mathsf{Ab})$ are exact, which is done in [Wei94, Theorem 2.6.15].

[37] Such a morphism b_i represents an element in $\mathrm{H}_n(X)$ via Proposition 4.3.3.3.

Proof. Represent X by a chain complex. Unpacking and using the natural equivalence from Proposition 4.3.2.1 (5) and Proposition 4.3.3.2 we obtain that the morphism in question is represented by a quasiisomorphism of chain complexes and is thus an equivalence. □

Proposition 4.3.3.8. *Let n be an integer, $\mathcal{I}$ a small ∞-category, and $F\colon \mathcal{I} \to \mathcal{D}(k)$ a functor.*

Assume that $F(I)$ lies in $\mathcal{D}(k)_{\geq n}$ for every object I of $\mathcal{I}$. Then the canonical morphism

$$\operatorname*{colim}_{\mathcal{I}} \mathrm{H}_n(F(\bullet)) \to \mathrm{H}_n\left(\operatorname*{colim}_{\mathcal{I}} F\right)$$

is an isomorphism.

Analogously, if $F(I)$ lies in $\mathcal{D}(k)_{\leq n}$ for every object I of $\mathcal{I}$, then the canonical morphism

$$\mathrm{H}_n\left(\lim_{\mathcal{I}} F\right) \to \lim_{\mathcal{I}} \mathrm{H}_n(F(\bullet))$$

is an isomorphism. ♡

Proof. It suffices to consider the case $n = 0$. By [HA, 1.2.1.6], the colimit of F is again in $\mathcal{D}(k)_{\geq 0}$ in the first case and in $\mathcal{D}(k)_{\leq 0}$ in the second case, and thus forms the colimit in that full subcategory by [HTT, 1.2.13.7]. The statement now follows from the fact that $\tau_{\leq 0}\colon \mathcal{D}(k)_{\geq 0} \to \mathcal{D}(k)^{\heartsuit}$ is left adjoint and thus preserves colimits and $\tau_{\geq 0}\colon \mathcal{D}(k)_{\leq 0} \to \mathcal{D}(k)^{\heartsuit}$ is a right adjoint and thus preserves limits. □

4.3.4 Properties of the truncation functors

Let n be an integer. The categories $\mathcal{D}(k)_{\geq n}$ and $\mathcal{D}(k)_{\leq n}$ defined as in [HA, 1.2.1.4] with respect to the t-structure discussed in Proposition 4.3.2.1 are the full subcategories of objects X with $\mathrm{H}_m(X) \cong 0$ for $m < n$ and $m > n$, respectively. By [HA, 1.2.1.6 and 1.2.1.7] we obtain adjunctions

$$\begin{tikzcd} \mathcal{D}(k) \arrow[r, shift left, "\tau_{\leq n}"] \arrow[r, phantom, "\perp"] & \mathcal{D}(k)_{\leq n} \arrow[l, shift left, "\iota_{\leq n}"] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{D}(k)_{\geq n} \arrow[r, shift left, "\iota_{\geq n}"] \arrow[r, phantom, "\perp"] & \mathcal{D}(k) \arrow[l, shift left, "\tau_{\geq n}"] \end{tikzcd}$$

with $\iota_{\leq n}$ and $\iota_{\geq n}$ the inclusions of the respective full subcategories.

We will sometimes omit $\iota_{\leq n}$ and $\iota_{\geq n}$ from the notation and consider $\tau_{\leq n}$ and $\tau_{\geq n}$ as endofunctors of $\mathcal{D}(k)$.

As the t-structure on $\mathcal{D}(k)$ is compatible with the symmetric monoidal structure, we get more, as the following proposition records.

Proposition 4.3.4.1. *The following list of statements hold.*

(1) $\mathcal{D}(k)_{\geq 0}$ *inherits a symmetric monoidal structure from* $\mathcal{D}(k)$.

(2) The adjunction $\iota_{\geq 0} \dashv \tau_{\geq 0}$ *can be upgraded to an adjunction* $\iota^{\otimes}_{\geq 0} \dashv \tau^{\otimes}_{\geq 0}$ *of lax monoidal functors relative to* Fin_* *(in the sense of [HA, 7.3.2.3]).*

(3) The lax monoidal functor $\iota^{\otimes}_{\geq 0}$ *is symmetric monoidal.*

(4) For $n \geq 0$*, the full subcategory* $(\mathcal{D}(k)_{\geq 0})_{\leq n}$ *inherits a symmetric monoidal structure from* $\mathcal{D}(k)_{\geq 0}$.

(5) The adjunction $\tau_{\leq n} \dashv \iota_{\geq 0, \leq n}$*, where* $\iota_{\geq 0, \leq n} : (\mathcal{D}(k)_{\geq 0})_{\leq n} \to \mathcal{D}(k)_{\geq 0}$ *is the inclusion, can be upgraded to an adjunction* $\tau^{\otimes}_{\leq n} \dashv \iota^{\otimes}_{\geq 0, \leq n}$ *of lax monoidal functors relative to* $N(\mathsf{Fin}_*)$.

(6) The lax monoidal functor $\tau^{\otimes}_{\leq n} \colon \mathcal{D}(k)^{\otimes}_{\geq 0} \to (\mathcal{D}(k)_{\geq 0})^{\otimes}_{\leq n}$ *is symmetric monoidal.*

Let $\mathcal{O}^{\otimes}$ *be an* ∞*-operad. Then the following statements hold as well.*

(7) The adjunction $\iota^{\otimes}_{\geq 0} \dashv \tau^{\otimes}_{\geq 0}$ *induces an adjunction*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}(k)_{\geq 0}) \arrow[r, shift left, "\mathrm{Alg}_{\mathcal{O}}(\iota_{\geq 0})"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}(k)) \arrow[l, shift left, "\mathrm{Alg}_{\mathcal{O}}(\tau_{\geq 0})"] \end{tikzcd}$$

and $\mathrm{Alg}_{\mathcal{O}}(\iota_{\geq 0})$ *is fully faithful with essential image spanned by those* $\mathcal{O}$*-algebras* A *in* $\mathcal{D}(k)$ *such that for every object* X *of* $\mathcal{O}$*, the underlying object* $\mathrm{ev}_X(A)$ *of* A *lies in* $\mathcal{D}(k)_{\geq 0}$.

(8) The adjunction $\tau^{\otimes}_{\leq n} \dashv \iota^{\otimes}_{\geq 0, \leq n}$ *induces an adjunction*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}(k)_{\geq 0}) \arrow[r, shift left, "\mathrm{Alg}_{\mathcal{O}}(\tau_{\leq n})"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathcal{O}}\Big((\mathcal{D}(k)_{\geq 0})_{\leq n}\Big) \arrow[l, shift left, "\mathrm{Alg}_{\mathcal{O}}(\iota_{\geq 0, \leq n})"] \end{tikzcd}$$

and $\mathrm{Alg}_{\mathcal{O}}(\iota_{\geq 0, \leq n})$ *is fully faithful with essential image spanned by those* $\mathcal{O}$*-algebras* A *in* $\mathcal{D}(k)_{\geq 0}$ *such that for every object* X *of* $\mathcal{O}$*, the underlying object* $\mathrm{ev}_X(A)$ *of* A *lies in* $(\mathcal{D}(k)_{\geq 0})_{\leq n}$. ♡

Proof. By Proposition 4.3.2.1, the t-structure on $\mathcal{D}(k)$ is compatible with with the symmetric monoidal structure in the sense of [HA, 2.2.1.3], so the statements (1), (2), and (3) follow from [HA, 2.2.1.1], and the statements (4), (5), and (6) follow from [HA, 2.2.1.10 and 2.2.1.9].

That we obtain induced adjunctions on algebras as in (7) and (8) now follows from Proposition E.3.3.1, see also [HA, 2.2.1.5]. Finally, that the functors induced on algebra categories by the inclusions are again fully faithful as well as the descriptions of the essential images follow from Proposition E.3.5.1. □

We also record the following for later use.

Proposition 4.3.4.2 ([HA, 1.2.1.6]). *Let n be an integer.*

Then $\mathcal{D}(k)_{\leq n}$ is closed under small limits and coproducts. In particular, $\mathcal{D}(k)_{\leq n}$ admits all small limits and finite biproducts and $\iota_{\leq n}$ preserves them.

Analogously, $\mathcal{D}(k)_{\geq n}$ is closed under small colimits and finite products. In particular, $\mathcal{D}(k)_{\leq n}$ admits all small colimits and finite biproducts and $\iota_{\geq n}$ preserves them. ♡

Proof. The closure properties for limits and colimits are [HA, 1.2.1.6] and closure under finite biproducts follows from the definition using that $\mathrm{H}_m(-)$ commutes with finite biproducts.

The rest of the claims now follow from the closure claims by [HTT, 1.2.13.7] □

4.4 The ∞-category of mixed complexes

In Notation 4.2.2.10 we constructed a commutative diagram of forgetful functors as follows.

$$\begin{tikzcd} & \mathrm{Alg}(\mathsf{Mixed}) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}"'] \arrow[dr, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})"] & \\ \mathsf{Mixed} \arrow[dr, "\mathrm{ev}_{\mathfrak{m}}"'] & & \mathrm{Alg}(\mathsf{Ch}(k)) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}"] \\ & \mathsf{Ch}(k) & \end{tikzcd} \tag{4.19}$$

All four functors preserve weak equivalences by Proposition 4.2.2.12 so we obtain a commutative diagram on underlying ∞-categories. For this, let us use the following notation.

Notation 4.4.0.1. Denote by W_{Ch}, W_{Alg}, W_{Mixed} and $W_{\mathrm{Alg}(\mathsf{Mixed})}$ the classes of weak equivalences in $\mathsf{Ch}(k)$, $\mathrm{Alg}(\mathsf{Ch}(k))$, Mixed, and $\mathrm{Alg}(\mathsf{Mixed})$, respectively, where we use the weak equivalences from the model structures defined in Fact 4.1.3.1, Definition 4.2.2.2, and Proposition 4.2.2.9.

In contexts in which we only consider a full subcategory of those model categories, we will use the same notation for the class of weak equivalences between objects in that subcategory. ♢

Diagram (4.19) now induces a commutative diagram of ∞-categories as follows.

$$\begin{tikzcd}[column sep=small] & \mathrm{Alg}(\mathsf{Mixed})[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}\,\prime}"'] \arrow[dr, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})'"] & \\ \mathsf{Mixed}[W^{-1}_{\mathsf{Mixed}}] \arrow[dr, "\mathrm{ev}'_{\mathfrak{m}}"'] & & \mathrm{Alg}(\mathsf{Ch}(k))[W^{-1}_{\mathrm{Alg}}] \arrow[dl, "\mathrm{ev}'_{\mathfrak{a}}"] \\ & \mathsf{Ch}(k)[W^{-1}_{\mathsf{Ch}}] & \end{tikzcd} \tag{4.20}$$

$\mathsf{Ch}(k)[W^{-1}_{\mathsf{Ch}}]$ can be identified with the derived category, $\mathcal{D}(k)$[38]. The canonical symmetric monoidal functor $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ induces a functor on commutative and cocommutative bialgebras, so we can apply it to the cofibrant commutative and cocommutative bialgebra D (see Construction 4.2.1.1 and Proposition 4.2.2.4) to obtain a commutative and cocommutative bialgebra $\gamma(\mathrm{D})$ in $\mathcal{D}(k)$.

Notation 4.4.0.2. We will denote the object $\gamma(\mathrm{D})$ of $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Comm}}(\mathcal{D}(k))$ by D (or D_k if we want to make k explicit).

By the results of Section 3.4 we obtain an induced symmetric monoidal structure on $\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$. We will denote this symmetric monoidal ∞-category by $\mathcal{M}\mathrm{ixed}$, or, if we want to make the base ring k explicit, by $\mathcal{M}\mathrm{ixed}_k$. ◊

We can construct from the symmetric monoidal ∞-category $\mathcal{D}(k)$ and cocommutative bialgebra D in $\mathcal{D}(k)$ the following commutative diagram that is analogous to (4.19).

$$\begin{tikzcd}[column sep=small] & \mathrm{Alg}(\mathcal{M}\mathrm{ixed}) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}"'] \arrow[dr, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})"] & \\ \mathcal{M}\mathrm{ixed} \arrow[dr, "\mathrm{ev}_{\mathfrak{m}}"'] & & \mathrm{Alg}(\mathcal{D}(k)) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}"] \\ & \mathcal{D}(k) & \end{tikzcd} \tag{4.21}$$

The goal of this section is to show that diagram (4.21) can be identified with diagram (4.20).

For algebras, there is a relevant result: For a monoidal model category $\mathcal{A}$ with certain properties, [HA, 4.1.8.4] shows that there is an equivalence

$$\mathrm{Alg}(\mathcal{A})^{\mathrm{cof}}[W'^{-1}] \xrightarrow{\simeq} \mathrm{Alg}\big(\mathcal{A}^{\mathrm{cof}}[W^{-1}]\big)$$

[38] By Proposition 4.3.2.1 (1) $\mathcal{D}(k) \simeq \mathsf{Ch}(k)^{\mathrm{cof}}[W^{-1}]$, but the inclusion of $\mathsf{Ch}(k)^{\mathrm{cof}}$ into $\mathsf{Ch}(k)$ and the cofibrant replacement functor induce mutually inverse equivalences after inverting weak equivalences, see [HA, 1.3.4.16] and Proposition A.3.2.1.

where W and W' are the respective classes of weak equivalences. The reason only the full subcategory of cofibrant objects is considered is that we want the tensor product to be automatically derived. The pushout product axiom ensures that the tensor product of two cofibrant objects is again cofibrant, so the tensor product restricts to the full subcategory of cofibrant objects. A monoidal category also needs a unit object, so in order to ensure that the subcategory is again a monoidal category, Lurie requires that the unit object in $\mathcal{A}$ is cofibrant. Unfortunately, this does not hold for the monoidal model category $\mathsf{Mixed} = \mathrm{LMod}_D(\mathsf{Ch}(k))$ that we considered above[39], so we can not directly apply Lurie's result. However, we proved that Mixed satisfies the monoid axiom (Proposition 4.2.2.8), which ensures that even though the unit object is not cofibrant, tensoring with it nevertheless results in the correct derived tensor product. Another (related) viewpoint would be to note that the tensor product in $\mathsf{Mixed} = \mathrm{LMod}_D(\mathsf{Ch}(k))$ is calculated on the underlying chain complexes, and in $\mathsf{Ch}(k)$ the unit object *is* cofibrant. This will open the possibility of nevertheless proving a result similar to [HA, 4.1.8.4] for our situation.

We will start in Section 4.4.1 by constructing a comparison natural transformation from diagram (4.20) to diagram (4.21), and then show that the comparison functors are equivalences in Section 4.4.2. Finally, in the very short Section 4.4.3 we show that $\mathcal{M}\mathrm{ixed}$ is a stable ∞-category, and in the also short section Section 4.4.4 we discuss how strongly homotopy linear morphisms of strict mixed complexes induce morphisms in $\mathcal{M}\mathrm{ixed}$.

4.4.1 Construction of comparison functors

In this section we will construct a comparison natural transformation from diagram (4.20) to diagram (4.21).

Construction 4.4.1.1. By Fact 4.1.3.1, the subcategory $\mathsf{Ch}(k)^{\mathrm{cof}}$ inherits a symmetric monoidal structure from $\mathsf{Ch}(k)$. As the underlying chain complex of D is cofibrant by Proposition 4.2.2.4, we can view D as an object of $\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Comm}}(\mathsf{Ch}(k)^{\mathrm{cof}})$. By Proposition 3.4.1.15 we can thus consider the pair $(\mathsf{Ch}(k)^{\mathrm{cof}}, \mathrm{D})$ as an object of $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathsf{Comm}}$.

The symmetric monoidal functor $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ is a morphism in the ∞-category $\mathrm{Mon}_{\mathsf{Comm}}(\mathcal{C}\mathrm{at}_\infty)$. Denote by $\overline{\gamma}$ a $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathsf{Comm}}}$-cocartesian lift of γ with source $(\mathsf{Ch}(k)^{\mathrm{cof}}, \mathrm{D})$. By Proposition 3.4.1.15 we can identify the codomain of the morphism $\overline{\gamma}$ with the bialgebra $\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Comm}}(\gamma)(\mathrm{D})$, which we also denote by D.

Applying the natural transformation $\mathrm{ev}_{\mathsf{m}}\colon \mathrm{LMod} \to \mathrm{pr}$ of functors from $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathsf{Comm}}$ to $\mathrm{Mon}_{\mathsf{Comm}}(\mathcal{C}\mathrm{at}_\infty)$ from Definition 3.4.2.1 we obtain a com-

[39]See the discussion in Section 4.2.2.2.

mutative diagram of symmetric monoidal ∞-categories as follows.

$$\begin{tikzcd}
\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)^{\otimes} \arrow[r, "\mathrm{LMod}_{\mathrm{D}}(\gamma)^{\otimes}"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}^{\otimes}"'] & \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))^{\otimes} \arrow[d, "\mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \\
\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)^{\otimes} \arrow[r, "\gamma^{\otimes}"'] & \mathcal{D}(k)^{\otimes}
\end{tikzcd}$$

Applying the natural transformation

$$\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{Alg}(-) \to - \times_{\mathsf{Fin}_*} \{\langle 1\rangle\}$$

we obtain the following commutative cube.

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathrm{Alg}\big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\big) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))) \arrow[dl] \arrow[dd] \\
\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr, crossing over] \arrow[dd] & & \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) & \\
& \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr] \arrow[dl] & & \mathrm{Alg}(\mathcal{D}(k)) \arrow[dl] \\
\mathsf{Ch}(k)^{\mathrm{cof}} \arrow[rr] & & \mathcal{D}(k) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the horizontal functors are all induced by γ, and the left and right squares are made up of the various forgetful functors. ♢

Notation 4.4.1.2. We will also denote by γ_{Mixed} the functor

$$\mathsf{Mixed}_{\mathrm{cof}} = \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \xrightarrow{\mathrm{LMod}_{\mathrm{D}}(\gamma)} \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) = \mathcal{M}\mathrm{ixed}$$

induced by γ. ♢

Remark 4.4.1.3. Let $\varphi\colon k \to k'$ be a morphism of commutative rings.

Then the symmetric monoidal and weak-equivalence preserving functor

$$k' \otimes_k -\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ch}(k')^{\mathrm{cof}}$$

from Fact 4.1.5.1 maps by Construction 4.2.1.1 D_k to $\mathrm{D}_{k'}$ and thus induces a transformation from the cube constructed in Construction 4.4.1.1 with respect to k to the same cube with respect to k' (i. e. a four-dimensional hypercube). In particular, there is an induced commutative diagram of symmetric monoidal functors as follows.

$$\begin{tikzcd}
\mathsf{Mixed}_{k,\mathrm{cof}} \arrow[r, "k'\otimes_k -"] \arrow[d, "\gamma_{\mathsf{Mixed}}"'] & \mathsf{Mixed}_{k',\mathrm{cof}} \arrow[d, "\gamma_{\mathsf{Mixed}}"] \\
\mathcal{M}\mathrm{ixed}_k \arrow[r, "k'\otimes_k -"'] & \mathcal{M}\mathrm{ixed}_{k'}
\end{tikzcd}$$

See also Remark 4.2.1.3, Proposition 4.2.2.3, and Remark 4.3.2.2. ♢

Proposition 4.4.1.4. *The functors*

$$\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$$
$$\gamma_{\mathsf{Mixed}}\colon \mathsf{Mixed}_{\mathrm{cof}} \to \mathcal{M}\mathrm{ixed}$$
$$\mathrm{Alg}(\gamma)\colon\ \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathrm{Alg}(\mathcal{D}(k))$$
$$\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\colon\ \mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}}) \to \mathrm{Alg}(\mathcal{M}\mathrm{ixed})$$

all map the respective weak equivalences to equivalences.

In particular, the commutative cube constructed in Construction 4.4.1.1 induces a commutative cube as follows.

$$\begin{tikzcd}
& \mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}(\mathcal{M}\mathrm{ixed}) \arrow[dl] \arrow[dd] \\
\mathsf{Mixed}_{\mathrm{cof}}[W^{-1}_{\mathsf{Mixed}}] \arrow[rr] \arrow[dd] & & \mathcal{M}\mathrm{ixed} \arrow[dd] & \\
& \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W^{-1}_{\mathrm{Alg}}] \arrow[rr] \arrow[dl] & & \mathrm{Alg}(\mathcal{D}(k)) \arrow[dl] \\
\mathsf{Ch}(k)^{\mathrm{cof}}[W^{-1}_{\mathsf{Ch}}] \arrow[rr] & & \mathcal{D}(k) &
\end{tikzcd}$$

where the horizontal functors are all induced by γ and the functors on the left and right sides are (induced by) the various forgetful functors. ♡

Proof. The following discussion refers to the cube constructed in Construction 4.4.1.1. Note that by Proposition 4.2.2.12 all the functors on the left side preserve weak equivalences, so that we obtain a commutative square as claimed after inverting the respective classes of weak equivalences. It remains to show that the horizontal functors map weak equivalences to equivalences.

The two functors $\mathrm{ev}_{\mathfrak{a}}$ on the right detect equivalences by [HA, 3.2.2.6], and by [HA, 4.2.3.3] the left vertical functor $\mathrm{ev}_{\mathfrak{m}}$ on the right side also detects equivalences. It follows that equivalences on the right side are detected in $\mathcal{D}(k)$, so it suffices to show that the compositions from the four categories on the left side to $\mathcal{D}(k)$ map weak equivalences to equivalences. But as all functors (or compositions) to $\mathsf{Ch}(k)^{\mathrm{cof}}$ preserve weak equivalences as already mentioned, it actually suffices to show that $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ maps weak equivalences to equivalences. But this is true by definition, see Proposition 4.3.2.1 (1). □

The commutative cube from Proposition 4.4.1.4 is pretty close to being a comparison natural transformation from diagram (4.20) to diagram (4.21). However, the left side is not quite given as (4.20) as we are only considering cofibrant underlying chain complexes. The next proposition shows that this does not make a difference.

Construction 4.4.1.5. We obtain a commutative cube completely analogous to the one constructed in Construction 4.4.1.1 from the symmetric monoidal inclusion functor $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ch}(k)$. Using Proposition 4.2.2.12 we obtain the following induced commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}(\mathsf{Mixed})[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[dl] \arrow[dd] \\
\mathsf{Mixed}_{\mathrm{cof}}[W^{-1}_{\mathsf{Mixed}}] \arrow[rr, crossing over] \arrow[dd] & & \mathsf{Mixed}[W^{-1}_{\mathsf{Mixed}}] & \\
& \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W^{-1}_{\mathrm{Alg}}] \arrow[rr] \arrow[dl] & & \mathrm{Alg}(\mathsf{Ch}(k))[W^{-1}_{\mathrm{Alg}}] \arrow[dl] \\
\mathsf{Ch}(k)^{\mathrm{cof}}[W^{-1}_{\mathsf{Ch}}] \arrow[rr] & & \mathsf{Ch}(k)[W^{-1}_{\mathsf{Ch}}] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the horizontal functors are induced by the inclusion $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ch}(k)$ and the functors on the left and right are the various forgetful functors. ♢

Construction 4.4.1.6. By Proposition 4.2.2.12 the cofibrant objects in

$$\mathrm{Alg}(\mathsf{Mixed}), \qquad \mathsf{Mixed}, \qquad \mathrm{Alg}(\mathsf{Ch}(k)), \quad \text{and} \quad \mathsf{Ch}(k)$$

all have cofibrant underlying chain complex[40]. We thus obtain a commutative cube as follows

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathrm{Alg}(\mathsf{Mixed})^{\mathrm{cof}}[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[dl] \arrow[dd] \\
\mathsf{Mixed}^{\mathrm{cof}}[W^{-1}_{\mathsf{Mixed}}] \arrow[rr, crossing over] \arrow[dd] & & \mathsf{Mixed}_{\mathrm{cof}}[W^{-1}_{\mathsf{Mixed}}] & \\
& \mathrm{Alg}(\mathsf{Ch}(k))^{\mathrm{cof}}[W^{-1}_{\mathrm{Alg}}] \arrow[rr] \arrow[dl] & & \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W^{-1}_{\mathrm{Alg}}] \arrow[dl] \\
\mathsf{Ch}(k)^{\mathrm{cof}}[W^{-1}_{\mathsf{Ch}}] \arrow[rr] & & \mathsf{Ch}(k)^{\mathrm{cof}}[W^{-1}_{\mathsf{Ch}}] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the horizontal functors are induced by the inclusions and the functors on the left and right are the various forgetful functors. ♢

Proposition 4.4.1.7. *The horizontal functors in the commutative cubes of Construction 4.4.1.5 and Construction 4.4.1.6 are equivalences.* ♡

Proof. The proof is very similar for the eight functors, so we only discuss the functor

$$\mathsf{Mixed}_{\mathrm{cof}}[W^{-1}_{\mathsf{Mixed}}] \to \mathsf{Mixed}[W^{-1}_{\mathsf{Mixed}}]$$

as the example case.

[40]While $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ was not shown in Proposition 4.2.2.12 to preserve cofibrant objects, this is not a problem, as both $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ and $\mathrm{ev}_{\mathfrak{a}}$ preserve cofibrant objects by Proposition 4.2.2.12, so their composition does so too.

As already mentioned in Construction 4.4.1.6, by Proposition 4.2.2.12 the forgetful functor $\mathrm{ev}_{\mathfrak{m}}$ from Mixed to $\mathsf{Ch}(k)$ preserves cofibrant objects, so the cofibrant replacement functor of Mixed lands in $\mathsf{Mixed}_{\mathrm{cof}}$. Let

$$\iota\colon \mathsf{Mixed}_{\mathrm{cof}} \to \mathsf{Mixed}$$

be the inclusion functor and

$$-^{\mathrm{cof}}\colon \mathsf{Mixed} \to \mathsf{Mixed}_{\mathrm{cof}}$$

the cofibrant replacement functor. The compositions $\iota \circ -^{\mathrm{cof}}$ and $-^{\mathrm{cof}} \circ \iota$ come with natural transformations to the identity functors that are pointwise weak equivalences. As both ι and $-^{\mathrm{cof}}$ preserve weak equivalences, we obtain induced functors after inverting weak equivalences, and by Proposition A.3.2.1 the natural transformations just mentioned become natural equivalences. Thus the functor induced by ι,

$$\mathsf{Mixed}_{\mathrm{cof}}[W_{\mathsf{Mixed}}^{-1}] \to \mathsf{Mixed}[W_{\mathsf{Mixed}}^{-1}]$$

is an equivalence. □

Definition 4.4.1.8. By composing the cube from Proposition 4.4.1.4 with the inverse of the cube from Construction 4.4.1.5 (where the horizontal functors are equivalences by Proposition 4.4.1.7), we obtain the following commutative cube.

$$\begin{tikzcd}
& \mathrm{Alg}(\mathsf{Mixed})[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}(\mathcal{M}\mathrm{ixed}) \arrow[dl] \arrow[dd] \\
\mathsf{Mixed}[W_{\mathsf{Mixed}}^{-1}] \arrow[rr] \arrow[dd] & & \mathcal{M}\mathrm{ixed} \arrow[dd] & \\
& \mathrm{Alg}(\mathsf{Ch}(k))[W_{\mathrm{Alg}}^{-1}] \arrow[rr] \arrow[dl] & & \mathrm{Alg}(\mathcal{D}(k)) \arrow[dl] \\
\mathsf{Ch}(k)[W_{\mathsf{Ch}}^{-1}] \arrow[rr] & & \mathcal{D}(k) &
\end{tikzcd}$$

The horizontal functors are induced by the composition of the respective cofibrant replacement functors and γ, and the other functors are (induced by) the various forgetful functors. ♢

4.4.2 The comparison functors are equivalences

In this section we show that the horizontal functors in the cube of Definition 4.4.1.8 are all equivalences.

Proposition 4.4.2.1 ([HA, 4.1.8.4])**.** *The functor*

$$\mathrm{Alg}(\mathsf{Ch}(k))[W_{\mathrm{Alg}}^{-1}] \to \mathrm{Alg}(\mathcal{D}(k))$$

from Definition 4.4.1.8 is an equivalence. ♡

Proof. By Proposition 4.4.1.7 it suffices to show that the related functor

$$\mathrm{Alg}(\mathsf{Ch}(k))^{\mathrm{cof}}[W_{\mathrm{Alg}}^{-1}] \to \mathrm{Alg}(\mathcal{D}(k))$$

induced by γ is an equivalence.

By Fact 4.1.3.1 $\mathsf{Ch}(k)$ is a combinatorial symmetric monoidal model category with cofibrant unit object, satisfies the monoid axiom, is left proper, and the class of cofibrations is generated by cofibrations between cofibrant objects[41]. The statement thus follows from [HA, 4.1.8.4, variant (B)]. □

Proposition 4.4.2.2 ([HA, 4.3.3.17])**.** *The functor*

$$\mathsf{Mixed}[W_{\mathsf{Mixed}}^{-1}] \to \mathcal{M}\mathrm{ixed}$$

from Definition 4.4.1.8 is an equivalence. ♡

Proof. The proof is very similar to the proof of Proposition 4.4.2.1. Again it suffices by Proposition 4.4.1.7 to show that the functor

$$\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))^{\mathrm{cof}}[W_{\mathsf{Mixed}}^{-1}] \to \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$$

is an equivalence.

By Fact 4.1.3.1 $\mathsf{Ch}(k)$ is a combinatorial monoidal model category with cofibrant unit object, and by Proposition 4.2.2.4 D is cofibrant. The statement thus follows from [HA, 4.3.3.17]. □

We now come to the last functor from Definition 4.4.1.8 that we still need to prove is an equivalence. As mentioned in the introduction to Section 4.4, we will not be able to merely cite an appropriate result from [HA], as the unit of Mixed is not cofibrant. We explain in more detail in Remark 4.4.2.4 below how the condition of the unit being cofibrant is used in the proof of [HA, 4.1.8.4].

Proposition 4.4.2.3. *The functor*

$$\mathrm{Alg}(\mathsf{Mixed})[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}] \to \mathrm{Alg}(\mathcal{M}\mathrm{ixed})$$

from Definition 4.4.1.8 is an equivalence. ♡

Proof. This proof will follow the proof of [HA, 4.1.8.4] closely. As in Proposition 4.4.2.1 and Proposition 4.4.2.2 it suffices by Proposition 4.4.1.7 to show that the functor

$$\mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)))^{\mathrm{cof}}[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}] \to \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)))$$

which we will call $\gamma_{\mathrm{Alg}(\mathsf{Mixed})}$ in this proof, is an equivalence.

[41] For this last bit see the description of the generating cofibrations in [Hov99, 2.3.11 and 2.3.3] in combination with the description of cofibrant objects in [Hov99, 2.3.6].

By Proposition 4.4.1.4 and Construction 4.4.1.6 there is a commutative square

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)))^{\mathrm{cof}}[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[r, "\gamma_{\mathrm{Alg}(\mathsf{Mixed})}"] \arrow[d, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}"'] & \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))) \arrow[d, "\mathrm{ev}_{\mathfrak{a}}"] \\
\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))[W^{-1}_{\mathsf{Mixed}}] \arrow[r, "\gamma_{\mathsf{Mixed}}"'] & \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))
\end{tikzcd}$$

where the horizontal functors are induced by γ, and $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}$ is induced by $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$. Proposition 4.4.2.2 shows that γ_{Mixed} is an equivalence.

Like the proof of [HA, 4.1.8.4], we will apply [HA, 4.7.3.16] to show that $\gamma_{\mathrm{Alg}(\mathsf{Mixed})}$ is an equivalence. For this it suffices to verify the following.

(1) $\mathrm{ev}_{\mathfrak{a}}$ has a left adjoint, which we will call $\mathrm{Free}_{\mathcal{M}\mathrm{ixed}}^{\mathrm{Alg}(\mathcal{M}\mathrm{ixed})}$.

(2) $\mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)))$ admits geometric realizations of simplicial objects.

(3) $\mathrm{ev}_{\mathfrak{a}}$ preserves geometric realizations of simplicial objects.

(4) $\mathrm{ev}_{\mathfrak{a}}$ is conservative.

(1') $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}$ has a left adjoint, which we will call $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})'}$.

(2') The ∞-category $\mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)))^{\mathrm{cof}}[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}]$ admits geometric realizations of simplicial objects.

(3') $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}$ preserves geometric realizations of simplicial objects.

(4') $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}$ is conservative.

(5) The push-pull natural transformation[42]

$$\mathrm{Free}_{\mathcal{M}\mathrm{ixed}}^{\mathrm{Alg}(\mathcal{M}\mathrm{ixed})} \circ \gamma_{\mathsf{Mixed}} \to \gamma_{\mathrm{Alg}(\mathsf{Mixed})} \circ \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})'}$$

is a natural equivalence.

Proof of claim (2) and (3): By Proposition 4.3.2.1 (1) $\mathcal{D}(k)$ is presentable symmetric monoidal ∞-category, so by the discussions leading to Definition 3.4.2.1, $\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$ is also a presentable symmetric monoidal ∞-category. The claims now follow from [HA, 3.2.3.1] and Proposition E.2.0.2.

Proof of claim (1): Follows from Proposition E.7.2.1, again using that $\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$ is presentable symmetric monoidal.

Proof of claim (4): This is [HA, 3.2.2.6].

Proof of claim (2'): By Proposition 4.2.2.9 the model category structure on $\mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)))$ is combinatorial, so it follows from [HA, 1.3.4.22]

[42] See [HTT, Beginning of 7.3.1].

that $\mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)))^{\mathrm{cof}}[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}]$ is presentable and hence in particular admits geometric realizations of simplicial objects.

Proof of claim (3'): This is the part of the proof where we need to do something differently than the proof of [HA, 4.1.8.4], as this is the point where the unit being cofibrant is used – see Remark 4.4.2.4 below for more details.

Consider the commutative diagram

$$\begin{tikzcd} & \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)))^{\mathrm{cof}}[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}\,\prime}"'] \arrow[dr, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})'"] & \\ \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))[W^{-1}_{\mathsf{Mixed}}] \arrow[dr, "\mathrm{ev}'_{\mathfrak{m}}"'] & & \mathrm{Alg}(\mathsf{Ch}(k))[W^{-1}_{\mathrm{Alg}}] \arrow[dl, "\mathrm{ev}'_{\mathfrak{a}}"] \\ & \mathsf{Ch}(k)[W^{-1}_{\mathsf{Ch}}] & \end{tikzcd}$$

that already appeared above as diagram (4.20)[43]. As the diagram commutes, it suffices to show the following three claims.

(a) The functor $\mathrm{ev}'_{\mathfrak{m}}$ in the above diagram detects geometric realizations of simplicial objects[44].

(b) The functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})'$ in the above diagram preserves geometric realizations of simplicial objects.

(c) The functor $\mathrm{ev}'_{\mathfrak{a}}$ in the above diagram preserves geometric realizations of simplicial objects.

Proof of claim (a): By Definition 4.4.1.8, Proposition 4.4.2.2, and Proposition 4.3.2.1 (1), we can identify the functor $\mathrm{ev}'_{\mathfrak{m}}$ in question with the functor

$$\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) \to \mathcal{D}(k)$$

which, as $\mathcal{D}(k)$ is presentable symmetric monoidal by Proposition 4.3.2.1 (1), detects small colimits by [HA, 4.2.3.5 (2)].

Proof of claim (b): By [HA, 1.3.4.24 and 1.3.4.25], it suffices to show that the functor

$$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}}) \colon \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))) \to \mathrm{Alg}(\mathsf{Ch}(k))$$

[43] With the tiny difference that we added a $-^{\mathrm{cof}}$ at the top, but by Proposition 4.4.1.7 this doesn't matter anyway.

[44] In other words it detects $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits.

preserves homotopy colimits. Homotopy colimits can be calculated by taking the colimit of a cofibrant replacement of the diagram with respect to the projective model structure on diagram categories, see [HTT, A.2.8]. As $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ preserves ordinary colimits and weak equivalences by Proposition 4.2.2.12 it hence suffices to show that

$$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})_*\colon \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)))) \to \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Alg}(\mathsf{Ch}(k)))$$

preserves generating cofibrations. But this follows from their description [HTT, A.2.8.5] and the fact that $\mathrm{Alg}(V)$ preserves colimits and cofibrations by Proposition 4.2.2.12.

Proof of claim (c): By Definition 4.4.1.8, Proposition 4.4.2.1, and Proposition 4.3.2.1 (1), we can identify the functor $\mathrm{ev}'_{\mathfrak{a}}$ in question with the functor

$$\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{D}(k)$$

which, as $\mathcal{D}(k)$ is presentable symmetric monoidal by Proposition 4.3.2.1 (1), preserves sifted colimits by [HA, 3.2.3.1].

Proof of claim (4'): It suffices to show that the induced functor on homotopy categories is conservative, i. e. reflects isomorphisms. By Proposition A.1.0.1 we can identify that functor with the functor induced by

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\colon \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))) \to \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$$

on homotopy categories of the model categories, i. e.

$$\mathrm{Ho}_{W_{\mathrm{Alg}(\mathsf{Mixed})}}(\mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)))) \to \mathrm{Ho}_{W_{\mathsf{Mixed}}}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)))$$

which is conservative by the classical constructions for homotopy categories[45], as $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ detects weak equivalences by Proposition 4.2.2.12.

Proof of claims (1') and (5): We consider the symmetric monoidal functor

$$\mathrm{LMod}_{\mathrm{D}}(\gamma)^{\otimes}\colon \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)^{\otimes} \to \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))^{\otimes}$$

from Construction 4.4.1.1. We want to show that the underlying functor preserves coproducts and that both $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$ and $\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$ admit coproducts and have tensor product functors that preserve coproducts in each variable separately.

That $\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$ is a presentable symmetric monoidal ∞-category was already mentioned above.

As the forgetful functor $\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)) \to \mathsf{Ch}(k)$ preserves colimits by Proposition 4.2.2.12, it follows that the subcategory $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$ is closed under coproducts[46] and hence admits coproducts, which are calculated in $\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$ (see [HTT, 1.2.13.7]). As $\mathrm{ev}_{\mathfrak{m}}$ detects colimits and

[45] See [Hov99, 1.2].

[46] Cofibrant objects in a model category are closed under coproducts, which can be checked using the lifting property that defines cofibrations, see [Hov99, 1.1.10].

is symmetric monoidal, and the tensor product in $\mathsf{Ch}(k)$ is compatible with colimits[47] we can conclude that the tensor product of $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$ preserves coproducts in each variable separately.

Finally, we show that the functor $\mathrm{LMod}_{\mathrm{D}}(\gamma)$ preserves coproducts. To see this, note that as argued in the proof of claim (a), the functor

$$\mathrm{ev}_{\mathfrak{m}}\colon\ \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) \to \mathcal{D}(k)$$

detects small colimits, and as by the discussion above the forgetful functor

$$\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathsf{Ch}(k)^{\mathrm{cof}}$$

preserves coproducts, it suffices to show that the functor

$$\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$$

preserves coproducts, which is true by Proposition 4.3.2.1 (3).

We have now verified that $\mathrm{LMod}_{\mathrm{D}}(\gamma)^{\otimes}$ satisfies the assumptions of variant (2) of Proposition E.7.2.2. We thus obtain a left adjoint

$$\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}\colon\ \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathrm{Alg}\big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\big)$$

to the forgetful functor $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$, which can be identified with a restriction of the functor of the same name defined in Notation 4.2.2.10. More crucially, Proposition E.7.2.2 shows that the push-pull transformation

$$\mathrm{Free}_{\mathcal{M}\mathrm{ixed}}^{\mathrm{Alg}(\mathcal{M}\mathrm{ixed})} \circ \mathrm{LMod}_{\mathrm{D}}(\gamma) \to \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\gamma)) \circ \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$$

is an equivalence.

The functor

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\colon\ \mathrm{Alg}\big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\big) \to \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$$

preserves weak equivalences by Proposition 4.2.2.12. We next show that the functor

$$\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}\colon\ \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathrm{Alg}\big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\big)$$

also preserves weak equivalences. As the functor

$$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\colon\ \mathrm{Alg}\big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\big) \to \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$$

detects weak equivalences by Proposition 4.2.2.12, it suffices to check that the composition preserves weak equivalences. This composition is by Proposition 4.2.2.11 naturally isomorphic to the composition of

$$\mathrm{ev}_{\mathfrak{m}}\colon\ \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathsf{Ch}(k)^{\mathrm{cof}}$$

[47]As the symmetric monoidal structure is closed by Definition 4.1.2.1.

with $\mathrm{Free}^{\mathrm{Alg}}$. But by Proposition 4.2.2.12, $\mathrm{ev}_{\mathfrak{m}}$ preserves weak equivalences, and $\mathrm{Free}^{\mathrm{Alg}}$ preserves weak equivalences between cofibrant objects as a left Quillen functor.

As $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ and $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$ preserve weak equivalences, they induce functors on the ∞-categories obtained by inverting weak equivalences. Additionally, unit and counit of the adjunction induce unit and counit of an adjunction as follows[48]

$$\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W_{\mathsf{Mixed}}^{-1}] \underset{\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}\prime}}{\overset{\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})\prime}}{\rightleftarrows}} \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\Big)[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}]$$

where we think of adjunctions in terms of units and counits as in Proposition D.2.1.1.

In the non-dashed commutative square

$$\begin{tikzcd}
\mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))^{\mathrm{cof}}\Big)[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}] \arrow[r, "\gamma_{\mathrm{Alg}(\mathsf{Mixed})}"] \arrow[d, shift left, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}\prime}"] & \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))) \arrow[d, shift left, "\mathrm{ev}_{\mathfrak{a}}"] \\
\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W_{\mathsf{Mixed}}^{-1}] \arrow[r, "\gamma_{\mathsf{Mixed}}"'] \arrow[u, dashed, shift left, "\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})\prime}"] & \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) \arrow[u, dashed, shift left, "\mathrm{Free}_{\mathcal{M}\mathrm{ixed}}^{\mathrm{Alg}(\mathcal{M}\mathrm{ixed})}"]
\end{tikzcd} \qquad (*)$$

from Proposition 4.4.1.4, there is thus an induced left adjoint of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}\prime}$ as indicated. Furthermore, as unit and counit of the adjunction on the left are induced by the unit and counit of the adjunction $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})} \dashv \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$, we can identify the push-pull transformation associated to the square with the natural transformation induced by the push-pull transformation

$$\mathrm{Free}_{\mathcal{M}\mathrm{ixed}}^{\mathrm{Alg}(\mathcal{M}\mathrm{ixed})} \circ \mathrm{LMod}_{\mathrm{D}}(\gamma) \to \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\gamma)) \circ \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$$

by passing from $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$ to $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W_{\mathsf{Mixed}}^{-1}]$. As the latter is a natural equivalence, it follow that the push-pull transformation associated to diagram $(*)$ is also a natural equivalence.

Finally, the functor

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}\prime}\colon\ \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))^{\mathrm{cof}}\Big)[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}] \to \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W_{\mathsf{Mixed}}^{-1}]$$

discussed so far is by Proposition 4.4.1.7 equivalent to the functor

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}\prime}\colon\ \mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k)))^{\mathrm{cof}}[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}] \to \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))[W_{\mathsf{Mixed}}^{-1}]$$

so this proves claims (1') and (5). □

[48] See the universal property of inverting morphisms in ∞-categories in [HA, 1.3.4.1].

Remark 4.4.2.4. While the statement [HA, 4.1.8.4] is formulated in such a way as to require the unit object to be cofibrant, thereby preventing us from using the result directly to show Proposition 4.4.2.3, let us remark on where this is used in the proof.

The main step in proving [HA, 4.1.8.4] is the lemma [HA, 4.1.8.13], which shows that if C is a monoidal model category satisfying certain assumptions and J is a small sifted category, then the forgetful functor $\mathrm{ev}_{\mathfrak{a}} \colon \mathrm{Alg}(\mathsf{C}) \to \mathsf{C}$ preserves J-indexed homotopy colimits.

The proof proceeds by showing that every projectively cofibrant object A of the functor category $\mathrm{Fun}(\mathsf{J}, \mathrm{Alg}(\mathsf{C}))$ is a retract of a certain transfinite composition with favorable properties[49]. What needs to be shown is that $(\mathrm{ev}_{\mathfrak{a}})_*(A)$ is *good*, an ad hoc property used in the proof, which is shown by transfinite induction.

The induction start needs that $(\mathrm{ev}_{\mathfrak{a}})_*(\mathrm{const}_{\mathbb{1}_{\mathsf{C}}})$ is good. The argument in [HA, (3) on page 500] shows that every constant functor whose value is a cofibrant object in C is good, so if one assumes that the unit $\mathbb{1}_{\mathsf{C}}$ is cofibrant in C, then this proves the induction start. Combining [HA, (3) on page 500] with the definition of good objects [HA, Middle of page 499] one sees that a constant functor $\mathsf{J} \to \mathsf{C}$ is actually good if and only if the constant value is cofibrant in C.

So if $\mathsf{C} = \mathsf{Mixed}$, where the unit is not cofibrant by Proposition 4.2.2.5, then the induction start fails, so $\mathrm{ev}_{\mathfrak{a}}$ preserving homotopy colimits needs to be proven in a different way than [HA, 4.1.8.13]. ♢

4.4.3 $\mathcal{M}$ixed is stable

In this section we show that $\mathcal{M}$ixed is a stable ∞-category.

Proposition 4.4.3.1. *The ∞-category $\mathcal{M}$ixed is stable*[50]*.* ♡

Proof. The statement follows by combining that $\mathcal{D}(k)$ is stable by Proposition 4.3.2.1 (2) with $\mathcal{M}$ixed admitting all small limits and colimits by [HA, 4.2.3.3 (1) and 4.2.3.5 (1)] and $\mathrm{ev}_{\mathfrak{m}} \colon \mathcal{M}\mathrm{ixed} \to \mathcal{D}(k)$ detecting small colimits and limits as well as equivalences by [HA, 4.2.3.3 (2) and 4.2.3.5 (2)]. □

4.4.4 Strongly homotopy linear morphisms

In Section 4.2.3 we introduced the notion of strongly homotopy linear morphisms between strict mixed complexes. In this short section we discuss how they induce morphisms in the ∞-category of mixed complexes.

Construction 4.4.4.1. Let X and Y be strict mixed complexes with cofibrant underlying chain complexes, and $f \colon X \to Y$ a strongly homotopy linear morphism. Recall from Proposition 4.2.3.7 and Definition 4.2.3.8 that f

[49] See [HA, End of page 500 and start of page 501].
[50] See [HA, 1.1.1.9] for a definition.

lifts to a morphism $f^{\mathrm{strict}}\colon X \to Y^{\mathrm{shl}}$ of strict mixed complexes, and from Proposition 4.2.3.5 that Y^{shl} comes with a quasiisomorphism of strict mixed complexes $\iota_Y^{\mathrm{shl}}\colon Y \to Y^{\mathrm{shl}}$.

We can't directly apply γ_{Mixed} to f^{strict}, as Y^{shl} might not have cofibrant underlying chain complex[51]. However we obtain a commutative diagram

$$\begin{tikzcd} X^{\mathrm{cof}} \arrow[r, "\left(f^{\mathrm{strict}}\right)^{\mathrm{cof}}"] \arrow[d] & \left(Y^{\mathrm{shl}}\right)^{\mathrm{cof}} \arrow[d] & Y^{\mathrm{cof}} \arrow[l, "\left(\iota_Y^{\mathrm{shl}}\right)^{\mathrm{cof}}"'] \arrow[d] \\ X \arrow[r, "f^{\mathrm{strict}}"'] & Y^{\mathrm{shl}} & Y \arrow[l, "\iota_Y^{\mathrm{shl}}"] \end{tikzcd} \tag{$*$}$$

in Mixed, where the vertical morphisms are the cofibrant replacements in Mixed, and by Proposition 4.2.2.12 all strict mixed complexes except possibly Y^{shl} in this diagram have cofibrant underlying chain complex. We can thus apply γ_{strict} to the part of the diagram not involving Y^{shl}.

$$\begin{tikzcd} \gamma_{\mathsf{Mixed}}\left(X^{\mathrm{cof}}\right) \arrow[r, "\gamma_{\mathsf{Mixed}}\left(\left(f^{\mathrm{strict}}\right)^{\mathrm{cof}}\right)"] \arrow[d, "\simeq"'] & \gamma_{\mathsf{Mixed}}\left(\left(Y^{\mathrm{shl}}\right)^{\mathrm{cof}}\right) & \gamma_{\mathsf{Mixed}}\left(Y^{\mathrm{cof}}\right) \arrow[l, "\gamma_{\mathsf{Mixed}}\left(\left(\iota_Y^{\mathrm{shl}}\right)^{\mathrm{cof}}\right)"', "\simeq"] \arrow[d, "\simeq"] \\ \gamma_{\mathsf{Mixed}}(X) \arrow[rr, dashed, "\gamma_{\mathsf{Mixed}}(f)"'] & & \gamma_{\mathsf{Mixed}}(Y) \end{tikzcd} \tag{4.22}$$

As the vertical morphisms in diagram $(*)$ as well as $\left(\iota_Y^{\mathrm{shl}}\right)^{\mathrm{cof}}$ are quasiisomorphisms, the corresponding morphisms in diagram (4.22) are equivalences. We can thus form the composition from X to Y, yielding a morphism in $\mathcal{M}\mathrm{ixed}$ that we will denote by $\gamma_{\mathsf{Mixed}}(f)$ and call the *morphism in* $\mathcal{M}\mathrm{ixed}$ *induced by* f. ♢

Remark 4.4.4.2. Let X and Y be strict mixed complexes with cofibrant underlying chain complex, and let $f\colon X \to Y$ be a strongly homotopy linear quasiisomorphism[52]. Then the induced morphism

$$\gamma_{\mathsf{Mixed}}(f)\colon \gamma_{\mathsf{Mixed}}(X) \to \gamma_{\mathsf{Mixed}}(Y)$$

is an equivalence. Indeed, considering diagram (4.22) in Construction 4.4.4.1, it is enough to show that f^{strict} is a quasiisomorphism. As the underlying morphism of chain complexes of f is by definition the composition of f^{strict} with the underlying morphism of chain complexes of p_Y^{shl}, which is a quasiisomorphism by Proposition 4.2.3.6, this follows from the underlying morphism of chain complexes of f being a quasiisomorphism. ♢

[51] This problem is related to the fact that Y^{shl} involves an infinite product (rather than an infinite coproduct, which would not be a problem).

[52] By this we mean a strongly homotopy linear morphism whose underlying morphism of chain complexes is a quasiisomorphism.

Chapter 5

Mixed complexes and circle actions

In Section 6.2.1 we will see that Hochschild homology carries a natural action by the circle group $\mathbb{T}$, i. e. Hochschild homology forms a functor

$$\mathrm{HH}_{\mathbb{T}}\colon\ \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} = \mathrm{Fun}(\mathrm{B}\,\mathbb{T}, k)$$

where $\mathrm{B}\,\mathbb{T}$ can be thought of as the ∞-groupoid with one object $*$ and $\mathrm{Aut}_{\mathrm{B}\,\mathbb{T}}(*) \simeq \mathbb{T}$, where $\mathbb{T}$ can be defined as $\{ z \in \mathbb{C} \mid |z| = 1 \}$. We will define $\mathbb{T}$ properly in Section 5.2.1 and $\mathrm{B}\,\mathbb{T}$ in Section 5.3.

For calculations it will be helpful to have model categories available that represent the involved ∞-categories. We have seen in Section 4.3.2 that $\mathcal{D}(k)$ is the underlying ∞-category of $\mathsf{Ch}(k)$ with the projective model structure. By [HA, 4.1.8.4], the model structure on $\mathrm{Alg}(\mathsf{Ch}(k))$ discussed in Theorem 4.2.2.1 has $\mathrm{Alg}(\mathcal{D}(k))$ as underlying ∞-category. This takes care of the domain of $\mathrm{HH}_{\mathbb{T}}$. How about the codomain?

If $\mathrm{B}\,\mathbb{T}$ were a 1-category, then we could apply [HA, 1.3.4.25], which would then imply that $\mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{D}(k))$ is the underlying ∞-category of the injective or projective model structure on $\mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathsf{Ch}(k))$. This is however not the case – $\mathrm{B}\,\mathbb{T}$ is a 2-category, but not a 1-category. We must thus proceed differently.

In Section 5.2 we will define a cocommutative bialgebra $k \boxtimes \mathbb{T}$ in $\mathcal{D}(k)$, and in Section 5.3 we will show that there is a symmetric monoidal equivalence

$$\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k))$$

where the ∞-category $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ carries the pointwise symmetric monoidal structure and $\mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k))$ the one from Definition 3.4.2.1.

By [HA, 4.3.3.17] the model category $\mathrm{LMod}_A(\mathsf{Ch}(k))$, with model structure as in Theorem 4.2.2.1, has $\mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k))$ as its underlying ∞-category if A is a differential graded algebra with cofibrant underlying complex and such that $\gamma(A) \simeq k \boxtimes \mathbb{T}$ as associative algebras.

We will show in Section 5.1 that the differential graded algebra D defined in Construction 4.2.1.1 represents $k \boxtimes \mathbb{T}$ as an associative algebra. In fact, we show more – D even represents $k \boxtimes \mathbb{T}$ as an associative and coassociative *bialgebra*. There is thus a monoidal (though not symmetric monoidal!)

equivalence as follows.

$$\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \simeq \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) = \mathcal{M}\mathrm{ixed}$$

Let us end by briefly going over the contents of the individual sections. We will start in Section 5.1 by showing a formality statement for commutative and coassociative bialgebras in $\mathcal{D}(k)$ with homology isomorphic to the homology of D and $k \boxtimes \mathbb{T}$. We will actually define $\mathbb{T}$ and $k \boxtimes \mathbb{T}$ in Section 5.2, and then use the result of Section 5.1 to conclude in Section 5.2.4 that $\mathrm{D} \simeq k \boxtimes \mathbb{T}$ as bialgebras. We show that there are symmetric monoidal equivalences of the form $\mathrm{Fun}(\mathrm{B}\,G, \mathcal{C}) \simeq \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}\boxtimes G}(\mathcal{C})}$ for presentable ∞-categories $\mathcal{C}$ and grouplike associative monoids G in $\mathcal{S}$ in Section 5.3. Finally, we put everything together to obtain the monoidal equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ in Section 5.4.

5.1 Formality of certain $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebras

In this section we show that any two commutative and coassociative bialgebras in $\mathcal{D}(k)$ with homology concentrated in degrees 0 and 1, where it is k, are equivalent as commutative and coassociative bialgebras.

Let us summarize the strategy used to prove this, which was suggested by Achim Krause. Let R be a commutative and coassociative bialgebra with homology as described. Then it suffices to construct another such commutative and coassociative bialgebra independently of R and construct an equivalence between that commutative bialgebra and R.

How could we go about to construct a morphism of commutative bialgebras? Or more generally, of algebras or coalgebras over some ∞-operad? There is one class of algebras where is easy to define morphisms out of, the *free* algebras, using that the free algebra functor is left adjoint to the forgetful functor. Analogously, it is easy to define morphisms of coalgebras into *cofree* algebras. While these concepts are in principle dual to each other, (by passing to opposite ∞-categories), it is in practice easier to work with free algebras than with cofree coalgebras. This is because the theory of free algebras works particularly well when the tensor products are compatible with colimits, see [HA, 3.1.3.5], which is usually the case in the kind of examples that we are interested in. Analogously, we would want the tensor products to be compatible with limits in order to obtain a good theory of cofree coalgebras, but this is usually *not* the case in examples of interest.

The discussion so far points us towards trying to find some kind of free resolution of the commutative and coassociative bialgebra R. Unfortunately, free *commutative* algebras are not quite as easy to describe as free associative algebras[1], as imposing commutativity requires taking certain (homotopy) orbits of actions by the symmetric groups Σ_n. Commutative algebras being

[1] [HA, 3.1.3.13] offers a description of free commutative and free associative algebras. We discuss the special case of associative algebras in Proposition E.7.2.1, and will unpack the statement for commutative algebras in the proof of Proposition 5.1.5.3.

more difficult to deal with in some respects is also reflected in the following fact. Let C be a reasonably nice symmetric monoidal model category that one finds in nature. Then it is often the case that $\mathrm{Alg}(\mathsf{C})$ inherits a nice model structure from C such that its underlying ∞-category is the ∞-category of algebras in the underlying ∞-category of C. However it is unreasonable to expect the analogous statement to hold for *commutative* algebras, which has to do with Σ_n orbits of the action of Σ_n on $X^{\otimes n}$ not necessarily being homotopy orbits[2].

So we would prefer to work with free associative algebras. To do so, we dualize the problem: R is dualizable in the symmetric monoidal ∞-category $\mathcal{D}(k)$, and the functor mapping a dualizable object to its dual,

$$(-)^\vee \colon (\mathcal{D}(k)_{\mathrm{fd}})^{\mathrm{op}} \to \mathcal{D}(k)_{\mathrm{fd}}$$

is symmetric monoidal equivalence and thus induces an equivalence

$$\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k)_{\mathrm{fd}}) = \mathrm{coAlg}(\mathrm{CAlg}(\mathcal{D}(k)_{\mathrm{fd}})) \simeq \mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)_{\mathrm{fd}}))^{\mathrm{op}}$$

so that it actually suffices to show that $R^\vee$ is formal.

To do so we will define a diagram

$$\begin{tikzcd}
\underline{B}_2 \arrow[r] \arrow[d] & k \arrow[d] & \underline{B}_4 \arrow[r] \arrow[d] & k \arrow[d] & \\
A_1 \arrow[r] & A_2 \arrow[r] & A_3 \arrow[r] & A_4 \arrow[r] & \dots \\
& \underline{B}_3 \arrow[r] \arrow[u] & k \arrow[u] & &
\end{tikzcd} \tag{5.1}$$

[2]The relevant compatibility result for associative algebras is [HA, 4.1.8.4], and for commutative algebras [HA, 4.5.4.7]. The assumptions necessary for the result on associative algebras are mild enough to usually hold in examples one is interested in. The assumptions made for commutative algebras however include that every cofibration must be a power cofibration (see [HA, 4.5.4.2]). This is a strong condition that one can not expect to hold in general for otherwise nice examples found in nature. For example $\mathsf{Ch}(k)$ with the projective model structure (see Fact 4.1.3.1) does not in general have this property. The chain complex $k[0]$ is cofibrant, so we would need $k[0]$ to be power cofibrant. Let $n > 1$ and let X be the chain complex concentrated in degrees 0 and 1 with $X_0 = X_1 = k^{\oplus n}$, with $\partial_1^X = \mathrm{id}$, and with Σ_n acting by permutation, and let Y be the chain complex concentrated in degrees 0 and 1 with $Y_0 = Y_1 = k$, with $\partial_1^Y = \mathrm{id}$, and with Σ_n acting trivially. There is an Σ_n-equivariant chain morphism $f \colon X \to Y$ that maps a tuple $(a_1, \dots, a_n)$ to $\sum_{1 \le i \le n} a_i$. This morphism is an acyclic fibration in the projective model structure on $\mathsf{Ch}(k)^{\mathrm{B}\,\Sigma_n}$. Let $\varphi \colon k[0] \cong k[0]^{\otimes n} \to Y$ be the inclusion (i.e. the identity in level 0). If $k[0]$ were power cofibrant, then it would need to be possible to lift φ in a Σ_n-equivariant manner to a chain morphism $\overline{\varphi} \colon k[0] \to X$. Suppose $\overline{\varphi}$ is such a lift. Let $\overline{\varphi}(1) = (a_1, \dots, a_n)$. That $\overline{\varphi}$ is Σ_n equivariant implies that $a := a_1 = \dots = a_n$. We must then have

$$1 = \varphi(1) = f(\overline{\varphi})(1) = f((a, \dots, a)) = n \cdot a$$

in k, so n must be invertible in k. But there are many interesting commutative rings that do not contain $\mathbb{Q}$.

in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$ such that each square is a pushout square and the colimit of $A_1 \to A_2 \to \ldots$ has homology isomorphic to $\mathrm{H}_*(R^\vee)$. Furthermore, every $\underline{B}_n$ as well as A_1 will be free as an associative algebra on the underlying pointed object in $\mathrm{coCAlg}(\mathcal{D}(k))$.

It will then be possible to define a morphism $A_1 \to R^\vee$ that is surjective on homology, so that it suffices to show that this morphism can be lifted inductively to each A_n. As k is a zero object in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$ (this will be shown in Remark 5.1.2.9), this amounts to showing that the composites $\underline{B}_n \to A_{n-1} \to R^\vee$ are nullhomotopic in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$. Using freeness, dualizing again, and calculations that exploit the homology of $R^\vee$, it will actually be possible to show that in fact *any* morphism $\underline{B}_n \to R^\vee$ in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$ is nullhomotopic.

We now briefly summarize the content of the individual subsections. We start in Section 5.1.1 by discussing dualizable objects in symmetric monoidal ∞-categories and the symmetric monoidal duality functor. In Section 5.1.2 we will then construct diagram (5.1). In order to show that any two morphism $\underline{B}_n \to R^\vee$ are homotopic as discussed above, we will need a formality statement for certain associative algebras, which we show in Section 5.1.3, and of commutative algebras like R as commutative algebras in $\mathcal{D}(k)$, which we will show in Section 5.1.5. As the case of commutative algebras involves arguing about orbits of actions of Σ_n, there is also a short Section 5.1.4 discussing the relationship of orbits of group actions in $\mathcal{D}(k)$ with group homology. The result regarding mapping spaces that we discussed above will then be shown in Section 5.1.6, and everything will be put together in Section 5.1.7 to show formality of R as a commutative and coassociative bialgebra in $\mathcal{D}(k)$.

The subsections do not all depend on all the previous ones. The following diagram shows the dependencies.

$$\begin{tikzcd}
\text{Section 5.1.1} \arrow[dddrr, bend right] & \text{Section 5.1.2} \arrow[dddr, bend right] & \text{Section 5.1.4} \arrow[d] & \text{Section 5.1.3} \arrow[dl] \arrow[dddl] \arrow[dd] \\
 & & \text{Section 5.1.5} \arrow[dr] \arrow[dd] & \\
 & & & \text{Section 5.1.6} \arrow[dl] \\
 & & \text{Section 5.1.7} &
\end{tikzcd}$$

5.1.1 Duality

In this section we discuss the notion of dualizable objects in symmetric monoidal ∞-categories, and we mostly follow [HA, 4.6.1], [HA, 5.2.1 and 5.2.2], and [Lur18, 3.2]. We start by recalling the definition of dualizable objects.

Definition 5.1.1.1 ([HA, 4.6.1.7, see also 4.6.1.12])**.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let C be an object of $\mathcal{C}$. The object C is called *dualizable* if there exists an object B of $\mathcal{C}$ and morphisms $c\colon \mathbb{1}_\mathcal{C} \to C \otimes B$ and $e\colon B \otimes C \to \mathbb{1}$ such that the composites

$$C \simeq \mathbb{1}_\mathcal{C} \otimes C \xrightarrow{c \otimes \mathrm{id}_C} C \otimes B \otimes C \xrightarrow{\mathrm{id}_C \otimes e} C \otimes \mathbb{1}_\mathcal{C} \simeq C$$

and

$$B \simeq B \otimes \mathbb{1}_\mathcal{C} \xrightarrow{\mathrm{id}_B \otimes c} B \otimes C \otimes B \xrightarrow{e \otimes \mathrm{id}_B} \mathbb{1}_\mathcal{C} \otimes B \simeq B$$

are homotopic to the identity.

In this case, we call B the *dual* of C, and write $B = C^\vee$; by [HA, 4.6.1.6 and 4.6.1.10] $C^\vee$ as well as c and e are essentially uniquely determined by C. We will also call C together with B, c, e, and homotopies as above a *duality datum*.

We let $\mathcal{C}_{\mathrm{fd}}$ be the full subcategory of $\mathcal{C}$ spanned by the dualizable objects. ♢

Remark 5.1.1.2. It follows easily from the definition that if C and C' are dualizable objects in a symmetric monoidal ∞-category $\mathcal{C}$, with c and e as in Definition 5.1.1.1 exhibiting $C^\vee$ as the dual of C and similarly c' and e' exhibiting $C'^\vee$ as a dual of C', then the compositions

$$\mathbb{1}_\mathcal{C} \simeq \mathbb{1}_\mathcal{C} \otimes \mathbb{1}_\mathcal{C} \xrightarrow{c \otimes c'} C \otimes C^\vee \otimes C' \otimes C'^\vee \xrightarrow{\mathrm{id}_C \otimes \tau \otimes \mathrm{id}_{C'^\vee}} (C \otimes C') \otimes (C^\vee \otimes C'^\vee)$$

and

$$(C^\vee \otimes C'^\vee) \otimes (C \otimes C') \xrightarrow{\mathrm{id}_{C^\vee} \otimes \tau \otimes \mathrm{id}_{C'}} C^\vee \otimes C \otimes C'^\vee \otimes C' \xrightarrow{e \otimes e'} \mathbb{1}_\mathcal{C} \otimes \mathbb{1}_\mathcal{C} \simeq \mathbb{1}_\mathcal{C}$$

exhibit $C^\vee \otimes C'^\vee$ as a dual of $C \otimes C'$, where τ is the symmetry equivalence and $\mathbb{1}_\mathcal{C} \simeq \mathbb{1}_\mathcal{C} \otimes \mathbb{1}_\mathcal{C}$ is the unitality equivalence. In particular the tensor product of two dualizable objects is again dualizable. Furthermore, $\mathbb{1}_\mathcal{C}$ is dualizable with dual $\mathbb{1}_\mathcal{C}$, so it follows from [HA, 2.2.1.2] that $\mathcal{C}_{\mathrm{fd}}$ inherits a symmetric monoidal structure from $\mathcal{C}$ such that the inclusion can be upgraded to a symmetric monoidal functor. ♢

It is easy to see from the definition that if C is dualizable with dual $C^\vee$, then $C^\vee$ is again dualizable with dual $C^{\vee\vee} \simeq C$. It is also clear from the definition that a symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ maps $\mathcal{C}_{\mathrm{fd}}$ to $\mathcal{D}_{\mathrm{fd}}$ and so restricts to a symmetric monoidal functor $F\colon \mathcal{C}_{\mathrm{fd}} \to \mathcal{D}_{\mathrm{fd}}$. In fact, the following is true.

Fact 5.1.1.3 ([Lur18, 3.2.4])**.** *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the assignment $C \mapsto C^\vee$ sending an object of $\mathcal{C}$ to a dual can be upgraded to an equivalence of symmetric monoidal ∞-categories*

$$(-)^\vee\colon (\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} \to \mathcal{C}_{\mathrm{fd}}$$

with inverse $((-)^\vee)^{\mathrm{op}}$.

Furthermore, this equivalence is compatible with symmetric monoidal functors in the following sense. Let $F\colon \mathcal{C} \to \mathcal{D}$ *be a symmetric monoidal functor. Then there is a commutative diagram of symmetric monoidal functors as follows.*

$$\begin{tikzcd}
(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} \arrow[r, "(-)^\vee"] \arrow[d, "F^{\mathrm{op}}"'] & \mathcal{C}_{\mathrm{fd}} \arrow[d, "F"] \\
(\mathcal{D}_{\mathrm{fd}})^{\mathrm{op}} \arrow[r, "(-)^\vee"'] & \mathcal{D}_{\mathrm{fd}}
\end{tikzcd}$$

♧

Remark 5.1.1.4. While the part of the statement of Fact 5.1.1.3 about compatibility with symmetric monoidal functors is not stated explicitly in [Lur18, 3.2.4][3], this becomes clear by going though every step of the proof. In this remark we provide some pointers to the relevant parts of the proof of [Lur18, 3.2.4] as well as the relevant material in [HA, 5.2.1 and 5.2.2] that is relevant for checking this.

First, as F maps dualizable objects to dualizable objects, it suffices to consider the case in which every object in $\mathcal{C}$ and $\mathcal{D}$ is dualizable.

Then the construction of the pairing of ∞-categories

$$\lambda = \mathrm{pr}_1\colon (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/\mathbb{1}_{\mathcal{C}}} \to \mathcal{C} \times \mathcal{C}$$

as well as its upgrade to a pairing of symmetric monoidal ∞-categories, is compatible with F. Furthermore, the description of left and right universal objects in $(\mathcal{C} \times \mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/\mathbb{1}_{\mathcal{C}}}$ from the proof of [Lur18, 3.2.4] together with the fact that F preserves duality data implies that the morphism of pairings of ∞-categories induced by F is left and right representable (see [HA, 5.2.1.16]). The symmetric monoidal functor $(-)^\vee$ for $\mathcal{C}$ is constructed in [HA, 5.2.2.25] as a lax symmetric monoidal functor – it is the left duality functor $\mathfrak{D}_\lambda^\otimes$ that uses that λ is left representable. It is shown in [Lur18, 3.2.4] that this functor is actually symmetric monoidal, but as symmetric monoidal functors form a *full* subcategory of lax symmetric monoidal functors [HA, 2.1.3.7] it suffices to consider these functors as lax symmetric monoidal functors when discussing compatibility with F.

So one only needs to check that the construction of the lax symmetric monoidal left duality functors of left representable pairings of symmetric monoidal ∞-categories are compatible with left representable morphisms of left representable pairings of symmetric monoidal ∞-categories. The lax symmetric monoidal functor $\mathfrak{D}_\lambda^\otimes$ for $\mathcal{C}$ is constructed as the composition of the inverse of a symmetric monoidal equivalence $\varphi_{\mathcal{C}}\colon (\mathcal{C}_\lambda^0)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ with the lax monoidal inclusion $\iota_{\mathcal{C}}\colon (\mathcal{C}_\lambda^0)^{\mathrm{op}} \to (\mathcal{C}_\lambda)^{\mathrm{op}}$ and a symmetric monoidal functor

[3] Functoriality *is* however used and alluded to with [Lur18, 3.2.6].

$\psi_{\mathcal{C}}\colon (\mathcal{C}_\lambda)^{\mathrm{op}} \to \mathcal{C}$, so it suffices to check that each of those is suitably compatible with F.

The inclusion $\iota_{\mathcal{C}}$ is defined in [HA, 5.2.1.28], and can be upgraded to a lax symmetric monoidal functor by the discussion in [HA, 5.2.2.25] together with [HA, 2.2.1.9]. That it is compatible with F follows from the definition together with $\iota_{\mathcal{D}}^{\otimes}$ being fully faithful and [HA, 5.2.1.17].

The symmetric monoidal equivalence $\varphi_{\mathcal{C}}$ is the composition of $\iota_{\mathcal{C}}$ with the functor constructed in [HA, 5.2.1.29]. It is clear from definition that this latter functor is compatible with F.

Finally, $\psi_{\mathcal{C}}$ arises from the counit of an adjunction as discussed in [HA, 5.2.2.24] and is thus compatible with F. ♢

Remark 5.1.1.5. Let us give some hints regarding the opposite of the dualization functor being its inverse. Let us – as in Remark 5.1.1.4 – reduce to the case where every object of $\mathcal{C}$ is dualizable. The duality functor discussed so far, in particular in Remark 5.1.1.4 starts with the pairing of ∞-categories $\lambda = \mathrm{pr}_1\colon \mathcal{M} = (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/\mathbb{1}_{\mathcal{C}}} \to \mathcal{C} \times \mathcal{C}$ that is both left and right representable. This pairing can be upgraded to a pairing of symmetric monoidal ∞-categories $\lambda^{\otimes}$, and then left representability of λ is used to construct a lax symmetric monoidal morphism of pairings of symmetric monoidal ∞-categories

$$\begin{tikzcd}
\mathrm{TwArr}(\mathcal{C})^{\otimes} \arrow[r] \arrow[d] & \mathcal{M}^{\otimes} \arrow[d, "\lambda^{\otimes}"] \\
\mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} (\mathcal{C}^{\mathrm{op}})^{\otimes} \arrow[r, "\mathrm{id}_{\mathcal{C}^{\otimes}} \times \mathfrak{D}_\lambda^{\otimes}"'] & \mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathcal{C}^{\otimes}
\end{tikzcd}$$

where the bottom functor is on the second factor precisely the lax symmetric monoidal left duality functor that we are interested in and called $(-)^{\vee}$. See [HA, 5.2.2.25] and also Remark 5.1.1.4.

Now the important point is that the underlying morphism of pairings of ∞-categories is right representable. If we assume this for the moment, then we can use functoriality of *right* duality functors (which can be shown completely analogously to the case of left duality functors sketched in Remark 5.1.1.4) to obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[r, "\mathrm{id}"] \arrow[d, "(\mathfrak{D}_\lambda^{\mathrm{op}})^{\otimes}"'] & \mathcal{C}^{\otimes} \arrow[d, "\mathrm{id}"] \\
(\mathcal{C}^{\mathrm{op}})^{\otimes} \arrow[r, "\mathfrak{D}_\lambda'^{\otimes}"'] & \mathcal{C}^{\otimes}
\end{tikzcd}$$

where the top horizontal functor is the right duality functor of $\mathrm{TwArr}(\mathcal{C})^{\otimes}$, which can be identified with the identity. This shows that the opposite of the right duality functor of λ (which is symmetric monoidal by analogous considerations as the left duality functor) is an inverse to the left duality functor, as symmetric monoidal functors.

Next, there is a commutative diagram as follows.

$$\begin{tikzcd}
\big((\mathcal{C}\times\mathcal{C})\times_{\mathcal{C}}\mathcal{C}_{/\mathbb{1}_{\mathcal{C}}}\big)^{\otimes} \arrow[r, "\tau'"] \arrow[dd, "\mathrm{pr}_1^{\otimes}"'] & \big((\mathcal{C}\times\mathcal{C})\times_{\mathcal{C}}\mathcal{C}_{/\mathbb{1}_{\mathcal{C}}}\big)^{\otimes} \arrow[d, "\mathrm{pr}_1^{\otimes}"] \\
 & \mathcal{C}^{\otimes}\times_{\mathsf{Fin}_*}\mathcal{C}^{\otimes} \arrow[d, "\tau^{\otimes}"] \\
\mathcal{C}^{\otimes}\times_{\mathsf{Fin}_*}\mathcal{C}^{\otimes} \arrow[r, "\mathrm{id}"'] & \mathcal{C}^{\otimes}\times_{\mathsf{Fin}_*}\mathcal{C}^{\otimes}
\end{tikzcd}$$

where τ' maps a tuple $(C, D, C\otimes D\to\mathbb{1})$ to $(D, C, D\otimes C\simeq C\otimes D\to\mathbb{1})$, where we use the symmetry equivalence of $\mathcal{C}$, and τ swaps the two factors. As $\mathrm{pr}_1^{\otimes}$ was a pairing of symmetric monoidal ∞-categories with left representable underlying pairing, one can see that the composition on the right is a pairing of symmetric monoidal ∞-categories with right representable underlying pairing, and the right duality functor can be identified with the left duality functor of $\mathrm{pr}_1^{\otimes}$. Furthermore, it follows from the description of left and right universal objects in [Lur18, 3.2.4] that the morphism of pairings encoded in the diagram is right representable. By functoriality of right duality functors we thus obtain a commutative diagram of symmetric monoidal functors

$$\begin{tikzcd}
(\mathcal{C}^{\mathrm{op}})^{\otimes} \arrow[r, "\mathfrak{D}_{\lambda}'^{\otimes}"] \arrow[d, "\mathrm{id}"'] & \mathcal{C}^{\otimes} \arrow[d, "\mathrm{id}"] \\
(\mathcal{C}^{\mathrm{op}})^{\otimes} \arrow[r, "\mathfrak{D}_{\lambda}^{\otimes}"'] & \mathcal{C}^{\otimes}
\end{tikzcd}$$

that shows that $\mathfrak{D}_{\lambda}^{\otimes}\simeq\mathfrak{D}_{\lambda}'^{\otimes}$ as lax symmetric monoidal (and hence also as symmetric monoidal) functors. As we previously obtained an equivalence $(\mathfrak{D}_{\lambda}'^{\mathrm{op}})^{\otimes}\simeq(\mathfrak{D}_{\lambda}^{-1})^{\otimes}$, this shows that $(\mathfrak{D}_{\lambda}^{\mathrm{op}})^{\otimes}\simeq(\mathfrak{D}_{\lambda}^{-1})^{\otimes}$.

Finally, let us say a few words on why, given a perfect[4] pairing of ∞-categories[5] $\lambda\colon\mathcal{M}\to\mathcal{C}\times\mathcal{D}$, the morphism of pairings

$$\begin{tikzcd}
\mathrm{TwArr}(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\lambda"] \\
\mathcal{C}\times(\mathcal{C}^{\mathrm{op}}) \arrow[r, "\mathrm{id}_{\mathcal{C}}\times\mathfrak{D}_{\lambda}"'] & \mathcal{C}\times\mathcal{D}
\end{tikzcd}$$

constructed in [HA, 5.2.2.24 and 5.2.2.25] is right representable, which means that the top horizontal functor needs to preserve right universal objects, see

[4]See [HA, 5.2.1.20 and 5.2.1.22].

[5]Of which the λ we discussed so far is an example by the proof of [Lur18, 3.2.4].

[HA, 5.2.1.13 and 5.2.1.8]. To start, we first see that by unwrapping the definition[6] we have to show that the composition of two morphisms of pairings of ∞-categories as depicted in the follow diagram preserves right universal objects.

$$\begin{tikzcd}
\mathrm{TwArr}^0_\lambda(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{TwArr}_\lambda(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\lambda"] \\
\mathcal{C} \times \mathcal{C}^{0\,\mathrm{op}}_\lambda \arrow[r] & \mathcal{C} \times \mathcal{C}^{\mathrm{op}}_\lambda \arrow[r] & \mathcal{C} \times \mathcal{D}
\end{tikzcd}$$

Unpacking the definition using [HA, 5.2.1.24] and in particular [HA, 5.2.1.28] we see that we can describe objects of $\mathcal{C}^{\mathrm{op}}_\lambda$ as tuples (C_r, D, ϕ), with C_r an object of $\mathcal{C}^{\mathrm{op}}$, D an object of $\mathcal{D}$, and ϕ a morphism $D \to \mathfrak{D}_\lambda(C_r)$ in $\mathcal{D}$. The fiber in $\mathrm{TwArr}_\lambda(\mathcal{C})$ of a pair $(C_l, (C_r, D, \phi))$ in $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}_\lambda$ can be identified with $\mathrm{Map}_{\mathcal{C}}(C_l, C_r)$. An object in $\mathrm{TwArr}_\lambda(\mathcal{C})$ that is given by a morphism $f\colon C_l \to C_r$ as just described is then mapped to the object in $\mathcal{M}$ described as follows. As λ is left representable, there is a left universal object M_r over C_r in $\mathcal{M}$, lying over $(C_r, \mathfrak{D}_\lambda(C_r))$. A λ-cartesian lift of the morphism (f, ϕ) is then a morphism $M_l \to M_r$ in $\mathcal{M}$ where M_l lies over (C_l, D). f is mapped to this object M_l.

By definition (see [HA, 5.2.1.28]) $\mathcal{C}^{0\,\mathrm{op}}_\lambda$ is the full subcategory of $\mathcal{C}^{\mathrm{op}}_\lambda$ spanned by those tuples where ϕ is an equivalence, and the left square in the above commutative diagram is a pullback. One can then see that an object in $\mathrm{TwArr}^0_\lambda(\mathcal{C})$ is right universal precisely if the associated morphism $f\colon C_l \to C_r$ as before is an equivalence. This then implies that (f, ϕ) will be an equivalence, so the λ-cartesian lift $M_l \to M_r$ is also an equivalence, and hence M_l is left universal, as M_r is so by assumption. But as λ is perfect, this means that M_l is also right universal, see [HA, 5.2.1.22]. $\diamondsuit$

We make a bit more explicit how $(-)^\vee$ applies to morphisms in the following remark.

Remark 5.1.1.6. Let $f\colon C \to D$ be a morphism of dualizable objects in a symmetric monoidal ∞-category $\mathcal{C}$. Then the functor $(-)^\vee$ from Fact 5.1.1.3 sends f to a morphism $f^\vee\colon D^\vee \to C^\vee$. Unpacking the definitions[7], one can see that this morphism fits into a commutative diagram as follows

$$\begin{tikzcd}
D^\vee \arrow[r, "f^\vee"] \arrow[d, dash, "\simeq"'] & C^\vee \arrow[d, dash, "\simeq"] \\
D^\vee \otimes \mathbb{1}_{\mathcal{C}} \arrow[d, "\mathrm{id}\otimes c"'] & \mathbb{1}_{\mathcal{C}} \otimes C^\vee \\
D^\vee \otimes C \otimes C^\vee \arrow[r, "\mathrm{id}\otimes f\otimes \mathrm{id}"'] & D^\vee \otimes D \otimes C^\vee \arrow[u, "e\otimes \mathrm{id}"']
\end{tikzcd}$$

[6] See [HA, 5.2.2.24 and 5.2.2.25] and also Remark 5.1.1.4.
[7] See in particular [Lur18, 3.2.4] and [HA, 5.2.1.9].

where the top two vertical equivalences are the unitality equivalences of $\mathcal{C}$, the morphism c takes part in a duality datum for C, and e takes part in a duality datum for D. ♢

Applying Fact 5.1.1.3 to the symmetric monoidal functor

$$\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$$

(see Proposition 4.3.2.1) we obtain the following.

Corollary 5.1.1.7. *There is a commutative diagram of symmetric monoidal functors as follows*

$$\begin{tikzcd}
\left(\mathsf{Ch}(k)^{\mathrm{cof}}_{\mathrm{fd}}\right)^{\mathrm{op}} \arrow[r, "(-)^\vee"] \arrow[d, "\gamma^{\mathrm{op}}"'] & \mathsf{Ch}(k)^{\mathrm{cof}}_{\mathrm{fd}} \arrow[d, "\gamma"] \\
\left(\mathcal{D}(k)_{\mathrm{fd}}\right)^{\mathrm{op}} \arrow[r, "(-)^\vee"'] & \mathcal{D}(k)_{\mathrm{fd}}
\end{tikzcd}$$

and both horizontal functors are equivalences. ♡

Example 5.1.1.8. Consider the commutative and cocommutative bialgebra D in $\mathsf{Ch}(k)$ from Construction 4.2.1.1. Its underlying chain complex is $k \cdot \{1\} \oplus k \cdot \{\mathrm{d}\}$ with 1 in degree 0 and d in degree 1. This chain complex is dualizable with dual[8] $k \cdot \{1\} \oplus k \cdot \{\mathrm{d}^\vee\}$ with 1 in degree 0 and $\mathrm{d}^\vee$ in degree -1.

By Fact 5.1.1.3 the commutative and cocommutative bialgebra structure on D induces again a commutative and cocommutative bialgebra structure on $\mathrm{D}^\vee$, with unit the basis element we called 1 in degree 0 (see Remark 5.1.1.6). The rest of the bialgebra structure is then already uniquely determined just as in Construction 4.2.1.1, with in particular $\Delta(\mathrm{d}^\vee) = 1 \otimes \mathrm{d}^\vee + \mathrm{d}^\vee \otimes 1$. ♢

As $(-)^\vee$ is a symmetric monoidal equivalence, it induces an equivalence that converts algebras into coalgebras and vice versa, as we note next.

Remark 5.1.1.9. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $\mathcal{O}$ and $\mathcal{O}'$ two ∞-operads. Note that the symmetric monoidal duality functor

$$(-)^\vee\colon (\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} \to \mathcal{C}_{\mathrm{fd}}$$

from Fact 5.1.1.3 induces a symmetric monoidal equivalence

$$\begin{aligned}\mathrm{BiAlg}_{\mathcal{O},\mathcal{O}'}(\mathcal{C}_{\mathrm{fd}}) &\simeq \mathrm{Alg}_{\mathcal{O}'}\left(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}}\right)^{\mathrm{op}} \\ &\xrightarrow{(-)^\vee} \mathrm{Alg}_{\mathcal{O}'}\left(\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{C}_{\mathrm{fd}}^{\mathrm{op}}\right)^{\mathrm{op}}\right)^{\mathrm{op}} \simeq \mathrm{Alg}_{\mathcal{O}'}(\mathrm{coAlg}_{\mathcal{O}}(\mathcal{C}_{\mathrm{fd}}))^{\mathrm{op}}\end{aligned}$$ ♢

[8] A duality datum is given by defining e by $e(1 \otimes 1) = 1 = e(\mathrm{d}^\vee \otimes \mathrm{d})$ and c by $c(1) = \mathrm{d} \otimes \mathrm{d}^\vee + 1 \otimes 1$.

5.1.2 Construction of a resolution

The goal of this section is to construct diagram (5.1) that was discussed in the introduction to Section 5.1, and we refer to there for motivation. We will construct such a diagram in $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$ first and then show that its image in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$ has the required properties. While we are still discussing algebras and coalgebras in the symmetric monoidal 1-category $\mathsf{Ch}(k)$, one should keep in mind that, as explained in Section 3.3, there is a canonical isomorphism

$$\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k))) \cong \mathrm{coCAlg}(\mathrm{Alg}(\mathsf{Ch}(k))) = \mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Comm}}(\mathsf{Ch}(k))$$

so we will be justified in identifying these categories and talking about objects as cocommutative bialgebras.

Let us now briefly go over the content of the subsections. In order to make it easier to talk about certain differential graded algebras that have free underlying $\mathbb{Z}$-graded k-algebras, we start in Section 5.1.2.1 by introducing some convenient notation. We will then begin the actual construction of diagram (5.1) in Section 5.1.2.2 by constructing a sequence of cocommutative bialgebras

$$A_0 \to A_1 \to A_2 \to \dots$$

in $\mathsf{Ch}(k)$. Section 5.1.2.3 will then be devoted to calculating the homology of $\mathrm{colim}_n\, A_n$. In Section 5.1.2.4 we will construct pushout diagrams

$$\begin{tikzcd} \underline{B}_n \arrow[r] \arrow[d] & B_n \arrow[d] \\ A_{n-1} \arrow[r] & A_n \end{tikzcd}$$

of cocommutative bialgebras in $\mathsf{Ch}(k)$. The cocommutative bialgebra B_n itself is not isomorphic to k, but maps to a cocommutative bialgebra in $\mathcal{D}(k)$ that is equivalent to k, as we will see in Section 5.1.2.5. We then combine the previous results in Section 5.1.2.6 to describe the induced diagram (5.1) in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$ and show that it has the required properties. Finally, in Section 5.1.2.7 we describe A_1 and $\underline{B}_n$ as free associative algebras on underlying pointed cocommutative coalgebras.

5.1.2.1 Notation for freely generated differential graded algebras

In this short section we introduce some notation for differential graded algebras whose underlying $\mathbb{Z}$-graded k-algebra is free associative.

Notation 5.1.2.1. Let X be a set and let[9] $\deg_{\mathsf{Ch}}(x)$ be an integer for every element x of X. Then we can form a $\mathbb{Z}$-graded k-module with basis X as

[9] Ultimately we want to define differential graded algebras generated by X, and in this differential graded algebra the chain degree of an element x of X will of course be exactly what we (prematurely, to avoid introducing more temporary notation) call $\deg_{\mathsf{Ch}}(x)$ here, making this notation in the end compatible with the notation in Definition 4.1.1.1.

follows.

$$k \cdot X := \bigoplus_{x \in X} k[\deg_{\mathsf{Ch}}(x)]$$

We will denote the free associative $\mathbb{Z}$-graded k-algebra generated by $k \cdot X$ by

$$\mathrm{Free}^{\mathsf{Assoc}}(X)$$

and if $X = \{ x_1, x_2, \dots \}$ then we will often write

$$\mathrm{Free}^{\mathsf{Assoc}}(x_1, x_2, \dots) = \mathrm{Free}^{\mathsf{Assoc}}(X)$$

instead. A basis of $\mathrm{Free}^{\mathsf{Assoc}}(X)$ is given by elements of the form $x_{i_1} \cdots x_{i_n}$ for $n \geq 0$[10] with x_{i_j} elements of X for $1 \leq j \leq n$.

We can make $\mathrm{Free}^{\mathsf{Assoc}}(X)$ into an associative differential graded algebra by furnishing it with the zero boundary operator. But we will sometimes want to define associative differential graded algebras that have a free underlying $\mathbb{Z}$-graded k-algebra, but *do* have nontrivial boundaries. So assume that for every element x of X we are given an element $f(x)$ of $\mathrm{Free}^{\mathsf{Assoc}}(X)_{\deg_{\mathsf{Ch}}(x)-1}$. Then we use the notation

$$\mathrm{Free}^{\mathsf{Assoc}}(X \mid \partial(x) = f(x))$$

for the differential graded k-algebra whose underlying $\mathbb{Z}$-graded k-algebra is given by $\mathrm{Free}^{\mathsf{Assoc}}(X)$ and with boundary operator (uniquely) extended by k-linearity and the Leibniz rule from the prescription $\partial(x) = f(x)$ for every element x of X. This does not in general actually define a differential graded algebra, as in general there is no reason for the boundary operator to square to 0, so if we use this notation we will need to check that $\partial(\partial(x)) = 0$ for every element x of X.

Sometimes we will omit $\partial(x)$ in this notation for some elements x of X, in which case this is to be interpreted as $\partial(x) = 0$. ♢

5.1.2.2 Construction of A as a directed colimit

In this section we construct a sequence of cocommutative bialgebras

$$A_0 \to A_1 \to A_2 \to \dots$$

in $\mathsf{Ch}(k)$ and describe its colimit.

Construction 5.1.2.2. We will construct a cocommutative bialgebra in chain complexes A_n for every integer $n \geq 0$. Using Notation 5.1.2.1, we define the underlying differential graded k-algebra of A_n as

$$A_n := \mathrm{Free}^{\mathsf{Assoc}}\left(y_1, \dots, y_n \;\middle|\; \partial(y_k) = \sum_{i+j=k} y_i y_j \right)$$

[10] If $n = 0$ we interpret the product as 1.

where $\deg_{\mathsf{Ch}(k)}(y_i) = -1$ and where the sum should of course be interpreted to only be taken over those i and j for which y_i and y_j are defined[11]. For this to actually define a differential graded algebra structure the definition of ∂ needs to satisfy $\partial(\partial(y_l)) = 0$ for any $1 \leq l \leq n$, which is the case as the following basic calculation shows.

$$\begin{aligned}
\partial(\partial(y_l)) &= \partial\left(\sum_{i+j=l} y_i y_j\right) \\
&= \sum_{i+j=l} \partial(y_i) y_j - \sum_{i+j=l} y_i \partial(y_j) \\
&= \sum_{i+j+k=l} y_i y_j y_k - \sum_{i+j+k=l} y_i y_j y_k = 0
\end{aligned}$$

We next define a cocommutative coalgebra structure on A_n. As the underlying graded k-algebra of A_n is free, we can define the counit $\epsilon\colon A_n \to k$ as well as the comultiplication $\Delta\colon A_n \to A_n \otimes A_n$ to be the morphisms of graded k-algebras determined by

$$\begin{aligned}
\epsilon(y_k) &= 0 \\
\Delta(y_k) &= 1 \otimes y_k + y_k \otimes 1
\end{aligned}$$

for $1 \leq k \leq n$. By definition comultiplication and counit are morphisms of algebras, so if this defines a cocommutative coalgebra structure in $\mathsf{Ch}(k)$, then this will make A_n into a cocommutative bialgebra in $\mathsf{Ch}(k)$ as claimed.

As counit and unit of the presumptive coalgebra structure are morphisms of algebras, it suffices to check compatibility of ϵ and Δ with ∂, coassociativity, counitality, and cocommutativity on multiplicative generators. For example for the comultiplication being a morphism of chain complexes we can calculate

$$\begin{aligned}
\Delta(\partial(y_k)) &= \Delta\left(\sum_{i+j=k} y_i y_j\right) \\
&= \sum_{i+j=k} (1 \otimes y_i + y_i \otimes 1) \cdot (1 \otimes y_j + y_j \otimes 1) \\
&= \sum_{i+j=k} 1 \otimes y_i y_j - y_j \otimes y_i + y_i \otimes y_j + y_i y_j \otimes 1 \\
&= \sum_{i+j=k} 1 \otimes y_i y_j + y_i y_j \otimes 1 \\
&= \partial(1 \otimes y_k + y_k \otimes 1) \\
&= \partial(\Delta(y_k))
\end{aligned}$$

[11] So in particular, $\partial(y_1) = 0$ and $\partial(y_2) = y_1^2$.

and as another example the following calculation verifies coassociativity.

$$(\mathrm{id} \otimes \Delta)(\Delta(y_k)) = 1 \otimes 1 \otimes y_k + 1 \otimes y_k \otimes 1 + y_k \otimes 1 \otimes 1 = (\Delta \otimes \mathrm{id})(\Delta(y_k))$$

Compatibility of ϵ with ∂, counitality, and cocommutativity are similarly immediate.

We can completely analogously define a cocommutative bialgebra A in $\mathsf{Ch}(k)$ as

$$A \coloneqq \mathrm{Free}^{\mathsf{Assoc}}\left(y_1, y_2, \dots \,\middle|\, \partial(y_k) = \sum_{i+j=k} y_i y_j\right)$$

with counitality and comultiplication defined exactly as for A_n. ♢

Remark 5.1.2.3. There is a commutative diagram of cocommutative bialgebras in $\mathsf{Ch}(k)$ as follows

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[drr] & A_1 \arrow[r] \arrow[dr] & A_2 \arrow[r] \arrow[d] & \cdots \arrow[dl] \\
& & A &
\end{tikzcd}$$

where all morphisms are the obvious inclusions. This diagram exhibits A as the colimit of the directed system of inclusions in the top row, as can be seen using that directed colimits of cocommutative bialgebras in $\mathsf{Ch}(k)$ are calculated on underlying chain complexes by [HA, 3.2.2.5] and [HA, 3.2.3.1] in combination with [HTT, 5.5.8.3]. ♢

5.1.2.3 Homology of A

As described in the introduction to Section 5.1 we will later construct a morphism from the object in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$ represented by A to $R^\vee$, the dual of a commutative bialgebra in $\mathcal{D}(k)$ with prescribed homology. From the construction it will be clear that the induced morphism on homology is surjective, and we will want to conclude that the morphism is an equivalence, or equivalently that the induced morphism on homology is an isomorphism. In order to do this we should calculate the homology of A, which we do in this section.

Proposition 5.1.2.4. *The chain complex A that we constructed in Construction 5.1.2.2 has homology*

$$\mathrm{H}_n(A) \cong \begin{cases} k & \textit{if } n = 0 \textit{ or } n = -1 \\ 0 & \textit{otherwise} \end{cases}$$

and the unit 1 of A and y_1 are representatives of elements forming a basis of $\mathrm{H}_0(A)$ and $\mathrm{H}_1(A)$. ♡

Proof. A is freely generated as a $\mathbb{Z}$-graded k-module by words in the multiplicative generators y_i, i. e. by elements of the form

$$y_{i_1} \cdots y_{i_n}$$

with $n \geq 0$ (for $n = 0$ we interpret the product as 1) and i_j elements of $\mathbb{Z}_{\geq 1}$. For $m \geq 0$, let $A(m)$ be the sub $\mathbb{Z}$-graded k-module generated by elements of this form with $\sum_{j=1}^{n} i_j = m$. It follows directly from the definitions that $A(m)$ is in fact a subcomplex of A, and that furthermore

$$A \cong \bigoplus_{m \geq 0} A(m)$$

in $\mathsf{Ch}(k)$.

Note that $A(0)$ and $A(1)$ are both concentrated in a single degree and of rank 1, with $A(0)$ having a basis formed by 1 in degree 0 and $A(1)$ having a basis formed by y_1 in degree -1. To finish the proof it thus suffices to show that $A(m)$ is acyclic for $m > 1$.

For this, we fix $m > 1$ and define a chain homotopy h on $A(m)$ by extending k-linearly from the following definition on the basis.

$$h(y_{i_1} \cdots y_{i_n}) = \begin{cases} y_{i_2+1} y_{i_3} \cdots y_{i_n} & \text{if } n > 1 \text{ and } i_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

We can now check that h is indeed a contracting homotopy by checking on basis elements. For this we distinguish three cases. First, the only basis element for which $n \leq 1$ is y_m, and for it we have the following calculation.

$$(\partial h + h\partial)(y_m) = \partial(0) + h\left(\sum_{i+j=m} y_i y_j\right) = y_m$$

Next, for those basis elements for which $n > 1$ and $i_1 = 1$, we obtain the following.

$$\begin{aligned}
&(\partial h + h\partial)(y_1 y_{i_2} \cdots y_{i_n}) \\
&= \partial(y_{i_2+1} \cdots y_{i_n}) + h\left(-y_1\left(\sum_{k+l=i_2} y_k y_l y_{i_3} \cdots y_{i_n}\right) + y_1 y_{i_2} \partial(y_{i_3} \cdots y_{i_n})\right) \\
&= \left(\sum_{k+l=i_2+1} y_k y_l y_{i_3} \cdots y_{i_n}\right) - y_{i_2+1} \partial(y_{i_3} \cdots y_{i_n}) \\
&\quad - \left(\sum_{k+l=i_2} y_{k+1} y_l y_{i_3} \cdots y_{i_n}\right) + y_{i_2+1} \partial(y_{i_3} \cdots y_{i_n})
\end{aligned}$$

$$
\begin{aligned}
&= y_1 y_{i_2} \cdots y_{i_n} + \left(\sum_{k+l=i_2} y_{k+1} y_l y_{i_3} \cdots y_{i_n} \right) - y_{i_2+1} \partial(y_{i_3} \cdots y_{i_n}) \\
&\quad - \left(\sum_{k+l=i_2} y_{k+1} y_l y_{i_3} \cdots y_{i_n} \right) + y_{i_2+1} \partial(y_{i_3} \cdots y_{i_n}) \\
&= y_1 y_{i_2} \cdots y_{i_n}
\end{aligned}
$$

Finally, for the other basis elements, i. e. those with $n > 1$ and $i_1 \neq 1$, we have the following calculation.

$$
\begin{aligned}
&(\partial h + h\partial)(y_{i_1} y_{i_2} \cdots y_{i_n}) \\
&= h\left(\left(\sum_{k+l=i_1} y_k y_l y_{i_2} \cdots y_{i_n} \right) - y_{i_1} \partial(y_{i_2} \cdots y_{i_n}) \right) \\
&= y_{i_1-1+1} y_{i_2} \cdots y_{i_n} + \sum_{k+l=i_1, k>1} 0 + 0 \\
&= y_{i_1} y_{i_2} \cdots y_{i_n}
\end{aligned}
$$

□

5.1.2.4 Construction of A_{n+1} from A_n

In order to be able to lift a morphism from A_{n-1} to a morphism from A_n, we will describe A_n as a pushout of A_{n-1} in this section. We start by constructing the relevant commutative square, and show that this square is a pushout square at the end of this section.

Construction 5.1.2.5. Let $n \geq 1$. Using Notation 5.1.2.1 we define a morphism of differential graded algebras as

$$
\underline{B}_n = \mathrm{Free}^{\mathsf{Assoc}}\left(\underline{y}_n\right) \to \mathrm{Free}^{\mathsf{Assoc}}\left(\underline{y}_n, y_n \,\middle|\, \partial(y_n) = \underline{y}_n\right) = B_n
$$

with $\deg_{\mathsf{Ch}}(y_n) = -1$ and $\deg_{\mathsf{Ch}}(\underline{y}_n) = -2$.

We can upgrade this morphism of differential graded k-algebras to a morphism of cocommutative bialgebras in $\mathsf{Ch}(k)$, by defining counit ϵ and comultiplication Δ as follows on the multiplicative basis.

$$
\begin{aligned}
\epsilon(y_n) &= 0 \\
\epsilon\left(\underline{y}_n\right) &= 0 \\
\Delta(y_n) &= 1 \otimes y_n + y_n \otimes 1 \\
\Delta\left(\underline{y}_n\right) &= 1 \otimes \underline{y}_n + \underline{y}_n \otimes 1
\end{aligned}
$$

Checking that ϵ and Δ are compatible with ∂ as well as coassociativity, counitality, and cocommutativity are similar to Construction 5.1.2.2.

We can define a morphism of differential graded algebras

$$B_n \to A_n$$

by sending y_n to y_n and $\underline{y}_n$ to $\partial(y_n)$. It is easy to check that this is also compatible with the coalgebra structure, making this a morphism of cocommutative coalgebras.

Finally, the restriction to $\underline{B}_n$ factors through A_{n-1}, so that we obtain a commutative diagram

$$\begin{tikzcd} \underline{B}_n \arrow[r] \arrow[d] & B_n \arrow[d] \\ A_{n-1} \arrow[r] & A_n \end{tikzcd} \tag{5.2}$$

in $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$. ♢

In order to show that (5.2) is a pushout square, we will need to two preliminary results that allow us to detect colimits in $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$ on underlying algebras in $\mathsf{Ch}(k)$.

Proposition 5.1.2.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $\mathcal{O}$ be a reduced*[12] *∞-operad with $\mathfrak{o}$ the essentially unique object in the underlying ∞-category $\mathcal{O}$. Assume that $\mathcal{C}$ is cocomplete and the tensor product preserves colimits separately in each variable.*

Then $\mathrm{coAlg}_{\mathcal{O}}(\mathcal{C})$ is cocomplete and the induced symmetric monoidal structure on $\mathrm{coAlg}_{\mathcal{O}}(\mathcal{C})$ is also compatible with colimits. ♡

Proof. $\mathrm{coAlg}_{\mathcal{O}}(\mathcal{C})$ is cocomplete by [HA, 3.2.2.5]. Furthermore, the forgetful functor

$$\mathrm{ev}_{\mathfrak{o}}\colon\ \mathrm{coAlg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$$

is symmetric monoidal by Proposition E.4.2.3, conservative by [HA, 3.2.2.6] and preserves colimits by [HA, 3.2.2.5]. It thus follows that the symmetric monoidal structure on $\mathrm{coAlg}_{\mathcal{O}'}(\mathcal{C})$ is also compatible with colimits. ☐

Proposition 5.1.2.7. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $\mathcal{O}$ and $\mathcal{O}'$ be ∞-operads. Assume that $\mathcal{O}'$ is reduced and let $\mathfrak{o}$ be the essentially unique object in $\mathcal{O}'$.*

Then the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\mathfrak{o}})\colon\ \mathrm{Alg}_{\mathcal{O}}(\mathrm{coAlg}_{\mathcal{O}'}(\mathcal{C})) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \tag{5.3}$$

is conservative, i. e. reflects equivalences.

Assume additionally that $\mathcal{C}$ is cocomplete and the tensor product preserves colimits separately in each variable. Then $\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\mathfrak{o}})$ preserves colimits. In particular, also being conservative, $\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\mathfrak{o}})$ detects colimits. ♡

[12] See [HA, 2.3.4.1] for a definition.

Proof. The symmetric monoidal forgetful functor[13]

$$\mathrm{ev}_{\mathfrak{o}}\colon\ \mathrm{coAlg}_{\mathcal{O}'}(\mathcal{C}) \to \mathcal{C}$$

is by [HA, 3.2.2.6] conservative and preserves colimits by [HA, 3.2.2.5]. It thus follows from Proposition E.3.4.1 and Proposition E.7.3.1[14] that the forgetful functor (5.3) is also conservative and colimit-preserving, and hence detects colimits. □

Proposition 5.1.2.8. *The commutative square* (5.2) *constructed in Construction 5.1.2.5 is a pushout diagram in* $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$ ♡

Proof. It follows from Proposition 5.1.2.7[15] that the forgetful functor from cocommutative bialgebras to underlying algebras

$$\mathrm{Alg}\big(\mathrm{ev}_{\langle 1\rangle}\big)\colon\ \mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k))) \to \mathrm{Alg}(\mathsf{Ch}(k))$$

detects colimits. It thus suffices to show that the underlying square of differential graded k-algebras is a pushout square.

The functor from chain complexes of k-modules to $\mathbb{Z}$-graded k-modules is conservative, symmetric monoidal, and preserves colimits. It thus follows from Proposition E.3.4.1 and Proposition E.7.3.1 just as in the proof of Proposition 5.1.2.7 that the forgetful functor from differential graded k-algebras to $\mathbb{Z}$-graded k-algebras detects colimits, so it actually suffices to show that the underlying commutative square of $\mathbb{Z}$-graded k-algebras is a pushout square.

There is a pushout diagram of $\mathbb{Z}$-graded k-modules

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & k \cdot \{\, y_n \,\} \arrow[d] \\
k \cdot \left\{\, \underline{y}_n \,\right\} \arrow[r] & k \cdot \left\{\, \underline{y}_n, y_n \,\right\}
\end{tikzcd}$$

where all morphisms are the obvious inclusions, which induces the pushout diagram of $\mathbb{Z}$-graded k-algebras at the top of the following commutative diagram

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & \mathrm{Free}^{\mathsf{Assoc}}(y_n) \arrow[d] \\
\mathrm{Free}^{\mathsf{Assoc}}\left(\underline{y}_n\right) \arrow[r] \arrow[d] & \mathrm{Free}^{\mathsf{Assoc}}\left(\underline{y}_n, y_n\right) \arrow[d] \\
\mathrm{Free}^{\mathsf{Assoc}}(y_1, \ldots, y_{n-1}) \arrow[r] & \mathrm{Free}^{\mathsf{Assoc}}(y_1, \ldots, y_n)
\end{tikzcd}$$

[13] See Proposition E.4.2.3.

[14] $\mathrm{coAlg}_{\mathcal{O}'}(\mathcal{C})$ is cocomplete and its symmetric monoidal structure is compatible with colimits by Proposition 5.1.2.6.

[15] The tensor product of $\mathsf{Ch}(k)$ preserves colimits in each variable separately as the symmetric monoidal structure is closed by Definition 4.1.2.1.

where all morphisms are the obvious inclusions. We have to show that the bottom square is a pushout square. As the top square is a pushout square, it suffices to show that the big outer square is a pushout.

But the big outer square is $\mathrm{Free}^{\mathsf{Assoc}}$ applied to the following pushout diagram of $\mathbb{Z}$-graded k-modules

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & k \cdot \{ y_n \} \arrow[d] \\ k \cdot \{ y_1, \ldots, y_{n-1} \} \arrow[r] & k \cdot \{ y_1, \ldots, y_n \} \end{tikzcd}$$

and is thus a pushout diagram. □

5.1.2.5 Identification of B_n up to quasiisomorphism

In this section we show that the cocommutative bialgebras B_n defined in Construction 5.1.2.5 are quasiisomorphic to k. We start by remarking that k is a zero object in $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$.

Remark 5.1.2.9. Let $\mathcal{C}$ be a cocomplete and complete symmetric monoidal ∞-category such that the tensor product is compatible with colimits in each variable. By [HA, 3.2.2.4 and 3.2.3.1], $\mathrm{coCAlg}(\mathcal{C})$ is complete and cocomplete, and the induced symmetric monoidal structure is again compatible with colimits by Proposition 5.1.2.6. Another application of [HA, 3.2.2.4 and 3.2.3.1] yields that $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{C}))$ is complete and cocomplete.

By [HA, 3.2.1.8], an initial object is given by the monoidal unit. We want to show that this object is also final and thus a zero object in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{C}))$. As the forgetful functor

$$\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{Alg}(\mathrm{coCAlg}(\mathcal{C})) \to \mathrm{coCAlg}(\mathcal{C})$$

detects limits by [HA, 3.2.2.4] and is also symmetric monoidal by Proposition E.4.2.3, it suffices to show that the monoidal unit is a final object in $\mathrm{coCAlg}(\mathcal{C})$, which again follows from [HA, 3.2.1.8] (and passing to opposite categories twice). ♢

Proposition 5.1.2.10. *Let $n \geq 1$. The unique morphism in the 1-category $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$ from the monoidal unit k (see Remark 5.1.2.9) to B_n is a quasi-isomorphism.* ♡

Proof. The forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\langle 1 \rangle})$ is symmetric monoidal and detects colimits by Proposition 5.1.2.7. By [HA, 3.2.1.8] it thus suffices to show that the unique morphism in $\mathrm{Alg}(\mathsf{Ch}(k))$ from the monoidal unit k to $\mathrm{Free}^{\mathrm{Alg}}(B'_n)$ is a quasiisomorphism, where B'_n is the chain complex which as a $\mathbb{Z}$-graded k-module is $k \cdot \left\{ \underline{y}_n, y_n \right\}$, with $\deg_{\mathsf{Ch}}(y_n) = -1$ and $\deg_{\mathsf{Ch}}(\underline{y}_n) = -2$, and with boundary operator defined by $\partial(y_n) = \underline{y}_n$.

But the left adjoint $\mathrm{Free}^{\mathrm{Alg}}$ to the forgetful functor $\mathrm{ev}_{\mathfrak{a}}$ preserves initial objects, so this morphism is $\mathrm{Free}^{\mathrm{Alg}}$ applied to the unique morphism of chain complexes $0 \to B'_n$.

By Proposition E.7.2.1 we can thus identify $k \to \mathrm{Free}^{\mathrm{Alg}}(B'_n)$ with the following inclusion of the summand indexed by 0

$$k = B_n'^{\otimes 0} \to \bigoplus_{i \geq 0} B_n'^{\otimes i}$$

As the tensor product of a contractible chain complex with another chain complex is again contractible it hence suffices to show that B'_n is contractible, which is clear. □

5.1.2.6 The resolution in $\mathcal{D}(k)$

In this section we describe the image of the constructions discussed in Section 5.1.2.2 and Section 5.1.2.4 under the symmetric monoidal functor $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$. The important point is that the pushout diagram (5.2) is in fact a homotopy pushout and thus mapped under γ to a pushout in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$, and likewise for the colimit of $A_0 \to A_1 \to A_2 \to \ldots$.

Proposition 5.1.2.11. *The underlying differential graded k-algebras of A_n and A from Construction 5.1.2.2 and of $\underline{B}_n$ and B_n from Construction 5.1.2.5 are cofibrant. Furthermore the pushout square (see Construction 5.1.2.5 and Proposition 5.1.2.8)*

$$\begin{array}{ccc} \underline{B}_n & \longrightarrow & B_n \\ \downarrow & & \downarrow \\ A_{n-1} & \longrightarrow & A_n \end{array}$$

is a homotopy pushout in $\mathrm{Alg}(\mathsf{Ch}(k))$ *and the colimit of the directed system (see Remark 5.1.2.3)*

$$A_0 \to A_1 \to A_2 \to \ldots$$

is a homotopy colimit in $\mathrm{Alg}(\mathsf{Ch}(k))$. ♡

Proof. For the cofibrancy statements it suffices to show that A_0 and $\underline{B}_n$ (for $n \geq 1$) are cofibrant and that the morphisms $\underline{B}_n \to B_n$ are generating cofibrations. The former is the case as $A_0 \cong \mathrm{Free}^{\mathrm{Alg}}(0)$ and $\underline{B}_n \cong \mathrm{Free}^{\mathrm{Alg}}(k \cdot \{\underline{y_n}\})$, and the chain complexes 0 and $k \cdot \{\underline{y_n}\}$ are cofibrant. The latter is the case as the morphism in question is isomorphic to $\mathrm{Free}^{\mathrm{Alg}}$ applied to a generating cofibration in $\mathsf{Ch}(k)$, see [Hov99, 2.3.3], Fact 4.1.3.1, and Theorem 4.2.2.1 (2).

That the pushout square is a homotopy pushout now follows from [HTT, A.2.4.4], and that the directed colimit is a homotopy colimit follows from [HTT, A.2.9.24 (i)][16]. □

[16] The reference shows that the diagram is cofibrant in the projective model structure on $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathsf{Ch}(k))$ if and only if A_0 is cofibrant and $A_n \to A_{n+1}$ is a cofibration for every $n \geq 0$.

Notation 5.1.2.12. Recall from Proposition 4.3.2.1 that we denote the symmetric monoidal functor from $\mathsf{Ch}(k)^{\mathrm{cof}}$ to $\mathcal{D}(k)$ by γ.

Let $\mathcal{O}$ and $\mathcal{O}'$ be ∞-operads. Then we denote the induced functor on $\mathcal{O}$-algebras of $\mathcal{O}'$-coalgebras as follows.

$$\gamma_{\mathcal{O}}^{\mathcal{O}'}\colon \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{coAlg}_{\mathcal{O}'}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\big) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{coAlg}_{\mathcal{O}'}(\mathcal{D}(k))) \qquad \diamondsuit$$

Remark 5.1.2.13. As all involved objects have cofibrant underlying chain complexes by Propositions 5.1.2.11 and 4.2.2.12, the commutative squares and directed system constructed in Construction 5.1.2.5 and Remark 5.1.2.3 are mapped by $\gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}$ to commutative diagrams in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$. $\diamondsuit$

Corollary 5.1.2.14. *For $n \geq 1$, the commutative square*

$$\begin{tikzcd} \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(\underline{B}_n) \arrow[r] \arrow[d] & \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(B_n) \arrow[d] \\ \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_{n-1}) \arrow[r] & \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_n) \end{tikzcd}$$

in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k))))$ *is a pushout diagram and the morphisms*

$$\gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_n) \to \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A)$$

exhibit $\gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A)$ *as a colimit of*

$$\gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_0) \to \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_1) \to \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_2) \to \dots$$

in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$*.* $\heartsuit$

Proof. As $\mathcal{D}(k)$ is presentable symmetric monoidal by Proposition 4.3.2.1, it suffices by Proposition 5.1.2.7 to show that the underlying diagrams in $\mathrm{Alg}(\mathcal{D}(k))$ are colimit diagrams.

By Proposition 5.1.2.11 the diagrams of differential graded algebras are pointwise cofibrant (not just with cofibrant underlying chain complexes) as well as homotopy colimit diagrams, so the claim follows from combining this with Proposition 4.4.2.1 and [HA, 1.3.4.24]. $\square$

5.1.2.7 Free generation of certain associative algebras

In order to be able work with morphisms out of $\gamma_{\mathbb{E}_0}^{\mathbb{E}_\infty}(\underline{B}_n)$, we will show in this section that $\gamma_{\mathbb{E}_0}^{\mathbb{E}_\infty}(\underline{B}_n)$ is the free associative algebra on an object in $\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg}(\mathcal{D}(k)))$.

We start by constructing the morphism that exhibits $\gamma_{\mathbb{E}_0}^{\mathbb{E}_\infty}(\underline{B}_n)$ as a free associative algebra.

Construction 5.1.2.15. Let $n \geq 1$. We define $\underline{B}'_n$ to be the sub $\mathbb{Z}$-graded k-module of $\underline{B}_n$ (see Construction 5.1.2.5) generated by 1 and $\underline{y}_n$. Note that

$\underline{B}'_n$ is closed under ∂ as well as Δ, and the unique morphism $k \to \underline{B}_n$ in $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$ (see Remark 5.1.2.9) factors over $\underline{B}'_n$.

We can thus consider $\underline{B}'_n$ as an object of $\mathrm{coCAlg}(\mathsf{Ch}(k))_{k/}$. The underlying chain complexes of $\underline{B}'_n$ and $\underline{B}_n$ are cofibrant by [Hov99, 2.3.6], so we can consider the inclusion of $\underline{B}'_n$ into $\underline{B}_n$ as a morphism in $\mathrm{coCAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})_{k/}$.

By [HA, 2.1.3.10] there is an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_0}\big(\mathrm{coCAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\big) \xrightarrow{\simeq} \mathrm{coCAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)_{k/}$$

under which we can consider the inclusion

$$\underline{B}'_n \to \underline{B}_n \tag{5.4}$$

as a morphism in $\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coAlg}_{\mathbb{E}_\infty}(\mathsf{Ch}(k)^{\mathrm{cof}}))$.

Completely analogously we define A'_1 to be the sub $\mathbb{Z}$-graded k-module of A_1 (see Construction 5.1.2.2) spanned by 1 and y_1 and consider the inclusion $A'_1 \to A_1$ as a morphism in $\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coAlg}_{\mathbb{E}_\infty}(\mathsf{Ch}(k)^{\mathrm{cof}}))$. ♢

Remark 5.1.2.16. By [HA, 2.1.3.9] there is a unique morphism of ∞-operads

$$\mathbb{E}_0^{\otimes} \to \mathsf{Assoc}^{\otimes}$$

which can be interpreted as follows. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the induced forgetful functor

$$\mathrm{Alg}_{\mathsf{Assoc}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C}) \simeq \mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$$

(where the equivalence is the one from [HA, 2.1.3.10]) sends an associative algebra A to the unit morphism $\mathbb{1}_{\mathcal{C}} \to A$. ♢

Notation 5.1.2.17. By [HA, 3.1.3.5][17], the forgetful functor

$$\mathrm{Alg}_{\mathsf{Assoc}}(\mathrm{coCAlg}(\mathcal{D}(k))) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg}(\mathcal{D}(k)))$$

from Remark 5.1.2.16 has a left adjoint that we will denote as follows.

$$\mathrm{Free}^{\mathrm{Alg}(\mathrm{coCAlg})}_{\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg})}\colon \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg}(\mathcal{D}(k))) \to \mathrm{Alg}_{\mathsf{Assoc}}(\mathrm{coCAlg}(\mathcal{D}(k)))$$

We use the analogous notation $\mathrm{Free}^{\mathrm{Alg}}_{\mathrm{Alg}_{\mathbb{E}_0}}$ for the left adjoint of the forgetful functor $\mathrm{Alg}_{\mathsf{Assoc}}(\mathcal{D}(k)) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{D}(k))$. ♢

Proposition 5.1.2.18. *In this proposition we use Notation 5.1.2.12.*

Let $n \geq 1$. The morphism

$$\gamma^{\mathbb{E}_\infty}_{\mathbb{E}_0}\big(\underline{B}'_n\big) \to \gamma^{\mathbb{E}_\infty}_{\mathbb{E}_0}\big(\underline{B}_n\big)$$

[17] Using Proposition 5.1.2.6 and Proposition 4.3.2.1 (1).

induced by the inclusion (5.4) *in* $\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg}(\mathcal{D}(k)))$ *induces a morphism*

$$\mathrm{Free}^{\mathrm{Alg}(\mathrm{coCAlg})}_{\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg})}\left(\gamma^{\mathbb{E}_\infty}_{\mathbb{E}_0}(\underline{B}'_n)\right) \to \gamma^{\mathbb{E}_\infty}_{\mathsf{Assoc}}(\underline{B}_n) \tag{5.5}$$

in $\mathrm{Alg}\big(\mathrm{coAlg}_{\mathbb{E}_\infty}(\mathcal{D}(k))\big)$*. This morphism is an equivalence.*

The analogously defined morphism

$$\mathrm{Free}^{\mathrm{Alg}(\mathrm{coCAlg})}_{\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg})}\left(\gamma^{\mathbb{E}_\infty}_{\mathbb{E}_0}(A'_1)\right) \to \gamma^{\mathbb{E}_\infty}_{\mathsf{Assoc}}(A_1)$$

is also an equivalence. ♡

Proof. We only discuss the case of $\underline{B}_n$, as the case of A_1 is completely analogous.

By Proposition 5.1.2.7 the functor

$$\mathrm{Alg}\big(\mathrm{ev}_{\langle 1\rangle}\big)\colon \mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k))) \to \mathrm{Alg}(\mathcal{D}(k))$$

is conservative, and hence it suffices to show that the underlying morphism in $\mathrm{Alg}(\mathcal{D}(k))$ of (5.5) is an equivalence.

The functor $\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{coCAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$ is symmetric monoidal and preserves colimits[18], so we can apply Proposition E.7.2.2 to conclude that the underlying morphism in $\mathrm{Alg}(\mathcal{D}(k))$ of morphism (5.5) is the morphism[19]

$$\mathrm{Free}^{\mathrm{Alg}}_{\mathrm{Alg}_{\mathbb{E}_0}}\big(\gamma_{\mathbb{E}_0}(\underline{B}'_n)\big) \to \gamma_{\mathsf{Assoc}}(\underline{B}_n)$$

adjoint to the morphism $\gamma_{\mathbb{E}_0}(\underline{B}'_n) \to \gamma_{\mathbb{E}_0}(\underline{B}_n)$.

Now consider the subcomplex $\underline{B}''_n$ of $\underline{B}'_n$ generated as a free $\mathbb{Z}$-graded k-module by $\underline{y}_n$. This complex is cofibrant and the morphism $\underline{B}''_n \to \underline{B}'_n$ in $\mathsf{Ch}(k)^{\mathrm{cof}}$ exhibits $\underline{B}'_n$ as the free $\mathbb{E}_0$-algebra generated by $\underline{B}''_n$, see Proposition E.7.2.1.

The symmetric monoidal functor $\gamma\colon \mathsf{Ch}(k) \to \mathcal{D}(k)$ preserves coproducts by Proposition 4.3.2.1 (3) so by Proposition E.7.2.2 variant (3) we can identify $\gamma_{\mathbb{E}_0}(\underline{B}'_n)$ with $\mathrm{Free}^{\mathrm{Alg}_{\mathbb{E}_0}}\big(\gamma(\underline{B}''_n)\big)$, and the equivalence

$$\mathrm{Free}^{\mathrm{Alg}_{\mathbb{E}_0}}\big(\gamma(\underline{B}''_n)\big) \xrightarrow{\simeq} \gamma_{\mathbb{E}_0}(\underline{B}'_n)$$

is adjoint to the inclusion $\gamma(\underline{B}''_n) \to \gamma(\underline{B}'_n)$. Using composability of adjoints [HTT, 5.2.2.6] we can identify $\mathrm{Free}^{\mathrm{Alg}}_{\mathrm{Alg}_{\mathbb{E}_0}} \circ \mathrm{Free}^{\mathrm{Alg}_{\mathbb{E}_0}}$ with $\mathrm{Free}^{\mathrm{Alg}}$, and under this identification the morphism

$$\mathrm{Free}^{\mathrm{Alg}}\big(\gamma(\underline{B}''_n)\big) \simeq \mathrm{Free}^{\mathrm{Alg}}_{\mathrm{Alg}_{\mathbb{E}_0}}\big(\gamma_{\mathbb{E}_0}(\underline{B}'_n)\big) \to \gamma_{\mathsf{Assoc}}(\underline{B}_n) \tag{5.6}$$

[18] See the proof of Proposition 5.1.2.6.

[19] We are also using that the various functors induced by γ are compatible with the forgetful functors here, to e.g. identify the underlying associative algebra of $\gamma^{\mathbb{E}_\infty}_{\mathsf{Assoc}}(\underline{B}_n)$ with $\gamma_{\mathsf{Assoc}}(\underline{B}_n)$.

which we need to show is an equivalence, is adjoint to the following inclusion.

$$\gamma(\underline{B}''_n) \to \gamma(\underline{B}_n)$$

We finish by invoking Proposition E.7.2.2 again, this time variant (2) (using that γ preserves coproducts by Proposition 4.3.2.1 (3)), and noting that $\underline{B}''_n \to \underline{B}_n$ indeed exhibits $\underline{B}_n$ as the free differential graded algebra generated by $\underline{B}''_n$ by definition. □

5.1.3 Formality of certain associative algebras

Let $\mathcal{C}$ be a monoidal ∞-category and C an associative algebra in $\mathcal{C}$. By [HA, 3.2.1.8] (see also [HA, 3.2.1.4]) C is an initial object in $\mathrm{Alg}_{/\mathsf{Assoc}}(\mathcal{C})$ if and only if the unit morphism $\mathbb{1}_{\mathcal{C}} \to C$ is an equivalence. In this section we show that this is the case if and only if there exists *any* equivalence $\mathbb{1}_{\mathcal{C}} \simeq C$ in $\mathcal{C}$. In particular, this implies that any two associative algebras in $\mathcal{C}$ whose underlying objects in $\mathcal{C}$ are equivalent to $\mathbb{1}_{\mathcal{C}}$ are already equivalent as associative algebras.

Notation 5.1.3.1. Let $\mathcal{C}$ be a monoidal ∞-category and $\mathbb{1}$ a unit of $\mathcal{C}$. We will use the following notation in this section.

As part of the monoidal structure on $\mathcal{C}$, there are equivalences, natural in X,

$$\lambda_{\mathbb{1},X} \colon \mathbb{1} \otimes X \xrightarrow{\simeq} X$$

and

$$\rho_{X,\mathbb{1}} \colon X \otimes \mathbb{1} \xrightarrow{\simeq} X$$

for $\mathbb{1}$ any unit object in $\mathcal{C}$ and X any object in $\mathcal{C}$, called the *left unitor* and *right unitor*, respectively.

The reason why we let $\mathbb{1}$ be part of the notation is that we will consider morphisms between two unit objects that might not (a priori) be equivalences, so it will be important to distinguish them. ♢

Proposition 5.1.3.2. *Let $\mathcal{C}$ be a monoidal ∞-category, let $\mathbb{1}$ be a unit object in $\mathcal{C}$, and f and g two endomorphisms of $\mathbb{1}$. Then $f \circ g$ and $g \circ f$ are homotopic.* ♡

Proof. Two morphisms in an ∞-category are homotopic if and only if their images in the homotopy category are equal. It thus suffices to show that the monoid structure induced by composition on

$$\pi_0(\mathrm{Map}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})) = \mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1}, \mathbb{1})$$

is commutative.

Note that the monoidal structure on the ∞-category $\mathcal{C}$ induces the structure of an ordinary monoidal category on the homotopy category $\mathrm{Ho}(\mathcal{C})$, see

[HA, 4.1.1.12]. We can define a binary operation $\star$ on $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1}, \mathbb{1})$ by letting $f \star g$ for f and g in $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1}, \mathbb{1})$ be given by conjugating $f \otimes g$ with the left unitor $\lambda_{\mathbb{1},\mathbb{1}}$ as depicted below.

$$\begin{tikzcd} \mathbb{1} \arrow[d, dashed, "f\star g"'] & \mathbb{1}\otimes\mathbb{1} \arrow[l, "\lambda_{\mathbb{1},\mathbb{1}}"', "\cong"] \arrow[d, "f\otimes g"] \\ \mathbb{1} & \mathbb{1}\otimes\mathbb{1} \arrow[l, "\cong"', "\lambda_{\mathbb{1},\mathbb{1}}"] \end{tikzcd}$$

Naturality of $\lambda_{\mathbb{1},-}$ immediately implies that $\mathrm{id}_\mathbb{1}$ is a left unit for the binary operation $\star$. We could similarly define $\star'$ using the right unitor $\rho_{\mathbb{1},\mathbb{1}}$, for which $\mathrm{id}_\mathbb{1}$ would be a *right* unit. As the composition

$$\mathbb{1} \xrightarrow{\lambda^{-1}_{\mathbb{1},\mathbb{1}}} \mathbb{1}\otimes\mathbb{1} \xrightarrow{\rho_{\mathbb{1},\mathbb{1}}} \mathbb{1}$$

is the identity[20], so $\star = \star'$, and hence we can conclude that $\mathrm{id}_\mathbb{1}$ is a two-sided unit for the binary operation $\star$ on $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1}, \mathbb{1})$.

As $(f \otimes g) \circ (h \otimes i) = (f \circ h) \otimes (g \circ i)$ in $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1}, \mathbb{1})$ by functoriality of the tensor product for f, g, h, and i endomorphisms of $\mathbb{1}$, we have $(f \star g) \circ (h \star i) = (f \circ h) \star (g \circ i)$ and can thus apply the Eckmann-Hilton argument to conclude that composition is commutative in $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1}, \mathbb{1})$. $\square$

Proposition 5.1.3.3. *Let $\mathcal{C}$ be a monoidal ∞-category and R an* Assoc*-algebra in $\mathcal{C}$ such that the underlying object in $\mathcal{C}$ is a monoidal unit. Let $\mathbb{1}$ be another, fixed, unit object. Then the unit morphism $\iota\colon \mathbb{1}_\mathcal{C} \to R$, that is part of the data of R as an* Assoc*-algebra, is an equivalence.* ♡

Proof. As part of the data of R as an Assoc-algebra there is also a multiplication morphism $\mu\colon R \otimes R \to R$, as well as a commutative diagram exhibiting (part of) unitality for R, depicted in the top half of the following diagram.

$$\begin{tikzcd} R \arrow[rrr, bend left, "\mathrm{id}_R"] \arrow[dr, "\mathrm{id}_R"'] & \mathbb{1}\otimes R \arrow[l, "\lambda_{\mathbb{1},R}"', "\simeq"] \arrow[r, "\iota\otimes\mathrm{id}_R"] \arrow[d, "\lambda_{\mathbb{1},R}", "\simeq"'] & R\otimes R \arrow[r, "\mu"] \arrow[d, "\lambda_{R,R}"', "\simeq"] & R \\ & R \arrow[r, dashed, "\varphi"'] & R \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

The morphisms φ and ψ are defined as the induced morphisms that make the diagrams commute.

[20] In [Mac98, VII.1] this is required as an axiom for the definition of monoidal categories, but Kelly showed in [Kel64, Theorems 6 and 7] that this in fact follows from the now usual list of axioms.

There is also a commutative diagram by naturality of $\rho_{-,R}$ as follows.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\iota"] & R \\
\mathbb{1} \otimes R \arrow[u, "\rho_{\mathbb{1},R}", "\simeq"'] \arrow[r, "\iota\otimes\mathrm{id}_R"'] & R \otimes R \arrow[u, "\simeq", "\rho_{R,R}"']
\end{tikzcd}$$

Thus ι is an equivalence if and only if $\iota \otimes \mathrm{id}_R$ is, which is an equivalence if and only if φ is. But for φ we already have a left inverse ψ, i. e. $\psi \circ \varphi$ is homotopic to id_R. It follows from Proposition 5.1.3.2 that $\varphi \circ \psi$ is then also homotopic to id_R, so φ is an equivalence. □

5.1.4 Group homology

Let G be a (discrete) group. The goal of this section is to discuss how to calculate orbits of G-objects in $\mathcal{D}(k)$ and discuss the relation to classical notions. The category of G-objects in $\mathcal{D}(k)$ is defined as

$$\mathcal{D}(k)^{\mathrm{B}\,G} := \mathrm{Fun}(\mathrm{B}\,G, \mathcal{D}(k))$$

where $\mathrm{B}\,G$ is the 1-groupoid with a single object $*$ and $\mathrm{Aut}_{\mathrm{B}\,G} := G$. If $F\colon \mathrm{B}\,G \to \mathcal{D}(k)$ is a functor that we think of as an object in $\mathcal{D}(k)$ with G-action, then we will often not distinguish notationally between F and $F(*)$.

Let X be a G-object in $\mathcal{D}(k)$. Then the *G-orbits* X_G of X is the colimit of X considered as a functor $\mathrm{B}\,G \to \mathcal{D}(k)$.

We want to relate the construction of orbits of G-objects in $\mathcal{D}(k)$ to classical notions of homological algebra. To start we note that by [HA, 1.3.4.25] every G-object in $\mathcal{D}(k)$ is represented by a G-object in $\mathsf{Ch}(k)$ that is cofibrant in the projective model structure on $\mathrm{Fun}(\mathrm{B}\,G, \mathsf{Ch}(k))$. Let X be a G-object in $\mathsf{Ch}(k)$ with cofibrant underlying chain complex. We can then apply [HA, 1.3.4.24] to conclude that $\gamma(X)_G \simeq \mathrm{hocolim}_{\mathrm{B}\,G} X$.

The category of G-objects in $\mathsf{Ch}(k)$ can be identified with $\mathsf{Ch}(kG)$, where kG is the group ring of G over k, see [Wei94, Section 6.1]. This isomorphism of categories is compatible with the respective forgetful functors to $\mathsf{Ch}(k)$, from which it immediately follows that the respective weak equivalences and projective fibrations coincide[21], so that this is even an equivalence of combinatorial model categories.

The colimit functor $\mathrm{Fun}(\mathrm{B}\,G, \mathsf{Ch}(k)) \to \mathsf{Ch}(k)$ is a left Quillen functor that is left adjoint to the functor const, the homotopy colimit functor is its derived

[21] For the projective model structure on $\mathrm{Fun}(\mathrm{B}\,G, \mathsf{Ch}(k))$, which we take with respect to the projective model structure on $\mathsf{Ch}(k)$, see [HTT, A.2.8.2], and for the projective model structure on $\mathsf{Ch}(kG)$ see Fact 4.1.3.1 – while we did not specifically mention it there, the assumption that the ring over which we take chain complexes is commutative is unnecessary for merely obtaining a combinatorial model category (commutativity is needed if we want to talk about the symmetric monoidal structure).

functor. Under the equivalence $\mathrm{Fun}(\mathrm{B}\,G, \mathsf{Ch}(k)) \cong \mathsf{Ch}(kG)$, the functor const corresponds to the restriction of scalars functor $\mathsf{Ch}(k) \to \mathsf{Ch}(kG)$ that is induced by restriction along the ring homomorphism $kG \to k$ that maps every element of G to 1. The left adjoint of this functor is given by extension of scalars, so $k \otimes_{kG} -$, see also the discussion in [Wei94, Exercise 6.1.1 2 and Lemma 6.1.1].

The upshot is the following: If X is is a G-object in $\mathsf{Ch}(k)$, then there is an equivalence

$$\gamma(X)_G \simeq \gamma\big(k \otimes^{\mathrm{L}}_{kG} X'\big)$$

where on the right we take the derived tensor product and X' is the object in $\mathsf{Ch}(kG)$ associated to X.

The homology k-modules of this derived tensor product is by definition given by Tor, and this particular case this is what is called the *group homology* of G with coefficients in X (or X'), and denoted by $\mathrm{H}_*(G; X)$, see [Wei94, Definition 6.1.2 and Exercise 6.1.2]. We can summarize the discussion as follows, using Proposition 4.3.3.2.

Proposition 5.1.4.1. *Let G be a discrete group and X a G-object in $\mathsf{Ch}(k)$. Then there are isomorphisms*

$$\mathrm{H}_i(\gamma(X)_G) \cong \mathrm{Tor}_i^{kG}(k, X') \cong \mathrm{H}_i(G; X)$$

for every integer i, where X' is the kG-chain complex associated to X under the isomorphism discussed above. These isomorphisms are natural in X. ♡

We can conclude the following from this.

Proposition 5.1.4.2. *Let G be a discrete group and X a G-object in $\mathcal{D}(k)$. Assume that n is an integer such that the homology of X vanishes in degrees below n. Then the homology of X_G also vanishes below degree n, and $\mathrm{H}_n(X_G) \cong \mathrm{H}_n(X)_G$.* ♡

Proof. One way to prove this is to use represent X by a G-object in $\mathsf{Ch}(k)$ concentrated in degrees n and above, and then the statement follows from Proposition 5.1.4.1.

Another way would be to note that $\mathcal{D}(k)_{\geq n}$ is by [HA, 1.2.1.6] closed under colimits, from which it follows that the homology vanishes below degree n, and use Proposition 4.3.3.8 for homology in degree n. □

5.1.5 Formality of certain commutative algebras

The goal of Section 5.1 is to show that any two commutative bialgebras in $\mathcal{D}(k)$ whose homology is concentrated in degrees 0 and 1, where it is isomorphic to k, are equivalent. As a stepping stone we show in this section the analogous and significantly easier statement for commutative algebras, so forgetting the coalgebra structure.

We start in the following construction by constructing a comparison morphism from a "standard" commutative algebra with the prescribed homology (that the homology is the correct one will be shown below in Proposition 5.1.5.3). We will later show that this morphism is an equivalence of commutative algebras.

Construction 5.1.5.1. Let R be an object of $\mathrm{CAlg}(\mathcal{D}(k))$ and

$$\vartheta\colon \mathrm{ev}_{\langle 1\rangle}(R) \xrightarrow{\simeq} k \oplus k[n]$$

an equivalence for some $n > 0$. Note that this equivalence is not assumed to have anything to do with the algebra structure on R, this is only an assumption on the equivalence class of the underlying object in $\mathcal{D}(k)$ of R.

As the underlying object of R is in $(\mathcal{D}(k)_{\geq 0})_{\leq n}$ it follows from Proposition 4.3.4.1 (7) and (8) that we can consider R as an object of the ∞-category $\mathrm{CAlg}((\mathcal{D}(k)_{\geq 0})_{\leq n})$.

Denote the inclusions that are part of $k \oplus k[n]$ being a coproduct by $\iota_0\colon k \to k \oplus k[n]$ and $\iota_n\colon k[n] \to k \oplus k[n]$, and let $g\colon k[n] \to \mathrm{ev}_{\langle 1\rangle}(R)$ be $g \coloneqq \vartheta^{-1} \circ \iota_n$.

By [HA, 1.2.1.6], [HTT, 1.2.13.7], Proposition 4.3.2.1 (1), and [HA, 3.1.3.5], the forgetful functor

$$\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CAlg}(\mathcal{D}(k)_{\geq 0}) \to \mathcal{D}(k)_{\geq 0}$$

admits a left adjoint $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}$. We thus obtain an induced map of commutative algebras in $\mathcal{D}(k)_{\geq 0}$

$$f'\colon \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}(k[n]) \to R$$

that is adjoint to g.

Note that as the inclusion $\iota_{\geq 0}\colon \mathcal{D}(k)_{\geq 0} \to \mathcal{D}(k)$ is symmetric monoidal (Proposition 4.3.4.1 (3)) and also preserves colimits ([HA, 1.2.1.6] with [HTT, 1.2.13.7]), we can use Proposition E.7.2.2 to identify $\mathrm{CAlg}(\iota_{\geq 0})(f')$ with the morphism

$$f''\colon \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]) \to R$$

that is adjoint to g.

Finally, as R lies in $\mathrm{CAlg}((\mathcal{D}(k)_{\geq 0})_{\leq n})$, the morphism f' is by Proposition 4.3.4.1 (8) adjoint to a morphism

$$f\colon \mathrm{CAlg}(\tau_{\leq n})\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}(k[n])\Big) \to R$$

of commutative algebras in $(\mathcal{D}(k)_{\geq 0})_{\leq n}$. ♢

The equivalence $\vartheta^{-1}\colon k \oplus k[n] \xrightarrow{\simeq} \mathrm{ev}_{\langle 1\rangle}(R)$ in Construction 5.1.5.1 could be anything on the summand k. However, we already have a candidate morphism $k \to \mathrm{ev}_{\langle 1\rangle}(R)$ – the unit morphism of the commutative algebra structure of R. In the next proposition we show that we can replace ϑ^{-1} on the first summand by the unit morphism without losing the property of being an equivalence.

Proposition 5.1.5.2. *In the situation of Construction 5.1.5.1, the morphism*

$$\iota \amalg g \colon k \oplus k[n] \to \mathrm{ev}_{\langle 1 \rangle}(R)$$

is an equivalence in $\mathcal{D}(k)$, where ι is the unit morphism of the algebra structure on R. ♡

Proof. It suffices to show that the composition $\vartheta \circ (\iota \amalg g)$ is an equivalence. Using the definition of g we can write this morphism as

$$k \oplus k[n] \xrightarrow{\begin{bmatrix} \iota' & 0 \\ \iota'' & \mathrm{id}_{k[n]} \end{bmatrix}} k \oplus k[n]$$

for some morphisms $\iota' \colon k \to k$ and $\iota'' \colon k \to k[n]$. It thus suffices to show that ι' is an equivalence, as then

$$\begin{bmatrix} \iota'^{-1} & 0 \\ -\iota''\iota'^{-1} & \mathrm{id}_{k[n]} \end{bmatrix}$$

will be an inverse.

While we do not need this, we note that ι'' must actually be nullhomotopic, as

$$\pi_0\Big(\mathrm{Map}_{\mathcal{D}(k)}(k, k[n])\Big) \cong H_0(k[n]) \cong 0$$

by Proposition 4.3.2.1 (5) and (4).

Applying the natural transformation $\mathrm{id}_{\mathcal{D}(k)} \to \iota_{\leq 0} \circ \tau_{\leq 0}$ (see Section 4.3.4) we obtain a commuting diagram as follows[22]

$$\begin{tikzcd}
k \arrow[r, "\vartheta\circ\iota"] \arrow[d] \arrow[rrr, bend left, "\iota'"] & k \oplus k[n] \arrow[rr, "\mathrm{pr}_0"] \arrow[d] & & k \arrow[d] \\
\tau(k) \arrow[r, "\tau(\vartheta\circ\iota)"] \arrow[dr, "\tau(\iota)"'] & \tau(k \oplus k[n]) \arrow[r, "\tau(\mathrm{pr}_0)\times\tau(\mathrm{pr}_n)"] & \tau(k) \oplus \tau(k[n]) \arrow[r, "\mathrm{pr}_0"] & \tau(k) \\
& \tau\big(\mathrm{ev}_{\langle 1 \rangle}(R)\big) \arrow[u, "\tau(\vartheta)"'] & &
\end{tikzcd}$$

in $\mathcal{D}(k)$ where the morphisms pr_0 and pr_n are the projections onto the first and second factor, respectively.

[22] To save space we write τ instead of $\iota_{\leq 0}\tau_{\leq 0}$.

We have to show that ι' is an equivalence. As k is in $\mathcal{D}(k)_{\leq 0}$, the leftmost and rightmost vertical morphisms are equivalences. It thus suffices to show that the composite from left to right in the middle row is an equivalence.

As a left adjoint $\tau_{\leq 0}$ preserves colimits and hence finite biproducts, and $\iota_{\leq 0}$ preserves finite biproducts as well by Proposition 4.3.4.2. Thus the morphism

$$\iota_{\leq 0}\tau_{\leq 0}(\mathrm{pr}_0) \times \iota_{\leq 0}\tau_{\leq 0}(\mathrm{pr}_n)$$

in the middle is an equivalence. The morphism pr_0 on the right (in the middle row) is an equivalence as $\tau_{\leq 0}(k[n]) \simeq 0$[23]. As ϑ is an equivalence, $\iota_{\leq 0}\tau_{\leq 0}(\vartheta)$ is also an equivalence.

It thus remains to show that $\iota_{\leq 0}\tau_{\leq 0}(\iota)$ is an equivalence. As we have already seen that domain and codomain of this morphism is equivalent to k and hence in $\mathcal{D}(k)_{\geq 0}$, this morphism is equivalent to $\iota_{\geq 0}\tau_{\geq 0}\iota_{\leq 0}\tau_{\leq 0}(\iota)$, which by [HA, 1.2.1.10] can be identified with $\iota_{\geq 0}\iota_{\geq 0, \leq 0}\tau_{\leq 0}\tau_{\geq 0}(\iota)$. As all four involved functors are lax symmetric monoidal by Proposition 4.3.4.1, this is the unit morphism of a commutative algebra in $\mathcal{D}(k)$ whose underlying object is equivalent to k. We can thus apply Proposition 5.1.3.3 to conclude that $\iota_{\leq 0}\tau_{\leq 0}(\iota)$ is an equivalence. □

Before we can show that the morphism f from Construction 5.1.5.1 is an equivalence, we need to determine the homology of $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$ in low degrees. We do this in the following proposition, where we actually calculate the homology in a wider range than would be necessary in this section – the calculations in the extra degrees will be used in later sections.

Proposition 5.1.5.3. *Let $n \geq 1$ and let*

$$\varphi\colon k[n] \to \mathrm{ev}_{\langle 1\rangle}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right)$$

be the morphism in $\mathcal{D}(k)$ exhibiting $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$ as the free commutative algebra generated by $k[n]$ and let

$$i\colon k \to \mathrm{ev}_{\langle 1\rangle}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right)$$

be the unit morphism.

[23] This can be easily seen using the fiber sequence

$$\iota_{\geq 1}\tau_{\geq 1}(k[n]) \to k[n] \to \iota_{\leq 0}\tau_{\leq 0}(k[n])$$

from [HA, 1.2.1.8] in which the first morphism is an equivalence as $k[n]$ lies in $\mathcal{D}(k)_{\geq 1}$.

Then the following holds for the homology of $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$.

$$\mathrm{H}_i\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\Big) \cong \begin{cases} 0 & \text{if } i<0 \\ k & \text{if } i=0 \\ 0 & \text{if } 0<i<n \\ k & \text{if } i=n \\ 0 & \text{if } n<i<2n \\ k & \text{if } i=2n \text{ and } n \text{ is even} \\ k/(2) & \text{if } i=2n \text{ and } n \text{ is odd} \end{cases}$$

Furthermore, a basis of the homology in degrees 0 *and* n *is given by* i *and* φ, *i. e.* $i \amalg \varphi\colon k \oplus k[n] \to \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$ *induces an isomorphism on homology in degrees smaller than* $2n$. ♡

Proof. Using [HA, 3.1.3.13] and unpacking the definition of the relevant ∞-groupoids $\mathcal{P}(m)$ for $\mathcal{O}^\otimes = \mathsf{Comm}^\otimes$[24] we obtain that there is an equivalence[25]

$$\mathrm{ev}_{\langle 1\rangle}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\Big) \simeq \coprod_{m\geq 0}\big(k[n]^{\otimes m}\big)_{\Sigma_m} \simeq k \amalg k[n] \amalg \coprod_{m\geq 2}\big(k[n]^{\otimes m}\big)_{\Sigma_m}$$

in $\mathcal{D}(k)$ and under this equivalence the unit morphism and the morphism φ exhibiting it as the free commutative algebra generated by $k[n]$ are the inclusions of the summands indexed by 0 and 1, respectively.

By Proposition 4.3.3.5 H_i preserves coproducts, so it suffices to show the following.

(1) $\mathrm{H}_i((k[n]^{\otimes m})_{\Sigma_m}) \cong 0$ for $m \geq 2$ and $i < nm$.

(2) $\mathrm{H}_{2n}((k[n]^{\otimes 2})_{\Sigma_2}) \cong k$ if n is even and $\mathrm{H}_{2n}((k[n]^{\otimes 2})_{\Sigma_2}) \cong k/(2)$ if n is odd.

Proof of Claim (1): Note that if $m \geq 2$ then $k[n]^{\otimes m} \simeq k[nm]$ has homology concentrated in degree nm and is hence in $\mathcal{D}(k)_{\geq nm}$. As $\mathcal{D}(k)_{\geq nm}$ is stable under colimits in $\mathcal{D}(k)$ (see [HA, 1.2.1.6]) we can conclude that $(k[n]^{\otimes m})_{\Sigma_m}$ is also in $\mathcal{D}(k)_{\geq nm}$ and hence has vanishing homology in degrees smaller than nm.

Proof of Claim (2): Going through [HA, 3.1.3.13] and [HA, 3.1.3.9] to identify the action of Σ_2 on $k[n]\otimes k[n]$, we see that the nontrivial element acts via the symmetry equivalence that is part of the structure of $\mathcal{D}(k)$ as a symmetric monoidal ∞-category, and which is induced by the symmetry isomorphism of the symmetric monoidal structure on $\mathsf{Ch}(k)$, see Proposition 4.3.2.1 (1)

[24] We get an equivalence of ∞-groupoids $\mathcal{P}(m) \simeq \mathrm{B}\,\Sigma_m$, where $\mathrm{B}\,\Sigma_m$ is the 1-groupoid with a single object and the symmetric group on m elements as automorphism group.

[25] The subscript Σ_m denotes a (homotopy) orbit, i. e. a colimit of a functor from $\mathrm{B}\,\Sigma_m$.

and Definition 4.1.2.1. We can thus represent the Σ_2-object $k[n] \otimes k[n]$ in $\mathcal{D}(k)$ by the Σ_2-object $k[n] \otimes k[n]$ in $\mathsf{Ch}(k)$ where the non-trivial element acts via the symmetry isomorphism. There is an isomorphism $k[n] \otimes k[n] \cong k[2n]$ mapping $1 \otimes 1$ to 1, and we obtain an induced Σ_2-action on $k[2n]$. If n is odd, then the non-trivial element of Σ_2 acts as $-\mathrm{id}$, which reflects the fact that if x is an element in odd degree of a commutative differential graded algebra, then we have $x^2 = -x^2$. If n is even, then the non-trivial element acts as id.

The claim now follows from Proposition 5.1.4.2. □

Proposition 5.1.5.4. *In the situation of Construction 5.1.5.1, the morphism f is an equivalence.*

In particular, if R' is another commutative algebra in $\mathcal{D}(k)$ such that the underlying objects $\mathrm{ev}_{\langle 1 \rangle}(R')$ and $\mathrm{ev}_{\langle 1 \rangle}(R)$ are equivalent, then R and R' are also equivalent as commutative algebras. ♡

Proof. The adjoint f of f' is by definition given by the composition

$$\mathrm{CAlg}(\tau_{\leq n})\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}(k[n])\Big) \xrightarrow{\mathrm{CAlg}(\tau_{\leq n})(f')} \mathrm{CAlg}(\tau_{\leq n})(\mathrm{CAlg}(\iota_{\geq 0, \leq n})(R))$$
$$\longrightarrow R$$

where the second morphism the the counit of the following adjunction.

$$\mathrm{CAlg}(\tau_{\leq n}) \dashv \mathrm{CAlg}(\iota_{\geq 0, \leq n})$$

This counit is homotopic to the identity by construction[26], so it suffices to show that $\mathrm{CAlg}(\tau_{\leq n})(f')$ is an equivalence. As $\iota_{\geq 0, \leq n}$ and $\iota_{\geq 0}$ are fully faithful and hence conservative, and $\mathrm{ev}_{\langle 1 \rangle}$ is also conservative [HA, 3.2.2.6], it suffices to show that

$$\begin{aligned}
&\big(\iota_{\geq 0} \circ \iota_{\geq 0, \leq n} \circ \mathrm{ev}_{\langle 1 \rangle} \circ \mathrm{CAlg}(\tau_{\leq n})\big)(f') \\
\simeq\ &\big(\iota_{\geq 0} \circ \iota_{\geq 0, \leq n} \circ \tau_{\leq n} \circ \mathrm{ev}_{\langle 1 \rangle}\big)(f') \\
\simeq\ &\big(\iota_{\leq n} \circ \tau_{\leq n} \circ \iota_{\geq 0} \circ \mathrm{ev}_{\langle 1 \rangle}\big)(f') \\
\simeq\ &\big(\iota_{\leq n} \circ \tau_{\leq n} \circ \mathrm{ev}_{\langle 1 \rangle} \circ \mathrm{CAlg}(\iota_{\geq 0})\big)(f') \\
\simeq\ &\big(\iota_{\leq n} \circ \tau_{\leq n} \circ \mathrm{ev}_{\langle 1 \rangle}\big)(f'')
\end{aligned}$$

is an equivalence.

Recall from Construction 5.1.5.1 that

$$f''\colon\ \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]) \to R$$

[26] See [HA, 1.2.1.5] and [HTT, 5.2.7.6, 5.2.7.7, and 5.2.7.8].

is the morphism in $\mathrm{CAlg}(\mathcal{D}(k))$ adjoint to g. There is thus a commutative diagram

$$\begin{tikzcd} & k[n] \arrow[dl, "\varphi"'] \arrow[dr, "g"] & \\ \mathrm{ev}_{\langle 1\rangle}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right) \arrow[rr, "\mathrm{ev}_{\langle 1\rangle}(f'')"'] & & \mathrm{ev}_{\langle 1\rangle}(R) \end{tikzcd}$$

where φ exhibits $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$ as the free commutative algebra generated by $k[n]$. If we let i be the unit morphism of $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$ and ι the unit morphism of R, then $f'' \circ i \simeq \iota$ as f'' is a morphism of commutative algebras. We can thus extend this commutative diagram to a commutative diagram as follows.

$$\begin{tikzcd} & k \oplus k[n] \arrow[dl, "i \amalg \varphi"'] \arrow[dr, "\iota \amalg g"] & \\ \mathrm{ev}_{\langle 1\rangle}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right) \arrow[rr, "\mathrm{ev}_{\langle 1\rangle}(f'')"'] & & \mathrm{ev}_{\langle 1\rangle}(R) \end{tikzcd}$$

The morphism on the right is an equivalence by Proposition 5.1.5.2. We have to show that $\tau_{\leq n}$ of the bottom morphism is an equivalence, so it suffices to show that $\tau_{\leq n}$ of the left morphism is an equivalence. But this follows from Proposition 5.1.5.3. $\square$

5.1.6 Identification of some mapping spaces

As explained in the introduction to Section 5.1, it will be important for us to show that

$$\pi_0\Big(\mathrm{Map}_{\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))}\big(\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n), R^\vee\big)\Big)$$

is trivial for certain commutative bialgebras R. We saw in Section 5.1.2.7 that $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n)$ is free on the pointed cocommutative algebra $\gamma^{\mathsf{Comm}}_{\mathbb{E}_0}(\underline{B}'_n)$, so we are led to consider path components of mapping spaces in

$$\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg}(\mathcal{D}(k))) \simeq \mathrm{coCAlg}(\mathcal{D}(k))_{k/}$$

and after dualizing of mapping spaces in $\mathrm{CAlg}(\mathcal{D}(k))_{/k}$.

This section concerns the steps needed to show that the sets of path components of such mapping spaces that are of interest to us are indeed trivial. In Section 5.1.6.1 we will show that the relevant mapping spaces in $\mathrm{CAlg}(\mathcal{D}(k))_{/k}$ can be calculated as the mapping spaces between the underlying objects in $\mathrm{CAlg}(\mathcal{D}(k))$. In Section 5.1.6.3 we will then show that π_0 of the relevant mapping spaces in $\mathrm{CAlg}(\mathcal{D}(k))$ are trivial. In order to do so, we will need to construct a commutative algebra with prescribed homology. We will define such a commutative algebra as a pushout of free commutative algebras and show that its homology has the required description in Section 5.1.6.2.

5.1.6.1 Identification of a mapping space in an overcategory

In this section we show that, under certain assumptions, mapping spaces in the ∞-category $\mathrm{CAlg}(\mathcal{D}(k))_{/k}$ are equivalent to the mapping spaces between the respective underlying objects in $\mathrm{CAlg}(\mathcal{D}(k))$.

Proposition 5.1.6.1. *Let $R \to k$ and $S \to k$ be objects of $\mathrm{CAlg}(\mathcal{D}(k))_{/k}$, and assume that there is an equivalence $\tau_{\leq 0}(\mathrm{ev}_{\langle 1\rangle}(R)) \simeq k$ in $\mathcal{D}(k)$.*

Then the map induced by the canonical forgetful functor on mapping spaces

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))_{/k}}(R,S) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R,S)$$

is an equivalence. ♡

Proof. By (the dual of) Proposition D.1.3.2 there is a pullback diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))_{/k}}(R,S) \arrow[r] \arrow[d] & \{R \to k\} \arrow[d] \\
\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R,S) \arrow[r, "(S\to k)_*"'] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R,k)
\end{tikzcd}$$

in $\mathcal{S}$, where the left vertical map is the one induced by the forgetful functor. It thus suffices to prove that $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R,k)$ is contractible.

k as well as the underlying object $\mathrm{ev}_{\langle 1\rangle}(R)$ of R are in $\mathcal{D}(k)_{\geq 0}$[27], so using that by Proposition 4.3.4.1 (7) $\mathrm{CAlg}(\iota_{\geq 0})$ is fully faithful with essential image spanned by those commutative algebras whose underlying object is in $\mathcal{D}(k)_{\geq 0}$, it suffices to show that

$$\mathrm{Map}_{\mathrm{CAlg}\left(\mathcal{D}(k)_{\geq 0}\right)}(R,k) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R,k)$$

is contractible.

As k actually lies in $(\mathcal{D}(k)_{\geq 0})_{\leq 0}$ we can use the adjunction

$$\mathrm{CAlg}(\tau_{\leq 0}) \dashv \mathrm{CAlg}(\iota_{\geq 0, \leq 0})$$

[27] By [HA, 1.2.1.8] there is a cofiber sequence

$$\iota_{\geq 0}\tau_{\geq 0}R \to R \to \iota_{\leq -1}\tau_{\leq -1}R$$

and

$$\tau_{\leq -1}R \simeq \tau_{\leq -1}\tau_{\leq 0}R \simeq \tau_{\leq -1}k \simeq 0$$

so $\iota_{\geq 0}\tau_{\geq 0}R \simeq R$ is in $\mathcal{D}(k)_{\geq 0}$.

with fully faithful right adjoint discussed in Proposition 4.3.4.1 (8) to obtain equivalences

$$\begin{aligned} &\mathrm{Map}_{\mathrm{CAlg}\left(\mathcal{D}(k)_{\geq 0}\right)}(R, k) \\ \simeq\; &\mathrm{Map}_{\mathrm{CAlg}\left(\left(\mathcal{D}(k)_{\geq 0}\right)_{\leq 0}\right)}(\mathrm{CAlg}(\tau_{\leq 0})(R), k) \\ \simeq\; &\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(\mathrm{CAlg}(\iota_{\leq 0})\,\mathrm{CAlg}(\tau_{\leq 0})(R), k) \end{aligned}$$

By assumption the underlying object

$$\left(\mathrm{ev}_{\langle 1\rangle} \circ \mathrm{CAlg}(\iota_{\leq 0}) \circ \mathrm{CAlg}(\tau_{\leq 0})\right)(R) \simeq \left(\iota_{\leq 0} \circ \tau_{\leq 0} \circ \mathrm{ev}_{\langle 1\rangle}\right)(R)$$

of $\mathrm{CAlg}(\iota_{\leq 0})\,\mathrm{CAlg}(\tau_{\leq 0})(R)$ is equivalent to k, so by Proposition 5.1.3.3 the unit morphism $k \to \mathrm{CAlg}(\iota_{\leq 0})\,\mathrm{CAlg}(\tau_{\leq 0})(R)$ is an equivalence. [HA, 3.2.1.9] then implies that $\mathrm{CAlg}(\iota_{\leq 0})\,\mathrm{CAlg}(\tau_{\leq 0})(R)$ is an initial object of $\mathrm{CAlg}(\mathcal{D}(k))$, so the mapping space

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(\mathrm{CAlg}(\iota_{\leq 0})\,\mathrm{CAlg}(\tau_{\leq 0})(R), k)$$

is contractible. □

5.1.6.2 The homology of a pushout of commutative algebras

Let $n > 0$ be an integer, and let R be a commutative algebra in $\mathcal{D}(k)$ with homology concentrated in degree 0 and n, where it is isomorphic to k. In Section 5.1.6.3 we want to show that the mapping space in $\mathrm{CAlg}(\mathcal{D}(k))$ from R to another commutative algebra S with certain restrictions on its homology is contractible. To do so, we construct a commutative algebra for which it is easier to calculate mapping spaces out of, and such that its homology is isomorphic to that of R in degrees smaller than or equal to $2n$. We can start with the free commutative algebra generated by one generator in degree n. We calculated the homology in the relevant degrees in Proposition 5.1.5.3, and it is already nearly as we want, except that the homology might not vanish in degree $2n$, where it is generated by a single element. To divide out that unwanted element we can form a pushout over the free commutative algebra with a generator in degree $2n$.

We will start by carrying out this construction in Construction 5.1.6.2, and then spend the remainder of this section proving that the homology is as we require in Proposition 5.1.6.3. One way to do this calculation would be to use the Tor spectral sequence, see [HA, 7.2.1.19], but we have opted for a more direct approach with a concrete resolution that suffices in order to calculate the homology groups in the necessary degrees.

Construction 5.1.6.2. Let $n > 0$ be an integer. In Proposition 5.1.5.3 we showed that

$$\mathrm{H}_{2n}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right) \cong k$$

if n is even and
$$\mathrm{H}_{2n}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right) \cong k/(2)$$
if n is odd. In both cases, this k-module can be generated by a single element. Let
$$f'\colon k[2n] \to \mathrm{ev}_{\langle 1\rangle}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right)$$
be a morphism in $\mathcal{D}(k)$ representing a generator[28] of $\mathrm{H}_{2n}(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]))$. We obtain an induced morphism
$$f\colon\ \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]) \to \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$$
in $\mathrm{CAlg}(\mathcal{D}(k))$ that is adjoint to f'.

The zero morphism $k[2n] \to k$ similarly induces a morphism of commutative algebras $p\colon\ \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]) \to k$.

Define P to be the pushout in $\mathrm{CAlg}(\mathcal{D}(k))$ as in the following diagram.

$$\begin{tikzcd}
\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]) \arrow[r, "p"] \arrow[d, "f"'] & k \arrow[d, "i"] \\
\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]) \arrow[r, "j"'] & P
\end{tikzcd}$$

We will use the notation P, f, p, i, and j elsewhere where we explicitly refer to this construction. ♢

Proposition 5.1.6.3. *Let $n > 0$ be an integer. For P as in Construction 5.1.6.2, the following holds for the homology of P.*

$$\mathrm{H}_i(P) \cong \begin{cases} 0 & \text{if } i < 0 \\ k & \text{if } i = 0 \\ 0 & \text{if } 0 < i < n \\ k & \text{if } i = n \\ 0 & \text{if } n < i \leq 2n \end{cases}$$

Furthermore, the morphism $j\colon\ \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]) \to P$ from Construction 5.1.6.2 induces an isomorphism on H_i for $i < 2n$. ♡

Proof. To improve readability in the formulas we will use the following shorthand notation in this proof. We write F_{2n} for $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n])$ and F_n for

[28] If 2 is invertible in k and n is odd, then we have $k/(2) \cong 0$, which is of course still generated by a single element 0, so we can carry out this construction also in this case, even though the construction is not really necessary for applications. However, we would like to avoid special handling of this one case.

$\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$. Furthermore, we will omit writing forgetful functors and will instead always make explicit in which ∞-category objects and morphisms are considered. We will also use the notation from Construction 5.1.6.2.

The strategy of this calculation is as follows. By construction P is a pushout of commutative algebras, so by Proposition E.8.0.5 can be calculated as a relative tensor product. We thus resolve k as a left-F_{2n}-module in a manner that suffices to extract the homology groups we are interested in from the long exact sequences in homology that we obtain.

Let $g\colon k[2n] \to F_{2n}$ be the morphism in $\mathcal{D}(k)$ that exhibits F_{2n} as the free commutative algebra generated by $k[2n]$. We first consider the following composition in $\mathrm{LMod}_{F_{2n}}(\mathcal{D}(k))$

$$\begin{tikzcd} F_{2n}[2n] \arrow[r, "\simeq"] \arrow[rrr, bend left, "g'"] & F_{2n} \otimes k[2n] \arrow[r, "\mathrm{id}_{F_{2n}} \otimes g"] & F_{2n} \otimes F_{2n} \arrow[r, "\mu"] & F_{2n} \arrow[r, "p"] & k \end{tikzcd} \tag{$*$}$$

where μ is the multiplication, g' is defined as the composition indicated in the diagram, and F_{2n} acts on $F_{2n} \otimes F_{2n}$ and $F_{2n} \otimes k[2n]$ via the the left tensor factor, and on k via p[29].

We claim that the composition pg' from $F_{2n}[2n]$ to k in $(*)$ is nullhomotopic as a morphism in $\mathrm{LMod}_{F_{2n}}(\mathcal{D}(k))$. In fact, every morphism of F_{2n}-algebras

[29] Here are some more details on obtaining these morphisms as morphisms in $\mathrm{LMod}_{F_{2n}}(\mathcal{D}(k))$.

There is a commutative diagram in $\mathrm{CAlg}(\mathcal{D}(k))$

$$\begin{tikzcd} & F_{2n} \otimes F_{2n} \arrow[dr, "\mu"] & & \\ F_{2n} \arrow[ur, "\mathrm{id}_{F_{2n}} \otimes 1"] \arrow[rr, "\mathrm{id}_{F_{2n}}"'] & & F_{2n} \arrow[r, "p"'] & k \end{tikzcd}$$

where $\mathrm{id}_{F_{2n}} \otimes 1$ is the composition $F_{2n} \simeq F_{2n} \otimes k$ with the identity tensor the unit of F_{2n} – this is the inclusion of the first summand of the coproduct $F_{2n} \otimes F_{2n} \simeq F_{2n} \amalg F_{2n}$ in $\mathrm{CAlg}(\mathcal{D}(k))$.

We can now forget down to associative algebras and then use the section $\mathrm{Alg}(\mathcal{D}(k)) \to \mathrm{LMod}(\mathcal{D}(k))$ from [HA, 4.2.1.17] that carries an algebra to the underlying object as a module over the algebra itself. We can then restrict the actions to obtain a commutative diagram of F_{2n}-modules. This constructs the morphisms μ and p in $(*)$. See also Construction E.8.0.4 for more details for this kind of construction.

The morphism

$$k[2n] \xrightarrow{1 \otimes g} F_{2n} \otimes F_{2n}$$

in $\mathcal{D}(k)$ is adjoint to a morphism of left-F_{2n}-modules $F_{2n} \otimes k[2n] \to F_{2n} \otimes F_{2n}$ (here $F_{2n} \otimes k[2n]$ is the free left-F_{2n}-module generated by $k[2n]$, see [HA, 4.2.4]). The morphism of $\mathcal{D}(k)$ underlying this morphism is then by definition given by the composition

$$F_{2n} \otimes k[2n] \xrightarrow{\mathrm{id}_{F_{2n}} \otimes 1 \otimes g} F_{2n} \otimes F_{2n} \otimes F_{2n} \xrightarrow{\mu \otimes \mathrm{id}_{F_{2n}}} F_{2n} \otimes F_{2n}$$

which is homotopic to $\mathrm{id}_{F_{2n}} \otimes g$.

$F_{2n}[2n] \to k$ is nullhomotopic, as we have by [HA, 4.2.4.6] an equivalence

$$\mathrm{Map}_{\mathrm{LMod}_{F_{2n}}(\mathcal{D}(k))}(F_{2n}[2n], k) \simeq \mathrm{Map}_{\mathcal{D}(k)}(k[2n], k)$$

which is contractible as $k[2n]$ is concentrated in degree $2n > 0$ and k is concentrated in degree 0.

The nullhomotopy of g' induces a morphism in $\mathrm{LMod}_{F_{2n}}(\mathcal{D}(k))$ from the cofiber of g' to k and a commutative triangle as in the following diagram

$$\begin{tikzcd} F_{2n}[2n] \arrow[r, "g'"] & F_{2n} \arrow[r, "\varphi"] \arrow[dr, "p"'] & C \arrow[d, dashed, "\psi"] \\ & & k \end{tikzcd} \qquad (**)$$

where the top row is a cofiber sequence.

Note that the forgetful functor $\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod}_{F_{2n}}(\mathcal{D}(k)) \to \mathcal{D}(k)$ preserves colimits by [HA, 4.2.3.5]. Using the long exact homology sequence for the cofiber sequence in $\mathcal{D}(k)$ underlying the one from $(**)$, together with the calculation of the lower homology groups of F_{2n} from Proposition 5.1.5.3, we obtain that φ induces an isomorphism

$$k \cong \mathrm{H}_0(F_{2n}) \xrightarrow{\mathrm{H}_0(\varphi)} \mathrm{H}_0(C)$$

and that for $i < 4n$ with $i \neq 0$ the homology group $\mathrm{H}_i(C)$ is zero[30]. As $\mathrm{H}_0(p)$ is an isomorphism (p underlies a morphism of commutative algebras and hence preserves the unit morphism) it follows that $\mathrm{H}_0(\psi)$ must be an isomorphism as well.

We now take the fiber of ψ to we obtain another cofiber sequence of left-F_{2n}-modules in $\mathcal{D}(k)$ as follows.

$$D \xrightarrow{\theta} C \xrightarrow{\psi} k \qquad (***)$$

Again using the long exact sequence in homology we can conclude that $\mathrm{H}_i(D) \cong 0$ for $i < 4n$.

Let us now get back to what we actually need to do, calculate the homology of P in low degrees. As $\mathcal{D}(k)$ is presentable symmetric monoidal by Proposition 4.3.2.1 (1), we can apply Proposition E.8.0.5, which tells us that P is

[30] The only nonzero homology groups of $F_{2n}[2n]$ and F_{2n} in degrees smaller than $4n$ are $\mathrm{H}_0(F_{2n}) \cong k$, $\mathrm{H}_{2n}(F_{2n}[2n])$, and $\mathrm{H}_{2n}(F_{2n})$, so the only thing that needs to be done is check that $\mathrm{H}_{2n}(g')$ is an isomorphism. By Proposition 5.1.5.3 the homology group $\mathrm{H}_{2n}(F_{2n}[2n])$ has a basis represented by the morphism $k[2n] \xrightarrow{1 \otimes \mathrm{id}_{k[2n]}} F_{2n} \otimes k[2n]$. Composing this morphism with g' we obtain by definition the morphism

$$\mu \circ \left(\mathrm{id}_{F_{2n}} \otimes g\right) \circ \left(1 \otimes \mathrm{id}_{k[2n]}\right) \simeq \mu \circ (1 \otimes g) \simeq g$$

which also by Proposition 5.1.5.3 forms a basis of $\mathrm{H}_{2n}(F_{2n})$.

equivalent to the relative tensor product[31] $F_n \otimes_{F_{2n}} k$, where we consider F_n and k as right and left modules over F_{2n}, which is considered as an associative algebra in $\mathrm{CAlg}(\mathcal{D}(k))$. The forgetful functor $\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{CAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$ is symmetric monoidal and preserves $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits by [HA, 3.2.3.2]. We can thus apply Proposition E.8.0.1 to conclude that the underlying object of $F_n \otimes_{F_{2n}} k$ in $\mathcal{D}(k)$ is equivalent to the relative tensor product $F_n \otimes_{F_{2n}} k$, where we consider F_{2n} as just an associative algebra in $\mathcal{D}(k)$.

Tensoring cofiber sequence $(***)$ with the right-F_{2n}-module F_n we obtain by [HA, 4.4.2.15] a cofiber sequence in $\mathcal{D}(k)$ as follows.

$$F_n \otimes_{F_{2n}} D \xrightarrow{\mathrm{id} \otimes_{\mathrm{id}} \theta} F_n \otimes_{F_{2n}} C \xrightarrow{\mathrm{id} \otimes_{\mathrm{id}} \psi} F_n \otimes_{F_{2n}} k$$

As $\mathrm{H}_i(P) \cong \mathrm{H}_i(F_n \otimes_{F_{2n}} k)$ for any integer i, we can use the long exact homology sequence associated to the above cofiber sequence to evaluate the homology groups of P. As remarked before, D lies in $\mathcal{D}(k)_{\geq 4n}$, and as F_n and F_{2n} are both in $\mathcal{D}(k)_{\geq 0}$ and taking colimits can only increase connectivity [HA, 1.2.1.6], it follows that

$$F_n \otimes_{F_{2n}} D \simeq \left| F_n \otimes F_{2n}^{\otimes \bullet} D \right|$$

is an object of $\mathcal{D}(k)_{\geq 4n}$ as well[32].

We can thus conclude that for $i \leq 2n$ the morphism $\mathrm{id}_{F_n} \otimes_{\mathrm{id}_{F_{2n}}} \psi$ induces an isomorphism as follows.

$$\mathrm{H}_i(F_n \otimes_{F_{2n}} C) \xrightarrow{\cong} \mathrm{H}_i(F_n \otimes_{F_{2n}} k) \cong \mathrm{H}_i(P)$$

To evaluate the homology groups of $F_n \otimes_{F_{2n}} C$ we can use the long exact homology sequence associated to the cofiber sequence

$$F_n \otimes_{F_{2n}} F_{2n}[2n] \xrightarrow{\mathrm{id}_{F_n} \otimes_{\mathrm{id}_{F_{2n}}} g'} F_n \otimes_{F_{2n}} F_{2n} \xrightarrow{\mathrm{id}_{F_n} \otimes_{\mathrm{id}_{F_{2n}}} \varphi} F_n \otimes_{F_{2n}} C$$

which we obtain by applying $F_n \otimes_{F_{2n}} -$ to the cofiber sequence in the top row of $(**)$. Using unitality of the relative tensor product [HA, 4.4.3.16] we can identify this cofiber sequence with the top row in the following commutative diagram[33] in $\mathcal{D}(k)$

$$\begin{tikzcd}
F_n[2n] \simeq F_n \otimes k[2n] \arrow[r, "\mu' \circ (\mathrm{id}_{F_n} \otimes (f \circ g))"] & F_n \arrow[r, "\lambda"] \arrow[d, "j"'] & F_n \otimes_{F_{2n}} C \arrow[d, "\mathrm{id}_{F_n} \otimes_{\mathrm{id}_{F_{2n}}} \psi"] \\
& P \arrow[r, no head, "\simeq"] & F_n \otimes_{F_{2n}} k
\end{tikzcd}$$

[31] See Construction E.8.0.4 for an explanation of the relevant module structures.

[32] See [HA, 4.4.2.8] for this description of the relative tensor product. That the bar construction really looks like this in the individual levels follows from unpacking the definition [HA, 4.4.2.7].

[33] The identification of the top left morphism arises from unpacking the definitions. For j fitting into the commutative diagram, note that the composition $F_{2n} \xrightarrow{\varphi} C \xrightarrow{\psi} k$ is by definition homotopic to p, and then use the identification of the pushout diagram from Construction 5.1.6.2 with the one from Proposition E.8.0.5.

where f and j are as in Construction 5.1.6.2, μ' is the multiplication morphism for F_n, and λ is a newly introduced name. It thus suffices to show that $\mathrm{H}_i(\lambda)$ is an isomorphism for $i < 2n$ and that additionally $\mathrm{H}_{2n}(F_n \otimes_{F_{2n}} C) \cong 0$.

In the range we are interested in $F_n[2n]$ has only homology in degree $2n$ (see Proposition 5.1.5.3), so that it immediately follows using the long exact sequence in homology that $\mathrm{H}_i(\lambda)$ is an isomorphism for $i < 2n$, and the statements for the homology of P in this range now follow from the calculation of the homology in low degrees of F_n, see Proposition 5.1.5.3.

It remains to show that $\mathrm{H}_{2n}(F_n \otimes_{F_{2n}} C) \cong 0$. By the long exact sequence in homology we have to show for this that $\mu' \circ (\mathrm{id}_{F_n} \otimes (f \circ g))$ induces a surjection on H_{2n}. Let $\iota\colon k \to F_n$ be the unit morphism. Then by Proposition 5.1.5.3 there is an isomorphism $\mathrm{H}_{2n}(F_n \otimes k[2n]) \cong k$, and this homology group has a basis formed by by $(\iota \otimes \mathrm{id}_{k[2n]}) \circ \eta$, where $\eta\colon k[2n] \simeq k \otimes k[2n]$ is the unitality equivalence of $\mathcal{D}(k)$. Composing with $\mu' \circ (\mathrm{id}_{F_n} \otimes (f \circ g))$ we obtain[34]

$$\begin{aligned} &\mu' \circ (\mathrm{id}_{F_n} \otimes (f \circ g)) \circ \left(\iota \otimes \mathrm{id}_{k[2n]}\right) \circ \eta \\ \simeq\ &\mu' \circ (\iota \otimes (f \circ g)) \circ \eta \\ \simeq\ &f \circ g \\ \simeq\ &f' \end{aligned}$$

which by definition is a generator of $\mathrm{H}_{2n}(F_n)$. □

5.1.6.3 On a mapping space of commutative algebras

In this section we show that a mapping space relevant in Section 5.1.7 has only a single path component.

Proposition 5.1.6.4. *Let $n > 0$ be an integer. Let R and S be commutative algebras in $\mathcal{D}(k)$, and assume that the homology of R is concentrated in degrees 0 and n, where it is isomorphic to k, that the homology of S is concentrated in degrees i with $0 \leq i \leq 2n$, and that $\mathrm{H}_n(S) \cong 0$.*

Then

$$\pi_0\left(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R, S)\right) \cong * \tag{5.7}$$

So up to homotopy, there is a unique morphism of commutative algebras $R \to S$. ♡

Proof. Consider the commutative algebra P that was constructed in Construction 5.1.6.2. Proposition 5.1.6.3 implies that $\tau_{\leq 2n}(P)$ has the same homology as R. As the homology is free (as a $\mathbb{Z}$-graded k-module) it follows from Proposition 4.3.3.7 that $\tau_{\leq 2n}(P)$ and R are equivalent as objects of $\mathcal{D}(k)$. It then follows from Proposition 5.1.5.4 that $\tau_{\leq 2n}(P)$ and R are even equivalent as commutative algebras in $\mathcal{D}(k)$.

[34] The last step is by definition, see Construction 5.1.6.2.

We thus obtain an equivalence as follows.

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R, S) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(\tau_{\leq 2n}(P), S)$$

$\tau_{\leq 2n}(P)$ and S both lie in $(\mathrm{CAlg}(\mathcal{D}(k))_{\geq 0})_{\leq 2n}$ by Proposition 4.3.4.1 (7) and (8), and as the inclusion is fully faithful we obtain another equivalence as follows.

$$\simeq \mathrm{Map}_{(\mathrm{CAlg}(\mathcal{D}(k))_{\geq 0})_{\leq 2n}}(\tau_{\leq 2n}(P), S)$$

We can now continue with the adjunction from Proposition 4.3.4.1 (8).

$$\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))_{\geq 0}}(P, S)$$

Finally, we use that $\mathrm{CAlg}(\iota_{\geq 0})$ is fully faithful and obtain the following equivalence.

$$\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(P, S)$$

As P was defined as a pushout in $\mathrm{CAlg}(\mathcal{D}(k))$, we obtain a pullback diagram in $\mathcal{S}$ (using notation from Construction 5.1.6.2) as follows.

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(P, S) \arrow[r, "j^*"] \arrow[d, "i^*"'] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]), S\Big) \arrow[d, "f^*"] \\
\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(k, S) \arrow[r, "p^*"'] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]), S\Big)
\end{tikzcd}$$

k is initial as a commutative algebra by [HA, 3.2.1.9], so $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(k, S)$ is contractible. This implies that[35]

$$\begin{aligned}\mathrm{Map}_{\mathrm{CAlg}}(P, S) &\xrightarrow{j^*} \mathrm{Map}_{\mathrm{CAlg}}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]), S\Big) \\ &\xrightarrow{f^*} \mathrm{Map}_{\mathrm{CAlg}}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]), S\Big)\end{aligned}$$

is a homotopy fiber sequence of which we can take the long exact sequence of homotopy groups. To show that $\pi_0\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(P, S)\Big) \cong *$ it then suffices to show that both

$$\pi_0\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]), S\Big)\Big)$$

and

$$\pi_1\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]), S\Big)\Big)$$

are trivial.

[35] We shorten $\mathrm{CAlg}(\mathcal{D}(k))$ as CAlg.

We can use the adjunction $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)} \dashv \mathrm{ev}_{\langle 1\rangle}$ to rewrite these homotopy groups as follows.

$$\begin{aligned}
\pi_0\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]), S\Big)\Big) &\cong \pi_0\big(k[n], \mathrm{ev}_{\langle 1\rangle}(S)\big) \cong \mathrm{H}_n(S) \cong 0\\
\pi_1\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]), S\Big)\Big) &\cong \pi_1\big(k[2n], \mathrm{ev}_{\langle 1\rangle}(S)\big)\\
&\cong \pi_0\big(k[2n+1], \mathrm{ev}_{\langle 1\rangle}(S)\big)\\
&\cong \mathrm{H}_{2n+1}(S) \cong 0
\end{aligned}$$

□

5.1.7 Formality of certain $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebras

In this section we finally put together the various results from sections Sections 5.1.1, 5.1.2, 5.1.3, 5.1.4, 5.1.5 and 5.1.6 and show formality of commutative bialgebras with homology concentrated in degrees 0 and 1, where it is isomorphic to k.

Proposition 5.1.7.1. *Let R be on object of* $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ *such that*

$$\mathrm{H}_i(R) \cong \begin{cases} k & \text{for } i = 0 \text{ and } i = 1\\ 0 & \text{otherwise}\end{cases}$$

Then the underlying object of R in $\mathcal{D}(k)$ is dualizable[36]*.*

Let furthermore[37]

$$f_1\colon \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_1) \to R^\vee$$

be some morphism in the ∞-category $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$*, where A_1 is as in Construction 5.1.2.2*[38]*, and $R^\vee$ is the dual of R, see Remark 5.1.1.9. Then f_1 can be extended to a morphism*

$$\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A) \to R^\vee$$

where A is as in Construction 5.1.2.2. ♡

Proof. That the underlying object of R is dualizable follows immediately from the assumptions on the homology together with the formality statement Proposition 4.3.3.7, see also Example 5.1.1.8.

By Corollary 5.1.2.14 the morphisms $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_n) \to \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A)$ exhibit the object $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A)$ as a colimit of

$$\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_1) \to \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_2) \to \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_3) \to \dots$$

in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$. It hence suffices to prove inductively that given an integer $n > 1$ and a morphism $f_{n-1}\colon \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_{n-1}) \to R^\vee$ there exists an

[36] See Definition 5.1.1.1.

[37] Recall Notation 5.1.2.12.

[38] A_1 is cofibrant as a chain complex by Proposition 5.1.2.11 Proposition 4.2.2.12.

extension to a morphism $f_n\colon \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_n) \to R^\vee$. Also by Corollary 5.1.2.14, it suffices for this to construct a commutative square

$$\begin{tikzcd}
\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n) \arrow[r] \arrow[d] & \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(B_n) \arrow[d, dashed] \\
\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_{n-1}) \arrow[r, "f_{n-1}"'] & R^\vee
\end{tikzcd}$$

in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$, where the morphism on the left and top are the ones constructed in Construction 5.1.2.5. Proposition 5.1.2.10 and Remark 5.1.2.9 imply that $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(B_n)$ is a zero object in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$, so there is an essentially unique morphism $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(B_n) \to R^\vee$ we can fill in on the right.

What remains is to construct a homotopy between the two possible composites from $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n)$ to $R^\vee$ in the diagram. For this it suffices to show that *any* two morphisms from $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n)$ to $R^\vee$ are homotopic, i. e. that

$$\pi_0\Big(\mathrm{Map}_{\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))}\big(\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n), R^\vee\big)\Big) \cong *$$

In Proposition 5.1.2.18 it was shown that

$$\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n) \simeq \mathrm{Free}^{\mathrm{Alg}(\mathrm{coCAlg})}_{\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg})}\big(\gamma^{\mathsf{Comm}}_{\mathbb{E}_0}(\underline{B}'_n)\big)$$

where $\gamma^{\mathsf{Comm}}_{\mathbb{E}_0}(\underline{B}'_n)$ is an object in

$$\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg}(\mathcal{D}(k)))$$

with underlying object equivalent to $k \oplus k[-2]$, see Construction 5.1.2.15. We thus obtain an isomorphism as follows.

$$\begin{aligned}
&\pi_0\Big(\mathrm{Map}_{\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))}\big(\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n), R^\vee\big)\Big)\\
\cong\ &\pi_0\Big(\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg}(\mathcal{D}(k)))}\big(\gamma^{\mathsf{Comm}}_{\mathbb{E}_0}(\underline{B}'_n), R^\vee\big)\Big)
\end{aligned}$$

By [HA, 2.1.3.10], the ∞-category of $\mathbb{E}_0$-algebras in a monoidal ∞-category $\mathcal{C}$ can be identified with $\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$, so applying this and dualizing (see Fact 5.1.1.3), we obtain the following isomorphisms.

$$\begin{aligned}
&\pi_0\Big(\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg}(\mathcal{D}(k)))}\big(\gamma^{\mathsf{Comm}}_{\mathbb{E}_0}(\underline{B}'_n), R^\vee\big)\Big)\\
\cong\ &\pi_0\Big(\mathrm{Map}_{\mathrm{coCAlg}(\mathcal{D}(k))_{k/}}\big(\gamma^{\mathsf{Comm}}(\underline{B}'_n), R^\vee\big)\Big)\\
\cong\ &\pi_0\Big(\mathrm{Map}_{(\mathrm{CAlg}(\mathcal{D}(k)))_{/k}}\big(R, \gamma^{\mathsf{Comm}}(\underline{B}'_n)^\vee\big)\Big)
\end{aligned}$$

By the assumptions on R, the truncation $\tau_{\leq 0}(R)$ has homology groups concentrated in degree 0 and $\mathrm{H}_0(R)$ is free of rank 1. Using Proposition 4.3.3.7 we can thus apply Proposition 5.1.6.1 to obtain the following isomorphism.

$$\pi_0\Big(\mathrm{Map}_{(\mathrm{CAlg}(\mathcal{D}(k)))_{/k}}\big(R, \gamma^{\mathsf{Comm}}(\underline{B}'_n)^\vee\big)\Big)$$

$$\cong \pi_0\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\big(R, \gamma^{\mathsf{Comm}}(\underline{B}'_n)^\vee\big)\Big)$$

As the dual of $k[l]$ is $k[-l]$, the underlying object in $\mathcal{D}(k)$ of $\gamma^{\mathsf{Comm}}(\underline{B}'_n)^\vee$ is equivalent to $k \oplus k[2]$. Now we can apply Proposition 5.1.6.4 to conclude that this set has exactly one element. □

Proposition 5.1.7.2. *Let R be on object of* $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ *such that*

$$\mathrm{H}_i(R) \cong \begin{cases} k & \text{for } i = 0 \text{ and } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $g\colon k[1] \to R$ be a morphism in $\mathcal{D}(k)$ representing a basis of $\mathrm{H}_1(R)$*. Let x be an element of k. Then there exists a morphism*[39]

$$\varphi\colon R \to \gamma(A)^\vee$$

in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ *that induces an isomorphism on* H_0 *and is such that* $\mathrm{H}_1(\varphi)$ *maps the element represented by g to $x \cdot y_1^\vee$ (see Proposition 5.1.2.4).* ♡

Proof. Consider the commutative algebra $\gamma(A'_1)^\vee$. (see Construction 5.1.2.15 for a definition of A'_1). The underlying object of $\gamma(A'_1)$ in $\mathcal{D}(k)$ is by definition equivalent to $k \oplus k[-1]$, so

$$\mathrm{H}_i(\gamma(A'_1)^\vee) \cong \begin{cases} k & \text{for } i = 0 \text{ and } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

with the homology group in degree 1 generated by $y_1^\vee$.

Define a morphism $\varphi''_1\colon \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}(k[1]) \to \gamma(A'_1)^\vee$ such that composing the morphism $k[1] \to \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[1])$ exhibiting $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[1])$ as the free commutative algebra generated by $k[1]$ with φ''_1 represents the element $x \cdot y_1^\vee$ in $\mathrm{H}_1(\gamma(A'_1)^\vee)$. As a morphism of commutative algebras, the unit morphisms must be preserved, so φ''_1 induces an isomorphism on H_0 by Proposition 5.1.3.3.

We obtain an induced morphism φ'_1 as in the following diagram

$$\begin{tikzcd} R \arrow[r, dash, "\simeq"] \arrow[d, dashed, "\varphi'_1"'] & \mathrm{CAlg}(\tau_{\leq 1})\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}(k[1])\Big) \arrow[d, "\mathrm{CAlg}(\tau_{\leq 1})(\varphi''_1)"] \\ \gamma(A'_1)^\vee \arrow[r, dash, "\simeq"'] & \mathrm{CAlg}(\tau_{\leq 1})(\gamma(A'_1)^\vee) \end{tikzcd}$$

[39] For a definition of A, see Construction 5.1.2.2. For the duality functor see Fact 5.1.1.3.

where the top horizontal equivalence is the one from Proposition 5.1.5.4[40] and the bottom horizontal equivalence is the one arising from $\gamma(A_1')^\vee$ already being concentrated in degrees 0 and 1. φ_1' then induces an isomorphism on H_0 and satisfies $\mathrm{H}_1(\varphi_1')(g) = x \cdot y_1^\vee$.

Applying [HA, 2.1.3.10] and Proposition 5.1.6.1 we can upgrade φ_1' to a morphism in $\mathrm{BiAlg}_{\mathsf{Comm},\mathbb{E}_0}(\mathcal{D}(k))$. Next, applying Proposition 5.1.2.18 and dualizing, we can lift this morphism to a morphism

$$\varphi_1 \colon R \to \gamma(A_1)^\vee$$

in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ such that the triangle

$$\begin{tikzcd} R \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_1'"'] & \gamma(A_1)^\vee \arrow[d] \\ & \gamma(A_1')^\vee \end{tikzcd}$$

of underlying morphisms of commutative algebras commutes, with the vertical morphism being the dual of γ applied to the inclusion $A_1' \to A_1$. Applying Proposition 5.1.7.1 (and dualizing twice), we can further lift φ_1 to a morphism φ that fits into a commuting triangle in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ as follows.

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "\varphi_1"'] & \gamma(A)^\vee \arrow[d] \\ & \gamma(A_1)^\vee \end{tikzcd}$$

By Proposition 5.1.2.4 (and dualizing) the homology of $\gamma(A)^\vee$ is k in degrees 0 and 1 and 0 in other degrees, and a basis is formed by $1^\vee$ in degree 0 and by $y_1^\vee$ in degree 1. As the inclusion $A_1' \to A$ sends 1 to 1 and y_1 to y_1, it follows that the induced morphisms $H_i(\gamma(A)^\vee) \to H_i(\gamma(A_1')^\vee)$ send $1^\vee$ to $1^\vee$ and $y_1^\vee$ to $y_1^\vee$ and are thus in particular isomorphisms. That φ satisfies the required properties now follows from this together with the description of φ_1' discussed above. $\square$

Proposition 5.1.7.3. *Let R and S be objects in* $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ *such that*

$$\mathrm{H}_i(R) \cong \begin{cases} k & \text{for } i = 0 \text{ and } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

[40] We choose this equivalence to be such that the morphism $k[1] \to \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[1])$ exhibiting $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[1])$ as the free commutative algebra generated by $k[1]$ composed with the equivalence is homotopic to g.

and

$$\mathrm{H}_i(S) \cong \begin{cases} k & \text{for } i = 0 \text{ and } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $\{g_R\}$ *and* $\{g_S\}$ *be a basis of* $\mathrm{H}_1(R)$ *and* $\mathrm{H}_1(S)$*, respectively. Let* x *be an element of* k*.*

Then there exists a morphism

$$\varphi\colon R \to S$$

in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ *such that* $\mathrm{H}_0(\varphi)$ *is an isomorphism and such that* $\mathrm{H}_1(\varphi)(g_R) = x \cdot g_S$*.*

In particular, φ *is an equivalence if and only if* x *is a invertible in* k*.* ♡

Proof. By Proposition 5.1.7.2 we can construct morphisms

$$R \xrightarrow{\varphi_R} \gamma(A)^\vee \xleftarrow{\varphi_S} S$$

in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ such that both φ_R and φ_S induce an isomorphism on H_0 and

$$\mathrm{H}_1(\varphi_R)(g_R) = x \cdot y_1^\vee \qquad \text{and} \qquad \mathrm{H}_1(\varphi_S)(g_S) = y_1^\vee$$

It follows from Proposition 5.1.2.4 and [HA, 3.2.2.6] that φ_S is an equivalence and φ_R is an equivalence if and only if x is invertible. We now define φ as the composition $(\varphi_S)^{-1} \circ \varphi_R$. □

5.2 The k-linear circle as an $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebra

The goal of this section is to define the circle group $\mathbb{T}$ as well as its k-linear version $k \boxtimes \mathbb{T}$ as commutative and cocommutative bialgebras, for $\mathbb{T}$ in $\mathcal{S}$, and for $k \boxtimes \mathbb{T}$ in $\mathcal{D}(k)$.

$\mathbb{T}$ will be defined in Section 5.2.1. We will then discuss the linearization functor $k \boxtimes -\colon \mathcal{S} \to \mathcal{D}(k)$ in Section 5.2.2, and apply it to define $k \boxtimes \mathbb{T}$ in the very short Section 5.2.3.

5.2.1 The circle group

Let W be the class of weak equivalences in the model structure on sSet discussed in [Hov99, Chapter 3] and [HTT, After A.2.7.3] – these are the morphisms whose geometric realization is a homotopy equivalence of topological spaces. The infinity category of spaces $\mathcal{S}$ can then be defined by inverting those weak equivalences of simplicial sets, so as

$$\mathcal{S} \coloneqq \mathsf{sSet}[W^{-1}]$$

see [HTT, 1.2.16.1] in combination with [HA, 1.3.4.20]. The canonical functor $\mathsf{sSet} \to \mathcal{S}$ preserves finite products, as finite products in sSet are automatically homotopy products[41]. The functor $\mathrm{Sing}\colon \mathsf{Top} \to \mathsf{sSet}$ also preserves products as a right adjoint, so that the composition $\mathsf{Top} \to \mathcal{S}$ also preserves finite products. Giving both involved ∞-categories the cartesian symmetric monoidal structure [HA, 2.4.1] upgrades this functor to a symmetric monoidal functor, and so induces an (again symmetric monoidal) functor of ∞-categories of commutative algebras $\mathrm{CAlg}(\mathsf{Top}) \to \mathrm{CAlg}(\mathcal{S})$. This allows us to construct commutative algebras in $\mathcal{S}$ by giving an explicit commutative topological monoid, which we will use in the following construction.

Construction 5.2.1.1. We let the *circle group* $\mathbb{T}$ refer to the object in $\mathrm{CAlg}(\mathcal{S})$ obtained by applying the above functor $\mathrm{CAlg}(\mathsf{Top}) \to \mathrm{CAlg}(\mathcal{S})$ to the (multiplicative) commutative submonoid $\{\, z \in \mathbb{C} \mid |z| = 1 \,\}$ of $\mathbb{C}$.

Note that every commutative topological monoid can be upgraded to a commutative and cocommutative topological bimonoid, with comultiplication given by the diagonal map. This phenomenon is in fact more general, as we saw in Proposition 3.3.1.2 that any commutative algebra in a cartesian symmetric monoidal ∞-category can be upgraded in an essentially unique way to a commutative and cocommutative bialgebra.

In particular, we can upgrade $\mathbb{T}$ in an essentially unique way to an $\mathbb{E}_\infty, \mathbb{E}_\infty$-bialgebra in spaces. $\diamondsuit$

5.2.2 The linearization functor

In Section 5.2.1 we considered $\mathcal{S}$ as a symmetric monoidal ∞-category via the cartesian symmetric monoidal structure. There is also a different way of defining the symmetric monoidal structure on $\mathcal{S}$, as we discuss in the following remark.

Remark 5.2.2.1. The ∞-category $\mathcal{S}$ is the unit object in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ by [HA, 4.8.1.20], and hence can be upgraded to a presentable symmetric monoidal ∞-category that is initial in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ by [HA, 3.2.1.9] in combination with [HA, 4.8.1.9 and 4.8.1.15].

To show that the so obtained symmetric monoidal structure is equivalent to the cartesian symmetric monoidal structure, it suffices in light of [HA, 4.8.1.12] to show that the product functor $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ preserves colimits separately in each variable, which is shown in [HTT, 6.1.3.14]. $\diamondsuit$

The characterization of $\mathcal{S}$ as an initial object in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ allows the following definition.

[41] As the geometric realization functor $|-|\colon \mathsf{sSet} \to \mathsf{Top}$ is the left adjoint of a Quillen equivalence, this follows from every object in sSet being cofibrant, $|-|$ preserving products [Hov99, 3.1.8], and every object in Top being fibrant.

Definition 5.2.2.2. Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category. Then we obtain an essentially unique colimit preserving symmetric monoidal functor that we denote as follows.

$$\mathbb{1}_{\mathcal{C}} \boxtimes - \colon \mathcal{S} \to \mathcal{C}$$

As $\mathcal{D}(k)$ is a presentable symmetric monoidal ∞-category by Proposition 4.3.2.1 (1), we hence obtain a colimit preserving symmetric monoidal functor

$$k \boxtimes - \colon \mathcal{S} \to \mathcal{D}(k)$$

that we sometimes call the *k-linearization functor.* ♢

Remark 5.2.2.3. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then universality of the functors defined in Definition 5.2.2.2 imply that we obtain a commuting triangle

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "k\boxtimes -"'] \arrow[dr, "k'\boxtimes -"] & \\ \mathcal{D}(k) \arrow[rr, "k' \otimes_k -"'] & & \mathcal{D}(k') \end{tikzcd}$$

where $k' \otimes_k -$ is the colimit-preserving symmetric monoidal functor discussed in Remark 4.3.2.2. ♢

Let X be an object of $\mathcal{S}$. In Section 4.3.3 we discussed the homology functors H_n on $\mathcal{D}(k)$, which we could thus apply to $k \boxtimes X$. In the rest of this section we show that this is compatible with the classical notions of homology of spaces. We begin by reviewing the definition of homology of simplicial sets.

Construction 5.2.2.4. We construct a functor

$$k \cdot - \colon \mathsf{sSet} \to \mathsf{Ch}(k)$$

as follows. There is a functor, which we also call $k \cdot -$, from Set to $\mathrm{LMod}_k(\mathsf{Ab})$ that maps a set X to the free k-module on the basis X. This functor induces a functor as follows.

$$\mathsf{sSet} \cong \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set}) \xrightarrow{(k\cdot -)_*} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{LMod}_k(\mathsf{Ab}))$$

The functor $k \cdot - \colon \mathsf{sSet} \to \mathsf{Ch}(k)$ is then to be the composition of this functor with the functor

$$C\colon \; \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{LMod}_k(\mathsf{Ab})) \to \mathsf{Ch}(k)$$

that maps a functor F to the chain complex $C(F)$ for which $C(F)_n \coloneqq F([n])$ and $\partial_n^{C(F)} \coloneqq \sum_{i=0}^{n} F(\delta_i)$. ♢

Classically, one defines homology for simplicial sets X with coefficients in the commutative ring k as $\mathrm{H}_n(X, k) := \mathrm{H}_n(k \cdot X)$. For topological spaces one then defines homology as the homology of their singular simplicial set.

What we would like to show is that there is a commutative diagram

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, "k\cdot -"] \arrow[d] & \mathsf{Ch}(k) \arrow[d, "\gamma"'] \arrow[dr, "\mathrm{H}_n"] & \\ & & \mathrm{LMod}_k(\mathsf{Ab}) \\ \mathcal{S} \arrow[r, "k\boxtimes -"'] & \mathcal{D}(k) \arrow[ur, "\mathrm{H}_n"'] & \end{tikzcd}$$

where the left vertical functor is the canonical one. That there is a filler for the right triangle was shown in Proposition 4.3.3.2. It thus remains to show that there is a filler for the left square. The strategy will be to use that colimit-preserving functors out of $\mathcal{S}$ are determined by their value on the one-point-space $*$. So we will show that $k \cdot -$ induces a colimit-preserving functor on underlying ∞-categories that maps $*$ to k. This functor will then by definition fit into such a commutative square but also be homotopic to $k \boxtimes -$.

Proposition 5.2.2.5. *The functor*

$$k \cdot - \colon \mathsf{sSet} \to \mathsf{Ch}(k)$$

from Construction 5.2.2.4 preserves weak equivalences as well as cofibrations, where sSet *carries the model structure discussed in [Hov99, Chapter 3] and [HTT, After A.2.7.3], and* $\mathsf{Ch}(k)$ *carries the projective model structure from Fact 4.1.3.1.* ♡

Proof. Weak equivalences in sSet are those maps whose geometric realization is a homotopy equivalence of spaces, and that singular homology maps homotopy equivalences to isomorphisms is classical[42].

Now let $f \colon X \to Y$ be a cofibration in sSet, i. e. the map $f_n \colon X_n \to Y_n$ is injective for every $n \geq 0$. To show that $k \cdot f$ is a cofibration we have by [Hov99, 2.3.9] to show that $k \cdot f$ is a levelwise split injection and that $k \cdot f$ has cofibrant cokernel.

But the morphism $(k \cdot f)_n$ is a morphism of free k-modules induced by an injection among the basis sets, so is a split injection. The cokernel can then be identified with a chain complex that is concentrated in nonnegative degrees and that in level $n \geq 0$ is given by the free k-module with basis $Y_n \setminus f_n(X_n)$. Thus the cokernel of $k \cdot f$ is cofibrant by [Hov99, 2.3.6]. ☐

Definition 5.2.2.6. By Proposition 5.2.2.5 the functor $k \cdot -$ from Construction 5.2.2.4 induces a functor

$$k \cdot - \colon \mathsf{sSet}^{\mathrm{cof}} \to \mathsf{Ch}(k)^{\mathrm{cof}}$$

[42]For a discussion in a textbook see for example [Bre93, 16.5]

preserving weak equivalences and thus a functor on underlying ∞-categories[43]

$$\mathcal{S} \simeq \mathsf{sSet}[W^{-1}] \to \mathsf{Ch}(k)^{\mathrm{cof}}[W'^{-1}] \simeq \mathcal{D}(k)$$

that we also call $k \cdot -$.

By construction this functor comes with a commutative square

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, "k\cdot -"] \arrow[d] & \mathsf{Ch}(k)^{\mathrm{cof}} \arrow[d, "\gamma"] \\ \mathcal{S} \arrow[r, "k\cdot -"'] & \mathcal{D}(k) \end{tikzcd} \tag{5.8}$$

of ∞-categories, where the left vertical functor is the canonical one. ♢

Proposition 5.2.2.7. *The functor*

$$k \cdot -\colon \mathsf{sSet} \to \mathsf{Ch}(k)$$

from Construction 5.2.2.4 preserves small colimits. ♡

Proof. Colimits in both sSet as well as $\mathsf{Ch}(k)$ are calculated levelwise. The statement thus boils down to the functor $k \cdot -\colon \mathsf{Set} \to \mathrm{LMod}_k(\mathsf{Ab})$ preserving colimits. But this functor is left adjoint to the forgetful functor. □

Proposition 5.2.2.8. *The functor*

$$k \cdot -\colon \mathcal{S} \to \mathcal{D}(k)$$

from Definition 5.2.2.6 preserves small colimits. ♡

Proof. By Fact 4.1.3.1 and [HTT, After A.2.7.3] sSet and $\mathsf{Ch}(k)$ are combinatorial model categories. Furthermore, by Proposition 5.2.2.7, [HTT, 5.5.2.9][44], and Proposition 5.2.2.5, the functor

$$k \cdot -\colon \mathsf{sSet} \to \mathsf{Ch}(k)$$

is a left Quillen functor between combinatorial model categories.

The claim thus follows from [HA, 1.3.4.26]. □

Proposition 5.2.2.9. *The functors $k \cdot -$ from Definition 5.2.2.6 and $k \boxtimes -$ from Definition 5.2.2.2 are homotopic as functors of infinity categories from $\mathcal{S}$ to $\mathcal{D}(k)$.* ♡

[43] W is to be the class of weak equivalences in sSet and W' the class of weak equivalences in $\mathsf{Ch}(k)$.

[44] As both sSet and $\mathsf{Ch}(k)$ have combinatorial model structures they are presentable.

Proof. $k \boxtimes -$ preserves small colimits by definition and $k \cdot -$ by Proposition 5.2.2.8. Then [HTT, 5.1.5.6] implies that it suffices to check that $k \boxtimes * \simeq k \cdot *$, where $*$ is the one-point-space.

As $k \boxtimes -$ is by definition symmetric monoidal, it maps the monoidal unit $*$ of $\mathcal{S}$ to the monoidal unit k of $\mathcal{D}(k)$.

As $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ is also symmetric monoidal it thus suffices to show that the chain complex[45] $k \cdot *$ is quasiisomorphic to $k[0]$. But it can easily be seen from the definition that $k \cdot *$ is the chain complex[46]

$$\cdots \leftarrow 0 \leftarrow k \xleftarrow{0} k \xleftarrow{\mathrm{id}} k \xleftarrow{0} k \xleftarrow{\mathrm{id}} \cdots$$

and the obvious inclusion of $k[0]$ is a quasiisomorphism. ☐

We can now put everything together and summarize the previous results as follows.

Proposition 5.2.2.10. *There is a commutative diagram*

$$\begin{tikzcd}
\mathsf{sSet} \arrow[r, "k\cdot -"] \arrow[d] & \mathsf{Ch}(k) \arrow[d, "\gamma"] \arrow[dr, "\mathrm{H}_n"] & \\
& & \mathrm{LMod}_k(\mathsf{Ab}) \\
\mathcal{S} \arrow[r, "k\boxtimes -"'] & \mathcal{D}(k) \arrow[ur, "\mathrm{H}_n"'] &
\end{tikzcd}$$

where the left vertical functor is the canonical one. ♡

Proof. For the left commutative square combine Proposition 5.2.2.9 with the commutative square (5.8) from Definition 5.2.2.6. The right commutative triangle was constructed in Proposition 4.3.3.2. ☐

5.2.3 Definition of the k-linear circle

We can now define the k-linear circle as a bialgebra in $\mathcal{D}(k)$.

Definition 5.2.3.1. The *k-linear circle* is the $\mathbb{E}_\infty, \mathbb{E}_\infty$-bialgebra $k \boxtimes \mathbb{T}$ in $\mathcal{D}(k)$. ♢

5.2.4 Formality of the k-linear circle as an $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebra

In this section we apply the main result of Section 5.1, Proposition 5.1.7.3, to the commutative bialgebra $k \boxtimes \mathbb{T}$ that we defined in Section 5.2.3. We start by recording the homology of $k \boxtimes \mathbb{T}$.

[45] Here $*$ is the simplicial set $\mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ that is constant with value $*$. As pointed out in the introduction to Section 5.2.1, the canonical functor $\mathsf{sSet} \to \mathcal{S}$ preserves finite products, so this simplicial set $*$ maps to the space $*$ in $\mathcal{S}$.

[46] The leftmost k is in level 0.

Proposition 5.2.4.1. *The following holds for the homology of $k \boxtimes \mathbb{T}$ as defined in Definition 5.2.3.1.*

$$\mathrm{H}_i(\mathbb{T} \boxtimes k) \cong \begin{cases} k & \text{for } i = 0 \text{ and } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

♡

Proof. By Proposition 5.2.2.10 and using the definition of $\mathbb{T}$ in Construction 5.2.1.1 there is an isomorphism

$$\mathrm{H}_*(k \boxtimes \mathbb{T}) \cong \mathrm{H}_*(\{\, z \in \mathbb{C} \mid |z| = 1 \,\}; k) \cong \mathrm{H}_*\big(S^1; k\big)$$

where on the right we have the usual singular homology of the topological 1-sphere with coefficients in k. □

We can now put all the work of Section 5.1 to use to obtain an equivalence of commutative bialgebras between $k \boxtimes \mathbb{T}$ and D.

Proposition 5.2.4.2. *Let g be a basis element of $\mathrm{H}_1(k \boxtimes \mathbb{T})$. Then there exists an equivalence*[47]

$$\varphi\colon \mathrm{D} \to k \boxtimes \mathbb{T}$$

in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ *that sends the element* d *of* $\mathrm{H}_1(\mathrm{D})$ *to the element* g *in* $\mathrm{H}_1(k \boxtimes \mathbb{T})$. ♡

Proof. Follows directly from Proposition 5.2.4.1 and Proposition 5.1.7.3. □

From Proposition 5.2.4.2 we obtain an equivalence $\mathrm{D} \simeq k \boxtimes \mathbb{T}$ as commutative bialgebras. This equivalence is however not canonically determined – not even the induced isomorphism on homology is, it depends on the choice of a element g of $\mathrm{H}_1(k \boxtimes \mathbb{T})$ that forms a basis. If g_0 is one element that forms a basis, then the set of all elements forming a basis is given by the products $x \cdot g_0$ where x is an invertible element of k. So which element should we choose?

We can reduce the indeterminacy by varying the ground ring. It follows from Construction 4.2.1.1, Remark 4.3.2.2, and Remark 5.2.2.3 that an equivalence of commutative bialgebras $\mathrm{D}_{\mathbb{Z}} \simeq \mathbb{Z} \boxtimes \mathbb{T}$ in $\mathcal{D}(\mathbb{Z})$ induces an equivalence of commutative bialgebras as follows

$$\mathrm{D}_k \simeq k \otimes_{\mathbb{Z}} \mathrm{D}_{\mathbb{Z}} \simeq k \otimes_{\mathbb{Z}} \mathbb{Z} \boxtimes \mathbb{T} \simeq k \boxtimes \mathbb{T}$$

where the first equivalence is the one obtained from combining Construction 4.2.1.1 with Remark 4.3.2.2, the middle equivalence arises from applying $k \otimes \mathbb{Z}-$ to the equivalence $\mathrm{D}_{\mathbb{Z}} \simeq \mathbb{Z} \boxtimes \mathbb{T}$, and the last equivalence is the one from Remark 5.2.2.3. By choosing this equivalence for k, we have thus

[47] See Notation 4.4.0.2 and Construction 4.2.1.1 for a definition of D and Definition 5.2.3.1 for a definition of $k \boxtimes \mathbb{T}$.

reduced the indeterminacy of the isomorphism on H_1 to choosing one of the two generators of $\mathrm{H}_1(\mathbb{Z} \boxtimes \mathbb{T}) \cong \mathbb{Z}$.

So which generator of $\mathrm{H}_1(\mathbb{Z}\boxtimes\mathbb{T})$ should we choose? We will in Section 6.1.1 define a 1-category $\mathbf{\Lambda}$ and call functors from $\mathbf{\Lambda}^{\mathrm{op}}$ into an ∞-category *cyclic objects* in that ∞-category. We will consider two relevant constructions on cyclic objects. We will define a functor

$$|-|_{\mathsf{Mixed}} \colon \operatorname{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathsf{Mixed}_{\mathrm{cof}} = \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}^{\mathrm{cof}})$$

in Section 6.3.1.2 and a functor

$$|-| \colon \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

in Section 6.1.3. Note that there are automorphisms of D and $\mathbb{T}$ that introduce a sign. For D we can describe this automorphism by $\mathrm{d} \mapsto -\mathrm{d}$, and the automorphism of $\mathbb{T}$ is given by $z \mapsto z^{-1}$. These reflect choices that are made when defining the two functors we just mentioned – for example for $|-|_{\mathsf{Mixed}}$ there is no intrinsic reason to define d the way it is done rather than adding an extra sign. But in any case, there are choices that have been made for both $|-|_{\mathsf{Mixed}}$ and $|-|$.

The result [Hoy18, 2.3] can now be phrased as follows: There is a generator of $\mathrm{H}_1(\mathbb{Z} \boxtimes \mathbb{T})$ such that the following diagram commutes

$$\begin{tikzcd}
\operatorname{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rrr, "|-|_{\mathsf{Mixed}}"] \arrow[d, "\gamma_*"'] & & & \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}^{\mathrm{cof}}) \arrow[d, "\gamma_{\mathsf{Mixed}}"] \\
\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \arrow[r, "|-|"'] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[r, "\simeq"'] & \mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \arrow[r, "\simeq"'] & \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))
\end{tikzcd} \tag{5.9}$$

where the middle bottom horizontal equivalence is one we will construct in Section 5.3, the right bottom horizontal equivalence is the one induced by the equivalence $\mathrm{D} \simeq k \boxtimes \mathbb{T}$ arising as discussed above from the choice of generator of $\mathrm{H}_1(\mathbb{Z} \boxtimes \mathbb{T})$, and γ_{Mixed} is the functor $\mathsf{Mixed}_{\mathrm{cof}} \to \mathcal{M}\mathrm{ixed}$ from Notation 4.4.1.2. We thus make the following convention.

Convention 5.2.4.3. From now on, when we refer to *the* equivalence of commutative bialgebras in $\mathcal{D}(k)$

$$\mathrm{D} \xrightarrow{\simeq} k \boxtimes \mathbb{T}$$

then this is to be the equivalence that arises in the manner discussed above from the generator of $\mathrm{H}_1(\mathbb{Z} \boxtimes \mathbb{T})$ that is such that there is a commutative diagram (5.9). ♢

Remark 5.2.4.4. The equivalence of bialgebras from Convention 5.2.4.3 induces via the functor LMod from Definition 3.4.2.1 an equivalence of monoidal ∞-categories

$$\mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \simeq \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$$

that is compatible with the forgetful functors to $\mathcal{D}(k)$.

Furthermore, if $\varphi\colon k \to k'$ is a morphism of commutative rings, then there is a commutative diagram[48]

$$\begin{tikzcd}
\mathrm{LMod}_{k\boxtimes\mathbb{T}} \arrow[rr, dash, "\simeq"] \arrow[dr, "\mathrm{ev}_{\mathfrak{m}}"] \arrow[ddd, "k'\otimes_k -"'] & & \mathrm{LMod}_{\mathrm{D}_k}(\mathcal{D}(k)) \arrow[dl, "\mathrm{ev}_{\mathfrak{m}}"'] \arrow[ddd, "k'\otimes_k -"] \\
& \mathcal{D}(k) \arrow[d, "k'\otimes_k -"] & \\
& \mathcal{D}(k') & \\
\mathrm{LMod}_{k'\boxtimes\mathbb{T}} \arrow[rr, dash, "\simeq"'] \arrow[ur, "\mathrm{ev}_{\mathfrak{m}}"] & & \mathrm{LMod}_{\mathrm{D}_{k'}}(\mathcal{D}(k)) \arrow[ul, "\mathrm{ev}_{\mathfrak{m}}"']
\end{tikzcd}$$

of monoidal functors, where the horizontal equivalences are the ones just mentioned and the vertical functors are induced by the symmetric monoidal functor

$$k' \otimes_k -\colon\ \mathcal{D}(k) \to \mathcal{D}(k')$$

from Remark 4.3.2.2. $\diamondsuit$

5.3 Group actions and modules over group rings

Let G be a grouplike[49] associative monoid in $\mathcal{S}$. One important class of examples is supplied by pointed spaces X by taking the loop space ΩX, which has a multiplication arising from composition of loops. The details of this construction are discussed in [HA, Introduction to 5.2.6], where a functor

$$\beta_1\colon \mathcal{S}_* \to \mathrm{Mon}^{\mathrm{gp}}_{\mathsf{Assoc}}(\mathcal{S})$$

is constructed that implements this idea. It turns out that there are no other examples, and that the restriction of β_1 to the full subcategory $\mathcal{S}_*^{\geq 1}$ of $\mathcal{S}_*$ spanned by the path connected spaces is an equivalence

$$\beta_1\colon \mathcal{S}_*^{\geq 1} \xrightarrow{\simeq} \mathrm{Mon}^{\mathrm{gp}}_{\mathsf{Assoc}}(\mathcal{S})$$

[48] There is also supposed to be a filler for the outer diagram that is compatible with the forgetful functors, i. e. this is a three-dimensional diagram that we are looking at from the top.

[49] See [HA, 5.2.6.2] for a definition.

as shown in [HA, 5.2.6.10]. The inverse functor of this equivalence will be called B. If we interpret $\mathrm{B}\,G$ as an ∞-groupoid, then $\mathrm{B}\,G$ has (up to equivalence) a unique object, and that object's automorphism space is equivalent to $\Omega\,\mathrm{B}\,G \simeq G$.

Now if $\mathcal{C}$ is an ∞-category, then we can consider the ∞-category of objects with G-action in $\mathcal{C}$, which is defined as[50] follows.

$$\mathcal{C}^{\mathrm{B}\,G} := \mathrm{Fun}(\mathrm{B}\,G, \mathcal{C})$$

If $\mathcal{C}$ carries a symmetric monoidal structure, then $\mathcal{C}^{\mathrm{B}\,G}$ can be given the induced pointwise symmetric monoidal structure.

On the other hand, if $\mathcal{C}$ is presentable symmetric monoidal, then we can form out of the Assoc-algebra[51] G in $\mathcal{S}$ the Assoc-algebra $\mathbb{1}_{\mathcal{C}} \boxtimes G$ in $\mathcal{C}$ (see Remark 5.2.2.1), and hence consider the ∞-category $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C})$ of left-$\mathbb{1}_{\mathcal{C}} \boxtimes G$-modules in $\mathcal{C}$. In fact, G can be upgraded essentially uniquely to an object in $\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Comm}}(\mathcal{S})$ by Proposition 3.3.1.2, with comultiplication given by the diagonal map $G \xrightarrow{\mathrm{id}_G \times \mathrm{id}_G} G \times G$. We hence also obtain an $\mathsf{Assoc}, \mathsf{Comm}$-bialgebra structure on $\mathbb{1}_{\mathcal{C}} \boxtimes G$, and thus an induced symmetric monoidal structure on $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C})$ by Definition 3.4.2.1.

Let us remark that the diagonal map is also used behind the scenes when defining the pointwise symmetric monoidal structure on $\mathcal{C}^{\mathrm{B}\,G}$ – the pointwise tensor product of two functors F and G can be written as the composition

$$\mathrm{B}\,G \xrightarrow{\mathrm{id}_{\mathrm{B}\,G} \times \mathrm{id}_{\mathrm{B}\,G}} \mathrm{B}\,G \times \mathrm{B}\,G \xrightarrow{F \times G} \mathcal{C} \times \mathcal{C} \xrightarrow{- \otimes -} \mathcal{C}$$

and the diagonal functor of $\mathrm{B}\,G$ can on automorphism spaces be identified with the diagonal map of G.

We can now ask the question whether $\mathcal{C}^{\mathrm{B}\,G}$ and $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C})$ are equivalent as symmetric monoidal ∞-categories, which Proposition 5.3.0.8, which is the goal of this section, will answer affirmatively.

As technical input we need to start by discussing compatibility of the tensor product of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see [HA, 4.8.1.15]) with functor categories. We will need two natural comparison functors, one for presentable symmetric monoidal ∞-categories, and one for presentable ∞-categories, but we will show in Proposition 5.3.0.4 that these constructions are compatible with the forgetful functor $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$. We will then show in Proposition 5.3.0.6 that these comparison functors are equivalences.

Construction 5.3.0.1. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable symmetric monoidal ∞-categories and $\mathcal{I}$ and $\mathcal{J}$ small ∞-categories. By [HA, 4.8.1.9] we can interpret $\mathcal{C}$ and $\mathcal{D}$ as objects in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

The symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ induces a symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ such that the forgetful functor $\mathrm{ev}_{\langle 1 \rangle}$ can be upgraded

[50] See for example [HA, 6.1.6.2] for this definition.

[51] By [HA, 2.4.2.5] the ∞-categories of Assoc-monoids in $\mathcal{S}$ and Assoc-algebras in $\mathcal{S}$ are equivalent, as the symmetric monoidal structure on $\mathcal{S}$ is cartesian (see Remark 5.2.2.1).

to a symmetric monoidal functor (see [HA, 3.2.4.4]). By [HA, 3.2.4.10] this symmetric monoidal structure is cocartesian.

The functor categories $\mathrm{Fun}(\mathcal{I},\mathcal{C})$ and $\mathrm{Fun}(\mathcal{J},\mathcal{D})$ and $\mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C}\otimes\mathcal{D})$ can be given the induced pointwise symmetric monoidal structures (see [HA, 2.1.3.4]). By [HTT, 5.5.3.6] the underlying ∞-categories are presentable again and as both the tensor products as well as colimits are calculated pointwise (see [HTT, 5.1.2.3]), the tensor products again preserve colimits pointwise in each variable [52].

Let $\iota_{\mathcal{C}}\colon \mathcal{C}\to\mathcal{C}\otimes\mathcal{D}$ and $\iota_{\mathcal{D}}\colon \mathcal{C}\to\mathcal{C}\otimes\mathcal{D}$ be the two morphisms in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ exhibiting $\mathcal{C}\otimes\mathcal{D}$ as a coproduct of $\mathcal{C}$ and $\mathcal{D}$. Using that $\mathrm{Fun}(\mathcal{I},\mathcal{C})\otimes\mathrm{Fun}(\mathcal{J},\mathcal{D})$ is a coproduct of $\mathrm{Fun}(\mathcal{I},\mathcal{C})$ and $\mathrm{Fun}(\mathcal{J},\mathcal{D})$ in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ we can then define a morphism $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ as follows.

$$\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}\colon\ \mathrm{Fun}(\mathcal{I},\mathcal{C})\otimes\mathrm{Fun}(\mathcal{J},\mathcal{D})\xrightarrow{(\iota_{\mathcal{C}}\circ-\circ\mathrm{pr}_1)\amalg(\iota_{\mathcal{D}}\circ-\circ\mathrm{pr}_2)}\mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C}\otimes\mathcal{D})\quad\diamondsuit$$

We next construct a functor of presentable ∞-categories very analogous to $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ (and with the same name, which will be justified by Proposition 5.3.0.4), where we however do not consider any symmetric monoidal structures.

Construction 5.3.0.2. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories and $\mathcal{I}$ and $\mathcal{J}$ small ∞-categories.

Consider the following diagram, which will be explained below.

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I},\mathcal{C})\times\mathrm{Fun}(\mathcal{J},\mathcal{D}) \arrow[r, "-\times-"] \arrow[d, "\psi'"'] & \mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C}\times\mathcal{D}) \arrow[d, "\psi_*"] \\
\mathrm{Fun}(\mathcal{I},\mathcal{C})\otimes\mathrm{Fun}(\mathcal{J},\mathcal{D}) \arrow[r, dashed, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}"'] & \mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C}\otimes\mathcal{D})
\end{tikzcd}\tag{5.10}$$

First, as already mentioned in Construction 5.3.0.1 are by [HTT, 5.5.3.6] the various functor categories appearing in the diagram diagram representable again. ψ' is to be the functor exhibiting $\mathrm{Fun}(\mathcal{I},\mathcal{C})\otimes\mathrm{Fun}(\mathcal{J},\mathcal{D})$ as the tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of $\mathrm{Fun}(\mathcal{I},\mathcal{C})$ and $\mathrm{Fun}(\mathcal{J},\mathcal{D})$, and likewise ψ is to be the functor

[52] To be precise (considering the case of $\mathrm{Fun}(\mathcal{I},\mathcal{C})$): The pointwise symmetric monoidal structure comes with symmetric monoidal evaluation functors for every object I of $\mathcal{I}$. This means we have commutative diagrams as follows

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I},\mathcal{C})\times\mathrm{Fun}(\mathcal{I},\mathcal{C}) \arrow[r, "-\otimes-"] \arrow[d, "\mathrm{ev}_I\times\mathrm{ev}_I"'] & \mathrm{Fun}(\mathcal{I},\mathcal{C}) \arrow[d, "\mathrm{ev}_I"] \\
\mathcal{C}\times\mathcal{C} \arrow[r, "-\otimes-"] & \mathcal{C}
\end{tikzcd}$$

where the horizontal functors are the respective tensor product functors. The left vertical functor preserves colimits in each component, and the bottom horizontal functor preserves colimits separately in each variable by assumption. It follows that the composition from top left to the bottom right along the top right preserves colimits separately in each variable, and as this is the case for every object I in $\mathcal{I}$, it follows that this is also the case for the top horizontal functor.

exhibiting $\mathcal{C} \otimes \mathcal{D}$ as the tensor product[53]. We claim that the composite from the top left over the top right to the bottom right preserves colimits in each variable separately. For this it suffices by [HTT, 5.1.2.3] to check that the composition with $\mathrm{ev}_{(I,J)}$ preserves colimits in each variable separately for every object I of $\mathcal{I}$ and J of $\mathcal{J}$. But as there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I},\mathcal{C}) \times \mathrm{Fun}(\mathcal{J},\mathcal{D}) \arrow[r, "- \times -"] \arrow[d, "\mathrm{ev}_I \times \mathrm{ev}_J"'] & \mathrm{Fun}(\mathcal{I} \times \mathcal{J}, \mathcal{C} \times \mathcal{D}) \arrow[r, "\psi_*"] \arrow[d, "\mathrm{ev}_{(I,J)}"] & \mathrm{Fun}(\mathcal{I} \times \mathcal{J}, \mathcal{C} \otimes \mathcal{D}) \arrow[d, "\mathrm{ev}_{(I,J)}"] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "\mathrm{id}"'] & \mathcal{C} \times \mathcal{D} \arrow[r, "\psi"'] & \mathcal{C} \otimes \mathcal{D}
\end{tikzcd}$$

this follows from ev_I and ev_J preserving colimits by [HTT, 5.1.2.3] and ψ by definition preserving colimits separately in each variable.

It now follows from the universal property[54] of the tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ that there is an essentially unique way to complete (5.10) to a commutative diagram with a colimit preserving dashed functor $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$. $\diamondsuit$

Remark 5.3.0.3. The functors φ from Construction 5.3.0.2 are compatible with colimit preserving functors of presentable ∞-categories and functors of the indexing ∞-categories as we will argue now. Let $f\colon \mathcal{I}' \to \mathcal{I}$ and $g\colon \mathcal{J}' \to \mathcal{J}$ be functors of small ∞-categories and $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{D} \to \mathcal{D}'$ colimit preserving functors between presentable ∞-categories.

Then consider the following diagram

$$\begin{tikzcd}
 & \mathcal{C}^{\mathcal{I}} \times \mathcal{D}^{\mathcal{J}} \arrow[rr, "- \times -"] \arrow[dl, "(F\circ - \circ f)\times(G\circ - \circ g)"'] \arrow[dd] & & (\mathcal{C} \times \mathcal{D})^{\mathcal{I}\times\mathcal{J}} \arrow[dl, "(F\times G)\circ - \circ(f\times g)"'] \arrow[dd] \\
\mathcal{C}'^{\mathcal{I}'} \times \mathcal{D}'^{\mathcal{J}'} \arrow[rr, "- \times -"] \arrow[dd] & & (\mathcal{C}' \times \mathcal{D}')^{\mathcal{I}'\times\mathcal{J}'} \arrow[dd] & \\
 & \mathcal{C}^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}} \arrow[rr, dashed, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}"'] \arrow[dl, "(F\circ - \circ f)\otimes(G\circ - \circ g)"] & & (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I}\times\mathcal{J}} \arrow[dl, "(F\otimes G)\circ - \circ(f\otimes g)"] \\
\mathcal{C}'^{\mathcal{I}'} \otimes \mathcal{D}'^{\mathcal{J}'} \arrow[rr, dashed, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C}',\mathcal{D}'}"'] & & (\mathcal{C}' \otimes \mathcal{D}')^{\mathcal{I}'\times\mathcal{J}'} &
\end{tikzcd}$$

where the vertical functors are (induced) by the various canonical functors exhibiting a presentable ∞-category as a tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The top, left, and right sides commute by the respective naturalities, and the front and back commute by construction. The claim we want to show is that there is an essentially unique filler for the bottom side and the cube. But this follows

[53] Again see [HA, 4.8.1.2, 4.8.1.3, 4.8.1.4, and 4.8.1.15].

[54] See [HA, 4.8.1.2, 4.8.1.3, 4.8.1.4, and 4.8.1.15].

immediately from the universal property of the back left vertical functor using the fact that all functors on the bottom preserve colimits.

The functors φ from Construction 5.3.0.1 satisfy an analogous naturality property, which one can deduce directly from the definition using the universal property of coproducts. ♢

The next proposition justifies the overloading of notation in Construction 5.3.0.1 and Construction 5.3.0.2.

Proposition 5.3.0.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable symmetric monoidal ∞-categories and $\mathcal{I}$ and $\mathcal{J}$ small ∞-categories.*

As the forgetful functor $\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is symmetric monoidal, we can identify the underlying presentable ∞-categories of the domain and codomain of $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ from Construction 5.3.0.1 with the domain and codomain of $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ from Construction 5.3.0.2.

Under this identification there is an essentially unique homotopy of morphisms in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ between the underlying functor of $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ as defined in Construction 5.3.0.1 and $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ as in Construction 5.3.0.2. ♡

Proof. Let $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ be the underlying functor of the symmetric monoidal functor defined in Construction 5.3.0.1. By the universal property of the tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ it suffices to show that $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ fits into a commutative diagram as depicted in (5.10).

For this we ponder the following commutative diagram in $\mathcal{C}\mathrm{at}_\infty$[55].

$$
\begin{tikzcd}
\mathcal{C}^{\mathcal{I}} \times \mathcal{D}^{\mathcal{J}} \arrow[rr, "\psi'"] \arrow[dr, "(\iota_{\mathcal{C}} \circ - \circ \mathrm{pr}_1) \times (\iota_{\mathcal{D}} \circ - \circ \mathrm{pr}_2)"] \arrow[dd, "- \times -"'] & & \mathcal{C}^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}} \arrow[rr, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}"] \arrow[dr, "(\iota_{\mathcal{C}} \circ - \circ \mathrm{pr}_1) \otimes (\iota_{\mathcal{D}} \circ - \circ \mathrm{pr}_2)"] & & (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \\
& (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \times (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \arrow[rr, "\psi''"] \arrow[drr, "- \times -"] & & (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \otimes (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \arrow[ur, "- \otimes -"] & \\
(\mathcal{C} \times \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \arrow[rrr, "(\iota_{\mathcal{C}} \times \iota_{\mathcal{D}})_*"] \arrow[dr, "\simeq"] \arrow[ddrrrr, bend right, "\mathrm{id}"'] & & & ((\mathcal{C} \otimes \mathcal{D}) \times (\mathcal{C} \otimes \mathcal{D}))^{\mathcal{I} \times \mathcal{J}} \arrow[uur, bend right, "(- \otimes -)_*"'] & \\
& ((\mathcal{C} \times *) \times (* \times \mathcal{D}))^{\mathcal{I} \times \mathcal{J}} \arrow[rr, "(\mathrm{id} \times 1 \times 1 \times \mathrm{id})_*"'] & & ((\mathcal{C} \times \mathcal{D}) \times (\mathcal{C} \times \mathcal{D}))^{\mathcal{I} \times \mathcal{J}} \arrow[u, "(\psi \times \psi)_*"'] \arrow[r, "(- \otimes -)_*"'] & (\mathcal{C} \times \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \arrow[uuu, bend right, "\psi_*"'] \\
& & & &
\end{tikzcd}
$$

The composite outer diagram is the one that we are after. All the morphisms ψ with some decoration are to be the canonical morphisms exhibiting some presentable ∞-category as a tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (one could also say: these are the functors arising from lax symmetric monoidality of the inclusion of

[55] To save space we write e. g. $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ as $\mathcal{C}^{\mathcal{I}}$.

$\mathcal{P}\mathrm{r}^{\mathrm{L}}$ into $\mathcal{C}\mathrm{at}_\infty$), and $1\colon * \to \mathcal{C}$ is to be the unit morphism of the commutative algebra $\mathcal{C}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, i. e. the functor with image $\mathbb{1}_{\mathcal{C}}$, and similarly for $1\colon * \to \mathcal{D}$. The morphisms $\iota_{\mathcal{C}}$ and $\iota_{\mathcal{D}}$ are to be as in Construction 5.3.0.1. Finally, the functors $-\otimes-$ are the internal tensor product functors of the various symmetric monoidal ∞-categories.

Let us now explain how the individual pieces of the above diagram arise. The top right triangle uses that the tensor product functor is the coproduct $\mathrm{id}\amalg\mathrm{id}$ in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. The top left square arises from naturality of the functors denoted by ψ with a decoration – the functor on the right is in fact defined as the essentially unique colimit preserving functor fitting into a square like this. In the middle square below the two already discussed ones we can (again[56]) identify the composition of the top two functors with the tensor product functor of $(\mathcal{C}\otimes\mathcal{D})^{\mathcal{I}\times\mathcal{J}}$, and then commutativity of the square arises from the definition of the symmetric monoidal structure on $(\mathcal{C}\otimes\mathcal{D})^{\mathcal{I}\times\mathcal{J}}$ as the pointwise one. The square on the right arises from $\psi\colon \mathcal{C}\times\mathcal{D}\to\mathcal{C}\otimes\mathcal{D}$ being a symmetric monoidal functor, which is the case because the functor $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})\to\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$ induced by the lax symmetric monoidal inclusion of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ into $\mathcal{C}\mathrm{at}_\infty$ is again lax symmetric monoidal, see [HA, 4.8.1.4] and Proposition E.4.2.3 (7). The upper square on the left comes from functoriality of taking products of functors. The irregularly shaped square at the very bottom arises from unitality of the tensor product functors on $\mathcal{C}$ and $\mathcal{D}$ and the fact that the tensor product on $\mathcal{C}\times\mathcal{D}$ is defined componentwise. Finally, the bottom left square is constructed from the definitions of $\iota_{\mathcal{C}}$ and $\iota_{\mathcal{D}}$. For example for $\iota_{\mathcal{C}}$, the unit morphism $1\colon *\to\mathcal{C}$ induces a colimit preserving functor $1\colon \mathbb{1}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}}\simeq\mathcal{S}\to\mathcal{C}$ and we then obtain the dashed functor in the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "\simeq"'] \arrow[rrr, dotted, "\iota_{\mathcal{C}}"] & & & \mathcal{C}\otimes\mathcal{D} \\
\mathcal{C}\times * \arrow[r, "\mathrm{id}\times 1"] \arrow[d, "\mathrm{id}\times 1"'] & \mathcal{C}\times\mathcal{D} \arrow[urr, "\psi"] & & \\
\mathcal{C}\times\mathcal{S} \arrow[ur, "\mathrm{id}\times 1"'] \arrow[rrr, "\psi''''"'] & & & \mathcal{C}\otimes\mathcal{S} \arrow[uu, dashed]
\end{tikzcd}$$

The dotted functor $\iota_{\mathcal{C}}$ is then defined as the composition along the outside of the diagram, i. e. making the outer diagram commute, which obviously also

[56]One can think of it like this: The lax symmetric monoidal inclusion of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ into $\mathcal{C}\mathrm{at}_\infty$ induces a functor on commutative algebras, which is why a presentable symmetric monoidal ∞-category $\mathcal{E}$ comes with a commutative triangle

$$\begin{tikzcd}
\mathcal{E}\times\mathcal{E} \arrow[dr, "-\otimes-"] \arrow[d] & \\
\mathcal{E}\otimes\mathcal{E} \arrow[r, "-\otimes-"'] & \mathcal{E}
\end{tikzcd}$$

where the left vertical functor is the canonical one exhibiting $\mathcal{E}\otimes\mathcal{E}$ as a tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and where both functors $-\otimes-$ can be thought of as "the tensor product functor" – the one on the bottom encodes that colimits are preserved in each variable separately.

implies that there also exists a filler for the top square. $\square$

Notation 5.3.0.5. Given ∞-categories $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{D}$, with $\mathcal{C}$ and $\mathcal{C}'$ admitting all small colimits, we write $\operatorname{Fun}^{\mathrm{colim}}(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the colimit-preserving functors. We will also write $\operatorname{Fun}^{\mathrm{colim}\times\mathrm{colim}}(\mathcal{C}\times\mathcal{C}', \mathcal{D})$ for the full subcategory of $\operatorname{Fun}(\mathcal{C}\times\mathcal{C}', \mathcal{D})$ of functors preserving colimits in each variable separately. $\diamondsuit$

Proposition 5.3.0.6. *In both the situation of Construction 5.3.0.1 as well as the situation of Construction 5.3.0.2 is the functor $\varphi^{\mathcal{I};\mathcal{J}}_{\mathcal{C};\mathcal{D}}$ an equivalence of presentable (symmetric monoidal) ∞-categories.* $\heartsuit$

Proof. This proof will follow ideas of [HA, Proof of 4.8.1.15].

By [HA, 2.1.3.8] is a symmetric monoidal functor is equivalence of symmetric monoidal ∞-categories if and only if the underlying functor of ∞-categories is an equivalence. In light of Proposition 5.3.0.4 it thus suffices to discuss the case of Construction 5.3.0.2.

By [HTT, 5.5.1.1, 5.4.2.7, 5.5.4.2, and 5.5.4.15] any presentable ∞-category is equivalent to a localization $S^{-1}\operatorname{Fun}(\mathcal{K}, \mathcal{S})$ for some small ∞-category $\mathcal{K}$ and small set of morphisms S in $\operatorname{Fun}(\mathcal{K}, \mathcal{S})$. It will thus suffice to show the following claims.

(1) $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{S},\mathcal{S}}$ is an equivalence for all small ∞-categories $\mathcal{I}$ and $\mathcal{J}$.

(2) Suppose $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ is an equivalence for fixed presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, but arbitrary small ∞-categories $\mathcal{I}$ and $\mathcal{J}$. Then $\varphi^{\mathcal{I},\mathcal{J}}_{\operatorname{Fun}(\mathcal{I}',\mathcal{C}),\operatorname{Fun}(\mathcal{J}',\mathcal{D})}$ is an equivalence for all small ∞-categories $\mathcal{I}'$, $\mathcal{J}'$, $\mathcal{I}$, and $\mathcal{J}$.

(3) Suppose $\varphi^{\mathcal{I};\mathcal{J}}_{\mathcal{C};\mathcal{D}}$ is an equivalence for fixed presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$ and all small ∞-categories $\mathcal{I}$ and $\mathcal{J}$. Let S be a small set of morphisms of $\mathcal{C}$. Then $\varphi^{\mathcal{I},\mathcal{J}}_{S^{-1}\mathcal{C},\mathcal{D}}$ is also an equivalence.

(4) Suppose $\varphi^{\mathcal{I};\mathcal{J}}_{\mathcal{C};\mathcal{D}}$ is an equivalence for fixed presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$ and small ∞-categories $\mathcal{I}$ and $\mathcal{J}$. Then $\varphi^{\mathcal{J},\mathcal{I}}_{\mathcal{D},\mathcal{C}}$ is an equivalence as well.

Proof of claim (1): It suffices to show that the composition

$$\theta\colon \operatorname{Fun}(\mathcal{I}, \mathcal{S}) \times \operatorname{Fun}(\mathcal{J}, \mathcal{S}) \xrightarrow{-\times-} \operatorname{Fun}(\mathcal{I}\times\mathcal{J}, \mathcal{S}\times\mathcal{S}) \xrightarrow{\psi_*} \operatorname{Fun}(\mathcal{I}\times\mathcal{J}, \mathcal{S}\otimes\mathcal{S})$$

exhibits $\operatorname{Fun}(\mathcal{I}\times\mathcal{J}, \mathcal{S}\otimes\mathcal{S})$ as the tensor product of $\operatorname{Fun}(\mathcal{I}, \mathcal{S})$ and $\operatorname{Fun}(\mathcal{J}, \mathcal{S})$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, i. e. we have to show that for any ∞-category $\mathcal{E}$ admitting all colimits the induced functor

$$\operatorname{Fun}^{\mathrm{colim}}(\operatorname{Fun}(\mathcal{I}\times\mathcal{J}, \mathcal{S}\otimes\mathcal{S}), \mathcal{E}) \xrightarrow{\theta^*} \operatorname{Fun}^{\mathrm{colim}\times\mathrm{colim}}(\operatorname{Fun}(\mathcal{I}, \mathcal{S}) \times \operatorname{Fun}(\mathcal{J}, \mathcal{S}), \mathcal{E})$$

is an equivalence.

Using that mapping spaces in products of ∞-categories are the products of the respective mapping spaces we obtain the following commutative diagram of ∞-categories.

$$\begin{tikzcd}
\mathcal{I}^{\mathrm{op}} \times \mathcal{J}^{\mathrm{op}} \arrow[rr, dash, "\simeq"] \arrow[d] & & (\mathcal{I} \times \mathcal{J})^{\mathrm{op}} \arrow[d] \\
\mathcal{P}(\mathcal{I}^{\mathrm{op}}) \times \mathcal{P}(\mathcal{J}^{\mathrm{op}}) \arrow[d, equal] & & \mathcal{P}\big((\mathcal{I} \times \mathcal{J})^{\mathrm{op}}\big) \arrow[d, equal] \\
\mathrm{Fun}(\mathcal{I}, \mathcal{S}) \times \mathrm{Fun}(\mathcal{J}, \mathcal{S}) \arrow[d, "- \times -"'] \arrow[drr, "\theta"] & & \mathrm{Fun}(\mathcal{I} \times \mathcal{J}, \mathcal{S}) \\
\mathrm{Fun}(\mathcal{I} \times \mathcal{J}, \mathcal{S} \times \mathcal{S}) \arrow[rr, "\psi_*"'] & & \mathrm{Fun}(\mathcal{I} \times \mathcal{J}, \mathcal{S} \otimes \mathcal{S}) \arrow[u, "(- \times -)_*"']
\end{tikzcd}$$

where the two top vertical functors are (products of) Yoneda embeddings [HTT, 5.1.3], the top horizontal one is the canonical equivalence witnessing that $-^{\mathrm{op}}$ preserves products, and $- \times -\colon \mathcal{S} \otimes \mathcal{S} \to \mathcal{S}$ is the tensor product of the *cartesian* presentable symmetric monoidal structure on $\mathcal{S}$, see Remark 5.2.2.1.

By applying $\mathrm{Fun}(-, \mathcal{E})$ and passing to appropriate full subcategories we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I}^{\mathrm{op}} \times \mathcal{J}^{\mathrm{op}}, \mathcal{E}) \arrow[r, dash, "\simeq"] & \mathrm{Fun}\big((\mathcal{I} \times \mathcal{J})^{\mathrm{op}}, \mathcal{E}\big) \\
\mathrm{Fun}^{\mathrm{colim} \times \mathrm{colim}}(\mathcal{P}(\mathcal{I}^{\mathrm{op}}) \times \mathcal{P}(\mathcal{J}^{\mathrm{op}}), \mathcal{E}) \arrow[u] & \mathrm{Fun}^{\mathrm{colim}}\big(\mathcal{P}\big((\mathcal{I} \times \mathcal{I})^{\mathrm{op}}\big), \mathcal{E}\big) \arrow[u] \arrow[d, "\big((- \times -)_*\big)^*"] \\
& \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Fun}(\mathcal{I} \times \mathcal{J}, \mathcal{S} \otimes \mathcal{S}), \mathcal{E}) \arrow[ul, "\theta^*"]
\end{tikzcd}$$

The top horizontal functor is an equivalence as it is induced by one. The top left and right vertical functors are equivalences by [HTT, 5.1.5.6][57]. Finally, the bottom right vertical functor is an equivalence because it is induced by the equivalence $\mathcal{S} \otimes \mathcal{S} \to \mathcal{S}$ (see [HA, 4.8.1.20]). It follow that θ^* is an equivalence as well.

Proof of claim (2): Let $\mathcal{C}$ and $\mathcal{D}$ be as in the claim and $\mathcal{I}$, $\mathcal{I}'$, $\mathcal{J}$, $\mathcal{J}'$ small ∞-categories. We have to show that $\varphi^{\mathcal{I},\mathcal{J}}_{\mathrm{Fun}(\mathcal{I}',\mathcal{C}),\mathrm{Fun}(\mathcal{J}',\mathcal{D})}$ is an equivalence. For this, consider the following diagram where the unlabeled functors are induced by the unit and counit of the product-Fun-adjunction and symmetry

[57] For the top left functor, note that by passing to adjoints $\mathrm{Fun}^{\mathrm{colim} \times \mathrm{colim}}(\mathcal{P}(\mathcal{I}^{\mathrm{op}}) \times \mathcal{P}(\mathcal{J}^{\mathrm{op}}), \mathcal{E})$ is equivalent to $\mathrm{Fun}^{\mathrm{colim}}\big(\mathcal{P}(\mathcal{I}^{\mathrm{op}}), \mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{J}^{\mathrm{op}}), \mathcal{E})\big)$, and now one can apply [HTT, 5.1.5.6] twice and then pass back to adjoints again.

equivalences, and the functors ψ, ψ', ψ'', and ψ''' are the various functors exhibiting a presentable ∞-category as a tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

$$\begin{tikzcd}
\mathcal{C}^{\mathcal{I}\times\mathcal{I}'} \times \mathcal{D}^{\mathcal{J}\times\mathcal{J}'} \arrow[r, "\psi''"] \arrow[d, "\simeq"'] \arrow[dddd, bend right=60, "-\times-"'] & \mathcal{C}^{\mathcal{I}\times\mathcal{I}'} \otimes \mathcal{D}^{\mathcal{J}\times\mathcal{J}'} \arrow[d, "\simeq"] \arrow[dddd, dashed, bend left=60, "\varphi^{\mathcal{I}\times\mathcal{I}',\mathcal{J}\times\mathcal{J}'}_{\mathcal{C},\mathcal{D}}"] \\
\left(\mathcal{C}^{\mathcal{I}'}\right)^{\mathcal{I}} \times \left(\mathcal{D}^{\mathcal{J}'}\right)^{\mathcal{J}} \arrow[r, "\psi'''"] \arrow[d, "-\times-"'] & \left(\mathcal{C}^{\mathcal{I}'}\right)^{\mathcal{I}} \otimes \left(\mathcal{D}^{\mathcal{J}'}\right)^{\mathcal{J}} \arrow[d, "\phi^{\mathcal{I},\mathcal{J}}_{\mathcal{C}^{\mathcal{I}'},\mathcal{D}^{\mathcal{J}'}}"] \\
\left(\mathcal{C}^{\mathcal{I}'} \times \mathcal{D}^{\mathcal{J}'}\right)^{\mathcal{I}\times\mathcal{J}} \arrow[r, "\psi'_*"] \arrow[d, "(-\times-)_*"'] & \left(\mathcal{C}^{\mathcal{I}'} \otimes \mathcal{D}^{\mathcal{J}'}\right)^{\mathcal{I}\times\mathcal{J}} \arrow[d, "\left(\varphi^{\mathcal{I}',\mathcal{J}'}_{\mathcal{C},\mathcal{D}}\right)_*"] \\
\left((\mathcal{C}\times\mathcal{D})^{\mathcal{I}'\times\mathcal{J}'}\right)^{\mathcal{I}\times\mathcal{J}} \arrow[r, "(\psi_*)_*"] \arrow[d, "\simeq"'] & \left((\mathcal{C}\otimes\mathcal{D})^{\mathcal{I}'\times\mathcal{J}'}\right)^{\mathcal{I}\times\mathcal{J}} \arrow[d, "\simeq"] \\
(\mathcal{C}\times\mathcal{D})^{\mathcal{I}\times\mathcal{I}'\times\mathcal{J}\times\mathcal{J}'} \arrow[r, "\psi_*"] & (\mathcal{C}\otimes\mathcal{D})^{\mathcal{I}\times\mathcal{I}'\times\mathcal{J}\times\mathcal{J}'}
\end{tikzcd}$$

The two middle squares commute by definition of $\phi^{\mathcal{I},\mathcal{J}}_{\mathcal{C}^{\mathcal{I}'},\mathcal{D}^{\mathcal{J}'}}$ and $\varphi^{\mathcal{I}',\mathcal{J}'}_{\mathcal{C},\mathcal{D}}$, and the top and left square arise from respective naturalities. As the left rectangle on the left commutes we obtain from the universal property of ψ'' that the colimit preserving vertical composite on the right must be homotopic to $\varphi^{\mathcal{I}\times\mathcal{I}',\mathcal{J}\times\mathcal{J}'}_{\mathcal{C},\mathcal{D}}$. That $\phi^{\mathcal{I},\mathcal{J}}_{\mathcal{C}^{\mathcal{I}'},\mathcal{D}^{\mathcal{J}'}}$ is an equivalence now follows from all other functors in the commuting right long rectangle being equivalences.

Proof of claim (3): Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{I}$, $\mathcal{J}$, and S be as in the statement of the claim. We will write $\overline{S}$ for the strongly saturated collection of morphisms of $\mathcal{C}$ generated by S, see [HTT, 5.5.4.5 and 5.5.4.7]. By [HTT, 5.5.4.15] $S^{-1}\mathcal{C} \simeq (\overline{S})^{-1}\mathcal{C}$ is again presentable, so $\varphi^{\mathcal{I},\mathcal{J}}_{S^{-1}\mathcal{C},\mathcal{D}}$ is defined. We have to show that it is an equivalence.

Before we do so we need to discuss how localizations commute with tensor products in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and with $\mathrm{Fun}(\mathcal{K}, -)$ for small ∞-categories $\mathcal{K}$.

For interaction with tensor products we note the following, which is taken from the proof of [HA, 4.8.1.15]. Let $\mathcal{E}$ and $\mathcal{F}$ be any presentable ∞-categories, and T a strongly saturated class of small generation of morphisms of $\mathcal{E}$. Let W be the collection of morphisms of the form $s \otimes \mathrm{id}_F$ in $\mathcal{E} \otimes \mathcal{F}$ for any s in S and object F of $\mathcal{F}$. Then $\overline{W}$ is of small generation, as shown in [HA, Proof of 4.8.1.15]. Now consider the following diagram

$$\begin{tikzcd} T^{-1}\mathcal{E} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \times \mathcal{F} \arrow[r] & \mathcal{E} \otimes \mathcal{F} \arrow[d] \\ T^{-1}\mathcal{E} \otimes \mathcal{F} \arrow[rr, dashed] & & W^{-1}(\mathcal{E} \otimes \mathcal{F}) \end{tikzcd}$$

where the top left horizontal functor is induced by the inclusion $T^{-1}\mathcal{E} \to \mathcal{E}$, the top right horizontal functor and the left vertical functor are the canonical functors exhibiting the respective targets as tensor products in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, and the right vertical functor is the localization functor. $W^{-1}(\mathcal{E} \otimes \mathcal{F})$ is representable by [HTT, 5.5.4.15], and the composite functor from the top left to the bottom right preserves colimits in each variable separately[58]. We hence obtain the induced dashed colimit preserving functor that is an equivalence by [HA, Proof of 4.8.1.15].

We now turn to the interaction of localizations with taking functor categories. For this, let $\mathcal{E}$ be a presentable ∞-category, $\mathcal{K}$ a small ∞-category, and T a strongly saturated class of morphisms of $\mathcal{E}$ of small generation. Let $L\colon \mathcal{E} \to T^{-1}\mathcal{E}$ be the localization functor. Then by Proposition D.2.2.1 and $\mathrm{Fun}(\mathcal{K}, -)$ preserving fully faithful functors by Proposition B.3.0.1 it follows that the induced functor

$$L_*\colon \mathrm{Fun}(\mathcal{K}, \mathcal{E}) \to \mathrm{Fun}\big(\mathcal{K}, T^{-1}\mathcal{E}\big)$$

is a localization functor again. Furthermore, $\mathrm{Fun}(\mathcal{K}, T^{-1}\mathcal{E})$ is presentable again by [HTT, 5.5.3.6]. Let W be the class of morphisms in $\mathrm{Fun}(\mathcal{K}, \mathcal{E})$ that are pointwise in T. By combining [HTT, 5.5.4.15], [HTT, 5.5.4.2], and Proposition A.3.2.1 we see that W consists precisely of those morphisms that are mapped to equivalences by L_*. It then follows from [HTT, 5.5.4.2] that there is a canonical equivalence

$$\mathrm{Fun}\big(\mathcal{K}, T^{-1}\mathcal{E}\big) \simeq W^{-1}\,\mathrm{Fun}(\mathcal{K}, \mathcal{E})$$

that is compatible with the localization functors.

We now return to showing that $\varphi^{\mathcal{I},\mathcal{J}}_{S^{-1}\mathcal{C},\mathcal{D}}$ is an equivalence. Let T be the strongly generated class of morphisms in $\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \otimes \mathrm{Fun}(\mathcal{J}, \mathcal{D})$ that is generated by morphisms of the form $\eta \otimes \mathrm{id}_G$ for any object G in $\mathrm{Fun}(\mathcal{J}, \mathcal{D})$ and any morphism η in $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ such that $\eta(I)$ is in S for all objects I of $\mathcal{I}$. Let W be the strongly generated class of morphisms in $\mathrm{Fun}(\mathcal{I} \times \mathcal{J}, \mathcal{C} \otimes \mathcal{D})$ that is generated by those morphisms for which for every object I of $\mathcal{I}$ and J of $\mathcal{J}$ the evaluation at (I, J) is equivalent to a morphism of the form $s \otimes \mathrm{id}_D$ for s in S and D and object of $\mathcal{D}$.

[58] One can see this using that by [HTT, 5.2.7.5] a diagram $p\colon K^{\triangleright} \to T^{-1}\mathcal{E}$ is a colimit if and only if the induced morphism from the colimit taken in $\mathcal{E}$ to the cone object, $\operatorname{colim} p|_K \to p(\infty)$, is a T-equivalence.

Consider the following commutative diagram

$$\begin{tikzcd}
 & T^{-1}\big(\mathcal{C}^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}}\big) \\
\mathcal{C}^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}} \arrow[ur] \arrow[r, "L_*\otimes \mathrm{id}_*"'] \arrow[d, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}"'] & \big(S^{-1}\mathcal{C}\big)^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}} \arrow[u, no head, "\simeq"'] \arrow[d, "\varphi^{\mathcal{I},\mathcal{J}}_{S^{-1}\mathcal{C},\mathcal{D}}"] \\
(\mathcal{C}\otimes\mathcal{D})^{\mathcal{I}\times\mathcal{J}} \arrow[r, "(L\otimes \mathrm{id})_*"] \arrow[dr] & \big(S^{-1}\mathcal{C}\otimes\mathcal{D}\big)^{\mathcal{I}\times\mathcal{J}} \arrow[d, no head, "\simeq"] \\
 & W^{-1}\Big((\mathcal{C}\otimes\mathcal{D})^{\mathcal{I}\times\mathcal{J}}\Big)
\end{tikzcd}$$

where L denotes the localization functor $\mathcal{C} \to S^{-1}\mathcal{C}$, the middle square arises from naturality of the functors φ with respect to the colimit preserving functor L (see Remark 5.3.0.3), and the top and bottom triangles use the compatibility of the tensor product and functor categories with localization as discussed above, with the top and bottom functors being the respective localization functors.

By assumption $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ is an equivalence, and it is clear from the definitions that the strongly saturated classes of morphisms T and W correspond under this equivalence, i. e. $\overline{\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}(T)} = W$. It then follows from [HTT, 5.5.4.20] that $\varphi^{\mathcal{I},\mathcal{J}}_{S^{-1}\mathcal{C},\mathcal{D}}$ is also an equivalence.

Proof of claim (4): One can show in a manner analogous to Remark 5.3.0.3 that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}} \arrow[r, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}"] \arrow[d, "\tau"', "\simeq"] & (\mathcal{C}\otimes\mathcal{D})^{\mathcal{I}\times\mathcal{J}} \arrow[d, "\simeq"', "\tau'_*"] \\
\mathcal{D}^{\mathcal{J}} \otimes \mathcal{C}^{\mathcal{I}} \arrow[r, "\varphi^{\mathcal{J},\mathcal{I}}_{\mathcal{D},\mathcal{C}}"'] & (\mathcal{D}\otimes\mathcal{C})^{\mathcal{J}\times\mathcal{I}}
\end{tikzcd}$$

where τ and τ' are the symmetry equivalences of the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The claim immediately follows from this. ☐

The proof of Proposition 5.3.0.8 below is also sketched in [Rak20, 2.2.9]. We need a small prerequisite before stating the result.

Proposition 5.3.0.7. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category, $\mathcal{O}$ an ∞-operad, and $\mathcal{O}'$ a reduced ∞-operad*[59]*. Then the unit of the induced symmetric monoidal structure on* $\mathrm{BiAlg}_{\mathcal{O},\mathcal{O}'}(\mathcal{C})$ *is a final object.* ♡

[59]See [HA, 2.3.4.1] for a definition.

Proof. By definition there is an equivalence as follows.

$$\mathrm{BiAlg}_{\mathcal{O},\mathcal{O}'}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}'}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{op})^{\mathrm{op}}$$

The unit is an initial object in $\mathrm{Alg}_{\mathcal{O}'}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{op})$ by [HA, 3.2.1.8] and hence final in $\mathrm{BiAlg}_{\mathcal{O},\mathcal{O}'}(\mathcal{C})$. $\square$

Proposition 5.3.0.8 ([Rak20, 2.2.9])**.** *Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category and G an object in $\mathrm{Mon}^{\mathrm{gp}}_{\mathsf{Assoc}}(\mathcal{S})$. Consider G as a cocommutative bialgebra in $\mathcal{S}$, and give $\mathbb{1}_{\mathcal{C}} \boxtimes G$ the induced cocommutative bialgebra structure, as discussed in the introduction to Section 5.3.*

Then there is a commutative diagram of presentable symmetric monoidal ∞-categories and colimit preserving symmetric monoidal functors[60] *as follows*

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{B}\,G} \arrow[rr, "\Psi^G_{\mathcal{C}}", "\simeq"'] \arrow[dr, "\mathrm{ev}_*"'] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes G}(\mathcal{C}) \arrow[dl, "\mathrm{ev}_{\mathrm{m}}"] \\
& \mathcal{C} &
\end{tikzcd} \tag{5.11}$$

where $\mathcal{C}^{\mathrm{B}\,G}$ carries the pointwise symmetric monoidal structure discussed in the introduction to Section 5.3 and $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes G}(\mathcal{C})$ the one from Definition 3.4.2.1. As indicated in the diagram, $\Psi^G_{\mathcal{C}}$ is an equivalence of presentable symmetric monoidal ∞-categories.

Furthermore, these equivalences can be chosen in such a way as to be compatible with morphisms $f\colon G \to H$ in $\mathrm{Mon}^{\mathrm{gp}}_{\mathsf{Assoc}}(\mathcal{S})$ and $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, *in the sense that for such f and F there is a commutative diagram in* $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ *as follows.*

$$\begin{tikzcd}
& \mathcal{C}^{\mathrm{B}\,H} \arrow[rr, "\Psi^H_{\mathcal{C}}"] \arrow[dl, "F\circ-\circ\mathrm{B}\,f"'] \arrow[ddrr, "\mathrm{ev}_*"] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes H}(\mathcal{C}) \arrow[dl, "\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes f}(F)"'] \arrow[dd, "\mathrm{ev}_{\mathrm{m}}"] \\
\mathcal{D}^{\mathrm{B}\,G} \arrow[rr, "\Psi^G_{\mathcal{D}}"] \arrow[ddr, "\mathrm{ev}_*"'] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes G}(\mathcal{D}) \arrow[ddl, "\mathrm{ev}_{\mathrm{m}}"] & \\
& & & \mathcal{C} \arrow[dll, "F"] \\
& \mathcal{D} & &
\end{tikzcd} \tag{5.12}$$

$\heartsuit$

[60]In other words, a commutative diagram in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

Remark 5.3.0.9. In the situation of Proposition 5.3.0.8, let $f\colon G \to *$ be the essentially unique morphism of grouplike associative monoids in $\mathcal{S}$. The induced morphism of cocommutative bialgebras in $\mathcal{C}$ given by

$$\mathbb{1}_{\mathcal{C}} \boxtimes f\colon \mathbb{1}_{\mathcal{C}} \boxtimes G \to \mathbb{1}_{\mathcal{C}} \boxtimes * \simeq \mathbb{1}_{\mathcal{C}}$$

is also the essentially unique one, see Proposition 5.3.0.7.

Then there is a commutative diagram by Proposition 5.3.0.8 as follows

$$\begin{tikzcd}
& \mathcal{C} & \\
\mathcal{C}^{\mathrm{B}\,*} \arrow[ur, "\mathrm{ev}_*"] \arrow[rr, "\Psi_{\mathcal{C}}^*"] \arrow[d, "(\mathrm{B}\,f)^*"'] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}}(\mathcal{C}) \arrow[ul, "\mathrm{ev}_{\mathfrak{m}}"'] \arrow[d, "\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes f}(\mathcal{C})"] \\
\mathcal{C}^{\mathrm{B}\,G} \arrow[rr, "\Psi_{\mathcal{C}}^G"'] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes G}(\mathcal{C})
\end{tikzcd}$$

Note that the functors ev_* and $\mathrm{ev}_{\mathfrak{m}}$ are equivalences[61], and we can interpret the composites from the top to the bottom left and bottom right as the functors that map an object of $\mathcal{C}$ to that same object equipped with the trivial action by G. ♢

Proof of Proposition 5.3.0.8. We start by noting that ignoring the horizontal functors, the rest of diagrams (5.11) and (5.12) are indeed diagrams in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. The ∞-category $\mathcal{C}^{\mathrm{B}\,G}$ with the pointwise symmetric monoidal structure is indeed presentable symmetric monoidal, as is explained in Construction 5.3.0.1. That $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes G}(\mathcal{C})$ is presentable symmetric monoidal is by construction, see Definition 3.4.2.1 and the propositions referenced there. $(F \circ - \circ \mathrm{B}\,f)\colon \mathrm{Fun}(\mathrm{B}\,H, \mathcal{C}) \to \mathrm{Fun}(\mathrm{B}\,G, \mathcal{D})$ can be upgraded to a symmetric monoidal functor and preserves colimits as both the symmetric monoidal structure as well as colimits are pointwise. Similarly, the evaluation functor ev_* is symmetric monoidal and preserves colimits. $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes f}(F)$ as well as $\mathrm{ev}_{\mathfrak{m}}$ are symmetric monoidal and colimit preserving by construction, see Definition 3.4.2.1. Finally, the left and right squares in (5.12) arise from naturality of the respective evaluation functors.

The commutative triangle we have to construct will be given as the composite outer triangle in a commutative diagram in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ as indicated below; we will individually construct each part together with the relevant

[61] See [HA, 4.2.4.9] for $\mathrm{ev}_{\mathfrak{m}}$ being an equivalence.

compatibility with respect to $f\colon G \to H$ and $F\colon \mathcal{C} \to \mathcal{D}$.

$$\begin{tikzcd}
\mathcal{C}^{BG} \arrow[rr, dash, "\simeq"] \arrow[dddrrr, bend right, "\mathrm{ev}_*"'] \arrow[rrrrrr, bend left, "\Psi^G_{\mathcal{C}}"] & & \mathcal{C} \otimes \mathcal{S}^{BG} \arrow[rr, dash, "\simeq"] \arrow[dr, "\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_*"] & & \mathcal{C} \otimes \mathrm{LMod}_G(\mathcal{S}) \arrow[rr, dash, "\simeq"] \arrow[dl, "\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_{\mathfrak{m}}"'] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C}) \arrow[dddlll, bend left, "\mathrm{ev}_{\mathfrak{m}}"] \\
& & & \mathcal{C} \otimes \mathcal{S} \arrow[dd, "\rho_{\mathcal{C}}", "\simeq"'] & & & \\
& & & & & & \\
& & & \mathcal{C} & & &
\end{tikzcd} \tag{5.13}$$

The tensor product is the tensor product induced on $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ by the tensor product of presentable ∞-categories[62]. The bottom vertical equivalence $\rho_{\mathcal{C}}$ is the right unitor, using that $\mathcal{S}$ is the monoidal unit in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see [HA, 4.8.1.20]).

Construction of the left square: The square arises as the composite outer square in the following commutative diagram in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{B}\,G} \arrow[d, "\mathrm{ev}_*"] & (\mathcal{C} \otimes \mathcal{S})^{* \times \mathrm{B}\,G} \arrow[l, "\rho_{\mathcal{C}} \circ - \circ (\mathrm{pr}_2)^{-1}"'] \arrow[d, "\mathrm{ev}_{(*,*)}"] & \mathcal{C}^* \otimes \mathcal{S}^{\mathrm{B}\,G} \arrow[l, "\varphi^{*,\mathrm{B}\,G}_{\mathcal{C},\mathcal{S}}"'] \arrow[r, "\mathrm{ev}_* \otimes \mathrm{id}_{\mathcal{S}^{\mathrm{B}\,G}}"] \arrow[d, "\mathrm{ev}_* \otimes \mathrm{ev}_*"] & \mathcal{C} \otimes \mathcal{S}^{\mathrm{B}\,G} \arrow[d, "\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_*"'] \\
\mathcal{C} & \mathcal{C} \otimes \mathcal{S} \arrow[l, "\rho_{\mathcal{C}}"] & \mathcal{C} \otimes \mathcal{S} \arrow[l, "\mathrm{id}_{\mathcal{C} \otimes \mathcal{S}}"] \arrow[r, "\mathrm{id}_{\mathcal{C} \otimes \mathcal{S}}"'] & \mathcal{C} \otimes \mathcal{S}
\end{tikzcd}$$

Here, the left square is induced by the unitality equivalences

$$\mathrm{pr}_2\colon\ * \times \mathrm{B}\,G \to \mathrm{B}\,G$$

(in $\mathcal{C}\mathrm{at}_\infty$) and

$$\rho_{\mathcal{C}}\colon \mathcal{C} \otimes \mathcal{S} \to \mathcal{C}$$

(in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$), which is clearly compatible with f and F. The equivalence $\varphi^{*,\mathrm{B}\,G}_{\mathcal{C},\mathcal{S}}$ is the one from Construction 5.3.0.1, and the middle square as well as the commutative cube for compatibility with f and F can be constructed directly using the definition and the universal property of coproducts in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. Finally, the right square arises directly from functoriality of the tensor product of $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, and ev_* is clearly an equivalence.

Construction of the right square: This square arises as the composite outer square obtained by combining the following two commutative diagrams in

[62] See [HA, 4.8.1.15]

$\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

$$\begin{tikzcd}
\mathcal{C} \otimes \mathrm{LMod}_G(\mathcal{S}) \arrow[d, "\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_{\mathfrak{m}}"] & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}}(\mathcal{C}) \otimes \mathrm{LMod}_G(\mathcal{S}) \arrow[l, "\mathrm{ev}_{\mathfrak{m}} \otimes \mathrm{id}"'] \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}} \otimes \mathrm{ev}_{\mathfrak{m}}"] & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \otimes G}(\mathcal{C} \otimes \mathcal{S}) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\
\mathcal{C} \otimes \mathcal{S} & \mathcal{C} \otimes \mathcal{S} \arrow[l, "\mathrm{id}_{\mathcal{C} \otimes \mathcal{S}}"] \arrow[r, "\mathrm{id}"'] & \mathcal{C} \otimes \mathcal{S}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \otimes G}(\mathcal{C} \otimes \mathcal{S}) \arrow[r, "\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \otimes G}(\rho_{\mathcal{C}})"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"'] & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C}) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\
\mathcal{C} \otimes \mathcal{S} \arrow[r, "\rho_{\mathcal{C}}"'] & \mathcal{C}
\end{tikzcd}$$

The left square of the first diagram arises from functoriality of the tensor product, and $\mathrm{ev}_{\mathfrak{m}}$ is an equivalence by [HA, 4.2.4.9 and 2.1.3.8]. Compatibility with f and F follows from $\mathrm{ev}_{\mathfrak{m}}$ being a natural transformation, see Definition 3.4.2.1. The right square of the first diagram as well as its compatibility with f and F is the one arising from $\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{LMod} \to \mathrm{pr}$ being a natural transformation of *symmetric monoidal* functors

$$\mathrm{LMod}\colon \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_{\mathsf{Comm}} \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$$

by Remark 3.4.2.2. Finally, the second diagram as well as its compatibility with f and F arises from the naturality of the right unitor ρ and $\mathrm{ev}_{\mathfrak{m}}$. That there is an equivalence $\rho_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}} \otimes G) \simeq \mathbb{1}_{\mathcal{C}} \boxtimes G$ that is compatible with f and F follows immediately from $\mathcal{S}$ being initial in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ (see Remark 5.2.2.1), so that there is a essentially unique natural equivalence between the composition of the inclusion[63] $\mathcal{S} \to \mathcal{C} \otimes \mathcal{S}$, which sends G to $\mathbb{1}_{\mathcal{C}} \otimes G$, with $\rho_{\mathcal{C}}$, and $\mathbb{1}_{\mathcal{C}} \boxtimes -$.

Construction of the middle triangle: It suffices to construct a commutative triangle

$$\begin{tikzcd}
\mathcal{S}^{\mathrm{B}\,G} \arrow[rr, dashed, "\Psi^G_{\mathcal{S}}", "\simeq"'] \arrow[dr, "\mathrm{ev}_*"'] & & \mathrm{LMod}_G(\mathcal{S}) \arrow[dl, "\mathrm{ev}_{\mathfrak{m}}"] \\
& \mathcal{S} &
\end{tikzcd} \tag{5.14}$$

in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ that is compatible with f, as the middle triangle in Equation (5.13) we need to construct can then be obtained by tensoring with $\mathcal{C}$.

As both ev_* and $\mathrm{ev}_{\mathfrak{m}}$ are symmetric monoidal as well as limit preserving and detecting[64], it follows from the symmetric monoidal structure on $\mathcal{S}$ being cartesian that the symmetric monoidal structures on $\mathcal{S}^{\mathrm{B}\,G}$ and $\mathrm{LMod}_G(\mathcal{S})$ are cartesian as well[65]. By [HA, 2.4.1.8], any filler for the horizontal functor and

[63]This is also the functor we could call $\mathbb{1}_{\mathcal{C} \otimes \mathcal{S}} \boxtimes -$, see Definition 5.2.2.2.
[64]See [HTT, 5.1.2.3] for ev_* and [HA, 4.2.3.3] for $\mathrm{ev}_{\mathfrak{m}}$.
[65]See [HA, 2.4.0.1] for the definition.

the triangle (5.14) in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ such that the horizontal functor is an equivalence[66], can then be lifted in an essentially unique way to a filler for the triangle as a diagram in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. It thus suffices to construct a commuting triangle (5.14) in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ in which the horizontal functor is an equivalence.

In [BP21, 3.9] an equivalence $\mathcal{S}^{\mathrm{B}\,G} \simeq \mathrm{LMod}_{\beta_1\,\mathrm{B}\,G}(\mathcal{S})$ is constructed as a sequence of equivalences[67]. See the introduction of Section 5.3 for a discussion of β_1 – the underlying space of $\beta_1\,\mathrm{B}\,G$ is $\Omega\,\mathrm{B}\,G$. As B is defined as the inverse functor to (the appropriately restricted) β_1, there is a canonical equivalence $\beta_1\,\mathrm{B}\,G \simeq G$, so that we obtain an equivalence $\mathrm{LMod}_{\beta_1\,\mathrm{B}\,G}(\mathcal{S}) \simeq \mathrm{LMod}_G(\mathcal{S})$.

Let us now go through the individual steps to say something about compatibility with forgetful functors to $\mathcal{S}$ and compatibility with f.

For the first step, let $j\colon\ \mathrm{B}\,G \to \mathrm{Fun}(\mathrm{B}\,G^{\mathrm{op}}, \mathcal{S})$ be the Yoneda embedding[68], and consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{B}\,G, \mathcal{S}) \arrow[dr, "\mathrm{ev}_*"'] & & \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Fun}(\mathrm{B}\,G^{\mathrm{op}}, \mathcal{S}), \mathcal{S}) \arrow[ll, "j^*", "\simeq"'] \arrow[dl, "\mathrm{ev}_{j(*)}"] \\
& \mathcal{S} &
\end{tikzcd}$$

where j^* is an equivalence by [HTT, 5.1.5.6]. Compatibility with f follows from naturality of the Yoneda embedding.

Before we discuss the second step, we first need to note something regarding right fibrations over ∞-groupoids[69]. Let X be an object of $\mathcal{S}$ and consider it as an ∞-groupoid. The ∞-category $\mathcal{RF}\mathrm{ib}(X)$ of right fibrations over X is the full subcategory of $\mathcal{CF}\mathrm{ib}(X)$ spanned by those cartesian fibrations whose fibers are ∞-groupoids. $\mathcal{CF}\mathrm{ib}(X)$ in turn is the subcategory of $\mathcal{C}\mathrm{at}_{\infty/X}$ spanned by the cartesian fibrations and morphisms of cartesian fibrations. Note that by [HTT, 2.4.2.4], if $p\colon \mathcal{E} \to X$ is a right fibration, then every morphism of $\mathcal{E}$ is p-cocartesian, so morphisms among cartesian fibrations over X (i. e. morphisms in $\mathcal{C}\mathrm{at}_{\infty/X}$) are automatically morphisms of cartesian fibrations. $\mathcal{RF}\mathrm{ib}(X)$ is thus the full subcategory of $\mathcal{C}\mathrm{at}_{\infty/X}$ spanned by the right fibrations. That X is an ∞-groupoid together with [HTT, 2.4.2.4 and 2.4.1.5] implies that a functor of ∞-categories $\mathcal{E} \to X$ is a right fibration if and only if $\mathcal{E}$ is an ∞-groupoid.

The inclusion $\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$ is also fully faithful, so induces by Proposition D.1.2.1 a fully faithful functor $\mathcal{S}_{/X} \to \mathcal{C}\mathrm{at}_{\infty/X}$ with the same essential image. We thus obtain a canonical equivalence $\mathcal{RF}\mathrm{ib}(X) \simeq \mathcal{S}_{/X}$, see Proposition B.4.3.1.

[66] Note that ev_* and $\mathrm{ev}_{\mathfrak{m}}$ are known to preserve products as already noted, so if the horizontal functor is an equivalence and hence also preserves products, (5.14) will be a commutative triangle of product preserving functors.

[67] [BP21, 3.9] contains an unnecessary use of $\mathrm{B}\,G^{\mathrm{op}} \simeq \mathrm{B}\,G$, which likely stems from a misreading of the definition of $\mathcal{P}(\mathrm{B}\,G)$ used in [HTT, 5.1.5.6], which is defined as $\mathrm{Fun}(\mathrm{B}\,G^{\mathrm{op}}, \mathcal{S})$ in [HTT, 5.1.0.1], not $\mathrm{Fun}(\mathrm{B}\,G, \mathcal{S})$.

[68] See [HTT, Introduction of 5.1.3] for a definition and discussion of j – it can be described as the functor $\mathrm{Map}_{\mathcal{S}}(\bullet, -)$.

[69] See also [HTT, 5.1.1.1] for a related discussion.

Now we can tackle the second step, for which we consider the following composite equivalence

$$\mathrm{Fun}(\mathrm{B}\,G^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{Gr}} \mathcal{RF}\mathrm{ib}(\mathrm{B}\,G) \simeq \mathcal{S}_{/\,\mathrm{B}\,G}$$

where the first equivalence is the Grothendieck construction. This equivalence is natural in G[70] and hence induces a commutative triangle

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Fun}(\mathrm{B}\,G^{\mathrm{op}}, \mathcal{S}), \mathcal{S}) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_{j(*)}"'] & & \mathrm{Fun}^{\mathrm{colim}}\big(\mathcal{S}_{/\,\mathrm{B}\,G}, \mathcal{S}\big) \arrow[dl, "\mathrm{ev}_{\mathrm{Gr}(j(*))}"] \\
& \mathcal{S} &
\end{tikzcd}$$

that is compatible with f.

$\mathrm{Gr}(j(*))\colon X \to \mathrm{B}\,G$ is the right fibration classified by $j(*)$. By [HTT, 4.4.4.5] the ∞-groupoid X has a final object and is thus contractible, so that we can identify $\mathrm{Gr}(j(*))$ with the inclusion of the basepoint of $\mathrm{B}\,G$.

For the third step the equivalence

$$\mathcal{S}_{/\,\mathrm{B}\,G} \xrightarrow{\simeq} \mathrm{RMod}_{\beta_1\,\mathrm{B}\,G}(\mathcal{S})$$

is used that is described in [HTT, 5.2.6.28 and 5.2.6.29], and which is compatible with f. By [HTT, 5.2.6.29] this equivalence fits into a commutative diagram

$$\begin{tikzcd}
& \mathcal{S} \arrow[ddr, "\mathrm{Free}"] & \\
\mathcal{S}_{/*} \arrow[ur, "\mathrm{pr}", "\simeq"'] \arrow[d, "(*\to\mathrm{B}\,G)_*"'] & & \\
\mathcal{S}_{/\,\mathrm{B}\,G} \arrow[rr, "\simeq"] & & \mathrm{RMod}_{\beta_1\,\mathrm{B}\,G}(\mathcal{S})
\end{tikzcd}$$

where $* \to \mathrm{B}\,G$ refers to the inclusion of the basepoint. It follows that $* \to \mathrm{B}\,G$ is mapped to the free right-$\beta_1\,\mathrm{B}\,G$-module generated by $*$, so to $\beta_1\,\mathrm{B}\,G$ considered as a right module over itself, under the equivalence $\mathcal{S}_{/\,\mathrm{B}\,G} \simeq \mathrm{RMod}_{\beta_1\,\mathrm{B}\,G}(\mathcal{S})$. By definition of B we also have a canonical equivalence $\beta_1\,\mathrm{B}\,G \simeq G$. We thus obtain a commuting triangle, compatible with f, as follows.

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{colim}}\big(\mathcal{S}_{/\,\mathrm{B}\,G}, \mathcal{S}\big) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_{(*\to\mathrm{B}\,G)}"'] & & \mathrm{Fun}^{\mathrm{colim}}(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}) \arrow[dl, "\mathrm{ev}_G"] \\
& \mathcal{S} &
\end{tikzcd}$$

For the fourth step, it is explained in [BP21, 3.9] that the forgetful functor

$$\mathrm{LinFun}^{\mathrm{colim}}_{\mathcal{S}}(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}) \to \mathrm{Fun}^{\mathrm{colim}}(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S})$$

[70] For naturality of the Grothendieck construction see [GHN17, A.32].

is an equivalence, so that we obtain a commutative triangle

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{colim}}(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}) \arrow[dr, "\mathrm{ev}_G"'] & & \mathrm{LinFun}^{\mathrm{colim}}_{\mathcal{S}}(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}) \arrow[ll, "\simeq"'] \arrow[dl, "\mathrm{ev}_G"] \\ & \mathcal{S} & \end{tikzcd}$$

that is compatible with f.

Finally, for the fifth step, [HA, 4.8.4.1] is used, where it is shown that there is an equivalence as indicated by the top horizontal functor in the following diagram.

$$\begin{tikzcd} \mathrm{LinFun}^{\mathrm{colim}}_{\mathcal{S}}(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_G"'] & & \mathrm{LMod}_G(\mathcal{S}) \arrow[dl, "\mathrm{ev}_{\mathrm{m}}"] \\ & \mathcal{S} & \end{tikzcd}$$

That there also is a commutative triangle as indicated follows from unpacking the definition of the top horizontal equivalence, from which one also sees that this commutative triangle is also compatible with f, see [HA, 4.8.4.1 and 4.6.2.9].

Combining everything yields a commutative triangle (5.14) in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ in a manner compatible with f. □

5.4 The monoidal equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$

We can now combine the main result of Section 5.3 with the equivalence between the bialgebras $k \boxtimes \mathbb{T}$ and D in $\mathcal{D}(k)$ to obtain an equivalence as follows.

$$\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \simeq \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$$

This equivalence is only (Assoc-)monoidal, not $\mathbb{E}_2$-monoidal or even symmetric monoidal, see Warning 5.4.0.2 below.

Construction 5.4.0.1. The ∞-category $\mathcal{D}(k)$ is a presentable symmetric monoidal ∞-category by Proposition 4.3.2.1 (1), and as the circle group $\mathbb{T}$ is path connected, it follows from [HA, 5.2.6.4] that $\mathbb{T}$ is grouplike as an associative monoid in $\mathcal{S}$. Hence we can apply Proposition 5.3.0.8 and Remark 5.3.0.9

to obtain a commutative diagram in $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ as follows

$$
\begin{tikzcd}
\mathcal{D}(k)^{*} \arrow[d, "(\mathrm{B}\,\mathbb{T}\to *)^{*}"'] & \mathcal{D}(k) \arrow[l, "(\mathrm{ev}_{*})^{-1}"'] \arrow[r, "(\mathrm{ev}_{\mathfrak{m}})^{-1}"] & \mathrm{LMod}_{\mathbb{1}_{\mathcal{D}(k)}}(\mathcal{D}(k)) \arrow[d, "\mathrm{LMod}_{\left(k\boxtimes\mathbb{T}\to\mathbb{1}_{\mathcal{D}(k)}\right)}(\mathcal{D}(k))"'] \\
\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_{*}"'] & & \mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \arrow[dl, "\mathrm{ev}_{\mathfrak{m}}"] \\
& \mathcal{D}(k) &
\end{tikzcd}
\tag{5.15}
$$

where the middle horizontal morphism is an equivalence and the morphisms of bialgebras $\mathrm{B}\,\mathbb{T} \to *$ and $\mathrm{D} \to \mathbb{1}_{\mathcal{D}(k)}$ are the essentially unique ones, see Proposition 5.3.0.7.

Proposition 5.2.4.2 and Convention 5.2.4.3 provide us with an equivalence of bialgebras in $\mathcal{D}(k)$

$$\varphi\colon \mathrm{D} \to k \boxtimes \mathbb{T}$$

and as k is a final object in $\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Assoc}}(\mathcal{D}(k))$ by Proposition 5.3.0.7, we can extend this to a commutative triangle of bialgebras in $\mathcal{D}(k)$ as follows.

$$
\begin{tikzcd}
\mathrm{D} \arrow[rr, "\varphi"] \arrow[dr] & & k \boxtimes \mathbb{T} \arrow[dl] \\
& k &
\end{tikzcd}
$$

Applying the functor LMod from Definition 3.4.2.1 we obtain a commutative diagram in $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$

$$
\begin{tikzcd}
& \mathrm{LMod}_{\mathbb{1}_{\mathcal{D}(k)}}(\mathcal{D}(k)) \arrow[dl, "\mathrm{LMod}_{\left(k\boxtimes\mathbb{T}\to\mathbb{1}_{\mathcal{D}(k)}\right)}(\mathcal{D}(k))"'] \arrow[dr, "\mathrm{LMod}_{\left(\mathrm{D}\to\mathbb{1}_{\mathcal{D}(k)}\right)}(\mathcal{D}(k))"] & \\
\mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \arrow[rr, "\mathrm{LMod}_{\varphi}(\mathcal{D}(k))", "\simeq"'] \arrow[dr, "\mathrm{ev}_{\mathfrak{m}}"'] & & \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) \arrow[dl, "\mathrm{ev}_{\mathfrak{m}}"] \\
& \mathcal{D}(k) &
\end{tikzcd}
\tag{5.16}
$$

where the top triangle is the one induced by the previous diagram, and the bottom one uses that $\mathrm{ev}_{\mathfrak{m}}$ is a natural transformation.

Combining (5.15) and (5.16) we obtain a commutative diagram in $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, i. e. of presentable monoidal ∞-categories with monoidal colimit preserving functors, as follows

$$\begin{tikzcd}
\mathcal{D}(k)^{*} \arrow[d, "(\mathrm{B}\,\mathbb{T}\to *)^{*}"'] & \mathcal{D}(k) \arrow[l, "(\mathrm{ev}_{*})^{-1}"'] \arrow[r, "(\mathrm{ev}_{\mathrm{m}})^{-1}"] & \mathrm{LMod}_{\mathbb{1}_{\mathcal{D}(k)}}(\mathcal{D}(k)) \arrow[d, "\mathrm{LMod}_{(\mathrm{D}\to\mathbb{1}_{\mathcal{D}(k)})}(\mathcal{D}(k))"] \\
\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_{*}"'] & & \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) = \mathcal{M}\mathrm{ixed} \arrow[dl, "\mathrm{ev}_{\mathrm{m}}"] \\
& \mathcal{D}(k) &
\end{tikzcd}$$

such that the middle horizontal functor is an equivalence. ♢

Warning 5.4.0.2. While both $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ and $\mathcal{M}\mathrm{ixed} = \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$ carry a symmetric monoidal structure, the equivalence between them is only Assoc-monoidal.

For this reason one should be careful to distinguish between "objects of $\mathcal{D}(k)$ with $\mathbb{T}$-action" (or "$\mathbb{T}$-objects in $\mathcal{D}(k)$") on the one hand and "mixed complexes" on the other hand whenever the symmetric monoidal structures might be relevant. ♢

Remark 5.4.0.3. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Combining the compatibility statement with colimit preserving symmetric monoidal functors between presentable symmetric monoidal ∞-categories that is part of Proposition 5.3.0.8 with Remark 5.2.4.4 we obtain a commutative diagram of monoidal colimit preserving functors[71]

$$\begin{tikzcd}
\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[rr, dash, "\simeq"] \arrow[dr, "\mathrm{ev}_{*}"] \arrow[ddd, "(k'\otimes_k -)_{*}"'] & & \mathrm{LMod}_{\mathrm{D}_k}(\mathcal{D}(k)) \arrow[dl, "\mathrm{ev}_{\mathrm{m}}"'] \arrow[ddd, "k'\otimes_k -"] \\
& \mathcal{D}(k) \arrow[d, "k'\otimes_k -"] & \\
& \mathcal{D}(k') & \\
\mathcal{D}(k')^{\mathrm{B}\,\mathbb{T}} \arrow[rr, dash, "\simeq"'] \arrow[ur, "\mathrm{ev}_{*}"] & & \mathrm{LMod}_{\mathrm{D}_{k'}}(\mathcal{D}(k)) \arrow[ul, "\mathrm{ev}_{\mathrm{m}}"']
\end{tikzcd}$$

where the horizontal equivalences are the ones from Construction 5.4.0.1. ♢

[71]Like with the diagram in Remark 5.2.4.4, there is also supposed to be a filler for the outer diagram that is compatible with the forgetful functors.

Chapter 6

Hochschild homology

In this chapter we introduce the main object of study of this text, *Hochschild homology*. We will give both a modern account, in which the main construction is a functor

$$\mathrm{HH}_{\mathbb{T}} \colon \operatorname{Alg}(\mathcal{D}(k)) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

called *Hochschild homology* that will be defined and discussed in Section 6.2, as well as a description of the classical constructions, where one considers a functor

$$\mathrm{C} \colon \operatorname{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathsf{Mixed}_{\mathrm{cof}}$$

called *standard Hochschild complex*. The latter construction will be discussed in Section 6.3, where we will also show that the two constructions are related – the standard Hochschild complex can be considered as a model for Hochschild homology. For both the definitions the first step is to apply the *cyclic bar construction*, which takes an associative algebra in an some monoidal ∞-category $\mathcal{C}$, and produces a *cyclic object* in $\mathcal{C}$, i. e. a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$, where $\mathbf{\Lambda}$ is Connes' cyclic category. For this reason, we start this chapter in Section 6.1 with a discussion of the cyclic bar construction as well as the geometric realization of cyclic objects.

6.1 The cyclic bar construction and geometric realization of cyclic objects

In this section we discuss the *cyclic bar construction*. Given a presentable symmetric monoidal ∞-category $\mathcal{C}$, this is a (symmetric monoidal) functor

$$\mathrm{B}^{\mathrm{cyc}} \colon \operatorname{Alg}(\mathcal{C}) \to \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$$

that constructs an object in $\mathcal{C}$ with $\mathbb{T}$-action out of every (associative) algebra in $\mathcal{C}$.

The construction proceeds in two main steps. Starting with an algebra R in $\mathcal{C}$, one first constructs a *cyclic object* in $\mathcal{C}$, denoted by $\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)$, and also called the cyclic bar construction[1], which is a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$, where $\mathbf{\Lambda}$ is

[1] In fact, we will almost exclusively refer to *this* construction as the cyclic bar construction in the remainder of the text.

Connes' cyclic category. We will review $\mathbf{\Lambda}$ in Section 6.1.1, and define the symmetric monoidal functor

$$\mathrm{B}^{\mathrm{cyc}}_{\bullet}\colon \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

in Section 6.1.2.

Given a cyclic object X in $\mathcal{C}$, one can then take the *geometric realization* $|X|$, which yields an object in $\mathcal{C}$ with $\mathbb{T}$-action, as will be discussed in Section 6.1.3. The cyclic bar construction $\mathrm{B}^{\mathrm{cyc}}$ of an associative algebra R can then be defined as $\mathrm{B}^{\mathrm{cyc}}(R) \coloneqq |\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)|$.

As main references for the material below we use [NikSch], [Hoy18], and [Lod98].

6.1.1 Connes' cyclic category $\mathbf{\Lambda}$

In this section we discuss Connes' cyclic category $\mathbf{\Lambda}$, which has the simplex category $\mathbf{\Delta}$ as a subcategory and is mainly of interest because it encodes circle actions. More concretely, if $\mathcal{C}$ is a presentable ∞-category and $X\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$ a diagram, then the geometric realization (i. e. colimit) of the restriction of X to $\mathbf{\Delta}^{\mathrm{op}}$ naturally acquires the action of the circle group[2] $\mathbb{T}$, as we will see as Fact 6.1.3.6 in Section 6.1.3.2.

We will start by reviewing the two different approaches towards defining the simplex category $\mathbf{\Delta}$ (one via generators and relations, one more abstract) in Section 6.1.1.1, before discussing analogous definitions of the cyclic category $\mathbf{\Lambda}$ in Sections 6.1.1.2 and 6.1.1.3. We will show that the two definitions we give for $\mathbf{\Lambda}$ are equivalent in Section 6.1.1.4. Finally, we will introduce the notion of *cyclic objects* in Section 6.1.1.5 and describe the self-duality functor of $\mathbf{\Lambda}$ in Section 6.1.1.6, which will be relevant for the definition of the cyclic bar construction in Section 6.1.2.

6.1.1.1 The simplex category $\mathbf{\Delta}$

Recall that there are two approaches towards defining the simplex category $\mathbf{\Delta}$.

- $\mathbf{\Delta}$ can be defined as the category of totally ordered non-empty finite sets together with (weakly) order-preserving maps.
- $\mathbf{\Delta}$ can be constructed as the category with objects $[n]$ for $n \geq 0$ and morphisms generated by $\delta_i\colon [n-1] \to [n]$ (for $n \geq 1$ and $0 \leq i \leq n$) and $\sigma_i\colon [n+1] \to [n]$ (for $n \geq 0$ and $0 \leq i \leq n$) satisfying the *simplicial identities*[3].

[2] $\mathbb{T}$ was defined in Construction 5.2.1.1.

[3] They can be found for example in [Lod98, B.3] or [Mac98, Page 177]. See also Remark 6.1.1.8 below.

If we temporarily refer to the second definition as $\mathbf{\Delta}'$, then we can relate $\mathbf{\Delta}'$ and $\mathbf{\Delta}$ with a functor $\mathbf{\Delta}' \to \mathbf{\Delta}$ that can be described as follows.

- $[n]$ is mapped to the totally ordered set $\{0 < 1 < \cdots < n\}$.
- $\delta_i \colon [n-1] \to [n]$ is mapped to the injective order-preserving map that does not have i in the image.
- $\sigma_i \colon [n+1] \to [n]$ is mapped to the order-preserving map that is surjective and maps both i and $i+1$ to i.

This functor is an equivalence of categories, as shown in [Mac98, Proposition 2 on page 178][4]. We will thus usually identify $\mathbf{\Delta}$ and $\mathbf{\Delta}'$ and use whatever description is most appropriate for the occasion.

Notation 6.1.1.1. Let $\mathcal{C}$ be an ∞-category. A functor

$$X \colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$$

will be called a *simplicial object* in $\mathcal{C}$. We will write X_n instead of $X([n])$ and accordingly often also use $X_\bullet$ for X if we want to emphasize X being a simplicial object. We will refer to the morphism induced by the opposite of δ_i as d_i, and to the morphism induced by the opposite of σ_i as s_i. ♢

Completely analogously to the situation for the simplex category, there are two approaches to Connes' cyclic category $\mathbf{\Lambda}$. We will discuss an abstract definition first in Section 6.1.1.2 and then discuss a definition using generators and relations in Section 6.1.1.3, before showing that they are equivalent in Section 6.1.1.4.

6.1.1.2 Definition of $\mathbf{\Lambda}$ via posets

Definition 6.1.1.2 ([NikSch, page 380])**.** We denote by PoSet the category of partially ordered sets with (weakly) order preserving maps. We furthermore define

$$\mathbb{Z}\mathsf{PoSet} \coloneqq \mathrm{Fun}(\mathrm{B}\,\mathbb{Z}, \mathsf{PoSet})$$

to be the category of objects in PoSet with $\mathbb{Z}$-action.

An example for an object in $\mathbb{Z}\mathsf{PoSet}$ is $(1/n) \cdot \mathbb{Z}$ for $n \geq 1$; as a subset of $\mathbb{Q}$ this set inherits a partial order, and an integer k acts by addition.

We now define $\mathbf{\Lambda}_\infty$ to be the full subcategory of $\mathbb{Z}\mathsf{PoSet}$ spanned by the objects isomorphic to $(1/n) \cdot \mathbb{Z}$ for $n \geq 1$. The category $\mathbf{\Lambda}_\infty$ is called the *paracyclic category.* ♢

---

[4] What is referred to as $\mathbf{\Delta}$ in [Mac98] is not what we refer to as $\mathbf{\Delta}$, but also includes the empty set. What we refer to as $\mathbf{\Delta}$ is denoted by $\mathbf{\Delta}^+$ in [Mac98] and discussed in [Mac98, Bottom of page 178]. But while the statement of [Mac98, Proposition 2 on page 178] does not directly deal with our $\mathbf{\Delta}$, it nevertheless directly implies the result, as there are no maps from a non-empty set to an empty set.

Recall the equivalence

$$\mathcal{S}_*^{\geq 1} \underset{\mathrm{B}}{\overset{\beta_1}{\rightleftarrows}} \mathrm{Mon}^{\mathrm{gp}}_{\mathsf{Assoc}}(\mathcal{S})$$

from [HA, 5.2.6, in particular 5.2.6.10] that was discussed in Section 5.3. The functors β_1 and B induce mutually inverse equivalences on the respective ∞-categories of commutative monoids, so as $\mathbb{Z}$ is commutative $\mathrm{B}\,\mathbb{Z}$ acquires an induced commutative monoid structure. $\mathrm{B}\,\mathbb{Z}$ can in fact be identified, as an object of $\mathrm{CMon}(\mathcal{S}_*^{\geq 1})$, with the circle group $\mathbb{T}$ (see Construction 5.2.1.1). To see this it suffices to check that $\beta_1(\mathbb{T}) \simeq \mathbb{Z}$ as commutative monoids in $\mathcal{S}$, but as the underlying spaces are discrete this is just a classical exercise using the Eckmann-Hilton argument[5].

As $\mathbb{T}$ is path connected, it is grouplike as a monoid in $\mathcal{S}$ by [HA, 5.2.6.4], so we can form $\mathrm{B}\,\mathbb{T}$ and consider objects with $\mathbb{T}$-action in some ∞-category $\mathcal{D}$, i. e. functors $\mathrm{B}\,\mathbb{T} \to \mathcal{D}$ – see the introduction to Section 5.3. The ∞-groupoid $\mathrm{B}\,\mathbb{T} \simeq \mathrm{B}\,\mathrm{B}\,\mathbb{Z}$ can be interpreted as the ∞-groupoid with a unique object $*$, unique morphism, and with $\mathbb{Z}$ being the space of 2-morphisms $\mathrm{id}_* \to \mathrm{id}_*$. A $\mathbb{T} \simeq \mathrm{B}\,\mathbb{Z}$-action on an ∞-category $\mathcal{C}$, i. e. a functor $\mathrm{B}\,\mathbb{T} \to \mathcal{C}\mathrm{at}_\infty$ mapping $*$ to $\mathcal{C}$, then essentially consists of a natural equivalence $\mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ corresponding to the generator 1 of $\mathbb{Z}$.

If $\mathcal{C} = \mathsf{C}$ is a 1-category, then this amounts to giving an automorphism $\varphi_X \colon X \to X$ for every object X of C in such a way that these automorphisms are compatible with every morphism of C, i. e. for every morphism $f \colon X \to Y$ of C it must hold $\varphi_Y \circ f = f \circ \varphi_X$. This data is in turn equivalent to a natural action of $\mathbb{Z}$ on the morphism sets of C: We can let n act on $\mathrm{Mor}_{\mathsf{C}}(X, Y)$ by $\varphi_Y^n \circ -$. If instead we have a natural action of $\mathbb{Z}$ on the morphism sets given, then we can recover the automorphisms φ_X as the result of letting 1 act on the element id_X in $\mathrm{Mor}_{\mathsf{C}}(X, X)$.

We can now state the definition of the cyclic category $\mathbf{\Lambda}$ as it is given in [NikSch, page 380].

Definition 6.1.1.3 ([NikSch, page 380])**.** There is an action of $\mathbb{Z}$ on the morphisms spaces of $\mathbf{\Lambda}_\infty$ such that the action of an integer k on a morphism f yields the morphism $f(-) + k = f(- + k)$.

Dividing out this action, i. e. identifying a morphism f with $f + k$ for any integer k, we obtain a category that we denote by $\mathbf{\Lambda}$ and call *Connes' cyclic category.* ♢

Notation 6.1.1.4. We will use the notation $[n]_{\mathbf{\Lambda}}$ for $(1/(n+1)) \cdot \mathbb{Z}$ when considered as an object in $\mathbf{\Lambda}$ as described in Definition 6.1.1.2. Up to isomorphism, the objects of $\mathbf{\Lambda}$ are thus given by $[n]_{\mathbf{\Lambda}}$ for $n \geq 0$. ♢

[5] The underlying space of $\beta_1(\mathbb{T})$ is $\Omega\mathbb{T}$. This loop space has two monoid structures – an associative via composition of loops, and a commutative one via pointwise multiplication using the commutative monoid structure on $\mathbb{T}$.

Warning 6.1.1.5. Notation 6.1.1.4 deviates from the notation in [NikSch], where $[n]_{\mathbf{\Lambda}}$ is defined to be $1/n \cdot \mathbb{Z}$.

The notation we use is chosen to be more consistent with the notation used for objects of $\mathbf{\Delta}$ – it also matches the notation used in [Lod98], see [Lod98, 6.1.1]. ♢

The category $\mathbf{\Lambda}$ contains $\mathbf{\Delta}$ as a subcategory, as we note next.

Construction 6.1.1.6 ([NikSch, page 382])**.** Consider $\mathbf{\Delta}$ as the category of totally ordered non-empty finite sets. We can then define a functor

$$\mathbf{\Delta} \to \mathbb{Z}\mathsf{PoSet}$$

by mapping a totally ordered non-empty finite set S to $\mathbb{Z} \times S$, equipped with the lexicographic order and action by $\mathbb{Z}$ via addition on the first component. If $S = \{s_0 < s_1 < \cdots < s_n\}$, then there is an isomorphism $\mathbb{Z} \times S \cong (1/(n+1)) \cdot \mathbb{Z}$ in $\mathbb{Z}\mathsf{PoSet}$ that maps (k, s_i) to $k + (i/(n+1))$, so the functor factors through $\mathbf{\Lambda}_\infty$.

Following [NikSch, page 382], we will denote the resulting functor $\mathbf{\Delta} \to \mathbf{\Lambda}_\infty$ by j_∞^{op}[6] and the composition

$$\mathbf{\Delta} \to \mathbf{\Lambda}_\infty \to \mathbf{\Lambda}$$

by j^{op}. It is not difficult to check that j_∞^{op} and j^{op} are faithful and induce bijections on isomorphism classes of objects. ♢

6.1.1.3 Definition of $\mathbf{\Lambda}$ via generators and relations

We now describe $\mathbf{\Lambda}$ with generators and relations.

Construction 6.1.1.7 ([Lod98, 6.1.1])**.** We define the 1-category $\mathbf{\Lambda}'$ to have objects $[n]_{\mathbf{\Lambda}'}$ for integers $n \geq 0$, and morphisms generated by

$\delta_i : [n-1]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}$	for $n \geq 1$ and $0 \leq i \leq n$
$\sigma_i : [n+1]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}$	for $n \geq 0$ and $0 \leq i \leq n$
$\tau : [n]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}$	for $n \geq 0$

subject to the following relations[7].

$\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$	for $i < j$
$\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$	for $i \leq j$
$\sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1}$	for $i < j$
$\sigma_j \circ \delta_i = \mathrm{id}$	for $i = j$ or $i = j+1$
$\sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j$	for $i > j+1$
$\tau \circ \delta_i = \delta_{i-1} \circ \tau$	for $i > 0$

[6] The reason for the $^{\mathrm{op}}$ is that the opposite of this functor is more important (or at least more often used) and hence gets to have the name with least decorations.

$$\begin{aligned}
&\tau \circ \delta_0 = \delta_n && \text{where } \tau\colon [n]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}\\
&\tau \circ \sigma_i = \sigma_{i-1} \circ \tau && \text{for } i > 0\\
&\tau \circ \sigma_0 = \sigma_n \circ \tau^2 && \text{where } \sigma_0\colon [n+1]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}\\
&\tau^{n+1} = \mathrm{id}_{[n]_{\mathbf{\Lambda}'}} && \text{where } \tau\colon [n]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}
\end{aligned}$$
♢

Remark 6.1.1.8. If we remove the morphisms τ as generators in Construction 6.1.1.7 (as well as the relations involving them), then we obtain precisely the definition of $\mathbf{\Delta}$ via generators and relations. We thus obtain a functor $j'^{\mathrm{op}}\colon \mathbf{\Delta} \to \mathbf{\Lambda}'$. ♢

6.1.1.4 Comparison of the two definitions of $\mathbf{\Lambda}$

To show that $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ are equivalent, we first construct a comparison functor.

Proposition 6.1.1.9. *There is a functor $\Phi\colon \mathbf{\Lambda}' \to \mathbf{\Lambda}$ defined as follows.*

- $[n]_{\mathbf{\Lambda}'}$ *is mapped to to* $[n]_{\mathbf{\Lambda}}$.
- $\delta_i\colon [n-1]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}$ *is mapped to the unique morphism that sends* 0 *to* 0 *and has* $\frac{0}{n+1}, \ldots, \frac{i-1}{n+1}, \frac{i+1}{n+1}, \ldots, \frac{n}{n+1}$ *in its image.*
- $\sigma_i\colon [n+1]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}$ *is mapped to the unique morphism that sends* 0 *to* 0, *is surjective, and sends* $\frac{i}{n+2}$ *and* $\frac{i+1}{n+2}$ *to* $\frac{i}{n+1}$.
- $\tau\colon [n]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}$ *is mapped to the unique morphism that is surjective and sends* $\frac{1}{n+1}$ *to* $\frac{0}{n+1}$.

Furthermore, this functor fits into a commutative square

$$\begin{tikzcd}
\mathbf{\Delta}' \arrow[r, "j'^{\mathrm{op}}"] \arrow[d, "\Phi_{\mathbf{\Delta}}"'] & \mathbf{\Lambda}' \arrow[d, "\Phi"] \\
\mathbf{\Delta} \arrow[r, "j^{\mathrm{op}}"'] & \mathbf{\Lambda}
\end{tikzcd}$$

that commutes up to natural isomorphism $\varphi\colon j^{\mathrm{op}} \circ \Phi_{\mathbf{\Delta}} \to \Phi \circ j'^{\mathrm{op}}$ where $\Phi_{\mathbf{\Delta}}$ is the equivalence from Section 6.1.1.1, and j'^{op} and j^{op} are as in Remark 6.1.1.8 and Construction 6.1.1.6. The components of the natural isomorphism φ are to be the isomorphisms

$$\varphi_{[n]}\colon j^{\mathrm{op}}([n]) = \mathbb{Z} \times [n] \xrightarrow{\cong} (1/(n+1)) \cdot \mathbb{Z} = [n]_{\mathbf{\Lambda}}$$

that were discussed in Construction 6.1.1.6. ♡

[7] As we do not specify the n as part of the notation of the three types of morphisms, notation like δ_i refers to more than a single morphism. The relations below are to be satisfied for all choices where the morphisms can be composed as indicated and both sides of the equation have same domain and codomain.

Proof. Easy but a bit tedious exercise checking the relations. □

For both $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ one can show that morphisms decompose uniquely as the composition of a power of τ with a morphism in the image of the inclusion of $\mathbf{\Delta}$, as we will see next. This is what will imply that the functor $\mathbf{\Lambda}' \to \mathbf{\Lambda}$ from Proposition 6.1.1.9 is an equivalence.

Proposition 6.1.1.10. *Let $f\colon [n]_{\mathbf{\Lambda}'} \to [m]_{\mathbf{\Lambda}'}$ be a morphism in $\mathbf{\Lambda}'$. Then there exists a unique morphism $g\colon [n] \to [m]$ in $\mathbf{\Delta}$ and integer k with $0 \leq k \leq n$ such that $f = j'^{\mathrm{op}}(f) \circ \tau^k$.*

An analogous statement also holds for $\mathbf{\Lambda}$. Let $f\colon [n]_{\mathbf{\Lambda}} \to [m]_{\mathbf{\Lambda}}$ be a morphism in $\mathbf{\Lambda}$. Then there is a unique morphism $g\colon [n] \to [m]$ in $\mathbf{\Delta}$ and integer k with $0 \leq k \leq n$ such that $f = \varphi_{[m]} \circ j^{\mathrm{op}}(f) \circ \varphi_{[n]}^{-1} \circ \Phi(\tau)^k$, where we use notation from Proposition 6.1.1.9. ♡

Proof. The statement for $\mathbf{\Lambda}'$ is precisely [Lod98, 6.1.3].

For $\mathbf{\Lambda}$ note that there is a unique $0 \leq k \leq n$ and morphism $f'\colon [n]_{\mathbf{\Lambda}} \to [m]_{\mathbf{\Lambda}}$ such that $f = f' \circ \Phi(\tau)^k$ and such that f' maps $\mathbb{Z}$ to $\mathbb{Z}$. The claim now follows from the observation that a morphism $\mathbb{Z} \times [n] \to \mathbb{Z} \times [m]$ in $\mathbf{\Lambda}_\infty$ that maps $(0,0)$ to $(0,0)$ must be of the form $\mathrm{id}_{\mathbb{Z}} \times g$ for a unique morphism $g\colon [n] \to [m]$ in $\mathbf{\Delta}$. □

Corollary 6.1.1.11. *The functor Φ from Proposition 6.1.1.9 is an equivalence.* ♡

Proof. Φ is by definition essentially surjective. That Φ is also fully faithful follows immediately from Proposition 6.1.1.10. □

We will from now on not distinguish between $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ and use the description best adapted for each individual situation.

6.1.1.5 Cyclic objects

Notation 6.1.1.12 ([Lod98, 6.1.2.1])**.** Let $\mathcal{C}$ be an ∞-category. We call a functor

$$X\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$$

a *cyclic object* in $\mathcal{C}$. We will use the same notational conventions as explained in Notation 6.1.1.1 for simplicial objects, and will refer to the image of $[n]_{\mathbf{\Lambda}}$ under X as X_n (and sometimes write $X_\bullet$ for X), to the morphism induced by the opposite of δ_i as d_i, to the morphism induced by the opposite of σ_i as s_i, and to the morphism induced by the opposite of τ as t. ♢

6.1.1.6 Self-duality of $\mathbf{\Lambda}$

We record that $\mathbf{\Lambda}$ has a self-duality functor, which will be needed later.

Fact 6.1.1.13 ([Lod98, 6.1.11])**.** *There is an equivalence $-^{\circ}\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathbf{\Lambda}$ that maps*

- $[n]_{\mathbf{\Lambda}}$ *to* $[n]_{\mathbf{\Lambda}}$,
- δ_i^{op} *to* σ_i,
- σ_i^{op} *to* δ_{i+1},
- τ^{op} *to* τ^{-1},

where $\sigma_{n+1}\colon [n+1]_{\mathbf{\Lambda}} \to [n]_{\mathbf{\Lambda}}$ *is what is called the* extra degeneracy *defined as* $\sigma_{n+1} = \sigma_0 \circ \tau^{-1}$. $\clubsuit$

The above is also proven in [NikSch, page 381] using the definition of $\mathbf{\Lambda}$ via posets[8], and one can check that the two functors agree by unpacking the definitions.

6.1.2 The cyclic bar construction as a cyclic object

In this section we discuss the cyclic bar construction of associative algebras. Let C be a symmetric monoidal 1-category and A an associative algebra in C. Then one can construct a simplicial object in C

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A\otimes A\otimes A \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A\otimes A \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[r, shift left=1] \arrow[r, shift right=1] & A \arrow[l] \end{tikzcd}$$

where the structure morphisms $d_i\colon A^{\otimes n} \to A^{\otimes(n-1)}$ and $s_i\colon A^{\otimes n} \to A^{\otimes(n+1)}$ can be described as follows[9]:

1. If $i \le n-2$, then d_i is $\mathrm{id}_A^{\otimes i} \otimes \mu \otimes \mathrm{id}_A^{\otimes(n-2-i)}$, where $\mu\colon A\otimes A \to A$ is the multiplication morphism.
2. d_{n-1} is the postcomposition of the symmetry isomorphism that brings the last tensor factor to the front with $\mu \otimes \mathrm{id}_A^{\otimes(n-2)}$.
3. s_i is $\mathrm{id}_A^{i+1} \otimes \iota \otimes \mathrm{id}_A^{\otimes(n-i-1)}$, where $\iota\colon \mathbb{1}_{\mathsf{C}} \to A$ is the unit morphism.

Making use of cyclic permutations of the tensor factors, we can even extend the above simplicial object to a cyclic object

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A\otimes A\otimes A \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[loop above] & A\otimes A \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[loop above] & A \arrow[l] \arrow[loop above] \end{tikzcd}$$

where the structure morphism $t\colon A^{\otimes n} \to A^{\otimes n}$ is the symmetry isomorphism moving the last tensor factor to the front.

[8] Whereas [Lod98, 6.1.11] uses the definition via generators and relations.

[9] We omit making explicit any associativity or unitality isomorphisms from the symmetric monoidal structure on C.

The goal of this section is to rigorously define a cyclic object implementing this idea for associative algebras in any symmetric monoidal ∞-category $\mathcal{C}$. Furthermore, we will also show that the resulting functor

$$\mathrm{B}_\bullet^{\mathrm{cyc}}\colon \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

can be upgraded to a symmetric monoidal functor, where $\mathrm{Alg}(\mathcal{C})$ carries the induced symmetric monoidal structure from Proposition E.4.2.3 and $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ the pointwise symmetric monoidal structure from [HA, 2.1.3.4].

$\mathrm{B}_\bullet^{\mathrm{cyc}}$ will be defined as a composition

$$\begin{aligned}\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) &\xrightarrow{A} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)\\ &\xrightarrow{(\otimes)_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)\\ &\xrightarrow{(V\circ(-^{\circ}))^*} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})\end{aligned}$$

and we will define individual ingredients one by one[10].

Let us now give a brief overview over the subsections below. We will start in Section 6.1.2.1 by discussing the symmetric monoidal envelope of an ∞-operad, which will explain what symmetric monoidal structure we consider on $\mathcal{C}^{\otimes}_{\mathrm{act}}$. In Section 6.1.2.2 we will then construct the first row (i. e. the first two functors) in the composition above that will define $\mathrm{B}_\bullet^{\mathrm{cyc}}$, and show that the composition of those two functor is lax symmetric monoidal. We will then define the symmetric monoidal functor $\otimes\colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ in Section 6.1.2.3 and show that the composition of the lax symmetric monoidal functor

$$\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)$$

from Section 6.1.2.2 with the symmetric monoidal functor $(\otimes)_*$ is not just lax symmetric monoidal, but symmetric monoidal. For the last step in the definition of $\mathrm{B}_\bullet^{\mathrm{cyc}}$, we have already defined the functor $(-)^{\circ}$, in Fact 6.1.1.13, and will define the remaining functor $V\colon \mathbf{\Lambda} \to \mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ in Section 6.1.2.4. This will be the last ingredient that we need to define $\mathrm{B}_\bullet^{\mathrm{cyc}}$, and we will put everything together in Section 6.1.2.5. We will end this section by giving a more direct description for $\mathrm{CAlg}(\mathrm{B}_\bullet^{\mathrm{cyc}})$, the functor induced by $\mathrm{B}_\bullet^{\mathrm{cyc}}$ on commutative algebras, in Section 6.1.2.6, and showing that $\mathrm{B}_\bullet^{\mathrm{cyc}}$ preserves sifted colimits in Section 6.1.2.7.

6.1.2.1 The symmetric monoidal envelope

Let $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. In [HA, 2.2.4] what is called the *symmetric monoidal envelope* of $\mathcal{O}$ is discussed[11], which is defined in [HA,

[10] We warn though that while $\mathrm{B}_\bullet^{\mathrm{cyc}}$ will be shown to be symmetric monoidal, we do not claim that the individual functors in the above composition are symmetric monoidal functors of symmetric monoidal ∞-categories.

[11] The definitions in [HA, 2.2.4] are more general, but we only need the symmetric monoidal case.

2.2.4.1] as

$$\mathrm{Env}(\mathcal{O})^{\otimes} := \mathcal{O}^{\otimes} \times_{\mathsf{Fin}_*} \mathrm{Act}(\mathsf{Fin}_*) \tag{6.1}$$

where the functor $\mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ is given by $p_{\mathcal{O}}$, the ∞-category $\mathrm{Act}(\mathsf{Fin}_*)$ is defined as the full subcategory of $\mathrm{Fun}([1], \mathsf{Fin}_*)$ spanned by the active morphisms[12], and the functor $\mathrm{Act}(\mathsf{Fin}_*) \to \mathsf{Fin}_*$ is ev_0.

Like [NikSch, page 366] and [HA, 2.2.4.3], we will use the notation $\mathcal{O}^{\otimes}_{\mathrm{act}}$ to refer to the subcategory of $\mathcal{O}^{\otimes}$ spanned by all objects and the active morphisms[13], i. e. those morphisms mapped by $p_{\mathcal{O}}$ to an active morphism in Fin_*. Note that the inclusion

$$\mathcal{O}^{\otimes}_{\mathrm{act}} \to \mathcal{O}^{\otimes}$$

can be identified with the functor

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times_{\mathsf{Fin}_*} (\mathsf{Fin}_*)_{\mathrm{act}} \arrow[r] \arrow[rr, bend left, "\mathrm{pr}_1"] & \mathcal{O}^{\otimes} \times_{\mathsf{Fin}_*} \mathsf{Fin}_* \arrow[r, "\simeq", "\mathrm{pr}_1"'] & \mathcal{O}^{\otimes}
\end{tikzcd}$$

where the left functor is the one induced by the inclusion $(\mathsf{Fin}_*)_{\mathrm{act}} \to \mathsf{Fin}_*$ – this follows from Proposition B.5.2.1 and Proposition B.4.3.1, see also Remark B.6.0.1.

Let $p_{\mathrm{Env}(\mathcal{O})} \colon \mathrm{Env}(\mathcal{O})^{\otimes} \to \mathsf{Fin}_*$ be defined as $\mathrm{ev}_1 \circ \mathrm{pr}_2$. Unpacking the definition of $\mathrm{Env}(\mathcal{O})^{\otimes}$, we can then interpret an object lying over $\langle n \rangle$ as a pair (O, α) with O an object of $\mathcal{O}^{\otimes}$ and α an active morphism $p_{\mathcal{O}}(O) \to \langle n \rangle$ in Fin_* – see [HA, 2.2.4.2]. In particular, as there is a unique active morphism from any object of Fin_* to $\langle 1 \rangle$, one can identify $\mathrm{Env}(\mathcal{O})^{\otimes}_{\langle 1 \rangle}$ with $\mathcal{O}^{\otimes}_{\mathrm{act}}$ – see [HA, 2.2.4.3].

One important result about $\mathrm{Env}(\mathcal{O})$ that we will need is the following.

Fact 6.1.2.1 ([HA, 2.2.4.4 and 2.2.4.15])**.** *Let $p_{\mathcal{O}} \colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. Then $p_{\mathrm{Env}(\mathcal{O})} \colon \mathrm{Env}(\mathcal{O})^{\otimes} \to \mathsf{Fin}_*$ is a cocartesian fibration of ∞-operads, i. e. exhibits $\mathcal{O}^{\otimes}_{\mathrm{act}}$ as a symmetric monoidal ∞-category.*

Furthermore, a morphism in $\mathrm{Env}(\mathcal{O})^{\otimes}$ is $p_{\mathrm{Env}(\mathcal{O})}$-cocartesian if and only if pr_1 maps that morphism to an inert morphism in $\mathcal{O}^{\otimes}$. ♧

Let us describe $p_{\mathrm{Env}(\mathcal{O})}$-cocartesian lifts a bit more concretely. Let O be an object of $\mathcal{O}^{\otimes}$, $\alpha \colon \langle n \rangle \to \langle m \rangle$ an active morphism in Fin_*, and consider (O, α) as an object of $\mathrm{Env}(\mathcal{O})^{\otimes}_{\langle m \rangle}$. Let $\beta \colon \langle m \rangle \to \langle k \rangle$ be a morphism of Fin_*. Then we can factor $\beta \circ \alpha$ as a composition of an inert morphism $\gamma \colon \langle n \rangle \to \langle l \rangle$ and an active morphism $\delta \colon \langle l \rangle \to \langle k \rangle$ in a unique way, see [HA, 2.1.2.2]. We can

[12] So those morphisms for which the preimage of $*$ has a single element, see [HA, 2.1.2.1].
[13] See [HA, 2.1.2.1 and 2.1.2.3] for a definition.

then interpret the commutative diagram

$$\begin{tikzcd} \langle n\rangle \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & \langle l\rangle \arrow[d, "\delta"] \\ \langle m\rangle \arrow[r, "\beta"'] & \langle k\rangle \end{tikzcd} \tag{6.2}$$

as a morphism from α to δ in $\mathrm{Act}(\mathsf{Fin}_*)$. Let $\overline{\gamma}\colon O \to O'$ be a $p_{\mathcal{O}}$-cocartesian lift of γ. Then $\overline{\gamma}$ together with (6.2) determine a $p_{\mathrm{Env}(\mathcal{O})}$-cocartesian morphism

$$(O,\alpha) \to (O',\delta)$$

in $\mathrm{Env}(\mathcal{O})$ lying over β. One implication of this discussion is that if O and O' are two objects of $\mathcal{O}^{\otimes}_{\mathrm{act}}$, then their tensor product is given by $O \oplus O'$, see also [HA, 2.2.4.6]. The monoidal unit of $\mathcal{O}^{\otimes}_{\mathrm{act}}$ is given by the essentially unique object in $\mathcal{O}^{\otimes}_{\langle 0\rangle}$.

The identity functor of $\mathcal{O}^{\otimes}$ together with the functor $\mathcal{O}^{\otimes} \to \mathrm{Act}(\mathsf{Fin}_*)$ that maps an object O to the active morphism $\mathrm{id}_{p_{\mathcal{O}}(O)}$[14] define a functor[15] $\mathcal{O}^{\otimes} \to \mathrm{Env}(\mathcal{O})^{\otimes}$ over Fin_*. Using Fact 6.1.2.1 it follows immediately that this functor is a morphism of ∞-operads. We are now ready to state the crucial result concerning $\mathrm{Env}(\mathcal{O})^{\otimes}$.

Fact 6.1.2.2 ([HA, 2.2.4.9])**.** *Let $\mathcal{O} \to \mathsf{Fin}_*$ be an ∞-operad and $\mathcal{D}$ a symmetric monoidal ∞-category. Then restriction along the functor $\mathcal{O}^{\otimes} \to \mathrm{Env}(\mathcal{O})^{\otimes}$ discussed above induces an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}),\mathcal{D}) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$$

between the ∞-category of symmetric monoidal functors $\mathrm{Env}(\mathcal{O}) \to \mathcal{D}$ and the ∞-category of morphisms of ∞-operads $\mathcal{O} \to \mathcal{D}$. ♧

Remark 6.1.2.3. Let $\alpha\colon \mathcal{O}' \to \mathcal{O}$ be a morphism of ∞-operads and let $G\colon \mathcal{D} \to \mathcal{D}'$ be a symmetric monoidal functor between symmetric monoidal ∞-categories.

It follows from Fact 6.1.2.1 that the morphism of ∞-categories α induces a symmetric monoidal functor

$$\mathrm{Env}(\alpha)\colon \mathrm{Env}(\mathcal{O}') \to \mathrm{Env}(\mathcal{O})$$

fitting into a commutative square of morphisms of ∞-operads as in the left of the following diagram, where the left horizontal functors are the morphisms

[14] More rigorously, we consider the functor $(p_{\mathcal{O}})_* \circ \mathrm{const}\colon \mathcal{O}^{\otimes} \to \mathrm{Fun}([1],\mathsf{Fin}_*)$ that is adjoint to the composition

$$[1] \times \mathcal{O}^{\otimes} \xrightarrow{\mathrm{pr}_2} \mathcal{O}^{\otimes} \xrightarrow{p_{\mathcal{O}}} \mathsf{Fin}_*$$

and remark that it factors through $\mathrm{Act}(\mathsf{Fin}_*)$.

[15] This functor is also discussed in [HA, Before 2.2.4.9].

of ∞-operads discussed above.

$$\begin{tikzcd} \mathcal{O} \arrow[r] \arrow[rr, bend left, "F"] & \mathrm{Env}(\mathcal{O}) \arrow[r, "\widetilde{F}"'] & \mathcal{D} \arrow[d, "G"] \\ \mathcal{O}' \arrow[u, "\alpha"] \arrow[r] & \mathrm{Env}(\mathcal{O}') \arrow[u, "\mathrm{Env}(\alpha)"'] & \mathcal{D}' \end{tikzcd}$$

The symmetric monoidal functor $\widetilde{F}$ in the above diagram is to be the one corresponding to F via the equivalence from Fact 6.1.2.2, i. e. making the triangle at the top commute.

It then follows from commutativity of the above diagram and Fact 6.1.2.2 that there is an equivalence

$$(\widetilde{G \circ F \circ \alpha}) \simeq G \circ \widetilde{F} \circ \mathrm{Env}(\alpha)$$

where $(\widetilde{G \circ F \circ \alpha})$ is the symmetric monoidal functor $\mathrm{Env}(\mathcal{O}') \to \mathcal{D}'$ corresponding to $G \circ F \circ \alpha$ under the equivalence of Fact 6.1.2.2. ♢

6.1.2.2 From associative algebras to active diagrams

Let us denote by $p_{\mathsf{Assoc}}\colon \mathsf{Assoc}^{\otimes} \to \mathsf{Fin}_*$ the canonical morphism of ∞-operads and let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category. Recall from Proposition E.4.2.3 that $\mathrm{Alg}(\mathcal{C})$ inherits an induced symmetric monoidal structure $p_{\mathrm{Alg}(\mathcal{C})}\colon\ \mathrm{Alg}(\mathcal{C})^{\otimes} \to \mathsf{Fin}_*$. This comes with a canonical inclusion

$$\iota_{\mathrm{Alg}}\colon\ \mathrm{Alg}(\mathcal{C})^{\otimes} \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \tag{6.3}$$

where the functors with respect to which the pullback is taken are $(p_{\mathcal{C}})_*$ and the functor[16] adjoint to $\mathsf{Fin}_* \times \mathsf{Assoc}^{\otimes} \xrightarrow{\mathrm{id}_{\mathsf{Fin}_*} \times p_{\mathsf{Assoc}}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{- \wedge -} \mathsf{Fin}_*$. The functor $p_{\mathrm{Alg}(\mathcal{C})}$ is then given by the composition $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$.

The functor ι_{Alg} will be the first step in the definition of the symmetric monoidal functor $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$.

We next recall that the pointwise symmetric monoidal structure on

$$\mathrm{Fun}(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}})$$

is given by the cocartesian fibration of ∞-operads

$$\begin{aligned} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes} = \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \\ \xrightarrow{\mathrm{pr}_2} \mathsf{Fin}_* \end{aligned} \tag{6.4}$$

[16]See also Proposition E.6.0.1.

that exhibits $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}})$ as a symmetric monoidal ∞-category, where the pullback is formed with respect to the functors $(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}})_*$ and the functor const[17].

We are now ready to construct a functor

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes}$$

over Fin_* whose composition with ι_{Alg} will be a lax symmetric monoidal functor. To be able to understand what this functor does it will later turn out to be helpful to additionally construct a certain natural transformation $\mu\colon A^{\mathrm{const}} \to \mathrm{pr}_1 \circ A^{\otimes}$.

Construction 6.1.2.4. Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category, and let us use notation as above. We will construct a functor

$$A^{\otimes}\colon\ \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes}$$

over Fin_*, as well as a functor

$$A^{\mathrm{const}}\colon\ \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\big)$$

together with a natural transformation[18] $\mu\colon A^{\mathrm{const}} \to \mathrm{pr}_1 \circ A^{\otimes}$ such that the natural transformation[19] $(\mathrm{pr}_1)_* \circ \mu$ is a natural equivalence. The names μ, A^{const} and $A^{\otimes}$ will only be used where we directly refer to this construction. The letter A has been chosen as a reference to the word *active*, and $A^{\otimes}$ has the superscript $\otimes$ as its composition with ι_{Alg} will be shown in Proposition 6.1.2.5 below to be a morphism of ∞-operads, whereas A^{const} is not even a functor over Fin_*. The reason why A^{const} has superscript const and the natural transformation is called μ will become clear during the construction. We will later also use the notation $A^{\otimes}$ for the functor obtained by composing $A^{\otimes}$ as constructed here with ι_{Alg}, see Proposition 6.1.2.5.

By the definition[20] of $\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes}$ and the universal property of pullbacks, constructing $A^{\otimes}$, A^{const}, and μ as stated above is equivalent to

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[17] In other words the functor adjoint to $\mathrm{pr}_1\colon \mathsf{Fin}_* \times \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \to \mathsf{Fin}_*$.

[18] The functor pr_1 appearing in $\mathrm{pr}_1 \circ A^{\otimes}$ is the following functor.

$$\begin{aligned}\mathrm{Fun}\Big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\Big)^{\otimes} &= \mathrm{Fun}\Big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\Big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \\ &\xrightarrow{\mathrm{pr}_1} \mathrm{Fun}\Big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\Big)\end{aligned}$$

See (6.4).

[19] The functor pr_1 appearing in $(\mathrm{pr}_1)_*$ is the following functor.

$$(\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathrm{Act}(\mathsf{Fin}_*) \xrightarrow{\mathrm{pr}_1} \mathcal{C}^{\otimes}$$

See Section 6.1.2.1 and in particular (6.1).

[20] See the introduction of Section 6.1.2.2.

constructing a diagram as follows

$$\begin{tikzcd}
\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \arrow[r, "\mathrm{pr}_2"] \arrow[d, dashed, bend right, "A^{\mathrm{const}}"'] \arrow[d, dashed, bend left, "A'"] \arrow[d, phantom, "\overset{\mu}{\Rightarrow}"] & \mathsf{Fin}_* \arrow[d, "\mathrm{const}"] \\
\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\big) \arrow[r, "\left(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}\right)_*"'] & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\big)
\end{tikzcd}$$

where the ∞-category in the upper left is the pullback from (6.3), the two functors on the right and bottom are as explained around (6.4), and the square on the right[21] is to be a commutative square, while μ is a natural transformation from A^{const} to A' such that $(\mathrm{pr}_1)_* \circ \mu$ is a natural equivalence. Using the $\times$-Fun-adjunction and plugging in the definition of the symmetric monoidal envelope $\mathrm{Env}(\mathcal{C})^{\otimes} = (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}$ from (6.1) this is in turn equivalent to constructing a diagram

$$\begin{tikzcd}
\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \arrow[r, "\mathrm{id}_{\mathsf{Assoc}^{\otimes}_{\mathrm{act}}} \times \mathrm{pr}_2"] \arrow[d, dashed, bend right, "A''^{\mathrm{const}}"'] \arrow[d, dashed, bend left, "A''"] \arrow[d, phantom, "\overset{\mu''}{\Rightarrow}"] & \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathsf{Fin}_* \arrow[d, "\mathrm{pr}_2"] \\
\mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathrm{Act}(\mathsf{Fin}_*) \arrow[r, "p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}"'] & \mathsf{Fin}_*
\end{tikzcd} \tag{6.5}$$

where again the square is to come with a filler exhibiting it as a commutative square, while μ'' is merely a natural transformation such that $\mathrm{pr}_1 \circ \mu''$ is a natural equivalence.

As the composition from the top left along the top right to the bottom right is the projection to the last factor and using the definition of $p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}$ as $\mathrm{ev}_1 \circ \mathrm{pr}_2$, we can finally unpack this to see that we need to construct the following.

[21] So involving A', but not A^{const}.

(1) A commutative diagram as follows.

$$\begin{tikzcd}[column sep=small] & \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \arrow[dl, dashed, "A''_l"'] \arrow[dr, dashed, "A''_r"] & \\ \mathcal{C}^{\otimes} \arrow[dr, "p_{\mathcal{C}}"'] & & \mathrm{Act}(\mathsf{Fin}_*) \arrow[dl, "\mathrm{ev}_0"] \\ & \mathsf{Fin}_* & \end{tikzcd} \tag{6.6}$$

This diagram will then encode the functor A'' from (6.5).

(2) A natural transformation

$$\mu''_r \colon A''^{\,\mathrm{const}}_r \to A''_r$$

such that $\mathrm{ev}_0 \circ \mu''_r$ is an equivalence. Together with A''_l and the filler of the commutative diagram (6.6) this encodes a natural transformation $\mu'' \colon A''^{\,\mathrm{const}} \to A''$ such that $\mathrm{pr}_1 \circ \mu''$ can be identified with $\mathrm{id}_{A''_l}$.

(3) A natural equivalence $\mathrm{ev}_1 \circ A''_r \simeq \mathrm{pr}_3$, which then encodes a filler for the right square in (6.5).

Construction of A''_l: We start by giving a definition of A''_l. This is to be the composition

$$\begin{gathered} \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \\ \xrightarrow{\mathrm{pr}_1 \times \mathrm{pr}_2} \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \xrightarrow{\mathrm{ev}} \mathcal{C}^{\otimes} \end{gathered}$$

that maps a tuple $(\langle m \rangle, F, \langle n \rangle)$ to $F(\langle m \rangle)$, which will be an object in $\mathcal{C}^{\otimes}_{\langle n \rangle \wedge \langle m \rangle}$, as we will see properly next. Indeed, the equivalences[22]

$$\begin{aligned} p_{\mathcal{C}} \circ A''_l &= p_{\mathcal{C}} \circ \mathrm{ev} \circ (\mathrm{pr}_1 \times \mathrm{pr}_2) \\ &\simeq \mathrm{ev} \circ (\mathrm{pr}_1 \times (p_{\mathcal{C}})_*) \circ (\mathrm{pr}_1 \times \mathrm{pr}_2) \\ &\simeq \mathrm{ev} \circ (\mathrm{pr}_1 \times ((p_{\mathcal{C}})_* \circ \mathrm{pr}_2)) \\ &\simeq \mathrm{ev} \circ \Big(\mathrm{pr}_1 \times \Big((\widehat{\mathrm{id}_{\mathsf{Fin}_*} \wedge p_{\mathsf{Assoc}}}) \circ \mathrm{pr}_3\Big)\Big) \\ &\simeq \mathrm{pr}_3 \wedge (p_{\mathsf{Assoc}} \circ \mathrm{pr}_1) \end{aligned}$$

[22] From the first to the second line we use functoriality of evaluation, from the second to the third functoriality of products of functors, from the third to the fourth the equivalence that is part of the data of the pullback over $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)$, and from the fourth to the fifth the $\times$-Fun-adjunction and functoriality.

allows us to identify the composition $p_{\mathcal{C}} \circ A_l''$ with the functor that can be informally described as mapping a tuple $(\langle m\rangle, F, \langle n\rangle)$ to $\langle n\rangle \wedge \langle m\rangle$.

Construction of μ_r'': Let us now think about the functor A_r''. The constraints imposed by (1) and (3) imply that A_r'' needs to map a tuple $(\langle m\rangle, F, \langle n\rangle)$ to an active morphism $\langle n\rangle \wedge \langle m\rangle \to \langle n\rangle$. The idea is to use the active morphism

$$\langle n\rangle \wedge \langle m\rangle \xrightarrow{\mathrm{id}_{\langle n\rangle} \wedge \mu_m} \langle n\rangle \wedge \langle 1\rangle \cong \langle n\rangle$$

where μ_m is the unique active morphism $\langle m\rangle \to \langle 1\rangle$ and the isomorphism $\langle n\rangle \wedge \langle 1\rangle \cong \langle n\rangle$ is the unitality isomorphism, see [HA, 2.2.5.2].

For $A_r''^{\,\mathrm{const}}$ we have the same constraint regarding the domain, but no constraint on the codomain. We can thus let $A_r''^{\,\mathrm{const}}$ map a tuple $(\langle m\rangle, F, \langle n\rangle)$ to the active morphism

$$\mathrm{id}_{\langle n\rangle\wedge\langle m\rangle} \colon \langle n\rangle \wedge \langle m\rangle \to \langle n\rangle \wedge \langle m\rangle$$

which also explains why we are using the superscript const in the notation.

The component of μ_r'' at $(\langle m\rangle, F, \langle n\rangle)$ is then to be given by the commutative diagram

$$\begin{tikzcd}
\langle n\rangle \wedge \langle m\rangle \arrow[r, "\mathrm{id}_{\langle n\rangle\wedge\langle m\rangle}"] \arrow[d, "\mathrm{id}_{\langle n\rangle\wedge\langle m\rangle}"'] & \langle n\rangle \wedge \langle m\rangle \arrow[d, "\mathrm{id}_{\langle n\rangle} \wedge \mu_m"] \\
\langle n\rangle \wedge \langle m\rangle \arrow[r, "\mathrm{id}_{\langle n\rangle} \wedge \mu_m"'] & \langle n\rangle \wedge \langle 1\rangle
\end{tikzcd}$$

considered as a morphism from $\mathrm{id}_{\langle n\rangle\wedge\langle m\rangle}$ to $\mathrm{id}_{\langle n\rangle} \wedge \mu_m$ in $\mathrm{Act}(\mathsf{Fin}_*)$, whose evaluation at 0 is $\mathrm{id}_{\langle n\rangle\wedge\langle m\rangle}$, and whose evaluation at 1 is $\mathrm{id}_{\langle n\rangle} \wedge \mu_m$.

To actually construct such functors and such a natural transformation, we first note that $(\mathsf{Fin}_*)_{\mathrm{act}}$ has a final object $\langle 1\rangle$, so that there exists a section

$$s \colon (\mathsf{Fin}_*)_{\mathrm{act}} \to \big((\mathsf{Fin}_*)_{\mathrm{act}}\big)_{/\langle 1\rangle}$$

of the projection, sending $\langle m\rangle$ to μ_m. We thus obtain a composition

$$(\mathsf{Fin}_*)_{\mathrm{act}} \xrightarrow{s} ((\mathsf{Fin}_*)_{\mathrm{act}})_{/\langle 1\rangle} \xrightarrow{i} \mathrm{Fun}([1], (\mathsf{Fin}_*)_{\mathrm{act}})$$

where i is the inclusion. That s is a section means that we have an identification $\mathrm{ev}_0 \circ i \circ s \simeq \mathrm{id}_{(\mathsf{Fin}_*)_{\mathrm{act}}}$ As ev_0 is right adjoint[23] to the functor const, we thus obtain a natural transformation

$$\widetilde{\mu} \colon \mathrm{const} \to i \circ s$$

of functors $(\mathsf{Fin}_*)_{\mathrm{act}} \to \mathrm{Fun}([1], (\mathsf{Fin}_*)_{\mathrm{act}})$.

[23] Note that as 0 is an initial object of $[0]$, we can identify ev_0 with lim.

We can now define A''_r as the composition

$$\begin{aligned}
&\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \\
&\xrightarrow{\mathrm{pr}_3 \times \mathrm{pr}_1} \mathsf{Fin}_* \times \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \\
&\xrightarrow{\mathrm{id}_{\mathsf{Fin}_*} \times p_{\mathsf{Assoc}}} \mathsf{Fin}_* \times (\mathsf{Fin}_*)_{\mathrm{act}} \\
&\xrightarrow{\mathrm{id}_{\mathsf{Fin}_*} \times (i \circ s)} \mathsf{Fin}_* \times \mathrm{Fun}([1], (\mathsf{Fin}_*)_{\mathrm{act}}) \\
&\xrightarrow{\mathrm{const} \times i'} \mathrm{Act}(\mathsf{Fin}_*) \times \mathrm{Act}(\mathsf{Fin}_*) \\
&\xrightarrow{- \wedge -} \mathrm{Act}(\mathsf{Fin}_*)
\end{aligned}$$

where i' is the inclusion $\mathrm{Fun}([1], (\mathsf{Fin}_*)_{\mathrm{act}}) \to \mathrm{Act}(\mathsf{Fin}_*)$.

We similarly make the following definitions.

$$\begin{aligned}
A''^{\,\mathrm{const}}_r &:= (\mathrm{const} \circ \mathrm{pr}_3) \wedge (i' \circ \mathrm{const} \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1) \\
\mu''_r &:= (\mathrm{const} \circ \mathrm{pr}_3) \wedge (i' \circ \widetilde{\mu} \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1)
\end{aligned}$$

Construction of the commutative diagram (6.6) *in (1)*: We already obtained an identification

$$p_{\mathcal{C}} \circ A''_l \simeq \mathrm{pr}_3 \wedge (p_{\mathsf{Assoc}} \circ \mathrm{pr}_1)$$

above. For $\mathrm{ev}_0 \circ A''_r$ we obtain the following sequence of equivalences

$$\begin{aligned}
\mathrm{ev}_0 \circ A''_r &= \mathrm{ev}_0 \circ ((\mathrm{const} \circ \mathrm{pr}_3) \wedge (i' \circ i \circ s \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1)) \\
&\simeq (\mathrm{ev}_0 \circ \mathrm{const} \circ \mathrm{pr}_3) \wedge (\mathrm{ev}_0 \circ i' \circ i \circ s \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1) \\
&\simeq \mathrm{pr}_3 \wedge (\mathrm{ev}_0 \circ i \circ s \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1) \\
&\simeq \mathrm{pr}_3 \wedge (p_{\mathsf{Assoc}} \circ \mathrm{pr}_1)
\end{aligned}$$

where from the first to second line we use compatibility of ev_0 with the functor $- \wedge -$, from the second to the third we use the identification $\mathrm{ev}_0 \circ \mathrm{const} \simeq \mathrm{id}$ and compatibility of ev_0 with the inclusion i', and from the third to the fourth we use the identification $\mathrm{ev}_0 \circ i \circ s \simeq \mathrm{id}_{(\mathsf{Fin}_*)_{\mathrm{act}}}$.

On $\mathrm{ev}_0 \circ \mu''_r$ *being a natural equivalence, thereby completing (2)*: Using identifications as just done for $\mathrm{ev}_0 \circ A''_r$ we see that it suffices to show that $\mathrm{ev}_0 \circ \widetilde{\mu}$ is a natural equivalence. But by definition we can identify $\mathrm{ev}_0 \circ \widetilde{\mu}$ with $\mathrm{id}_{(\mathsf{Fin}_*)_{\mathrm{act}}}$.

Construction of a natural equivalence $\mathrm{ev}_1 \circ A''_r \simeq \mathrm{pr}_3$ *as in (3)*: There is a sequence of equivalences as follows

$$\begin{aligned}
\mathrm{ev}_1 \circ A''_r &\simeq \mathrm{pr}_3 \wedge (\mathrm{ev}_1 \circ i \circ s \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1) \\
&\simeq \mathrm{pr}_3 \wedge \big(\mathrm{const}_{\langle 1 \rangle} \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1\big) \\
&\simeq \mathrm{pr}_3 \wedge \big(\mathrm{const}_{\langle 1 \rangle}\big) \\
&\simeq \mathrm{pr}_3
\end{aligned}$$

where the first one is obtained just like for $\mathrm{ev}_0 \circ A''_r$, the equivalence from the first to the second line uses the definition of i as the inclusion of $(\mathsf{Assoc}^{\otimes}_{\mathrm{act}})_{/\langle 1\rangle}$, the equivalence from the second to the third line uses the canonical equivalences for precompositions of constant functors, and the last equivalence uses the natural unitality equivalence [HA, 2.2.5.2][24] $- \wedge \langle 1\rangle \cong \mathrm{id}_{\mathsf{Fin}_*}$. ♢

Proposition 6.1.2.5. *Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category. Then the composition of functors over Fin_**

$$\begin{aligned}\mathrm{Alg}(\mathcal{C})^{\otimes} &\xrightarrow{\iota_{\mathrm{Alg}}} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \\ &\xrightarrow{A^{\otimes}} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes}\end{aligned}$$

where ι_{Alg} is as discussed in the introduction to Section 6.1.2.2 in (6.3) *and $A^{\otimes}$ is as in Construction 6.1.2.4, is a lax symmetric monoidal functor.*

We will also denote this lax symmetric monoidal composition by $A^{\otimes}$. ♡

Proof. We have to show that the composition sends $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$-cocartesian morphisms over an inert morphism in Fin_* to pr_2-cocartesian morphisms[25]. So let $\varphi\colon R \to S$ be a $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$-cocartesian morphism in $\mathrm{Alg}(\mathcal{C})^{\otimes}$ lying over an inert morphism in Fin_*. We have to show that $(A^{\otimes} \circ \iota_{\mathrm{Alg}})(\varphi)$ is pr_2-cocartesian. By the result [HTT, 2.4.1.3 (2)] regarding cocartesian morphisms and pullbacks it suffices for this to show that $(\mathrm{pr}_1 \circ A^{\otimes} \circ \iota_{\mathrm{Alg}})(\varphi)$ is $(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}})_*$-cocartesian. Applying the result [HTT, 3.1.2.1] on cocartesian fibrations and functor categories and using that $p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}$ is a cocartesian fibration by Fact 6.1.2.1, we are further reduced to showing that for every object X of $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$, the morphism $(\mathrm{ev}_X \circ \mathrm{pr}_1 \circ A^{\otimes} \circ \iota_{\mathrm{Alg}})(\varphi)$ is $p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}$-cocartesian. Finally, using the description of $p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}$-cocartesian morphisms from Fact 6.1.2.1, we conclude that we need to show that for every object X of $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ the morphism $(\mathrm{pr}_1 \circ \mathrm{ev}_X \circ \mathrm{pr}_1 \circ A^{\otimes} \circ \iota_{\mathrm{Alg}})(\varphi)$ is an inert morphism in $\mathcal{C}^{\otimes}$.

Using notation from Construction 6.1.2.4 we have by construction a sequence of equivalences[26] as follows.

$$\begin{aligned}&\mathrm{pr}_1 \circ \mathrm{ev}_X \circ \mathrm{pr}_1 \circ A^{\otimes} \circ \iota_{\mathrm{Alg}} \\ \simeq\ &\mathrm{pr}_1 \circ \mathrm{ev}_X \circ A' \circ \iota_{\mathrm{Alg}} \\ \simeq\ &\mathrm{pr}_1 \circ A'' \circ (\mathrm{const}_X \times \iota_{\mathrm{Alg}}) \\ \simeq\ &A''_l \circ (\mathrm{const}_X \times \iota_{\mathrm{Alg}}) \\ \simeq\ &\mathrm{ev} \circ (\mathrm{pr}_1 \times \mathrm{pr}_2) \circ (\mathrm{const}_X \times \iota_{\mathrm{Alg}}) \\ \simeq\ &\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\end{aligned}$$

[24] Depending on the definition one takes, this might even by an equality, see [HA, 2.2.5.1].

[25] See the introduction to Section 6.1.2.2 for a discussion of the canonical morphisms of ∞-operads from the two symmetric monoidal ∞-categories to Fin_*. Without looking at the previous pages for reference it may be hard to follow what the various projections etc. in this proof refer to.

[26] The pr_1 in the last line corresponds to pr_2 in the second to last line.

The claim now follow directly from Proposition E.4.2.3 (2). □

We will later need the following proposition, which will allow us to deduce statements for $A^{\otimes}$ from A^{const}, for which we will also provide a simpler description in Proposition 6.1.2.7 below.

Proposition 6.1.2.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and X an object of*

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_*$$

i. e. of the domain of $A^{\otimes}$ and A^{const} from Construction 6.1.2.4. Then the morphism

$$\mu_X\colon A^{\mathrm{const}}(X) \to \big(\mathrm{pr}_1 \circ A^{\otimes}\big)(X)$$

in

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\big)$$

is $(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}})_*$-cocartesian. ♡

Proof. Let X be as in the statement. By [HTT, 3.1.2.1] and the description of $p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}$-cocartesian morphisms in Fact 6.1.2.1 it suffices to show that for every object Y in $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ the morphism $(\mathrm{pr}_1 \circ \mathrm{ev}_Y)(\mu_X) = (\mathrm{ev}_Y \circ (\mathrm{pr}_1))(\mu_X)$ is inert. But by Construction 6.1.2.4 that morphism is an equivalence, and hence in particular inert. □

We end this section by giving another, simpler, description for the functor A^{const} from Construction 6.1.2.4.

Proposition 6.1.2.7. *Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category. Then the functor*

$$A^{\mathrm{const}}\colon\ \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\big)$$

constructed in Construction 6.1.2.4 is equivalent to the composition

$$\begin{aligned} &\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \\ \xrightarrow{\mathrm{pr}_1} &\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \\ \xrightarrow{\mathrm{Fun}(\alpha, \iota_{\mathrm{act}})} &\mathrm{Fun}\Big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \big(\mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes}\Big) \end{aligned}$$

where $\alpha\colon \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \to \mathsf{Assoc}^{\otimes}$ is the inclusion, and $\iota_{\mathrm{act}}\colon \mathcal{C}^{\otimes} \to (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}$ is the functor described before Fact 6.1.2.2. ♡

Proof. In this proof we use notation from Construction 6.1.2.4, as well as the discussions of the relevant definitions at the start of Section 6.1.2.2.

It suffices to check that the adjoint functors

$$\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \to \mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathrm{Act}(\mathsf{Fin}_*)$$

are homotopic. For A^{const} this adjoint functor is by construction A''^{const}. For the composition given in the statement this adjoint is equivalent to the following composition, which we will call $\widetilde{A}''^{\mathrm{const}}$ for now.

$$\begin{aligned}
&\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \\
\xrightarrow{(\alpha\circ\mathrm{pr}_1)\times\mathrm{pr}_2} &\mathsf{Assoc}^{\otimes} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \\
\xrightarrow{\mathrm{ev}} &\mathcal{C}^{\otimes} \\
\xrightarrow{\iota_{\mathrm{act}}} &\mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathrm{Act}(\mathsf{Fin}_*)
\end{aligned}$$

To show that two such functor are equivalent we need to show that we can identify the two corresponding commutative diagrams of the following form.

$$\begin{tikzcd}
& \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \arrow[dl] \arrow[dr] & \\
\mathcal{C}^{\otimes} \arrow[dr, "p_{\mathcal{C}}"'] & & \mathrm{Act}(\mathsf{Fin}_*) \arrow[dl, "\mathrm{ev}_0"] \\
& \mathsf{Fin}_* &
\end{tikzcd} \tag{6.7}$$

To simplify this problem we first notice that $\mathrm{pr}_2 \circ \iota_{\mathrm{act}}$, and hence $\mathrm{pr}_2 \circ \widetilde{A}''^{\mathrm{const}}$, by definition factors though $\mathrm{const}\colon \mathsf{Fin}_* \to \mathrm{Act}(\mathsf{Fin}_*)$. Similarly, we have equivalences as follows.

$$\mathrm{pr}_2 \circ A''^{\mathrm{const}} = A''^{\mathrm{const}}_r$$

By definition we obtain the following.

$$= (\mathrm{const} \circ \mathrm{pr}_3) \wedge (i' \circ \mathrm{const} \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1)$$

Using functoriality of $- \wedge -$.

$$\simeq \mathrm{const} \circ (\mathrm{pr}_3 \wedge (p_{\mathsf{Assoc}} \circ \mathrm{pr}_1))$$

This shows that also $\mathrm{pr}_2 \circ A''^{\mathrm{const}}$ factors through const.

We claim that because of this it actually suffices to construct a homotopy between $\mathrm{pr}_1 \circ \widetilde{A}''^{\mathrm{const}}$ and $\mathrm{pr}_1 \circ A''^{\mathrm{const}}$, as we can then obtain a homotopy between $\mathrm{pr}_2 \circ \widetilde{A}''^{\mathrm{const}}$ and $\mathrm{pr}_2 \circ A''^{\mathrm{const}}$ in such a manner that there is an evident compatible homotopy between the fillers of the commutative squares (6.7) as follows.

$$\mathrm{pr}_2 \circ \widetilde{A}''^{\mathrm{const}}$$

Using that $\mathrm{const} \circ \mathrm{ev}_0 \circ \mathrm{const} \simeq \mathrm{const}$.

$$\simeq \mathrm{const} \circ \mathrm{ev}_0 \circ \mathrm{pr}_2 \circ \widetilde{A}''^{\mathrm{const}}$$

Using the canonical homotopy from the diagram (6.7) associated to $\widetilde{A}''^{\mathrm{const}}$.

$$\simeq \mathrm{const} \circ p_{\mathcal{C}} \circ \mathrm{pr}_1 \circ \widetilde{A}''^{\mathrm{const}}$$

Using the homotopy $\mathrm{pr}_1 \circ \widetilde{A}''^{\mathrm{const}} \simeq \mathrm{pr}_1 \circ A''^{\mathrm{const}}$ that we assume given.

$$\simeq \mathrm{const} \circ p_{\mathcal{C}} \circ \mathrm{pr}_1 \circ A''^{\mathrm{const}}$$

Using the canonical homotopy from the diagram (6.7) associated to A''^{const}.

$$\simeq \mathrm{const} \circ \mathrm{ev}_0 \circ \mathrm{pr}_2 \circ A''^{\mathrm{const}}$$

Using that $\mathrm{const} \circ \mathrm{ev}_0 \circ \mathrm{const} \simeq \mathrm{const}$.

$$\simeq \mathrm{pr}_2 \circ A''^{\mathrm{const}}$$

It thus suffices to show that $\mathrm{pr}_1 \circ \widetilde{A}''^{\mathrm{const}} \simeq \mathrm{pr}_1 \circ A''^{\mathrm{const}}$. But it follows immediately from unpacking the definitions that there is an equivalence as follows.

$$\begin{aligned} \mathrm{pr}_1 \circ \widetilde{A}''^{\mathrm{const}} &= \mathrm{id}_{\mathcal{C}^{\otimes}} \circ \mathrm{ev} \circ ((\alpha \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \\ &\simeq \mathrm{ev} \circ ((\alpha \circ \mathrm{pr}_1) \times \mathrm{pr}_2) \\ &= \mathrm{pr}_1 \circ A''^{\mathrm{const}} \end{aligned}$$

□

6.1.2.3 Tensoring active diagrams together

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. In Section 6.1.2.1 we discussed the symmetric monoidal structure on the ∞-category $\mathcal{C}^{\otimes}_{\mathrm{act}}$, where the tensor product can be described as follows.

$$\left(\bigoplus_{1 \le i \le n} X_i\right) \otimes \left(\bigoplus_{n+1 \le i \le n+m} X_i\right) \simeq \bigoplus_{1 \le i \le n+m} X_i$$

In Definition 6.1.2.8 below we will define a symmetric monoidal functor $\otimes\colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$, which can be described as mapping $\bigoplus_{1 \le i \le n} X_i$ to $\bigotimes_{1 \le i \le n} X_i$. Given the informal description of the symmetric monoidal structure on $\mathcal{C}^{\otimes}_{\mathrm{act}}$ it should be plausible that there is such a symmetric monoidal functor.

Definition 6.1.2.8. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We let

$$\otimes\colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$$

be the symmetric monoidal functor that corresponds to the lax symmetric monoidal functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ (which is actually symmetric monoidal, but we do not use that here) under the equivalence of Fact 6.1.2.2. ♢

Note that by definition, the underlying functor of $\otimes$ from Definition 6.1.2.8 maps objects X of $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$ to X, so by symmetric monoidality we obtain that $\bigoplus_{1 \le i \le n} X_i$ must be mapped to $\bigotimes_{1 \le i \le n} X_i$.

Remark 6.1.2.9. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor of symmetric monoidal ∞-categories. Combining Remark 6.1.2.3 with

$$F^*(\mathrm{id}_{\mathcal{D}}) = F_*(\mathrm{id}_{\mathcal{C}})$$

yields a commutative diagram of symmetric monoidal functors as follows

$$\begin{tikzcd} \mathcal{C}^{\otimes}_{\mathrm{act}} \arrow[r, "F^{\otimes}_{\mathrm{act}}"] \arrow[d, "\otimes"'] & \mathcal{D}^{\otimes}_{\mathrm{act}} \arrow[d, "\otimes"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

where the two functors denoted by $\otimes$ are those from Definition 6.1.2.8. ♢

As $\otimes\colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ is a symmetric monoidal functor, it induces a symmetric monoidal functor

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big) \xrightarrow{(\otimes)_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)$$

on functor categories with the pointwise symmetric monoidal structure[27]. Furthermore, the composition $(\otimes_*)^{\otimes} \circ A^{\otimes}$ of the lax symmetric monoidal functor $A^{\otimes}$ from Proposition 6.1.2.5 with this symmetric monoidal functor is not only lax symmetric monoidal, but actually symmetric monoidal, as we see in Proposition 6.1.2.11 below. Before doing so we will use Proposition 6.1.2.6 and Proposition 6.1.2.7 to describe the compositions $\mathrm{ev}_{\langle m\rangle} \circ \otimes_* \circ A$.

Proposition 6.1.2.10. *Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category.*

Then the composition[28]

$$\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \xrightarrow{\alpha^*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\big) \xrightarrow{(p_{\mathcal{C}})_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\big)$$

is the constant functor with image $p_{\mathsf{Assoc}} \circ \alpha$ and the composition[29]

$$\begin{aligned} \mathrm{Alg}(\mathcal{C}) \xrightarrow{A} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big) &\xrightarrow{\otimes_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \\ \xrightarrow{(\mathcal{C}\to\mathcal{C}^{\otimes})_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\big) &\xrightarrow{(p_{\mathcal{C}})_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\big) \end{aligned}$$

is the constant functor with image $\mathrm{const}_{\langle 1\rangle}$*.*

[27] This follows directly from the definition [HA, 2.1.3.4] together with Proposition C.1.1.1 and [HTT, 3.1.2.1].

[28] The functor $\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big)$ is to be the canonical one, i. e. inclusion into $\mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big)$ followed by the projection, and α is the inclusion of $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ into $\mathsf{Assoc}^{\otimes}$.

[29] A is the underlying functor of the lax symmetric monoidal functor from Proposition 6.1.2.5, and $\otimes$ is the functor defined in Definition 6.1.2.8.

Let $\mu^{\mathsf{Fin}_*}\colon p_{\mathsf{Assoc}} \circ \alpha \to \mathrm{const}_{\langle 1\rangle}$ be the unique natural transformation of functors $\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \to \mathsf{Fin}_*$ that is pointwise an active morphism. Then there is a homotopy between the composition

$$\mathrm{Alg}(\mathcal{C}) \xrightarrow{\alpha^*\circ\left(\mathrm{Alg}(\mathcal{C})\to\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes},\mathcal{C}^{\otimes}\right)\right)} \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\right)_{p_{\mathsf{Assoc}}\circ\alpha}$$
$$\xrightarrow{\left(\mu^{\mathsf{Fin}_*}\right)_!} \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\right)_{\mathrm{const}_{\langle 1\rangle}}$$

and the following functor.

$$\left(\mathcal{C} \to \mathcal{C}^{\otimes}\right)_* \circ \otimes_* \circ A\colon\ \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\right)_{\mathrm{const}_{\langle 1\rangle}}$$

In particular, there is a commutative diagram of ∞-categories as follows for every $m \geq 0$

$$\begin{tikzcd}
\mathrm{Alg}(\mathcal{C}) \arrow[r, "A"] \arrow[d] & \mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\right) \arrow[r, "\otimes_*"] & \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\right) \arrow[d, "\mathrm{ev}_{\langle m\rangle}"] \\
\mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right) \arrow[r, "\mathrm{ev}_{\langle m\rangle}"'] & \mathcal{C}^{\otimes}_{\langle m\rangle} \arrow[r, "(\mu_m)_!"'] & \mathcal{C}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{C}
\end{tikzcd} \tag{6.8}$$

where the left vertical functor is the canonical functor and and μ_m is the unique active morphism $\langle m\rangle \to \langle 1\rangle$ in Fin_*.

Now let R be an associative algebra in $\mathcal{C}$. Then $(\otimes_* \circ A)(R)(\langle m\rangle)$ can be identified with $R^{\otimes m}$ and if $f\colon \langle m\rangle \to \langle m'\rangle$ is an active morphism in $\mathsf{Assoc}^{\otimes}$, then we can identify $(\otimes_* \circ A)(R)(f)$ with the morphism $R^{\otimes m} \to R^{\otimes m'}$ induced by f, so for example for f the unique active morphism $\langle 0\rangle \to \langle 1\rangle$ we can identify $(\otimes_* \circ A)(R)(f)$ with the unit morphism $\mathbb{1}_{\mathcal{C}} \to R$. ♡

Proof. In this proof we use notation from Construction 6.1.2.4.

Recall the natural transformation[30] $\mu\colon A^{\mathrm{const}} \to \mathrm{pr}_1 \circ A^{\otimes}$ from Construction 6.1.2.4. We can define a natural transformation

$$\overline{\mu} \coloneqq \left(\otimes^{\otimes}\right)_* \circ \mu \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\right)$$

of functors from $\mathrm{Alg}(\mathcal{C})$ to $\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\right)$.

We first claim that it suffices to show the following.

(1) $\left(\otimes^{\otimes}\right)_* \circ A^{\mathrm{const}} \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\right) \simeq \alpha^* \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right)\right)$

(2) $\left(\otimes^{\otimes}\right)_* \circ \mathrm{pr}_1 \circ A^{\otimes} \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\right) \simeq \left(\mathcal{C} \to \mathcal{C}^{\otimes}\right)_* \circ \otimes_* \circ A$

(3) $(p_{\mathcal{C}})_* \circ \overline{\mu} \simeq \mathrm{const}_{\mu^{\mathsf{Fin}_*}}$

[30] We use A^{const} here as notation for the restriction of what was called A^{const} in Construction 6.1.2.4 to $\mathrm{Alg}(\mathcal{C})^{\otimes}$, and similarly for μ – like we do for $A^{\otimes}$.

(4) For every object R of $\mathrm{Alg}(\mathcal{C})$, the component $\overline{\mu}_R$ of $\overline{\mu}$ is $(p_{\mathcal{C}})_*$-cocartesian.

Let us now explain how the statements we need to prove follow from claims (1), (2), (3) and (4).

The claims regarding the images of the two functors to $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*)$ follow directly from claims (1), (2) and (3), and the identification of

$$\big(\mathcal{C} \to \mathcal{C}^{\otimes}\big)_* \circ \otimes_* \circ A$$

then follows from claims (1), (2), (3) and (4)[31]. The inclusion functor $\mathcal{C} \to \mathcal{C}^{\otimes}$ is fully faithful[32], so for construction of a commutative diagram (6.8) it suffices by Proposition B.4.3.1 to show that the two composite functors from the top left to the bottom right become homotopic after composing with the inclusion to $\mathcal{C}^{\otimes}$. But we have a chain of equivalences as follows.

$$\big(\mathcal{C} \to \mathcal{C}^{\otimes}\big) \circ \mathrm{ev}_{\langle m \rangle} \circ \otimes_* \circ A$$

Using compatibility of evaluation with postcomposition.

$$\simeq \mathrm{ev}_{\langle m \rangle} \circ \big(\mathcal{C} \to \mathcal{C}^{\otimes}\big)_* \circ \otimes_* \circ A$$

Postcomposing the already obtained equivalence with $\mathrm{ev}_{\langle m \rangle}$.

$$\simeq \mathrm{ev}_{\langle m \rangle} \circ \big(\mu^{\mathsf{Fin}_*}\big)_! \circ \alpha^* \circ \big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big)\big)$$

Using [HTT, 3.1.2.1 (2)].

$$\simeq (\mu_m)_! \circ \mathrm{ev}_{\langle m \rangle} \circ \alpha^* \circ \big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big)\big)$$

Finally, compatibility of evaluations with precomposing and (un)making the identification $\mathcal{C}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{C}$.

$$\simeq \big(\mathcal{C} \to \mathcal{C}^{\otimes}\big) \circ (\mu_m)_! \circ \mathrm{ev}_{\langle m \rangle} \circ \big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big)\big)$$

Finally, the concrete description of $(\otimes_* \circ A)(R)$ follows directly from the identification of $(\mathcal{C} \to \mathcal{C}^{\otimes})_* \circ \otimes_* \circ A$ by unpacking the definitions.

So let us now prove claims (1), (2), (3) and (4).

Proof of claim (1): We have equivalences as follows.

$$\big(\otimes^{\otimes}\big)_* \circ A^{\mathrm{const}} \circ \Big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\Big)$$

Using the description of A^{const} from Proposition 6.1.2.7.

$$\simeq \big(\otimes^{\otimes}\big)_* \circ (\iota_{\mathrm{act}})_* \circ \alpha^* \circ \big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big)\big)$$

Using that by definition of the functor $\otimes$ – see Definition 6.1.2.8 – there is an equivalence $\otimes^{\otimes} \circ \iota_{\mathrm{act}} \simeq \mathrm{id}_{\mathcal{C}}$.

$$\simeq \alpha^* \circ \big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big)\big)$$

[31] We remark that we do not need to worry about the equivalences in claims (1) and (2) lying over non-identity natural isomorphisms of functors to Fin_*, as the unique active morphism $\langle m \rangle \to \langle 1 \rangle$ in Fin_* stays unchanged if we pre- and postcompose it by isomorphisms.

[32] This follows from Proposition B.5.3.1 using that $\{\langle 1 \rangle\} \to \mathsf{Fin}_*$ is fully faithful.

Proof of claim (2): Follows immediately by using that lax monoidal functors such as A and $\otimes$ are compatible with the inclusion of the underlying ∞-category into the respective ∞-operad.

Proof of claim (3): It suffices to show that the adjoint natural transformations of functors

$$\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Alg}(\mathcal{C}) \to \mathsf{Fin}_*$$

are equivalent, i. e. that there is an equivalence between $p_{\mathcal{C}} \circ \breve{\overline{\mu}}$ and $\mu^{\mathsf{Fin}_*} \circ \mathrm{pr}_1$.

We first note that as $\otimes^{\otimes} \colon (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes} \to \mathcal{C}^{\otimes}$ is a functor over Fin_*, we have an equivalence as follows.

$$p_{\mathcal{C}} \circ \otimes^{\otimes} \simeq p_{\mathcal{C}^{\otimes}_{\mathrm{act}}} = \mathrm{ev}_1 \circ \mathrm{pr}_2$$

Unpacking the definition of μ in Construction 6.1.2.4 we thus obtain equivalences as follows.

$$\begin{aligned}
&p_{\mathcal{C}} \circ \breve{\overline{\mu}} \\
&= p_{\mathcal{C}} \circ \otimes^{\otimes} \circ \breve{\mu} \circ \Big(\mathrm{id}_{\mathsf{Assoc}^{\otimes}_{\mathrm{act}}} \times \Big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\Big)\Big) \\
&\simeq \mathrm{ev}_1 \circ \mathrm{pr}_2 \circ \breve{\mu} \circ \Big(\mathrm{id}_{\mathsf{Assoc}^{\otimes}_{\mathrm{act}}} \times \Big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\Big)\Big) \\
&\simeq \mathrm{ev}_1 \circ \mu''_r \circ \Big(\mathrm{id}_{\mathsf{Assoc}^{\otimes}_{\mathrm{act}}} \times \Big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\Big)\Big) \\
&\simeq \mathrm{ev}_1 \circ \big(\big(\mathrm{const}_{\langle 1\rangle}\big) \wedge (i' \circ \widetilde{\mu} \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1)\big) \\
&\simeq \mathrm{ev}_1 \circ i' \circ \widetilde{\mu} \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1 \\
&\simeq \mu^{\mathsf{Fin}_*} \circ \mathrm{pr}_1
\end{aligned}$$

Proof of claim (4): Follows immediately by combining that all components of μ are $\left(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}\right)_*$-cocartesian by Proposition 6.1.2.6, that $\otimes^{\otimes}$ is symmetric monoidal by definition, and [HTT, 3.1.2.1].

□

Proposition 6.1.2.11. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Consider the composition*

$$\mathrm{Alg}(\mathcal{C})^{\otimes} \xrightarrow{A^{\otimes}} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes} \xrightarrow{(\otimes_*)^{\otimes}} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)^{\otimes}$$

of functors over Fin_*, where $A^{\otimes}$ is as in Proposition 6.1.2.5 and $(\otimes_*)^{\otimes}$ is the symmetric monoidal functor induced by $\otimes$ from Definition 6.1.2.8 on functor categories with the pointwise symmetric monoidal structure.

Then this composition is a symmetric monoidal functor. ♡

Proof. We will use notation from Construction 6.1.2.4 in this proof[33], which will be similar to the proof of Proposition 6.1.2.10.

[33] We will though use A^{const} as notation for the restriction of what was called A^{const} in Construction 6.1.2.4 to $\mathrm{Alg}(\mathcal{C})^{\otimes}$, and similarly for μ, as we do for $A^{\otimes}$.

Just like in Proposition 6.1.2.5, it suffices to show that for every object $\langle m\rangle$ in $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$, the composition

$$\mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ (\otimes_*)^{\otimes} \circ A^{\otimes}$$

maps $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$-cocartesian morphisms to $p_{\mathcal{C}}$-cocartesian morphisms. Also like in Proposition 6.1.2.5, we use the definitions of the various functors to rewrite this composition into a more suitable form. We start by using the definition of $(\otimes_*)^{\otimes}$ and compatibility of evaluation with postcomposition of functors to obtain homotopies as follows.

$$\begin{aligned}
&\mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ (\otimes_*)^{\otimes} \circ A^{\otimes}\\
\simeq\ &\mathrm{ev}_{\langle m\rangle} \circ \left(\otimes^{\otimes}\right)_* \circ \mathrm{pr}_1 \circ A^{\otimes}\\
\simeq\ &\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ A^{\otimes}
\end{aligned}$$

Let $f\colon X \to Y$ be a $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$-cocartesian morphism in $\mathrm{Alg}(\mathcal{C})^{\otimes}$. From the natural transformation $\mu\colon A^{\mathrm{const}} \to \mathrm{pr}_1 \circ A^{\otimes}$ we obtain a commutative square as follows.

$$\begin{tikzcd}
\left(\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ A^{\mathrm{const}}\right)(X) \arrow[r, "\left(\otimes^{\otimes}\circ \mathrm{ev}_{\langle m\rangle}\right)(\mu_X)"] \arrow[d, "\left(\otimes^{\otimes}\circ \mathrm{ev}_{\langle m\rangle}\circ A^{\mathrm{const}}\right)(f)"'] & \left(\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ A^{\otimes}\right)(X) \arrow[d, "\left(\otimes^{\otimes}\circ \mathrm{ev}_{\langle m\rangle}\circ \mathrm{pr}_1\circ A^{\otimes}\right)(f)"'] \\
\left(\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ A^{\mathrm{const}}\right)(Y) \arrow[r, "\left(\otimes^{\otimes}\circ \mathrm{ev}_{\langle m\rangle}\right)(\mu_Y)"'] & \left(\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ A^{\otimes}\right)(Y)
\end{tikzcd}$$

We need to show that the right vertical morphism is $p_{\mathcal{C}}$-cocartesian. By Proposition 6.1.2.6 we know that μ_X and μ_Y are $(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}})_*$-cocartesian, so it follows from [HTT, 3.1.2.1] and $\otimes^{\otimes}$ being symmetric monoidal by definition that the top and bottom horizontal morphisms in the diagram are $p_{\mathcal{C}}$-cocartesian. It thus suffices by [HTT, 2.4.1.7] to show that the left vertical morphism is $p_{\mathcal{C}}$-cocartesian.

For this we use the description of A^{const} from Proposition 6.1.2.7 and that by definition $\otimes^{\otimes} \circ \iota_{\mathrm{act}} \simeq \mathrm{id}_{\mathcal{C}^{\otimes}}$ to obtain equivalences as follows.

$$\begin{aligned}
&\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ A^{\mathrm{const}}\\
\simeq\ &\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ \left(\iota_{\mathrm{act}}\right)_* \circ \alpha^* \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\\
\simeq\ &\otimes^{\otimes} \circ \iota_{\mathrm{act}} \circ \mathrm{ev}_{\langle m\rangle} \circ \alpha^* \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\\
\simeq\ &\mathrm{ev}_{\langle m\rangle} \circ \alpha^* \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\\
\simeq\ &\mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}
\end{aligned}$$

So what is left to show is that $\mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}$ maps $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$-cocartesian morphisms to $p_{\mathcal{C}}$-cocartesian morphisms. But this follows immediately from Proposition E.4.2.3 (4). □

Remark 6.1.2.12. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor of symmetric monoidal ∞-categories. Then going through the constructions and using Remark 6.1.2.3 it is straightforward to see that there is a commutative diagram of symmetric monoidal functors as follows

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{C})^{\otimes} \arrow[r, "(\otimes_*)^{\otimes}\circ A^{\otimes}\circ\iota_{\mathrm{Alg}}"] \arrow[d, "\mathrm{Alg}(F)^{\otimes}"'] & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)^{\otimes} \arrow[d, "(F_*)^{\otimes}"] \\ \mathrm{Alg}(\mathcal{D})^{\otimes} \arrow[r, "(\otimes_*)^{\otimes}\circ A^{\otimes}\circ\iota_{\mathrm{Alg}}"'] & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{D}\big)^{\otimes} \end{tikzcd}$$

where the horizontal functors are the compositions considered in Proposition 6.1.2.11 for $\mathcal{C}$ and $\mathcal{D}$, respectively. Furthermore, if $G\colon \mathcal{D} \to \mathcal{E}$ is another symmetric monoidal functor, then the composite of the compatibility diagrams for F and G as above can be identified with the compatibility diagram for $G \circ F$. $\diamondsuit$

6.1.2.4 The functor $V\colon \mathbf{\Lambda} \to \mathsf{Assoc}^{\otimes}_{\mathrm{act}}$

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. With Proposition 6.1.2.11 we have now constructed a symmetric monoidal functor

$$\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)$$

that is the first[34] step in the symmetric monoidal functor $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$. We already constructed the self-duality functor

$$-^{\circ}\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathbf{\Lambda}$$

in Section 6.1.1.6. We will now introduce a functor

$$V\colon \mathbf{\Lambda} \to \mathsf{Assoc}^{\otimes}_{\mathrm{act}}$$

so that precomposition with $V \circ (-^{\circ})$ induces a symmetric monoidal functor

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

with respect to the pointwise symmetric monoidal structures.

Fact 6.1.2.13 ([NikSch, B.1])**.** *There is a functor*

$$V\colon \mathbf{\Lambda} \to \mathsf{Assoc}^{\otimes}_{\mathrm{act}}$$

that maps

- $[n]_{\mathbf{\Lambda}}$ *to* $\langle n+1 \rangle$,

[34] Or the first two or three, however one wants to count.

- $\delta_j\colon [n-1]_{\mathbf{\Lambda}} \to [n]_{\mathbf{\Lambda}}$ *to the active map that sends* i *to* i *if* $i < j+1$ *and to* $i+1$ *otherwise*[35],
- $\sigma_j\colon [n+1]_{\mathbf{\Lambda}} \to [n]_{\mathbf{\Lambda}}$ *to the active map that sends* i *to* i *if* $i \leq j+1$ *and to* $i-1$ *otherwise, with ordering on the preimage of* $j+1$ *given by* $j+1 < j+2$,
- $\tau\colon [n]_{\mathbf{\Lambda}} \to [n]_{\mathbf{\Lambda}}$ *to the active map that sends* 1 *to* $n+1$ *and* i *to* $i-1$ *for* $i > 1$. ♧

Proposition 6.1.2.14. *Let* $\mathcal{C}$ *be a symmetric monoidal* ∞*-category. Then the functor*

$$\begin{aligned}&\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \\ &\xrightarrow{(V\circ(-^{\circ}))^* \times_{(V\circ(-^{\circ}))^*} \mathrm{id}} \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Fin}_*)} \mathsf{Fin}_*\end{aligned}$$

over Fin_* *upgrades the functor*

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \xrightarrow{(V\circ(-^{\circ}))^*} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

to a symmetric monoidal functor with respect to the pointwise symmetric monoidal structures (see [HA, 2.1.3.4]). ♡

Proof. Follows directly from the definition of the respective pointwise symmetric monoidal structures and Proposition C.1.1.1 and [HTT, 3.1.2.1]. □

Remark 6.1.2.15. The symmetric monoidal functor obtained in Proposition 6.1.2.14 is natural in $\mathcal{C}$. In particular, for $F\colon \mathcal{C} \to \mathcal{D}$ a symmetric monoidal functor between symmetric monoidal ∞-categories, we obtain a commutative square

$$\begin{tikzcd}
\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \arrow[r, "(V\circ(-^{\circ}))^*"] \arrow[d, "F_*"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \arrow[d, "F_*"] \\
\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{D}\big) \arrow[r, "(V\circ(-^{\circ}))^*"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D})
\end{tikzcd}$$

of symmetric monoidal functors. ♢

[35] For the reader confused by why it is $j+1$ and not j: This arises from the fact that we defined δ_j using elements $\frac{0}{n+1}, \dots, \frac{n}{n+1}$ (i.e. we start counting from 0), whereas the elements of $\langle n+1 \rangle$ are $1, \dots, n+1$ (i.e. we start counting from 1).

6.1.2.5 The definition of the cyclic bar construction as a cyclic object

We are now ready to define the cyclic bar construction $\mathrm{B}^{\mathrm{cyc}}_\bullet$.

Definition 6.1.2.16 ([NikSch, III.2.3])**.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We define the *cyclic bar construction* as the symmetric monoidal functor[36]

$$\mathrm{B}^{\mathrm{cyc}}_\bullet\colon\ \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}(\boldsymbol{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

that is given as the composition of the symmetric monoidal functor

$$\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)$$

from Proposition 6.1.2.11 and the symmetric monoidal functor

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \to \mathrm{Fun}(\boldsymbol{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

from Proposition 6.1.2.14. ♢

Remark 6.1.2.17. $\mathrm{B}^{\mathrm{cyc}}_\bullet$ is compatible with symmetric monoidal functors. If $F\colon \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor, then there is a commuting diagram

$$\begin{tikzcd}
\mathrm{Alg}(\mathcal{C}) \arrow[r, "\mathrm{B}^{\mathrm{cyc}}_\bullet"] \arrow[d, "\mathrm{Alg}(F)"'] & \mathrm{Fun}(\boldsymbol{\Lambda}^{\mathrm{op}}, \mathcal{C}) \arrow[d, "F_*"] \\
\mathrm{Alg}(\mathcal{D}) \arrow[r, "\mathrm{B}^{\mathrm{cyc}}_\bullet"'] & \mathrm{Fun}(\boldsymbol{\Lambda}^{\mathrm{op}}, \mathcal{D})
\end{tikzcd}$$

of symmetric monoidal functors. Furthermore, if $G\colon \mathcal{D} \to \mathcal{E}$ is another symmetric monoidal functor, then the composite of the compatibility squares as above for F and G can be identified with the compatibility square for $G \circ F$. This follows by combining Remark 6.1.2.12 with Remark 6.1.2.15. ♢

6.1.2.6 $\mathrm{B}^{\mathrm{cyc}}_\bullet$ for cocartesian symmetric monoidal ∞-categories

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. The cyclic bar construction

$$\mathrm{B}^{\mathrm{cyc}}_\bullet\colon\ \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}(\boldsymbol{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

is a symmetric monoidal functor and thus induces a functor as follows.

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{C}) \arrow[r, no head, "\simeq"] \arrow[d, dashed] & \mathrm{CAlg}(\mathrm{Alg}(\mathcal{C})) \arrow[d, "\mathrm{CAlg}(\mathrm{B}^{\mathrm{cyc}}_\bullet)"] \\
\mathrm{Fun}(\boldsymbol{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C})) \arrow[r, no head, "\simeq"'] & \mathrm{CAlg}(\mathrm{Fun}(\boldsymbol{\Lambda}^{\mathrm{op}}, \mathcal{C}))
\end{tikzcd} \tag{6.9}$$

[36] In the codomain with respect to the pointwise symmetric monoidal structure.

In this section we will give a different description of this dashed functor: It is the left adjoint of the forgetful functor $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$.

To prove this we will proceed as follows. We will first show in Proposition 6.1.2.18 that already $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ – so without passing to commutative algebras – is left adjoint to $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$, under the assumption that the symmetric monoidal structure on $\mathcal{C}$ is cocartesian. In order to apply this to the dashed composition in (6.9), we will then show in Proposition 6.1.2.19 how we can identify $\mathrm{CAlg}(\mathrm{B}^{\mathrm{cyc}}_{\bullet})$ (where the cyclic bar construction is taken of algebras in $\mathcal{C}$) with the cyclic bar construction for $\mathrm{CAlg}(\mathcal{C})$.

Proposition 6.1.2.18. *Let $\mathcal{C}$ a symmetric monoidal ∞-category and assume that the underlying ∞-category admits finite coproducts and that the symmetric monoidal structure is cocartesian in the sense of [HA, 2.4.0.1]. Under these assumptions the forgetful functor*

$$\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$$

is an equivalence by [HA, 2.4.3.9].

Then the composite

$$\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1} \colon \mathcal{C} \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

is left adjoint to the evaluation functor $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$. ♡

Proof. Let $i\colon \{[0]_{\mathbf{\Lambda}}\} \to \mathbf{\Lambda}^{\mathrm{op}}$ be the inclusion. We will identify the ∞-category $\mathcal{C}$ with $\mathrm{Fun}(\{[0]_{\mathbf{\Lambda}}\}, \mathcal{C})$ and consider $\mathrm{ev}_{\mathfrak{a}}$ as a functor to $\mathrm{Fun}(\{[0]_{\mathbf{\Lambda}}\}, \mathcal{C})$. Under this identification, the functor $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$ corresponds to precomposition with i.

We start by noting that we can use Proposition 6.1.2.10 to identify the composition $i^* \circ \mathrm{B}^{\mathrm{cyc}}_{\bullet}$ with $\mathrm{ev}_{\mathfrak{a}}$ and this identification provides for every object R of $\mathcal{C}$ a commutative triangle of ∞-categories as follows.

$$\begin{tikzcd}
\{[0]_{\mathbf{\Lambda}}\} \arrow[d, "i"'] \arrow[dr, "\mathrm{const}_R"] & \\
\mathbf{\Lambda}^{\mathrm{op}} \arrow[r, "(\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)"'] & \mathcal{C}
\end{tikzcd}$$

It now suffices to show that this triangle exhibits $(\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)$ as a left Kan extension of const_R – see [HA, 4.3.2, 4.3.3, and in particular 4.3.3.7][37].

[37] That we only need to check this pointwise for a single (though of course arbitrary) R boils down to the fact that induced natural transformations between left Kan extensions are defined essentially uniquely through the universal property of left Kan extensions and ultimately colimits.

For this we need to show by [HA, 4.3.2.2 and 4.3.1.3] that for every object $[n]_{\mathbf{\Lambda}}$ of $\mathbf{\Lambda}^{\mathrm{op}}$ the induced diagram

$$\begin{tikzcd} (\mathbf{\Lambda}^{\mathrm{op}})_{/[n]_{\mathbf{\Lambda}}} \times_{\mathbf{\Lambda}^{\mathrm{op}}} \{[0]_{\mathbf{\Lambda}}\} \arrow[r, "\mathrm{pr}"] \arrow[d] & \mathbf{\Lambda}^{\mathrm{op}} \arrow[r, "(\mathrm{B}^{\mathrm{cyc}}_\bullet \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)"] & \mathcal{C} \\ \left((\mathbf{\Lambda}^{\mathrm{op}})_{/[n]_{\mathbf{\Lambda}}} \times_{\mathbf{\Lambda}^{\mathrm{op}}} \{[0]_{\mathbf{\Lambda}}\}\right)^{\triangleright} \arrow[urr, "G"'] & & \end{tikzcd}$$

where the left vertical functor is the inclusion and G is the functor that is induced by $(\mathrm{B}^{\mathrm{cyc}}_\bullet \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)$, exhibits $G(\infty) = (\mathrm{B}^{\mathrm{cyc}}_\bullet \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)([n]_{\mathbf{\Lambda}})$ as a colimit of $(\mathrm{B}^{\mathrm{cyc}}_\bullet \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R) \circ \mathrm{pr}$.

Let us start by unpacking what the category $(\mathbf{\Lambda}^{\mathrm{op}})_{/[n]_{\mathbf{\Lambda}}} \times_{\mathbf{\Lambda}^{\mathrm{op}}} \{[0]_{\mathbf{\Lambda}}\}$ looks like. As $[0]_{\mathbf{\Lambda}}$ has no nontrivial endomorphisms $(\mathbf{\Lambda}^{\mathrm{op}})_{/[n]_{\mathbf{\Lambda}}} \times_{\mathbf{\Lambda}^{\mathrm{op}}} \{[0]_{\mathbf{\Lambda}}\}$ is actually a discrete category. Objects are morphisms $[0]_{\mathbf{\Lambda}} \to [n]_{\mathbf{\Lambda}}$ in $\mathbf{\Lambda}^{\mathrm{op}}$, so morphisms $[n]_{\mathbf{\Lambda}} \to [0]_{\mathbf{\Lambda}}$ in $\mathbf{\Lambda}$. There are $n+1$ such morphisms, namely f_m for $1 \leq m \leq n+1$, where f_m is the morphism $(1/(n+1)) \cdot \mathbb{Z} \to \mathbb{Z}$ in $\mathbf{\Lambda}$[38] that maps $l/(n+1)$ to 0 for $0 \leq l < m-1$ and to 1 for $m-1 \leq l \leq n$. In terms of the generators of $\mathbf{\Lambda}$[39] we can write f_m as $f_m \coloneqq \sigma_0^n \circ \tau^{m-1}$.

Hence what we need to show is that the morphism

$$\coprod_{1 \leq m \leq n+1} \left(\mathrm{B}^{\mathrm{cyc}}_\bullet \circ \mathrm{ev}_{\mathfrak{a}}^{-1}\right)(R)([0]_{\mathbf{\Lambda}}) \xrightarrow{\coprod_{1 \leq m \leq n+1} \left(\mathrm{B}^{\mathrm{cyc}}_\bullet \circ \mathrm{ev}_{\mathfrak{a}}^{-1}\right)(R)(f_m)} \left(\mathrm{B}^{\mathrm{cyc}}_\bullet \circ \mathrm{ev}_{\mathfrak{a}}^{-1}\right)(R)([n]_{\mathbf{\Lambda}}) \qquad (*)$$

is an equivalence in $\mathcal{C}$.

For this we need to understand what $(\mathrm{B}^{\mathrm{cyc}}_\bullet \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)$ maps the morphism f_m to. First we use Fact 6.1.1.13 to see that the self-duality functor $-^\circ$ of $\mathbf{\Lambda}$ maps $f_m = \sigma_0^n \circ \tau^{m-1}$ to $\tau^{1-m}\delta_1^n$. Next we need to apply the functor V from Fact 6.1.2.13, which maps this to the active morphism $\langle 1 \rangle \to \langle n+1 \rangle$ in $\mathsf{Assoc}^{\otimes}$ that sends 1 to m. Denote this morphism of $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ by f'_m.

We can then identify morphism $(*)$ with the morphism[40]

$$\coprod_{1 \leq m \leq n+1} \left(\otimes_* \circ A \circ \mathrm{ev}_{\mathfrak{a}}^{-1}\right)(R)(\langle 1 \rangle) \xrightarrow{\coprod_{1 \leq m \leq n+1} \left(\otimes_* \circ A \circ \mathrm{ev}_{\mathfrak{a}}^{-1}\right)(R)(f'_m)} \left(\otimes_* \circ A \circ \mathrm{ev}_{\mathfrak{a}}^{-1}\right)(R)(\langle n+1 \rangle)$$

in $\mathcal{C}$. With Proposition 6.1.2.10 we can further identify this morphism with

[38] See Section 6.1.1.2.
[39] See Section 6.1.1.3.
[40] We use notation like in Proposition 6.1.2.10.

the morphism

$$\coprod_{1\le m\le n+1} R \xrightarrow{\coprod_{1\le m\le n+1}\left(R\simeq \mathbb{1}_{\mathcal{C}}^{\otimes m-1}\otimes R\otimes \mathbb{1}_{\mathcal{C}}^{\otimes n-m}\xrightarrow{u^{\otimes m-1}\otimes \mathrm{id}_R\otimes u^{\otimes n-m}} R^{\otimes n+1}\right)} R^{\otimes n+1} \tag{$**$}$$

where $u\colon \mathbb{1}_{\mathcal{C}} \to R$ is the unit morphism of the associative algebra $\mathrm{ev}_{\mathfrak{a}}^{-1}(R)$. Morphism $(**)$ is an equivalence as the symmetric monoidal structure on $\mathcal{C}$ is cocartesian. ☐

Proposition 6.1.2.19. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We compare $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ for $\mathrm{CAlg}(\mathcal{C})$ and $\mathcal{C}$ in this proposition, so to distinguish them we will use superscripts such as $\mathrm{B}^{\mathrm{cyc},\mathcal{C}}_{\bullet}$.*

Then there is a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{Alg}(\mathcal{C})) \arrow[r, "\mathrm{CAlg}\left(\mathrm{B}^{\mathrm{cyc},\mathcal{C}}_{\bullet}\right)"] \arrow[d, dash, "\simeq"'] & \mathrm{CAlg}(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C})) \arrow[d, dash, "\simeq"] \\ \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \arrow[r, "\mathrm{B}^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}_{\bullet}"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathrm{CAlg}(\mathcal{C})) \end{tikzcd} \tag{6.10}$$

where the left and right vertical equivalences are the canonical ones[41]*.* ♡

Proof. The symmetric monoidal forgetful functor $\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ induces by Remark 6.1.2.17 a commuting diagram

$$\begin{tikzcd} \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \arrow[r, "\mathrm{B}^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}_{\bullet}"] \arrow[d, "\mathrm{Alg}\left(\mathrm{ev}_{\langle 1\rangle}\right)"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathrm{CAlg}(\mathcal{C})) \arrow[d, "\left(\mathrm{ev}_{\langle 1\rangle}\right)_*"] \\ \mathrm{Alg}(\mathcal{C}) \arrow[r, "\mathrm{B}^{\mathrm{cyc},\mathcal{C}}_{\bullet}"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C}) \end{tikzcd}$$

of symmetric monoidal functors. Applying CAlg to this diagram we obtain the bottom commutative square in the commutative diagram of ∞-categories

[41] For the left equivalence this is the composition

$$\mathrm{CAlg}(\mathrm{Alg}(\mathcal{C})) \simeq \mathrm{BiFunc}(\mathsf{Comm},\mathsf{Assoc};\mathcal{C}) \simeq \mathrm{BiFunc}(\mathsf{Assoc},\mathsf{Comm};\mathcal{C}) \simeq \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C}))$$

where the middle equivalence is given by precomposition with the symmetry equivalence and the other two are the ones from Proposition E.5.0.1. For the right vertical equivalence see [HA, 2.1.3.4].

below.

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \arrow[r, "\mathrm{B}_\bullet^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}"] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C})) \\
\mathrm{CAlg}(\mathrm{Alg}(\mathrm{CAlg}(\mathcal{C}))) \arrow[u, "\mathrm{ev}_{\langle 1\rangle}"] \arrow[r, "\mathrm{CAlg}\left(\mathrm{B}_\bullet^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}\right)"] \arrow[d, "\mathrm{CAlg}\left(\mathrm{Alg}\left(\mathrm{ev}_{\langle 1\rangle}\right)\right)"'] & \mathrm{CAlg}(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C}))) \arrow[u, "\mathrm{ev}_{\langle 1\rangle}"'] \arrow[d, "\mathrm{CAlg}\left(\left(\mathrm{ev}_{\langle 1\rangle}\right)_*\right)"] \\
\mathrm{CAlg}(\mathrm{Alg}(\mathcal{C})) \arrow[r, "\mathrm{CAlg}\left(\mathrm{B}_\bullet^{\mathrm{cyc},\mathcal{C}}\right)"'] & \mathrm{CAlg}(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}))
\end{tikzcd} \qquad (*)$$

By [HA, 3.2.4.7] the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is cocartesian, from which it follows that the induced symmetric monoidal structure on $\mathrm{Alg}(\mathrm{CAlg}(\mathcal{C}))$ is also cocartesian, and hence the left top vertical functor is an equivalence by [HA, 2.4.3.9]. To see that the lower left vertical functor is also an equivalence and that the composite left vertical equivalence can be identified with the one in diagram (6.10), we consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \arrow[d, dash, "\simeq"] & \mathrm{CAlg}(\mathrm{Alg}(\mathrm{CAlg}(\mathcal{C}))) \arrow[l, "\mathrm{ev}_{\langle 1\rangle}"', "\simeq"] \arrow[r, "\mathrm{CAlg}\left(\mathrm{Alg}\left(\mathrm{ev}_{\langle 1\rangle}\right)\right)"] \arrow[d, dash, "\simeq"] & \mathrm{CAlg}(\mathrm{Alg}(\mathcal{C})) \arrow[d, dash, "="] \\
\mathrm{CAlg}(\mathrm{Alg}(\mathcal{C})) & \mathrm{CAlg}(\mathrm{CAlg}(\mathrm{Alg}(\mathcal{C}))) \arrow[l, "\simeq"', "\mathrm{ev}_{\langle 1\rangle}"] \arrow[r, "\mathrm{CAlg}\left(\mathrm{ev}_{\langle 1\rangle}\right)"'] & \mathrm{CAlg}(\mathrm{Alg}(\mathcal{C}))
\end{tikzcd}$$

where the middle and left vertical equivalences are (induced by) the canonical equivalence exchanging the "inner" Alg and CAlg. By Proposition E.6.0.1, the bottom right horizontal functor is an equivalence, and the composite equivalence from the bottom left to the bottom right is homotopic to the identity functor. It follows that the bottom left vertical functor in diagram $(*)$ is an equivalence and that the composite left vertical equivalence can be identified with the left vertical equivalence in diagram (6.10).

We can argue completely analogously for the two right vertical functors in diagram $(*)$ being equivalences and the identification of the composite with the right vertical equivalence in diagram (6.10) – this time we need to exchange the "inner" $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, -)$ and CAlg. $\square$

Proposition 6.1.2.20. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Consider* *the composition*[42]

$$\mathrm{CAlg}(\mathcal{C}) \xrightarrow{\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})^{-1}} \mathrm{CAlg}(\mathrm{Alg}(\mathcal{C})) \xrightarrow{\mathrm{CAlg}(\mathrm{B}^{\mathrm{cyc}}_{\bullet})} \mathrm{CAlg}(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})) \xrightarrow{\simeq} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C}))$$

where the last functor is the canonical equivalence [HA, 2.1.3.4][43]*. This composition is left adjoint to the functor* $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$. ♡

Proof. Using Proposition 6.1.2.19 we can identify the composition in question with the following composition

$$\mathrm{CAlg}(\mathcal{C}) \xrightarrow{\mathrm{ev}_{\mathfrak{a}}^{-1}} \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \xrightarrow{\mathrm{B}^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}_{\bullet}} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C}))$$

where $\mathrm{B}^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}_{\bullet}$ is the cyclic bar construction with respect to the symmetric monoidal ∞-category $\mathrm{CAlg}(\mathcal{C})$. The claim now follows from Proposition 6.1.2.18, as the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is cocartesian. □

6.1.2.7 $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ and sifted colimits

The following statement concerning $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ and sifted colimits will be helpful later when we want to show that Hochschild homology is compatible with relative tensor products.

Proposition 6.1.2.21. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Let $\mathcal{I}$ be a small sifted ∞-category*[44]*, and assume that the symmetric monoidal structure of $\mathcal{C}$ is compatible with $\mathcal{I}$-indexed colimits in the sense of [HA, 3.1.1.18].*

Then the functor

$$\mathrm{B}^{\mathrm{cyc}}_{\bullet}\colon \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

from Definition 6.1.2.16 preserves $\mathcal{I}$-indexed colimits. ♡

Proof. Colimits in functor categories are detected pointwise by [HTT, 5.1.2.3], so it suffices to show that for every $m \geq 1$ the composition $\mathrm{ev}_{[m-1]_{\mathbf{\Lambda}}} \circ \mathrm{B}^{\mathrm{cyc}}_{\bullet}$ preserves $\mathcal{I}$-indexed colimits. Unpacking the definition of $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$, we can identify this composition with $\mathrm{ev}_{\langle m \rangle} \circ \otimes_* \circ A$, see Definition 6.1.2.16 and Proposition 6.1.2.11. Using Proposition 6.1.2.10 we can further identify this composition with

$$\mathrm{Alg}(\mathcal{C}) \xrightarrow{\mathrm{ev}_{\langle m \rangle}} \mathcal{C}^{\otimes}_{\langle m \rangle} \xrightarrow{(\mu_m)_!} \mathcal{C}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{C}$$

[42] $\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})$ can be identified with the composition

$$\mathrm{CAlg}(\mathrm{Alg}(\mathcal{C})) \simeq \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \xrightarrow{\mathrm{ev}_{\mathfrak{a}}} \mathrm{CAlg}(\mathcal{C})$$

and is thus an equivalence by [HA, 3.2.4.7 and 2.4.3.9].

[43] This equivalence arises from using that $\mathrm{Fun}(\mathsf{Fin}_*, -)$ preserves pullbacks and the $\times$-Fun-adjunction.

[44] See [HTT, 5.5.8.1] for a definition.

where $\mu_m \colon \langle m \rangle \to \langle 1 \rangle$ is the unique active morphism in Fin_*.

By [HA, 3.2.3.7], the functor $(\mu_m)_!$ appearing above preserves $\mathcal{I}$-indexed colimits, so it remains to show that

$$\mathrm{ev}_{\langle m \rangle} \colon \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}^{\otimes}_{\langle m \rangle}$$

also does so. The inert morphisms $\rho^i \colon \langle m \rangle \to \langle 1 \rangle$ determine natural transformations $\mathrm{ev}_{\rho^i} \colon \mathrm{ev}_{\langle m \rangle} \to \mathrm{ev}_{\langle 1 \rangle}$. By definition of $\mathrm{Alg}(\mathcal{C})$, these natural transformations will be componentwise inert morphisms in $\mathcal{C}^{\otimes}$ lying over ρ^i. It follows[45] that the natural transformation

$$\prod_{1 \leq i \leq m} \mathrm{ev}_{\rho^i} \colon \mathrm{ev}_{\langle m \rangle} \to \prod_{1 \leq i \leq m} \mathrm{ev}_{\langle 1 \rangle}$$

is a natural equivalence.

It thus suffices to show that

$$\prod_{1 \leq i \leq m} \mathrm{ev}_{\langle 1 \rangle} \colon \mathrm{Alg}(\mathcal{C}) \to \prod_{1 \leq i \leq m} \mathcal{C}$$

preserves $\mathcal{I}$-indexed colimits. As colimits in products of ∞-categories are detected componentwise by [HTT, 5.1.2.3], we are left to show that

$$\mathrm{ev}_{\langle 1 \rangle} \colon \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$$

preserves $\mathcal{I}$-indexed colimits, which is true by [HA, 3.2.3.1 (4)]. $\square$

6.1.3 Geometric realization of cyclic objects

Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category and $X \colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$ a cyclic object in $\mathcal{C}$. Recall from Construction 6.1.1.6 that there is a functor $j \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Lambda}^{\mathrm{op}}$, along which we can precompose X, obtaining a simplicial object j^*X. In this section we discuss how the extra automorphisms in $\mathbf{\Lambda}$ provide the structure of a $\mathbb{T}$-action on the geometric realization $|j^*X| = \operatorname{colim} j^*X$. We follow the approach of [Hoy18], but see also [NikSch, Appendix B].

We will start in Section 6.1.3.1 by briefly reviewing ∞-groupoid completions and the fact that the ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$ is $\mathrm{B}\,\mathbb{T}$, which will be needed to define the geometric realization functor for cyclic objects in Section 6.1.3.2. We will end in Section 6.1.3.3 by discussing monoidality of this construction.

6.1.3.1 The ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$

In this short section we recall that the ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$ is given by $\mathrm{B}\,\mathbb{T}$. We first introduce some notation.

[45] See Proposition A.3.2.1 and [HA, 2.1.1.14].

Notation 6.1.3.1. Let $\mathcal{C}$ be an ∞-category. We denote the *∞-groupoid completion* of $\mathcal{C}$ by $\mathcal{C}^{\mathrm{gpd}}$. Concretely $\mathcal{C}^{\mathrm{gpd}}$ is the ∞-groupoid obtained by inverting all morphisms of $\mathcal{C}$, and comes with a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{gpd}}$ that is initial among functors with domain $\mathcal{C}$ and whose codomain is an ∞-groupoid.

This construction can be made into a functor $-^{\mathrm{gpd}} \colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ that is left left adjoint to the inclusion, see [HTT, 1.2.5.6 and the preceding discussion] and [HA, 1.3.4.1]. ♢

We can now recall the following result about the ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$. The two references state their results as $\mathbf{\Lambda}^{\mathrm{gpd}} \simeq \mathrm{B}\,\mathbb{T}$, but Fact 6.1.3.2 can be immediately obtained from this by either using that $\mathbf{\Lambda}$ is self-dual by Fact 6.1.1.13 or using that $-^{\mathrm{gpd}}$ is compatible with passing to opposite ∞-categories and that ∞-groupoids are equivalent to their opposites.

Fact 6.1.3.2 ([Hoy18, 1.2], [NikSch, B.4])**.** *There is an equivalence*

$$(\mathbf{\Lambda}^{\mathrm{op}})^{\mathrm{gpd}} \simeq \mathrm{B}\,\mathbb{T}$$

of ∞-groupoids. ♧

6.1.3.2 Definition of the geometric realization

We now come to the definition of the geometric realization of cyclic objects. This will be defined as a left adjoint, so we start by showing that the left adjoint exists.

Proposition 6.1.3.3. *Let $\mathcal{C}$ be an ∞-category. Denote by $\phi\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathrm{B}\,\mathbb{T}$ the canonical functor exhibiting $\mathrm{B}\,\mathbb{T}$ as the ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$, see Fact 6.1.3.2. Then the following hold.*

(1) The functor
$$\phi^* \colon \operatorname{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{C}) \to \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$
is fully faithful, and its essential image is spanned by those functors that map every morphism in $\mathbf{\Lambda}^{\mathrm{op}}$ to an equivalence in $\mathcal{C}$.

(2) Assume that $\mathcal{C}$ is presentable. Then ϕ^* admits a left adjoint. ♡

Proof. Proof of claim (1): Holds by definition, see [HA, 1.3.4.1].

Proof of claim (2): By [HTT, 5.5.3.6], both $\operatorname{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{C})$ and $\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ are presentable. By the adjoint functor theorem [HTT, 5.5.2.9] it thus suffices to show that ϕ^* is accessible and preserves small limits. This follows immediately from the fact that limits and colimits in functor categories are calculated pointwise[46]. □

We can now make the following definition.

[46] See [HTT, 5.1.2.3] for the fact that (co)limits are calculated pointwise, and [HTT, 5.4.2.5 and 5.3.4.5] for the definition of accessible functors.

Definition 6.1.3.4 ([Hoy18, Page 2])**.** Let $\mathcal{C}$ be a presentable ∞-category. Then we denote the left adjoint to ϕ^* from Proposition 6.1.3.3 by

$$|-|\colon \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$$

and call it the *geometric realization* functor for cyclic objects. ♢

Remark 6.1.3.5. Let

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C}' \arrow[l, shift left, "G"] \end{tikzcd}$$

be an adjunction of ∞-categories, with $\mathcal{C}$ and $\mathcal{C}'$ both presentable.

Then compatibility of precomposing with postcomposing yields a commutative diagram

$$\begin{tikzcd} \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) & \mathcal{C}^{\mathrm{B}\,\mathbb{T}} \arrow[l, "\phi^*"'] \\ \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}') \arrow[u, "G_*"] & \mathcal{C}'^{\mathrm{B}\,\mathbb{T}} \arrow[l, "\phi^*"] \arrow[u, "G_*"'] \end{tikzcd}$$

so that, by passing to left adjoints and using Proposition D.2.2.1 and [HTT, 5.2.6.2] we obtain a commutative diagram

$$\begin{tikzcd} \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \arrow[r, "|-|"] \arrow[d, "F_*"'] & \mathcal{C}^{\mathrm{B}\,\mathbb{T}} \arrow[d, "F_*"] \\ \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}') \arrow[r, "|-|"'] & \mathcal{C}'^{\mathrm{B}\,\mathbb{T}} \end{tikzcd}$$

relating the geometric realization functors for $\mathcal{C}$ and $\mathcal{C}'$. ♢

We end this section with the following comparison between geometric realization of cyclic and simplicial objects, which gives a description of the underlying object of $|X|$ for a cyclic object X.

Fact 6.1.3.6 ([Hoy18, 1.1])**.** *Let $\mathcal{C}$ be a presentable ∞-category. Then there is a commutative square of ∞-categories as follows*

$$\begin{tikzcd} \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \arrow[r, "|-|"] \arrow[d, "j^*"'] & \mathcal{C}^{\mathrm{B}\,\mathbb{T}} \arrow[d, "\mathrm{ev}_*"] \\ \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}) \arrow[r, "|-|"'] & \mathcal{C} \end{tikzcd}$$

where ϕ is as in Construction 6.1.1.6, $*$ is the basepoint (i. e. the up to equivalence unique object) of $\mathrm{B}\,\mathbb{T}$, and the lower horizontal functor is the geometric realization functor for simplicial objects, so the functor $\operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}}$. ♧

6.1.3.3 Monoidality

If $\mathcal{C}$ is a presentable *symmetric monoidal* ∞-category, then $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ and $\mathcal{C}^{\mathrm{B}\,\mathbb{T}}$ can both be given the pointwise symmetric monoidal structure[47], with respect to which the functor ϕ^* from Proposition 6.1.3.3 can be upgraded to a symmetric monoidal functor. In this section we show that the geometric realization functor for cyclic objects can also be upgraded to a symmetric monoidal functor.

Proposition 6.1.3.7. *Let $\mathcal{O}$ be an ∞-operad and let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration of ∞-operads, and assume furthermore that $\mathcal{C}_X$ is presentable for every object X of $\mathcal{O}$, and that the $\mathcal{O}$-monoidal structure on $\mathcal{C}$ is compatible with small colimits in the sense of [HA, 3.1.1.18 and 3.1.1.19].*

Then the adjunctions $|-| \dashv \phi^*$ from Definition 6.1.3.4 for the presentable ∞-categories $\mathcal{C}_X$ for objects X of $\mathcal{O}$ can be upgraded to an adjunction relative to $\mathcal{O}^{\otimes}$ in the sense of [HA, 7.3.2.2]

$$\begin{tikzcd} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})^{\otimes} \arrow[rr, shift left, "(|-|)^{\otimes}"] \arrow[dr] & & \left(\mathcal{C}^{\mathrm{B}\,\mathbb{T}}\right)^{\otimes} \arrow[ll, shift left, "(\phi^*)^{\otimes}"] \arrow[dl] \\ & \mathcal{O}^{\otimes} & \end{tikzcd}$$

where the functors to $\mathcal{O}^{\otimes}$ are the canonical $\mathcal{O}$-monoidal functors that exhibit $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ and $\mathcal{C}^{\mathrm{B}\,\mathbb{T}}$ as equipped with the pointwise $\mathcal{O}$-monoidal structure.

Furthermore, both $(|-|)^{\otimes}$ and $(\phi^*)^{\otimes}$ are $\mathcal{O}$-monoidal functors. ♡

Proof. $(\phi^*)^{\otimes}$ is defined as the induced functor

$$\mathrm{Fun}\big(\mathrm{B}\,\mathbb{T}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{O}^{\otimes})} \mathcal{O}^{\otimes} \xrightarrow{\phi^* \times_{\phi^*} \mathrm{id}_{\mathcal{O}^{\otimes}}} \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{O}^{\otimes})} \mathcal{O}^{\otimes}$$

which by [HTT, 3.1.2.1] and Proposition C.1.1.1 preserves pr_2-cocartesian morphisms and is thus $\mathcal{O}$-monoidal. Furthermore, by Proposition 6.1.3.3 (1), the functors

$$\phi^*\colon \mathrm{Fun}\big(\mathrm{B}\,\mathbb{T}, \mathcal{C}^{\otimes}\big) \to \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}^{\otimes}\big)$$

and

$$\phi^*\colon \mathrm{Fun}\big(\mathrm{B}\,\mathbb{T}, \mathcal{O}^{\otimes}\big) \to \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{O}^{\otimes}\big)$$

are both fully faithful, with essential image spanned by those functors that map all morphisms to equivalences. It then follows from Proposition B.5.3.1 that $(\phi^*)^{\otimes}$ is also fully faithful, with essential image spanned by those objects which are mapped by pr_1 to functors that invert all morphisms. An object in $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})^{\otimes}$ lying over $X \simeq X_1 \oplus \cdots \oplus X_n$ in $\mathcal{O}^{\otimes}$ is mapped by pr_1 to a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}^{\otimes}$ that factors over the conservative inclusion of $\mathcal{C}^{\otimes}_X \simeq \mathcal{C}_{X_1} \times \cdots \times \mathcal{C}_{X_n}$. As morphisms in products of ∞-categories are equivalences if and only if their component morphisms are, we can hence identify

[47] See [HA, 2.1.3.4].

the essential image of $(\phi^*)^{\otimes}$ with the the induced ∞-operad structure as defined in [HA, Start of section 2.2.1] on the full subcategory $\mathrm{Fun}(\mathrm{B}\,\mathbb{T},\mathcal{C})$ of the the underlying ∞-category $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C})$ of the ∞-operad $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C})^{\otimes}$.

The claims will now follow from the conclusion of [HA, 2.2.1.9][48]. To verify the requirements to apply that result, it remains to show that the localization functors

$$\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C}_X) \xrightarrow{|-|} \mathrm{Fun}(\mathrm{B}\,\mathbb{T},\mathcal{C}_X) \xrightarrow{\phi^*} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C}_X)$$

for X an object of $\mathcal{O}$ are compatible with the $\mathcal{O}$-monoidal structure on $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C})^{\otimes}$ in the sense of [HA, 2.2.1.6].

So let $f\colon X_1 \oplus \cdots \oplus X_n \to Y$ be a morphism in $\mathcal{O}^{\otimes}$, with X_i and Y objects of $\mathcal{O}$. We obtain an induced functor on fibers

$$\prod_{1\le i\le n} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C}_{X_i}) \simeq \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C})^{\otimes}_{X_1\oplus\cdots\oplus X_n} \xrightarrow{f_!} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C})^{\otimes}_{Y} \simeq \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C}_Y)$$

and what we have to show is that if morphisms g_i are an mapped to equivalences by

$$|-|\colon \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C}_{X_i}) \to \mathrm{Fun}(\mathrm{B}\,\mathbb{T},\mathcal{C}_{X_i})$$

for each $1 \le i \le n$, then so is $f_!(g_1 \oplus \cdots \oplus g_n)$.

Using that the forgetful functor $\mathrm{ev}_*\colon \mathrm{Fun}(\mathrm{B}\,\mathbb{T},\mathcal{C}_Y) \to \mathcal{C}_Y$ detects equivalences by Proposition A.3.2.1, and combining this with Fact 6.1.3.6, this boils down to showing that

$$(\mathrm{ev}_* \circ |-|)(f_!(g_1 \oplus \cdots \oplus g_n)) \simeq \left(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \circ\, j^*\right)(f_!(g_1 \oplus \cdots \oplus g_n))$$

is an equivalence if $(\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \circ j^*)(g_i)$ is for every $1 \le i \le n$.

Let us unpack the functor $\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \circ j^* \circ f_!$. We have natural equivalences as follows, where C_i is an object of $\mathcal{C}_{X_i}$.

$$\left(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \circ\, j^* \circ f_!\right)(C_1 \oplus \cdots \oplus C_n)$$

Using that j^* is $\mathcal{O}$-monoidal with respect to the pointwise $\mathcal{O}$-monoidal structures on $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C})$ and $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathcal{C})$.

$$\simeq \left(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \circ f_!\right)(j^*C_1 \oplus \cdots \oplus j^*C_n)$$

Using the definition of the pointwise $\mathcal{O}$-monoidal structure.

$$\simeq \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \left(\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\prod_{1\le i\le n} \mathrm{id}_{\mathbf{\Delta}^{\mathrm{op}}}} \prod_{1\le i\le n} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\prod_{1\le i\le n} C_i \circ j} \prod_{1\le i\le n} \mathcal{C}_{X_i} \xrightarrow{f_!} \mathcal{C}_Y \right)$$

[48] That $|-|^{\otimes}_X$ will be given by $|-|\colon \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}},\mathcal{C}_X) \to \mathrm{Fun}(\mathrm{B}\,\mathbb{T},\mathcal{C}_X)$ for X an object of $\mathcal{O}$ follows from [HA, 7.3.2.5] and [HTT, 5.2.6].

Applying [HA, 3.2.3.7], which is applicable as the $\mathcal{O}$-monoidal structure of $\mathcal{C}$ is compatible with small colimits by assumption and $\mathbf{\Delta}^{\mathrm{op}}$ is sifted [HTT, 5.5.8.1 and 5.5.8.4].

$$\simeq f_!\left(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}\left(\prod_{1\leq i\leq n} C_i \circ j\right)\right)$$

Using that colimits in products are calculated pointwise [HTT, 5.1.2.3].

$$\simeq f_!\left(\bigoplus_{1\leq i\leq n} \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} C_i \circ j\right)$$

Thus the claim we need to show ultimately boils down to the following: If $g_i\colon C_i \to D_i$ induces an equivalence

$$\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(C_i \circ j) \to \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(D_i \circ j)$$

for every $1 \leq i \leq n$, then the induced morphism

$$f_!\left(\bigoplus_{1\leq i\leq n} \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(C_i \circ j)\right) \to f_!\left(\bigoplus_{1\leq i\leq n} \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(D_i \circ j)\right)$$

is an equivalence as well, which is clear. $\square$

Remark 6.1.3.8. Let $\mathcal{O}$ be an ∞-operad and let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ as well as $p_{\mathcal{C}'}\colon \mathcal{C}'^{\otimes} \to \mathcal{O}^{\otimes}$ be cocartesian fibrations of ∞-operads that both satisfy the conditions of Proposition 6.1.3.7. Let

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[rr, shift left, "F^{\otimes}"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p_{\mathcal{C}}"'] & & \mathcal{C}'^{\otimes} \arrow[ll, shift left, "G^{\otimes}"] \arrow[dl, "p_{\mathcal{C}}"] \\
& \mathcal{O}^{\otimes} &
\end{tikzcd}$$

be an adjunction relative to $\mathcal{O}^{\otimes}$ in the sense of [HA, 7.3.2.2 and 7.3.2.3], with both F and G being $\mathcal{O}$-monoidal.

Then proceeding like in Remark 6.1.3.5 and using Proposition 6.1.3.7, we can conclude that the commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \arrow[r, "|-|"] \arrow[d, "F_*"'] & \mathcal{C}^{\mathrm{B}\,\mathbb{T}} \arrow[d, "F_*"] \\
\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}') \arrow[r, "|-|"'] & \mathcal{C}'^{\mathrm{B}\,\mathbb{T}}
\end{tikzcd}$$

can be upgraded to a commutative diagram of $\mathcal{O}$-monoidal functors. $\diamondsuit$

6.2 Hochschild homology

In this section we finally define the functor

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{M}\mathrm{ixed}$$

that the chapters below will be about, and discuss some crucial first properties[49].

We will start with the definition in Section 6.2.1. In Section 6.2.2 we will then discuss different descriptions of Hochschild homology of *commutative* algebras. Finally, we will show in Section 6.2.3 that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserves relative tensor product, which will later be crucial for calculations.

6.2.1 Definition of Hochschild homology

We can now define Hochschild homology by specializing the general discussion of the cyclic bar construction and geometric realization of cyclic objects of Section 6.1 to the case of $\mathcal{D}(k)$. We can apply the definitions of $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ and $|-|$ to $\mathcal{D}(k)$ as it is a presentable symmetric monoidal ∞-category according to Proposition 4.3.2.1.

Definition 6.2.1.1. We define $\mathrm{HH}_{\mathbb{T}}$ to be the symmetric monoidal functor that is given as the composition

$$\mathrm{HH}_{\mathbb{T}}\colon \mathrm{Alg}(\mathcal{D}(k)) \xrightarrow{\mathrm{B}^{\mathrm{cyc}}_{\bullet}} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \xrightarrow{|-|} \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

where $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ is the symmetric monoidal functor from Definition 6.1.2.16 and $|-|$ is the symmetric monoidal functor from Definition 6.1.3.4 and Proposition 6.1.3.7.

We furthermore denote by

$$\mathrm{HH}\colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{D}(k)$$

the symmetric monoidal functor given by composing $\mathrm{HH}_{\mathbb{T}}$ with the symmetric monoidal functor ev_*.

We refer to both $\mathrm{HH}_{\mathbb{T}}$ and HH as the *Hochschild homology functor*. ◊

The reason we use the subscript $\mathbb{T}$ for $\mathrm{HH}_{\mathbb{T}}$ is to distinguish this functor from the composition with the equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ from Construction 5.4.0.1, as we will need to refer to both functors in later chapters. We thus also give the latter functor a name.

Definition 6.2.1.2. We define

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{M}\mathrm{ixed}$$

[49] We will compare $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ with the classical standard Hochschild complex in the next section, Section 6.3.

to be the monoidal functor obtained by composing the symmetric monoidal functor $\mathrm{HH}_{\mathbb{T}}$ from Definition 6.2.1.1 with the monoidal equivalence from Construction 5.4.0.1. ♢

Notation 6.2.1.3. If we evaluate HH, $\mathrm{HH}_{\mathbb{T}}$, or $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ at an object of the form $\mathrm{Alg}(\gamma)(R)$, with R an object of $\mathrm{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})$, then we will often omit γ from the notation and just write e. g. $\mathrm{HH}(R)$ instead of $\mathrm{HH}(\mathrm{Alg}(\gamma)(R))$. ♢

Warning 6.2.1.4. As the equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ from Construction 5.4.0.1 is only (associatively) monoidal, not symmetric monoidal, the same is true for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$. ♢

Remark 6.2.1.5. As the monoidal equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ that was constructed in Construction 5.4.0.1 is compatible with the forgetful functors to $\mathcal{D}(k)$, we obtain a homotopy

$$\mathrm{ev}_{\mathfrak{m}} \circ \mathrm{HH}_{\mathcal{M}\mathrm{ixed}} \simeq \mathrm{ev}_{*} \circ \mathrm{HH}_{\mathbb{T}} \simeq \mathrm{HH}$$

of monoidal functors. ♢

Remark 6.2.1.6. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then combining Remark 6.1.2.17 with Remark 6.1.3.8 applied to the adjunction from Remark 4.3.2.2 we obtain a commutative diagram of symmetric monoidal functors as follows.

$$\begin{tikzcd}
\mathrm{Alg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"] \arrow[d, "k'\otimes_k -"'] \arrow[rr, bend left, "\mathrm{HH}"] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[r, "\mathrm{ev}_{\mathfrak{m}}"] \arrow[d, "(k'\otimes_k -)_*"] & \mathcal{D}(k) \arrow[d, "k'\otimes_k -"] \\
\mathrm{Alg}(\mathcal{D}(k)') \arrow[r, "\mathrm{HH}_{\mathbb{T}}"'] \arrow[rr, bend right, "\mathrm{HH}"'] & \mathcal{D}(k')^{\mathrm{B}\,\mathbb{T}} \arrow[r, "\mathrm{ev}_{\mathfrak{m}}"'] & \mathcal{D}(k')
\end{tikzcd}$$

Combining the above with Remark 5.4.0.3 we also obtain a commutative diagram of monoidal functors as follows.

$$\begin{tikzcd}
\mathrm{Alg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}"] \arrow[d, "k'\otimes_k -"'] & \mathcal{M}\mathrm{ixed}_k \arrow[d, "k'\otimes_k -"] \\
\mathrm{Alg}(\mathcal{D}(k')) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}"'] & \mathcal{M}\mathrm{ixed}_{k'}
\end{tikzcd}$$

♢

6.2.2 Hochschild homology and commutative algebras

The functors $\mathrm{HH}_{\mathbb{T}}$ and HH defined in Definition 6.2.1.1 are symmetric monoidal functors and thus induce functors on ∞-categories of commutative algebras. In this section we will give different characterizations of those induced functors that will be of use later.

We will start in Section 6.2.2.1 by mostly fixing notation. In Section 6.2.2.3 we will show that if R is a commutative algebra in $\mathcal{D}(k)$, then $\mathrm{HH}_{\mathbb{T}}(R)$ can essentially be obtained as $R \boxtimes \underline{\mathbb{T}}$, i. e. tensoring R as an object of $\mathrm{CAlg}(\mathcal{D}(k))$ with $\mathbb{T}$, considered as a space with a $\mathbb{T}$-action. To properly discuss this, we will first introduce $- \boxtimes -$ and $\underline{\mathbb{T}}$ in Section 6.2.2.2. As an application of this description, we will show in Section 6.2.2.4 and Section 6.2.2.5 how interpret HH of commutative algebras as pushouts and relative tensor products in $\mathrm{CAlg}(\mathcal{D}(k))$.

6.2.2.1 HH for commutative algebras

As the functors HH and $\mathrm{HH}_{\mathbb{T}}$ from Definition 6.2.1.1 are both symmetric monoidal, they induce functors on ∞-categories of commutative algebras as well. By precomposing and postcomposing with canonical equivalences, we arrive at the following definitions.

Definition 6.2.2.1. We denote by $\mathrm{HH}_{\mathbb{T}}$ the composition

$$\mathrm{CAlg}(\mathcal{D}(k)) \xrightarrow{\simeq} \mathrm{CAlg}(\mathrm{Alg}(\mathcal{D}(k))) \xrightarrow{\mathrm{CAlg}(\mathrm{HH}_{\mathbb{T}})} \mathrm{CAlg}\Big(\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}\Big) \xrightarrow{\simeq} \mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\,\mathbb{T}}$$

where the individual functors are as follows.

- The first equivalence is the inverse of the following equivalence[50].

$$\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})\colon\ \mathrm{CAlg}(\mathrm{Alg}(\mathcal{D}(k))) \to \mathrm{CAlg}(\mathcal{D}(k))$$

[50]This functor can be identified with the composition of the equivalence

$$\mathrm{CAlg}(\mathrm{Alg}(\mathcal{D}(k))) \simeq \mathrm{BiFunc}(\mathsf{Comm}, \mathsf{Assoc}; \mathcal{D}(k))$$

from Proposition E.5.0.1, the equivalence

$$\mathrm{BiFunc}(\mathsf{Comm}, \mathsf{Assoc}; \mathcal{D}(k)) \simeq \mathrm{BiFunc}(\mathsf{Assoc}, \mathsf{Comm}; \mathcal{D}(k))$$

given by precomposing with the symmetry equivalence

$$\mathsf{Assoc}^{\otimes} \times \mathsf{Comm}^{\otimes} \simeq \mathsf{Comm}^{\otimes} \times \mathsf{Assoc}^{\otimes}$$

the equivalence

$$\mathrm{BiFunc}(\mathsf{Assoc}, \mathsf{Comm}; \mathcal{D}(k)) \simeq \mathrm{Alg}(\mathrm{CAlg}(\mathcal{D}(k)))$$

from Proposition E.5.0.1, and the functor

$$\mathrm{ev}_{\mathfrak{a}}\colon\ \mathrm{Alg}(\mathrm{CAlg}(\mathcal{D}(k))) \to \mathrm{CAlg}(\mathcal{D}(k))$$

that is an equivalence by [HA, 3.2.4.7 and 2.4.3.9].

- The functor $\mathrm{HH}_{\mathbb{T}}$ appearing in $\mathrm{CAlg}(\mathrm{HH}_{\mathbb{T}})$ refers to the symmetric monoidal functor from Definition 6.2.1.1.

- The second equivalence refers to the canonical equivalence, see [HA, 2.1.3.4].

We furthermore denote by HH the composition of the functor $\mathrm{HH}_{\mathbb{T}}$ above with the functor

$$\mathrm{ev}_*\colon \mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\,\mathbb{T}} \to \mathrm{CAlg}(\mathcal{D}(k))$$

that is given by evaluation at the basepoint. Equivalently, HH is the composition

$$\mathrm{CAlg}(\mathcal{D}(k)) \xrightarrow{\simeq} \mathrm{CAlg}(\mathrm{Alg}(\mathcal{D}(k))) \xrightarrow{\mathrm{CAlg}(\mathrm{HH})} \mathrm{CAlg}(\mathcal{D}(k))$$

where the equivalence is like above and the symmetric monoidal functor HH occurring in $\mathrm{CAlg}(\mathrm{HH})$ is the one from Definition 6.2.1.1. ♢

We next show that the definitions made in Definition 6.2.2.1 are compatible with the definitions from Definition 6.2.1.1 in the appropriate way.

Proposition 6.2.2.2. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"] \arrow[d, "p^*_{\mathsf{Assoc}}"'] & \mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\,\mathbb{T}} \arrow[d, "(\mathrm{ev}_{\langle 1\rangle})_*"] \\
\mathrm{Alg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"'] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}
\end{tikzcd} \tag{6.11}$$

where p_{Assoc} is the canonical morphism of ∞-operads $\mathsf{Assoc}^{\otimes} \to \mathsf{Comm}^{\otimes}$, the top horizontal functor is the one from Definition 6.2.2.1 and the bottom horizontal functor is the one from Definition 6.2.1.1.

Similarly, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}"] \arrow[d, "p^*_{\mathsf{Assoc}}"'] & \mathrm{CAlg}(\mathcal{D}(k)) \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"] \\
\mathrm{Alg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}"'] & \mathcal{D}(k)
\end{tikzcd} \tag{6.12}$$

in $\mathcal{C}\mathrm{at}_\infty$. ♡

Proof. Diagram (6.11) is obtained as the composite outer diagram of the following commutative diagram.

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"] \arrow[dd, bend right=60, "p^*_{\mathsf{Assoc}}"'] & \mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\,\mathbb{T}} \arrow[dd, bend left=60, "(\mathrm{ev}_{\langle 1\rangle})_*"] \\
\mathrm{CAlg}(\mathrm{Alg}(\mathcal{D}(k))) \arrow[u, "\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})"] \arrow[r, "\mathrm{CAlg}(\mathrm{HH}_{\mathbb{T}})"] \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"'] & \mathrm{CAlg}\Big(\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}\Big) \arrow[u, "\simeq"'] \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"] \\
\mathrm{Alg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}"'] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}
\end{tikzcd}$$

where the upper right vertical functor is the canonical equivalence. The top square commutes by definition of the top horizontal functor, the bottom square commutes by naturality of $\mathrm{ev}_{\langle 1\rangle}$, and commutativity of the right triangle is clear from the definition. It remains discuss the left triangle, which we obtain as the outer commutative triangle in the following commutative diagram

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{Alg}(\mathcal{D}(k))) \arrow[rrrr, bend left=30, "\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})"] \arrow[drr, "\mathrm{ev}_{\langle 1\rangle}"'] & & \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{D}(k))) \arrow[ll, "\mathrm{CAlg}(p^*_{\mathsf{Assoc}})"'] \arrow[rr, "\mathrm{CAlg}(\mathrm{ev}_{\langle 1\rangle})", "\mathrm{ev}_{\langle 1\rangle}"'] & & \mathrm{CAlg}(\mathcal{D}(k)) \arrow[dll, "p^*_{\mathsf{Assoc}}"] \\
& & \mathrm{Alg}(\mathcal{D}(k)) & &
\end{tikzcd}$$

where we use that $\mathrm{CAlg}(\mathrm{ev}_{\langle 1\rangle})$ and $\mathrm{ev}_{\langle 1\rangle}$ are homotopic and both equivalences by Proposition E.6.0.1 and that $\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})$, and hence $\mathrm{CAlg}(p^*_{\mathsf{Assoc}})$, are equivalences as well.

To obtain commutative diagram (6.12) from (6.11) it suffices to remark that there is an equivalence $\mathrm{ev}_* \circ (\mathrm{ev}_{\langle 1\rangle})_* \simeq \mathrm{ev}_{\langle 1\rangle} \circ \mathrm{ev}_*$. $\square$

6.2.2.2 Circle actions on tensor products with $\mathbb{T}$

There is one object with $\mathbb{T}$-action that is perhaps the most obvious non-trivial example: $\mathbb{T}$ acting on itself. Roughly, this action should be encoded in a functor $\mathrm{B}\,\mathbb{T} \to \mathcal{S}$ that maps the object $*$ to the underlying space of $\mathbb{T}$, and a morphism in $\mathrm{B}\,\mathbb{T}$, corresponding to an element t of $\mathbb{T}$, to the map $t \cdot - \colon \mathbb{T} \to \mathbb{T}$. A bit more rigorously, we could view $\mathbb{T}$ as an object in $\mathrm{LMod}_{\mathbb{T}}(\mathcal{S})$ using the morphism of ∞-operads $\mathsf{LM} \to \mathsf{Assoc}$ from [HA, 4.2.1.5], and then use the equivalence $\mathcal{S}^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{\mathbb{T}}(\mathcal{S})$ from Proposition 5.3.0.8. As yet another alternative approach, one can define the functor $\mathrm{B}\,\mathbb{T} \to \mathcal{S}$ as the left Kan extension along the inclusion $* \to \mathrm{B}\,\mathbb{T}$ of the functor $\mathrm{const}_*\colon * \to \mathcal{S}$, as discussed in [RSV21, Before 2.12]. We will follow [RSV21] in denoting this object of $\mathcal{S}^{\mathrm{B}\,\mathbb{T}}$ by $\underline{\mathbb{T}}$.

That $\underline{\mathbb{T}}$ defined as a left Kan extension is equivalent to the object with $\mathbb{T}$-action obtained from $\mathbb{T}$ as a left module over itself can be seen by using that the left Kan extension functor $\mathcal{S} \simeq \mathrm{Fun}(*, \mathcal{S}) \to \mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{S})$ is left adjoint to the forgetful functor ev_* by [HTT, 4.3.3.7], that the left-$\mathbb{T}$-module $\mathbb{T}$ can be described as the free $\mathbb{T}$-module generated by $*$ and so as the image of $*$ under the left adjoint of $\mathrm{ev}_{\mathfrak{m}}$ by [HA, 4.2.4.8], and that the equivalence $\mathcal{S}^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{\mathbb{T}}(\mathcal{S})$ is shown in Proposition 5.3.0.8 to be compatible with the respective forgetful functors to $\mathcal{S}$, and hence must also be compatible with their left adjoints.

Now let $\mathcal{C}$ be a presentable ∞-category. $\mathcal{S}$ is the unit object in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ by [HA, 4.8.1.20], so there is a unitality equivalence $\mathcal{C} \otimes \mathcal{S} \simeq \mathcal{C}$ that amounts to a functor

$$- \boxtimes - \colon \mathcal{C} \times \mathcal{S} \to \mathcal{C}$$

that preserves small colimits separately in each variable[51]. We thus obtain a colimit-preserving functor

$$- \boxtimes \mathbb{T} \colon \mathcal{C} \to \mathcal{C}$$

which we should lift to a functor as follows.

$$- \boxtimes \underline{\mathbb{T}} \colon \mathcal{C} \to \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$$

This is indeed the case, and this functor has in fact the following universal property.

Fact 6.2.2.3 ([RSV21, 2.12]). *Let $\mathcal{C}$ be a presentable ∞-category. Then there is an adjunction*

$$\mathcal{C} \underset{\mathrm{ev}_*}{\overset{-\boxtimes\underline{\mathbb{T}}}{\rightleftarrows}} \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$$

such that the composition $\mathrm{ev}_* \circ (- \boxtimes \underline{\mathbb{T}})$ *is equivalent to* $- \boxtimes \mathbb{T}$. ♧

6.2.2.3 HH of commutative algebras as a tensor product with $\mathbb{T}$

As $\mathcal{D}(k)$ is a presentable symmetric monoidal ∞-category by Proposition 4.3.2.1, we obtain that $\mathrm{CAlg}(\mathcal{D}(k))$ is presentable as well by [HA, 3.2.3.5 (2)]. We can thus apply Fact 6.2.2.3 and obtain a functor

$$- \boxtimes \underline{\mathbb{T}} \colon \ \mathrm{CAlg}(\mathcal{D}(k)) \to \mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\,\mathbb{T}}$$

which we will show is equivalent to the functor $\mathrm{HH}_{\mathbb{T}}$ from Definition 6.2.2.1[52].

[51] Compare with Section 5.2.2 for a more detailed related discussion.

[52] This claim also appears as Proposition IV.2.2 in [NikSch], but the proof only considers the underlying objects in $\mathcal{D}(k)$.

Proposition 6.2.2.4. *There is an adjunction*

$$\mathrm{CAlg}(\mathcal{D}(k)) \underset{\mathrm{ev}_*}{\overset{\mathrm{HH}_{\mathbb{T}}}{\rightleftarrows}} \mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\,\mathbb{T}}$$

where $\mathrm{HH}_{\mathbb{T}}$ *is the functor from Definition 6.2.2.1. Furthermore, there is a homotopy* $\mathrm{HH}_{\mathbb{T}} \simeq (- \boxtimes \underline{\mathbb{T}})$ *of functors from* $\mathrm{CAlg}(\mathcal{D}(k))$ *to* $\mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\,\mathbb{T}}$ *as well as* $\mathrm{HH} \simeq (- \boxtimes \mathbb{T})$ *of endofunctors of* $\mathrm{CAlg}(\mathcal{D}(k))$. ♡

Proof. It suffices to show the claim that $\mathrm{HH}_{\mathbb{T}}$ is left adjoint to ev_*, as the other two claims then follow immediately from Fact 6.2.2.3 by using uniqueness of left adjoints [HTT, 5.2.6] and the definition of HH as $\mathrm{ev}_* \circ \mathrm{HH}_{\mathbb{T}}$ in Definition 6.2.2.1.

Unpacking the definition of $\mathrm{HH}_{\mathbb{T}}$ in Definition 6.2.2.1 and Definition 6.2.1.1, the functor $\mathrm{HH}_{\mathbb{T}}$ of the statement is given by the composition

$$\begin{aligned}\mathrm{CAlg}(\mathcal{D}(k)) &\xrightarrow{\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})^{-1}} \mathrm{CAlg}(\mathrm{Alg}(\mathcal{D}(k)))\\ &\xrightarrow{\mathrm{CAlg}(\mathrm{B}^{\mathrm{cyc}}_{\bullet})} \mathrm{CAlg}(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)))\\ &\xrightarrow{\mathrm{CAlg}(|-|)} \mathrm{CAlg}\Big(\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}\Big)\\ &\xrightarrow{\simeq} \mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\,\mathbb{T}}\end{aligned} \tag{6.13}$$

where the last equivalence is the canonical one.

By Definition 6.1.3.4 the functor

$$|-|\colon\ \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

is left adjoint to ϕ^*, where $\phi\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathrm{B}\,\mathbb{T}$ is the canonical functor exhibiting $\mathrm{B}\,\mathbb{T}$ as the ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$, see Fact 6.1.3.2. Applying Proposition 6.1.3.7 and Proposition E.3.3.1 we obtain that $\mathrm{CAlg}(|-|)$ is left adjoint to $\mathrm{CAlg}(\phi^*)$. From the the commutative diagram

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k))) \arrow[d, "\simeq"'] & \mathrm{CAlg}\Big(\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}\Big) \arrow[l, "\mathrm{CAlg}(\phi^*)"'] \arrow[d, "\simeq"] \\
\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{D}(k))) & \mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\,\mathbb{T}} \arrow[l, "\phi^*"]
\end{tikzcd}$$

where the vertical equivalences are the canonical ones, together with uniqueness of adjoints, we obtain a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k))) \arrow[r, "\mathrm{CAlg}(|-|)"] \arrow[d, "\simeq"'] & \mathrm{CAlg}\Big(\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}\Big) \arrow[d, "\simeq"] \\
\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{D}(k))) \arrow[r, "|-|"'] & \mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\,\mathbb{T}}
\end{tikzcd}$$

We can thus identify the composition (6.13) with the following composition.

$$\begin{aligned}\mathrm{CAlg}(\mathcal{D}(k)) &\xrightarrow{\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})^{-1}} \mathrm{CAlg}(\mathrm{Alg}(\mathcal{D}(k)))\\ &\xrightarrow{\mathrm{CAlg}(\mathrm{B}^{\mathrm{cyc}}_{\bullet})} \mathrm{CAlg}(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)))\\ &\xrightarrow{\simeq} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{D}(k)))\\ &\xrightarrow{|-|} \mathrm{CAlg}(\mathcal{D}(k))^{\mathrm{B}\mathbb{T}}\end{aligned} \tag{6.14}$$

By Definition 6.1.3.4, $|-|$ is left adjoint to ϕ^* and by Proposition 6.1.2.20 the composition of the first three functors of (6.14) is left adjoint to $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$. It follows from composability of adjoints [HTT, 5.2.2.6] that the composition of all four functors of (6.14) is left adjoint to

$$\mathrm{ev}_{[0]_{\mathbf{\Lambda}}} \circ \phi^* \simeq \mathrm{ev}_{\phi([0]_{\mathbf{\Lambda}})} \simeq \mathrm{ev}_*$$

which is what needed to be shown. □

6.2.2.4 HH of commutative algebras as a pushout

The description of HH for commutative algebras from Proposition 6.2.2.4 allows us to derive the following alternative description that will be useful when comparing it to the classical standard Hochschild complex.

Proposition 6.2.2.5. *The functor*

$$\mathrm{HH}\colon \mathrm{CAlg}(\mathcal{D}(k)) \to \mathrm{CAlg}(\mathcal{D}(k))$$

from Definition 6.2.2.1 is homotopic to the functor that maps a commutative *algebra R to the pushout of*[53]

$$\begin{tikzcd} R \amalg R \arrow[r, "\mathrm{id}_R \amalg \mathrm{id}_R"] \arrow[d, "\mathrm{id}_R \amalg \mathrm{id}_R"'] & R \\ R & \end{tikzcd} \tag{6.15}$$

in $\mathrm{CAlg}(\mathcal{D}(k))$ *– the coproduct in the diagram is also to be taken in* $\mathrm{CAlg}(\mathcal{D}(k))$ *and is hence by [HA, 3.2.4.7] given by the tensor product.* ♡

Proof. By Proposition 6.2.2.4 the functor HH is homotopic to $- \boxtimes \mathbb{T}$. The underlying space of $\mathbb{T}$ is a 1-circle, and there is thus a pushout diagram

$$\begin{tikzcd} * \amalg * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathbb{T} \end{tikzcd}$$

in $\mathcal{S}$. As $- \boxtimes -$ preserves colimits in each variable separately[54], the claim immediately follows using that $- \boxtimes * \simeq \mathrm{id}$. ☐

6.2.2.5 HH of commutative algebras as a relative tensor product

As pushouts of commutative algebras can be calculated as relative tensor products, we obtain the following corollary of Proposition 6.2.2.5.

[53] Here is how to more rigorously define this functor. Let

$$\mathcal{I} = (\bullet \leftarrow \bullet \rightarrow \bullet) = [1] \amalg_{\{0\}} [1]$$

so that it suffices to construct a functor $\mathrm{CAlg}(\mathcal{D}(k)) \to \mathrm{Fun}(\mathcal{I}, \mathrm{CAlg}(\mathcal{D}(k)))$ that maps an object R to the diagram (6.15), for we can then compose this functor with the functor $\mathrm{colim}_{\mathcal{I}}$. Using the $\times$-Fun-adjunction, it suffices to construct a functor

$$\mathcal{I} \times \mathrm{CAlg}(\mathcal{D}(k)) \to \mathrm{CAlg}(\mathcal{D}(k))$$

for which it suffices to produce a commutative diagram as follows.

$$\begin{tikzcd} \{0\} \times \mathrm{CAlg}(\mathcal{D}(k)) \arrow[r] \arrow[d] & {[1]} \times \mathrm{CAlg}(\mathcal{D}(k)) \arrow[d, dashed] \\ {[1]} \times \mathrm{CAlg}(\mathcal{D}(k)) \arrow[r, dashed] & \mathrm{CAlg}(\mathcal{D}(k)) \end{tikzcd}$$

with the left vertical and top horizontal functor the inclusion. Each of the two other functors are to correspond to the natural transformation that sends R to $R \amalg R \xrightarrow{\mathrm{id} \amalg \mathrm{id}} R$, and taking the same functors there is an obvious filler for the diagram, so it suffices to construct this natural transformation.

The functor mapping R to $R \amalg R$ is the composition

$$\mathrm{CAlg}(\mathcal{D}(k)) \xrightarrow{\mathrm{const}} \mathrm{CAlg}(\mathcal{D}(k))^{* \amalg *} \xrightarrow{\mathrm{colim}} \mathrm{CAlg}(\mathcal{D}(k))$$

so as colim is left adjoint to the functor const (see [HTT, 4.2.4.3]) we obtain the required natural transformation as the counit of the adjunction.

[54] See Section 6.2.2.2.

Corollary 6.2.2.6. *The functor*

$$\mathrm{HH}\colon \mathrm{CAlg}(\mathcal{D}(k)) \to \mathrm{CAlg}(\mathcal{D}(k))$$

from Definition 6.2.2.1 is homotopic to the functor that maps a commutative algebra R to the relative tensor product in $\mathrm{CAlg}(\mathcal{D}(k))$

$$R \otimes_{R\otimes R} R$$

where the structure of R as a left and right $R \otimes R$-module arises from the morphism of commutative algebras

$$R \otimes R \simeq R \amalg R \xrightarrow{\mathrm{id}_R \amalg \mathrm{id}_R} R$$

– see Construction E.8.0.4 for more details on how to construct the necessary data to take the relative tensor product of of this. ♡

Proof. Follows immediately from combining Proposition 6.2.2.5 with Proposition E.8.0.5. □

Remark 6.2.2.7. If R is a commutative algebra in $\mathcal{D}(k)$, then the underlying morphism in $\mathcal{D}(k)$ of the morphism

$$R \otimes R \simeq R \amalg R \xrightarrow{\mathrm{id}_R \amalg \mathrm{id}_R} R$$

in $\mathrm{CAlg}(\mathcal{D}(k))$ can be identified with the multiplication morphism of R. This essentially follows from Proposition E.6.0.1[55]. ♢

6.2.3 Hochschild homology and relative tensor products

In this short section we show that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserves relative tensor products, which will be crucial later for calculating $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of certain quotients.

Proposition 6.2.3.1. *The functors* $\mathrm{HH}_{\mathbb{T}}$*,* $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$*, and* HH *from Definition 6.2.1.1 and Definition 6.2.1.2 preserve sifted colimits.*

In particular, all three functors being monoidal as well, they also preserve relative tensor products[56]. ♡

[55] Denote for the moment the functor

$$\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{D}(k))) \to \mathrm{CAlg}(\mathcal{D}(k))$$

by $\mathrm{ev}'_{\langle 1\rangle}$ to distinguish it from the following functor.

$$\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$$

Then the morphism in question is – as a morphism in $\mathrm{CAlg}(\mathcal{D}(k))$ – the multiplication morphism of the object R' in $\mathrm{CAlg}(\mathrm{CAlg}(\mathcal{D}(k)))$ corresponding to R under the equivalence $\mathrm{ev}'_{\langle 1\rangle}$. As $\mathrm{ev}_{\langle 1\rangle}$ is symmetric monoidal, it maps this morphism to the multiplication morphism of the commutative algebra in $\mathcal{D}(k)$ given by $\mathrm{CAlg}(\mathrm{ev}_{\langle 1\rangle})(R')$. We would like to identify this with R, and Proposition E.6.0.1 says that we can.

[56] See Remark E.8.0.2 for a discussion of what the statement that those functors preserve relative tensor products means.

Proof. As $\mathcal{D}(k)$ is presentable symmetric monoidal by Proposition 4.3.2.1 (1), the symmetric monoidal structure on $\mathcal{D}(k)$ is in particular compatible with sifted colimits, and hence we can apply Proposition 6.1.2.21 to conclude that

$$\mathrm{B}_\bullet^{\mathrm{cyc}} \colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathrm{Fun}(\boldsymbol{\Lambda}^{\mathrm{op}}, \mathcal{D}(k))$$

preserves sifted colimits. As a left adjoint, the geometric realization functor

$$|-| \colon \mathrm{Fun}(\boldsymbol{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

preserves all colimits, so in particular sifted colimits – see Definition 6.1.3.4 and [HTT, 5.2.3.5]. It thus follows that $\mathrm{HH}_{\mathbb{T}}$ and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserve sifted colimits, and as the forgetful functor $\mathrm{ev}_* \colon \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \to \mathcal{D}(k)$ preserves colimits by [HTT, 5.1.2.3] it also follows that HH preserves sifted colimits.

All three functors are monoidal by definition, so they also preserve relative tensor products by Proposition E.8.0.1. □

6.3 The standard Hochschild complex

In this section we review the classical definitions for Hochschild homology on the level of chain complexes. The main point is that if A is a differential graded algebra, then one can construct a strict mixed complex $\mathrm{C}(A)$ out of A, called the *standard Hochschild complex*, which represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(A)$. Similarly, when A is a *commutative* differential graded algebra, then the underlying chain complex of $\mathrm{C}(A)$ can be upgraded to a commutative differential graded algebra that represents $\mathrm{HH}(A)$.

We will start in Section 6.3.1 by reviewing the standard Hochschild complex for associative algebras, before treating the commutative case in Section 6.3.2. In Section 6.3.3 we will then discuss in what way $\gamma \colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ preserves relative tensor products, which will be relevant when we show that the standard Hochschild complex indeed represents Hochschild homology in Section 6.3.4.

6.3.1 The standard Hochschild complex for associative algebras

In Section 6.2.1 we defined a functor

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}} \colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{M}\mathrm{ixed}$$

called *Hochschild homology*. This was a definition on the level of the ∞-category $\mathcal{D}(k)$. There is also a classical definition of Hochschild homology constructed on the level of chain complexes, and we will recall the main definitions in this section[57]. We use the book [Lod98] as well as [Hoy18] the main references for this material.

[57] We will later see in Section 6.3.4.1 that the classical definition indeed represents the one from Section 6.2.1.

We will start in Section 6.3.1.1 by making concrete how the cyclic bar construction $\mathrm{B}_\bullet^{\mathrm{cyc}}$ looks like in the case of the symmetric monoidal category $\mathsf{Ch}(k)^{\mathrm{cof}}$. While in the definition of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ the next step would be the geometric realization functor for cyclic objects that would yield an object of $\left(\mathsf{Ch}(k)^{\mathrm{cof}}\right)^{\mathrm{B}\,\mathbb{T}}$, this is not sensible in this setting[58] – as $\mathsf{Ch}(k)^{\mathrm{cof}}$ is a 1-category, any functor $\mathrm{B}\,\mathbb{T} \to \mathsf{Ch}(k)^{\mathrm{cof}}$ factors though $\tau_{\leq 1}(\mathrm{B}\,\mathbb{T}) \simeq *$, so a $\mathbb{T}$-action on an object of $\mathsf{Ch}(k)^{\mathrm{cof}}$ yields no extra information. So in Section 6.3.1.2 we instead give a different construction that produces a strict mixed complex out of a cyclic object in chain complexes. We end in Section 6.3.1.3 by defining the standard Hochschild complex as the composite functor from $\mathrm{Alg}(\mathsf{Ch}(k))$ to Mixed.

6.3.1.1 The cyclic bar construction for chain complexes

$\mathsf{Ch}(k)$ is a symmetric monoidal category, so we can apply Definition 6.1.2.16 to obtain the cyclic bar construction functor $\mathrm{B}_\bullet^{\mathrm{cyc}}$. The next proposition makes this functor more concrete.

Proposition 6.3.1.1. *The functor*

$$\mathrm{B}_\bullet^{\mathrm{cyc}}\colon \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big)$$

from Definition 6.1.2.16 is given on a differential graded algebra A with cofibrant underlying complex by the following formulas[59]*.*

$$\begin{aligned}
\mathrm{B}_n^{\mathrm{cyc}}(A) &= A^{\otimes(n+1)} \\
d_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n \qquad \text{for } i < n \\
d_n(a_0 \otimes \cdots \otimes a_n) &= (-1)^{\deg_{\mathsf{Ch}}(a_n)\cdot\sum_{i=0}^{n-1}\deg_{\mathsf{Ch}}(a_i)} a_n \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \\
s_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \\
t(a_0 \otimes \cdots \otimes a_n) &= (-1)^{\deg_{\mathsf{Ch}}(a_n)\cdot\sum_{i=0}^{n-1}\deg_{\mathsf{Ch}}(a_i)} a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}
\end{aligned}$$

In particular, the restriction of $\mathrm{B}_\bullet^{\mathrm{cyc}}$ *to* $\mathrm{Alg}(\mathrm{LMod}_k(\mathsf{Ab}))$ *via the inclusion of chain complexes that are concentrated in degree 0*[60] *can be identified with the functor defined in [Lod98, 6.1.12]*[61] ♡

Proof. This amounts to unpacking the definition of the functors $-^\circ$ and V in Fact 6.1.1.13 and Fact 6.1.2.13 to see where the generators of $\mathbf{\Lambda}^{\mathrm{op}}$ are taken by $V \circ (-)^\circ$, and then applying Proposition 6.1.2.10[62]. □

[58] Even without asking for the construction to be compatible with $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$.

[59] See Notation 6.1.1.12 for the notation we use here.

[60] This implies that the signs in the formulas above vanish.

[61] Compare also to [Lod98, 1.6.1, 2.1.0, and 2.5.4] – there are though some differences in the signs, see [Lod98, 6.1.2.2].

[62] The signs arise from the signs in the symmetry isomorphism of the symmetric monoidal structure on $\mathsf{Ch}(k)$, see Definition 4.1.2.1.

6.3.1.2 Geometric realization of cyclic chain complexes

In Definition 6.2.1.2 we defined $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ as the composition of the cyclic bar construction with the geometric realization functor

$$\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

defined in Definition 6.1.3.4 and the equivalence

$$\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$$

from Construction 5.4.0.1. There is also a classical way of obtaining a strict mixed complex out of a cyclic chain complex, as we recall now.

Construction 6.3.1.2 ([Hoy18, Section 2] and [Lod98, 2.5.10])**.** Let $X_\bullet$ be an object in $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k))$. We then define a number of new operators on $X_\bullet$ as follows.

$$\begin{aligned}
&\partial^X\colon X_n \to X_{n-1}, && \partial^X \coloneqq \sum_{i=0}^{n} (-1)^i d_i\\
&s_{-1}\colon X_n \to X_{n+1}, && s_{-1} \coloneqq t \circ s_n\\
&t'\colon X_n \to X_n, && t' \coloneqq (-1)^n t\\
&N\colon X_n \to X_n, && N \coloneqq \sum_{i=0}^{n} t'^i\\
&\mathrm{d}\colon X_n \to X_{n+1}, && \mathrm{d} \coloneqq (\mathrm{id} - t') \circ s_{-1} \circ N
\end{aligned}$$

The operator ∂^X then satisfies $\partial^X \circ \partial^X = 0$ so that we can consider $X_\bullet$ together with ∂^X as a complex in $\mathsf{Ch}(k)$, i. e. a double complex[63], and hence can form the total complex, an object of $\mathsf{Ch}(k)$, by setting

$$\mathrm{Tot}\big(X_\bullet, \partial^X\big)_n \coloneqq \bigoplus_{i+j=n} (X_i)_j$$

and for x and element of $(X_i)_j$

$$\partial^{\mathrm{Tot}(X_\bullet, \partial^X)}(x) \coloneqq \partial^X(x) + (-1)^i \partial^{X_i}(x)$$

as the boundary operator[64].

[63] To be precise, we set

$$X_n \coloneqq \begin{cases} X_{[n]_{\mathbf{\Lambda}}} & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

so in particular, $(X_\bullet, \partial^X)$ is a upper half plane (or right half plane, depending on which way around one arranges the two indices) double complex.

[64] In the formula for the boundary operator, $\partial^X(x)$ is an element of $(X_{i-1})_j$ and $\partial^{X_i}(x)$ is an element of $(X_i)_{j-1}$.

The operator d induces morphisms $\mathrm{Tot}\big(X_\bullet, \partial^X\big)_* \to \mathrm{Tot}\big(X_\bullet, \partial^X\big)_{*+1}$ that we also denote by d, and the identities holding in $\mathbf{\Lambda}$ (see Construction 6.1.1.7) imply that d makes $\mathrm{Tot}\big(X_\bullet, \partial^X\big)$ into a strict mixed complex[65], see for example the arguments in [Lod98, Section 2.1].

This construction is functorial, and we denote the resulting functor

$$\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)) \to \mathsf{Mixed}$$

by $|-|_{\mathsf{Mixed}}$. Composing with the forgetful functor that maps strict mixed complexes to their underlying chain complexes we obtain a functor

$$\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)) \to \mathsf{Ch}(k)$$

that we denote by $|-|_{\mathsf{Ch}}$. ♢

Warning 6.3.1.3. Our notation deviates from the notation used in most previous work. We use ∂ and d instead of b and B, which is the notation used in for example [Lod98] and [Hoy18], which are the sources we have otherwise followed in Construction 6.3.1.2. The notation ∂ is widely used for the boundary operator of a chain complex[66], and d fits better with the relation to the mixed complex of de Rham forms, which will be introduced in Section 7.1.

Apart from the change of notation, the various operators in Construction 6.3.1.2 agree with the definitions in [Hoy18, Section 2]. The definitions also agree with the definitions given in [Lod98, 2.5.10] if we restrict to cyclic objects in $\mathrm{LMod}_k(\mathsf{Ab})$ (via the inclusion as chain complexes concentrated in degree 0). While the formulas in [Lod98, 2.5.10] differ by some signs, those arise from the fact that Loday does not actually define a mixed complex from the input of a cyclic object in chain complexes, but of a *cyclic module* as defined in [Lod98, 2.5.1]. While the data of a cyclic module and a cyclic object in $\mathrm{LMod}_k(\mathsf{Ab})$ are isomorphic, the isomorphism introduces some signs, see [Lod98, 6.1.2.2]. After composing Loday's construction with the isomorphism between cyclic objects in $\mathrm{LMod}_k(\mathsf{Ab})$ and cyclic modules, the signs cancel. ♢

Proposition 6.3.1.4. *If $X_\bullet$ is a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathsf{Ch}(k)$ that is pointwise cofibrant, then $|X_\bullet|_{\mathsf{Ch}}$ is cofibrant as well.*

We thus obtain a commutative diagrams as follows

$$\begin{tikzcd}
\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[r, "|-|_{\mathsf{Mixed}}"] \arrow[d] & \mathsf{Mixed}_{\mathrm{cof}} \arrow[r, "\mathrm{ev}_{\mathfrak{m}}"] \arrow[d] & \mathsf{Ch}(k)^{\mathrm{cof}} \arrow[d] \\
\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)) \arrow[r, "|-|_{\mathsf{Mixed}}"'] & \mathsf{Mixed} \arrow[r, "\mathrm{ev}_{\mathfrak{m}}"'] & \mathsf{Ch}(k)
\end{tikzcd}$$

[65]See Definition 4.2.1.2 and Remark 4.2.1.4 for the definition.

[66]So is d, but this would be very confusing when the mixed complex of de Rham forms shows up in Section 7.1.

where $\mathsf{Mixed}_{\mathrm{cof}}$ *is the full subcategory of* Mixed *spanned by those strict mixed complexes whose underlying chain complex is cofibrant (see Definition 4.2.1.2), and the vertical functors are (induced by) the inclusion of* $\mathsf{Ch}(k)^{\mathrm{cof}}$ *into* $\mathsf{Ch}(k)$. ♡

Proof. Let $X_\bullet$ be an object in $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}})$. Define a sequence

$$\ldots \to |X_\bullet|_{\mathsf{Ch}}^{\leq -1} \to |X_\bullet|_{\mathsf{Ch}}^{\leq 0} \to |X_\bullet|_{\mathsf{Ch}}^{\leq 1} \to \ldots$$

of sub-chain-complexes of $|X_\bullet|_{\mathsf{Ch}}$ by letting $|X_\bullet|_{\mathsf{Ch}}^{\leq m}$ be given by[67]

$$\left(|X_\bullet|_{\mathsf{Ch}}^{\leq m}\right)_n \coloneqq \bigoplus_{i+j=n, i\leq m} (X_i)_j$$

which one should think of as taking the total complex of the brutal truncation of $X_\bullet$ to degrees less than or equal to m.

Note that $|X_\bullet|_{\mathsf{Ch}}^{\leq m} \cong 0$ for $m < 0$, and $|X_\bullet|_{\mathsf{Ch}}$ is the colimit of the above sequence of inclusions. It thus suffices to show that $|X_\bullet|_{\mathsf{Ch}}^{\leq 0}$ is cofibrant and that each inclusion $|X_\bullet|_{\mathsf{Ch}}^{\leq m} \to |X_\bullet|_{\mathsf{Ch}}^{\leq m+1}$ is a cofibration.

That $|X_\bullet|_{\mathsf{Ch}}^{\leq 0}$ is cofibrant follows immediately from the assumption, as there is an obvious isomorphism $|X_\bullet|_{\mathsf{Ch}}^{\leq 0} \cong X_0$. So now let m be a nonnegative integer. Then there is a pushout diagram as follows

$$\begin{tikzcd}
S^m \otimes X_{m+1} \arrow[r, "\partial^X"] \arrow[d, "i\otimes \mathrm{id}_{X_{m+1}}"'] & {|X_\bullet|_{\mathsf{Ch}}^{\leq m}} \arrow[d] \\
D^{m+1} \otimes X_{m+1} \arrow[r] & {|X_\bullet|_{\mathsf{Ch}}^{\leq m+1}}
\end{tikzcd}$$

where S^m and D^{m+1} are as in [Hov99, 2.3.3][68] and i is the inclusion, ∂^X is to be understood as mapping $1 \otimes x$ to $\partial^X(x)$, which is defined as in Construction 6.3.1.2, and the right vertical morphism is the inclusion. As X_{m+1} was assumed to be a cofibrant chain complex and i is a cofibration, it follows from $\mathsf{Ch}(k)$ being a symmetric monoidal model category that the left vertical morphism, and hence also the right vertical morphism, are cofibrations. □

Remark 6.3.1.5. Construction 6.3.1.2 is clearly compatible with respect to extension of scalars. Specifically, let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the symmetric monoidal functor $k' \otimes_k -$ from $\mathsf{Ch}(k)^{\mathrm{cof}}$ to $\mathsf{Ch}(k')^{\mathrm{cof}}$

[67]The boundary operator of $|X_\bullet|_{\mathsf{Ch}}$ never increases i or j, so this indeed defines a sub-chain-complex.

[68]So S^m is $k[m]$, and D^{m+1} is concentrated in degrees m and $m+1$, with the boundary operator from degree $m+1$ to degree m being id_k.

(see Fact 4.1.5.1) induces an obvious commutative diagram

$$\begin{tikzcd} \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[r, "|-|_{\mathsf{Mixed}}"] \arrow[d, "(k'\otimes_k -)_*"'] & \mathsf{Mixed}_{k,\mathrm{cof}} \arrow[d, "k'\otimes_k -"] \\ \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[r, "|-|_{\mathsf{Mixed}}"'] & \mathsf{Mixed}_{k',\mathrm{cof}} \end{tikzcd}$$

of 1-categories. ♢

6.3.1.3 The standard Hochschild complex

Combining Sections 6.3.1.1 and 6.3.1.2 we obtain the following definition.

Definition 6.3.1.6. Composing the cyclic bar construction for associative algebras in $\mathsf{Ch}(k)^{\mathrm{cof}}$[69], with the functor $|-|_{\mathsf{Mixed}}$ from Construction 6.3.1.2 we obtain a functor

$$\mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathsf{Mixed}_{\mathrm{cof}}$$

that we denote by C and call the *standard Hochschild complex*. ♢

Remark 6.3.1.7. Combining functoriality of $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ (see Remark 6.1.2.17) and $|-|_{\mathsf{Mixed}}$ (see Remark 6.3.1.5) we can deduce that C is functorial in k. Concretely, if $\varphi\colon k \to k'$ is a morphism of commutative rings, then there is a commutative diagram

$$\begin{tikzcd} \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"] \arrow[d, "k'\otimes_k -"'] & \mathsf{Mixed}_{k,\mathrm{cof}} \arrow[d, "k'\otimes_k -"] \\ \mathrm{Alg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"'] & \mathsf{Mixed}_{k',\mathrm{cof}} \end{tikzcd}$$

in Cat. ♢

6.3.1.4 C for algebras concentrated in degree 0

In this section we discuss the standard Hochschild complex as defined in Definition 6.3.1.6 for k-algebras R with projective underlying k-module, which we consider as algebras in $\mathsf{Ch}(k)^{\mathrm{cof}}$ concentrated in degree 0.

Remark 6.3.1.8. The restriction of the standard Hochschild complex functor C as we defined it to k-algebras whose underlying k-module is projective agrees with the functor C defined in [Lod98], see [Lod98, Section 1.1, in particular 1.1.3, and section 2.1, in particular 2.1.7]. This follows from Proposition 6.3.1.1 and Warning 6.3.1.3. ♢

[69] See Proposition 6.3.1.1.

Going through the definitions, one obtains the following description.

Proposition 6.3.1.9. *Let R be a k-algebra with projective underlying k-module. Then the strict mixed complex $\mathrm{C}(R)$ is concentrated in nonnegative degrees and for $n \geq 0$ the following hold*[70]*.*

$$\begin{aligned}
\mathrm{C}_n(R) &= R^{\otimes(n+1)} \\
\partial(r_0 \otimes \cdots \otimes r_n) &= (-1)^n r_n \cdot r_0 \otimes r_1 \otimes \cdots \otimes r_{n-1} \\
&\quad + \sum_{i=0}^{n-1} (-1)^i r_0 \otimes \cdots \otimes r_i \cdot r_{i+1} \otimes \cdots \otimes r_n \\
\mathrm{d}(r_0 \otimes \cdots \otimes r_n) &= \sum_{i=0}^{n} (-1)^{in} 1 \otimes r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1} \\
&\quad + \sum_{i=0}^{n} (-1)^{in} r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1}
\end{aligned}$$

These formulas agree with the definitions used in [Lod98][71]*.* ♡

Proof. Follows directly by unpacking the definitions in Proposition 6.3.1.1 and Construction 6.3.1.2. Let us go through the steps for the last formula in a bit more detail. We use that d is defined as $(\mathrm{id} - t') \circ s_{-1} \circ N$, and go through the application of each composition factor individually. $r_0 \otimes \cdots \otimes r_n$ is mapped by N to the following element

$$\sum_{i=0}^{n} (-1)^{in} r_{n+1-i} \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{n-i}$$

where the summand indexed by $i = 0$ is to be interpreted as $r_0 \otimes \cdots \otimes r_n$. Using that $(n+1)n$ is even, we can replace i by $n+1-i$ to rewrite the above expression as

$$\sum_{i=1}^{n+1} (-1)^{in} r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1}$$

which is also equal to the following sum, as the summand for $i = 0$ is equal to the one for $i = n + 1$.

$$\sum_{i=0}^{n} (-1)^{in} r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1}$$

[70] For $n = 0$ we instead have $\partial(r_0) = 0$.

[71] For the boundary operator, see [Lod98, 1.1.1]. For the differential a formula is given in [Lod98, 2.1.7.3], which is though differing from our formula by the sign before the second sum, which is presumably due to a typo – the definition given in [Lod98, 2.1.7.1 and 2.1.0] yields the formula we have stated above. That there must be a typo in [Lod98] around this formula can also be seen by comparing with the formulas for $B(a_0)$ and $B(a_0, a_1)$ given just below [Lod98, 2.1.7.3], which are compatible with the sign as in the formula stated above, but not the sign in [Lod98, 2.1.7.3].

The effect of applying s_{-1} can be described as inserting a tensor factor 1 at the start, so the above expression is mapped by s_{-1} to the following.

$$\sum_{i=0}^{n}(-1)^{in}1\otimes r_i\otimes\cdots\otimes r_n\otimes r_0\otimes\cdots\otimes r_{i-1}$$

Finally, applying $\mathrm{id}-t'$ we obtain

$$\begin{aligned}&\sum_{i=0}^{n}(-1)^{in}1\otimes r_i\otimes\cdots\otimes r_n\otimes r_0\otimes\cdots\otimes r_{i-1}\\ -(-1)^{n+1}&\sum_{i=0}^{n}(-1)^{in}r_{i-1}\otimes 1\otimes r_i\otimes\cdots\otimes r_n\otimes r_0\otimes\cdots\otimes r_{i-2}\end{aligned}$$

and after replacing i by $i-1$ in the second sum to remove the sign due to

$$-(-1)^{n+1}(-1)^n=1$$

and using that the resulting summands for $i=0$ and $i=n+1$ are equal we finally obtain the following.

$$\begin{aligned}&=\sum_{i=0}^{n}(-1)^{in}1\otimes r_i\otimes\cdots\otimes r_n\otimes r_0\otimes\cdots\otimes r_{i-1}\\ &+\sum_{i=0}^{n}(-1)^{in}r_i\otimes 1\otimes r_{i+1}\otimes\cdots\otimes r_n\otimes r_0\otimes\cdots\otimes r_{i-1}\end{aligned}$$ □

6.3.1.5 The normalized standard Hochschild complex

To simplify formulas it is often useful to divide out a particularly easy to describe acyclic subcomplex of $\mathrm{C}(R)$, spanned by elements of the form $r_0\otimes\cdots\otimes r_n$ with one of the elements $r_1,\ldots,r_n$ being equal to 1. We only use this for the case where R is concentrated in degree 0 and refer to [Lod98, 1.1.14] for more details.

Proposition 6.3.1.10 ([Lod98, 1.1.14 and 1.1.15])**.** *Let R be a k-algebra with projective underlying k-module. We define $\overline{R}$ to be the quotient $R/(k\cdot 1)$ of k-modules, where $k\cdot 1$ is the k-submodule of R spanned by the unit 1. We will use the notation $\overline{r}$ for the image of an element r of R under the quotient map $R\to\overline{R}$. Define*

$$\overline{\mathrm{C}}_n(R):=\begin{cases}R\otimes\overline{R}^{\otimes n} & \text{if } n\geq 0\\ 0 & \text{otherwise}\end{cases}$$

for integers n and note that $\overline{\mathrm{C}}_n(R)$ is a quotient of $\mathrm{C}_n(R)$.

Then the strict mixed complex structure of $\mathrm{C}(R)$ *induces a strict mixed complex structure on* $\overline{\mathrm{C}}(R)$ *such that the following simplified formula holds for the differential.*

$$\mathrm{d}(r_0 \otimes \overline{r_1} \otimes \cdots \otimes \overline{r_n}) = \sum_{i=0}^{n} (-1)^{in} 1 \otimes \overline{r_i} \otimes \cdots \otimes \overline{r_n} \otimes \overline{r_0} \otimes \cdots \otimes \overline{r_{i-1}}$$

Furthermore, the morphism of strict mixed complexes

$$\mathrm{C}(R) \to \overline{\mathrm{C}}(R), \qquad r_0 \otimes r_1 \otimes \cdots \otimes r_n \mapsto r_0 \otimes \overline{r_1} \otimes \cdots \otimes \overline{r_n}$$

determines a natural transformation $\mathrm{C} \to \overline{\mathrm{C}}$ *of functors from the category of* k*-algebras with projective underlying* k*-module to* Mixed *that is pointwise a quasiisomorphism.*

We call $\overline{\mathrm{C}}(R)$ *the* normalized standard Hochschild complex. ♡

Proof. That $\overline{\mathrm{C}}(R)$ obtains an induced chain complex structure is [Lod98, 1.6.4] and that the quotient morphism $\mathrm{C}(R) \to \overline{\mathrm{C}}(R)$ is a quasiisomorphism is shown in [Lod98, 1.1.15 and 1.6.5]. That these quotient morphisms assemble to a natural transformation as claimed follows directly from the definition.

That the kernel of $\mathrm{C}(R) \to \overline{\mathrm{C}}(R)$ is closed under d is clear by looking at the formula given for d in Proposition 6.3.1.9, and the expression for the induced operator d on $\overline{\mathrm{C}}(R)$ also follows immediately. See also [Lod98, 2.1.9]. □

Remark 6.3.1.11. Functoriality of C with respect to change of scalars as discussed in Remark 6.3.1.7 passes to $\overline{\mathrm{C}}$. In particular, for $\varphi\colon k \to k'$ a morphism of commutative rings there exists a dashed natural isomorphism fitting into a commutative diagram

$$\begin{tikzcd}
\mathrm{C}(k' \otimes_k -) \arrow[r] \arrow[d, "\cong"'] & \overline{\mathrm{C}}(k' \otimes_k -) \arrow[d, dashed, "\cong"] \\
k' \otimes_k \mathrm{C}(-) \arrow[r] & k' \otimes_k \overline{\mathrm{C}}(-)
\end{tikzcd}$$

of functors from the category k-algebras with projective underlying k-module to $\mathsf{Mixed}_{k'}$. The top and bottom natural transformations are (induced by) the ones from Proposition 6.3.1.10 and the left natural isomorphism is the one from Remark 6.3.1.7. ♢

6.3.2 The standard Hochschild complex for commutative algebras

The functor

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{M}\mathrm{ixed}$$

is monoidal and hence induces a functor on ∞-categories of (associative) algebras. Unfortunately, the standard Hochschild complex functor

$$\mathrm{C}\colon \operatorname{Alg}\bigl(\mathsf{Ch}(k)^{\mathrm{cof}}\bigr) \to \mathsf{Mixed}_{\mathrm{cof}}$$

that was defined in Definition 6.3.1.6 is *not* monoidal and not even lax or colax monoidal, see [Lod98, 4.3.1] and [Kas87]. To get around this for Künneth-type-formulas, one can employ a weakened notion of morphism between strict mixed complexes that is called *strongly homotopy linear map* in [Kas87] and *S-morphism* in [Lod98] – see [Kas87, 2.2] and [Lod98, 2.5.14]. This is a morphism of underlying chain complexes that need not strictly commute with d, but only up to specified homotopy, which in turn also does not need to strictly commute with d, but up to specified homotopy, and so on. For a more detailed discussion of strongly homotopy linear morphisms see Section 4.2.3.

We take the necessity to consider these kind of sequences of higher homotopies as a hint that if one is interested in both the mixed structure as well as (symmetric) monoidal structure, then one should work at the level of ∞-categories and consider the functor $\mathrm{HH}_{\mathbb{T}}$. From this perspective, that C may not be fully adequate to consider both mixed and multiplicative structures can also be expected from the fact that while $\mathrm{HH}_{\mathbb{T}}$ and HH are symmetric monoidal, $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ has only been shown to be (associatively) monoidal – so it would be unexpected for C as a functor to $\mathsf{Mixed}_{\mathrm{cof}}$ to be *symmetric* monoidal[72].

To nevertheless be able to do some calculations on the level of chain complexes regarding multiplicative structures, we forget about the strict mixed complex structure, and only consider C as a functor to $\mathsf{Ch}(k)^{\mathrm{cof}}$.

To bring the standard Hochschild complex functor C, considered as a functor to $\mathsf{Ch}(k)^{\mathrm{cof}}$, into a form that is more amenable for our purposes, we discuss the *bar resolution* $\mathrm{C}^{\mathrm{Bar}}(A)$ of an associative algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$ in Section 6.3.2.1, which will allow us to rewrite $\mathrm{C}(A)$ as a relative tensor product $\mathrm{C}(A) \cong A \otimes_{A\otimes A^{\mathrm{op}}} \mathrm{C}^{\mathrm{Bar}}(A)$ in Section 6.3.2.2. We will also show that as a left-$A \otimes A^{\mathrm{op}}$-module, $\mathrm{C}^{\mathrm{Bar}}(A)$ is a *cofibrant* replacement of A, which will be relevant in Section 6.3.4.2, where we compare the standard Hochschild complex to HH. In Section 6.3.2.3 we then introduce the *shuffle product* on $\mathrm{C}^{\mathrm{Bar}}(A)$, and upgrade all the relevant constructions to commutative algebras – provided that A itself was commutative. This will allow us to describe the standard Hochschild complex of a commutative differential graded algebra with cofibrant underlying chain complex as an object of $\operatorname{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$ in Section 6.3.2.4.

6.3.2.1 The bar resolution

In this section we introduce the bar resolution, that will be used in Section 6.3.2.2 to give an alternative description of the standard Hochschild complex of Definition 6.3.1.6. We closely follow [Lod98, 1.1.11 to 1.1.13], though

[72] At least in a homotopically meaningful way that is compatible with $\mathrm{HH}_{\mathbb{T}}$.

we also consider differential graded algebras that are not concentrated in degree 0.

Construction 6.3.2.1. [Lod98, 1.1.11 to 1.1.13] Let A be an associative algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$.

We let $\mathrm{Bar}_A(A, A)_\bullet$ be the chain complex in $\mathsf{Ch}(k)$ (so a double complex) that is determined by the following formulas.

$$\mathrm{Bar}_A(A, A)_n \coloneqq A \otimes A^{\otimes n} \otimes A$$

$$\partial^{\mathrm{Bar}_A(A,A)_\bullet}(a_0 \otimes \cdots \otimes a_{n+1}) \coloneqq \sum_{i=0}^{n} (-1)^i (a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_{n+1})$$

We then define $\mathrm{C}^{\mathrm{Bar}}(A)$, called the *bar resolution* of A, to be the total complex of $\mathrm{Bar}_A(A, A)_\bullet$, so we let

$$\mathrm{C}^{\mathrm{Bar}}(A)_n \coloneqq \bigoplus_{i+j=n} (\mathrm{Bar}_A(A, A)_i)_j = \bigoplus_{i+j=n} \left(A^{\otimes(i+2)}\right)_j$$

and for a an element of $(\mathrm{Bar}_A(A, A)_i)_j$ we define the boundary operator as follows.

$$\partial^{\mathrm{C}^{\mathrm{Bar}}(A)}(a) \coloneqq \partial^{\mathrm{Bar}_A(A,A)_\bullet}(a) + (-1)^i \partial^{A \otimes A^{\otimes i} \otimes A}(a)$$

Note that if A is concentrated in degree 0, then $\mathrm{C}^{\mathrm{Bar}}(A)$ is precisely the complex C_*^{bar} defined in [Lod98, 1.1.11].

There are two important extra pieces of structure regarding $\mathrm{C}^{\mathrm{Bar}}(A)$ that we will also need.

The first is that there is a natural morphism of chain complexes

$$\mathrm{C}^{\mathrm{Bar}}(A) \to A$$

that is defined by the formula

$$(a_0 \otimes \cdots \otimes a_{i+1}) \mapsto \begin{cases} a_0 \cdot a_1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

an again, this is precisely the augmentation of C_*^{bar} as defined in [Lod98, 1.1.11] if A is concentrated in degree 0.

The second extra piece of structure is that $\mathrm{C}^{\mathrm{Bar}}(A)$ can be given the structure of a left module over $A \otimes A^{\mathrm{op}}$, where A^{op} refers to the opposite algebra of A, i. e. the differential graded algebra with the same underlying chain complex, but if we denote the multiplication in A with $\cdot$ and in A^{op} with $\star$, then $\star$ is defined as $a \star a' \coloneqq (-1)^{\deg_{\mathsf{Ch}}(a) \cdot \deg_{\mathsf{Ch}}(a')} a' \cdot a$. The left module structure is then defined via the following formula.

$$(a \otimes a') \cdot (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) \coloneqq$$
$$(-1)^{\left(\sum_{i=0}^{n+1} \deg_{\mathsf{Ch}}(a_i)\right) \deg_{\mathsf{Ch}}(a')} ((a \cdot a_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes (a_{n+1} \cdot a'))$$

One can similarly define a left-$A \otimes A^{\mathrm{op}}$-module structure on A, via

$$(a \otimes a') \cdot a'' \coloneqq (-1)^{\deg_{\mathsf{Ch}}(a') \deg_{\mathsf{Ch}}(a'')} a \cdot a'' \cdot a'$$

and this makes the morphism of chain complexes $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ into a morphism of left-$A \otimes A^{\mathrm{op}}$-modules. This structure is again (for A concentrated in degree 0) exactly the one considered in [Lod98, 1.1.13].

The above constructions can be summarized in the following diagram

$$\begin{tikzcd}
\mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr, bend left, "\mathrm{C}^{\mathrm{Bar}}"] \arrow[rr, bend right, "A \mapsto A"'] \arrow[rr, phantom, "\Downarrow"] \arrow[dr, "A \mapsto A \otimes A^{\mathrm{op}}"'] & & \mathrm{LMod}(\mathsf{Ch}(k)) \arrow[dl] \\
& \mathrm{Alg}(\mathsf{Ch}(k)) &
\end{tikzcd}$$

where the functor on the right is the forgetful functor, the bottom functor at the top maps A to A considered as a left-$A \otimes A^{\mathrm{op}}$-module as described above, and the natural transformation at the top lies over the identity natural transformation of $A \mapsto A \otimes A^{\mathrm{op}}$. ♢

To show that the terminology "bar *resolution*" is reasonable, we will now prove that $\mathrm{C}^{\mathrm{Bar}}(A)$ is cofibrant as a left-$A \otimes A^{\mathrm{op}}$-module, as well as quasiisomorphic to A.

Proposition 6.3.2.2 ([Lod98, 1.1.12]). *Let A be an associative algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$. Then the morphism*

$$\mathrm{C}^{\mathrm{Bar}}(A) \to A$$

of chain complexes constructed in Construction 6.3.2.1 is a quasiisomorphism. ♡

Proof. The proof is an immediate generalization of the proof of [Lod98, 1.1.12], though we need to add some signs to account for elements of A in odd degrees. So let ϕ denote the morphism $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ and let $\psi\colon A \to \mathrm{C}^{\mathrm{Bar}}(A)$ be the morphism of chain complexes that maps a to $1 \otimes a$. Then $\phi \circ \psi = \mathrm{id}_A$, so it suffices to construct a homotopy h between $\mathrm{id}_{\mathrm{C}^{\mathrm{Bar}}(A)}$ and $\psi \circ \phi$. For this, define h via

$$h(a_0 \otimes \cdots \otimes a_{n+1}) \coloneqq 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$$

by k-linearly extending.

If $n > 0$ we then have

$$\Big(\partial^{\mathrm{C}^{\mathrm{Bar}}(A)} \circ h + h \circ \partial^{\mathrm{C}^{\mathrm{Bar}}(A)}\Big)(a_0 \otimes \cdots \otimes a_{n+1})$$

$$
\begin{aligned}
&= 1 \cdot a_0 \otimes \cdots \otimes a_{n+1} - \sum_{i=0}^{n} (-1)^i (1 \otimes a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_{n+1}) \\
&\quad + (-1)^{n+1} \sum_{i=0}^{n+1} (-1)^{\sum_{j=0}^{i-1} \deg_{\mathsf{Ch}}(a_j)} \big(1 \otimes a_0 \otimes \cdots \otimes \partial^A(a_i) \otimes \cdots \otimes a_{n+1}\big) \\
&\quad + \sum_{i=0}^{n} (-1)^i (1 \otimes a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_{n+1}) \\
&\quad + (-1)^{n} \sum_{i=0}^{n+1} (-1)^{\sum_{j=0}^{i-1} \deg_{\mathsf{Ch}}(a_j)} \big(1 \otimes a_0 \otimes \cdots \otimes \partial^A(a_i) \otimes \cdots \otimes a_{n+1}\big) \\
&= a_0 \otimes \cdots \otimes a_{n+1}
\end{aligned}
$$

while for $n = 0$ the third term does not appear, so that we obtain

$$
\begin{aligned}
&\Big(\partial^{\mathrm{C}^{\mathrm{Bar}}(A)} \circ h + h \circ \partial^{\mathrm{C}^{\mathrm{Bar}}(A)}\Big)(a_0 \otimes a_1) \\
&= a_0 \otimes a_1 - 1 \otimes a_0 \cdot a_1 \\
&= (\mathrm{id} - \psi \circ \phi)(a_0 \otimes a_1)
\end{aligned}
$$

which shows that h is a homotopy as required. □

Proposition 6.3.2.3. *Let A be an associative algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$. Then $\mathrm{C}^{\mathrm{Bar}}(A)$ as defined in Construction 6.3.2.1 is cofibrant as a left-$A \otimes A^{\mathrm{op}}$-module with respect to the model structure of Theorem 4.2.2.1.*

In particular, the underlying chain complex of $\mathrm{C}^{\mathrm{Bar}}(A)$ is cofibrant. ♡

Proof. Let us begin by noting that the second claim, that the underlying chain complex of $\mathrm{C}^{\mathrm{Bar}}(A)$ is cofibrant, follows from the first claim by applying Theorem 4.2.2.1 (8), which is applicable as the underlying chain complex of A is cofibrant by assumption.

Let $\mathrm{Bar}_A^{\leq m}(A, A)_\bullet$ be the chain complex in $\mathsf{Ch}(k)$ defined as the brutal truncation to degrees smaller or equal to m of $\mathrm{Bar}_A(A, A)_\bullet$ from Construction 6.3.2.1, i. e.

$$
\mathrm{Bar}_A^{\leq m}(A, A)_n \coloneqq \begin{cases} A \otimes A^{\otimes n} \otimes A & \text{if } n \leq m \\ 0 & \text{otherwise} \end{cases}
$$

and with boundary operator defined by the same formula as in Construction 6.3.2.1.

We then let $\mathrm{C}^{\mathrm{Bar}}_{\leq m}(A)$ be the total complex of $\mathrm{Bar}_A^{\leq m}(A, A)_\bullet$, which concretely means that $\mathrm{C}^{\mathrm{Bar}}_{\leq m}(A)$ is given in level n by $\bigoplus_{i+j=n, i \leq m} (A^{\otimes (i+2)})_j$.

Note that the left-$A \otimes A^{\mathrm{op}}$-module structure restricts from $\mathrm{C}^{\mathrm{Bar}}(A)$ to $\mathrm{C}^{\mathrm{Bar}}_{\leq m}(A)$, and $\mathrm{C}^{\mathrm{Bar}}(A)$ is the colimit of the sequence

$$
\mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A) \to \mathrm{C}^{\mathrm{Bar}}_{\leq 1}(A) \to \mathrm{C}^{\mathrm{Bar}}_{\leq 2}(A) \to \dots
$$

so that it suffices to show that $\mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A)$ is cofibrant and each of the morphisms

$$\mathrm{C}^{\mathrm{Bar}}_{\leq m}(A) \to \mathrm{C}^{\mathrm{Bar}}_{\leq m+1}(A)$$

is a cofibration.

For $\mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A)$ we note that

$$\mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A) \cong A \otimes A^{\mathrm{op}}$$

as left-$A \otimes A^{\mathrm{op}}$-modules, so $\mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A)$ is isomorphic to the free left-$A \otimes A^{\mathrm{op}}$-module generated by k and hence cofibrant, as k is cofibrant in $\mathsf{Ch}(k)$.

For $m \geq 0$ there is an evident pushout diagram in $\mathsf{Ch}(\mathsf{Ch}(k))$

$$\begin{tikzcd}
A^{\otimes m+3} \otimes S^m \arrow[r] \arrow[d, "\mathrm{id}_{A^{\otimes m+3}} \otimes i'"] & \mathrm{Bar}^{\leq m}_A(A, A)_\bullet \arrow[d] \\
A^{\otimes m+3} \otimes D^{m+1} \arrow[r] & \mathrm{Bar}^{\leq (m+1)}_A(A, A)_\bullet
\end{tikzcd}$$

where $A^{\otimes m+3}$ is concentrated in degree 0 with respect to the "outer" chain degree, S^m is the chain complex in $\mathsf{Ch}(k)$ that is concentrated in degree m, where it is $k[0]$, the complex D^{m+1} is concentrated in degrees m and $m+1$, where it is $k[0]$, with the boundary operator from degree $m+1$ to degree m the identity morphism, and i' is the inclusion.

As the formation of the total complex preserves pushouts, we obtain a pushout diagram in $\mathsf{Ch}(k)$. It is not difficult to see that that square can be considered as a commutative square of left-$A \otimes A^{\mathrm{op}}$-modules of the following form

$$\begin{tikzcd}
\mathrm{Free}^{\mathrm{LMod}_{A\otimes A^{\mathrm{op}}}}\left(A^{\otimes m+1} \otimes S^m\right) \arrow[r] \arrow[d, "\mathrm{Free}^{\mathrm{LMod}_{A\otimes A^{\mathrm{op}}}}\left(\mathrm{id}_{A^{\otimes m+1}} \otimes i\right)"'] & \mathrm{C}^{\mathrm{Bar}}_{\leq m}(A) \arrow[d] \\
\mathrm{Free}^{\mathrm{LMod}_{A\otimes A^{\mathrm{op}}}}\left(A^{\otimes m+1} \otimes D^{m+1}\right) \arrow[r] & \mathrm{C}^{\mathrm{Bar}}_{\leq m+1}(A)
\end{tikzcd}$$

where i is the inclusion of $S^m = k[m]$ into the chain complex concentrated in degrees m and $m+1$ that is given by $(D^{m+1})_m = (D^{m+1})_{m+1} = k$, with boundary operator the identity, see [Hov99, 2.3.3]. As we assumed A to have cofibrant underlying complex and i is a cofibration in $\mathsf{Ch}(k)$, the tensor product $\mathrm{id}_{A^{\otimes m+1}} \otimes i$ is a cofibration as well, and it then follows that $\mathrm{Free}^{\mathrm{LMod}_{A\otimes A^{\mathrm{op}}}}(\mathrm{id}_{A^{\otimes m+1}} \otimes i)$ is a cofibration of left-$A \otimes A^{\mathrm{op}}$-modules, and thus so is the inclusion $\mathrm{C}^{\mathrm{Bar}}_{\leq m}(A) \to \mathrm{C}^{\mathrm{Bar}}_{\leq m+1}(A)$. □

6.3.2.2 C as a relative tensor product

Using the bar resolution from Section 6.3.2.1 we can now give a different description of the standard Hochschild complex that we defined in Section 6.3.1.3.

Proposition 6.3.2.4 ([Lod98, 1.1.13])**.** *The standard Hochschild complex functor*

$$\mathrm{C}\colon \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathsf{Ch}(k)^{\mathrm{cof}}$$

as defined in Definition 6.3.1.6 (but postcomposed with the forgetful functor from $\mathsf{Mixed}_{\mathrm{cof}}$ *to* $\mathsf{Ch}(k)^{\mathrm{cof}}$*) is naturally isomorphic to the functor*[73]

$$A \mapsto A \otimes_{A \otimes A^{\mathrm{op}}} \mathrm{C}^{\mathrm{Bar}}(A)$$

where $\mathrm{C}^{\mathrm{Bar}}(A)$ *is as in Construction 6.3.2.1 and* A *is a right-*$A \otimes A^{\mathrm{op}}$*-module via the action defined by* $a \cdot (a' \otimes a'') := a'' a a'$. ♡

Proof. Follows from unpacking the definitions and using isomorphisms of the following form.

$$\begin{aligned} A \otimes_{A \otimes A^{\mathrm{op}}} \big(A \otimes A^{\otimes n} \otimes A\big) &\cong A \otimes A^{\otimes n} \\ a \otimes (a_0 \otimes \cdots \otimes a_{n+1}) &\mapsto a_{n+1} \cdot a \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_n \end{aligned}$$

□

6.3.2.3 The shuffle product

In this section we assume that A is a *commutative* algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$, and upgrade the bar resolution $\mathrm{C}^{\mathrm{Bar}}(A)$ from Section 6.3.2.1 to a commutative differential graded algebra.

Definition 6.3.2.5 ([Lod98, 4.2.1] and [BACH, 1.2])**.** Let n and m be non-negative integers. Then we define

$$\begin{aligned} B_{n,m} &:= \Big\{\sigma \in \Sigma_{n+m} \;\Big|\; \sigma(1) < \cdots < \sigma(n) \text{ and } \sigma(n+1) < \cdots < \sigma(n+m)\Big\} \\ &= \Big\{\sigma \in \Sigma_{n+m} \;\Big|\; \sigma \text{ preserves the ordering of } \{1, \ldots, n\} \\ &\qquad\qquad \text{and } \{n+1, \ldots, n+m\}\Big\} \end{aligned}$$

where Σ_{n+m} is the symmetric group on $n+m$ elements, see Section 2.3 (34). ♢

Construction 6.3.2.6 ([Lod98, E.4.2.2] and [BACH, 1.2])**.** Let A be a commutative algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$. We then define a product on $\mathrm{C}^{\mathrm{Bar}}(A)$ from Section 6.3.2.1 by k-linearly extending the following formula

$$\begin{aligned} &(a_l \otimes a_1 \otimes \cdots \otimes a_n \otimes a_r) \cdot (a'_l \otimes a_{n+1} \otimes \cdots \otimes a_{n+m} \otimes a'_r) \\ &:= \sum_{\sigma \in B_{n,m}} (-1)^s \cdot \big(a_l \cdot a'_l \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n+m)} \otimes a_r a'_r\big) \end{aligned}$$

[73] We take the relative tensor product in $\mathsf{Ch}(k)$. That the relative tensor product is isomorphic to $\mathrm{C}(A)$ shows that it is indeed cofibrant and can thus be considered as functor to $\mathsf{Ch}(k)^{\mathrm{cof}}$.

where s is a sign (dependent on σ etc.) defined as follows.

$$\begin{aligned} s = {} & \operatorname{sgn}(\sigma) \\ & + \left(\deg_{\mathsf{Ch}}(a_r) \cdot \sum_{i=n+1}^{n+m} \deg_{\mathsf{Ch}}(a_i)\right) + \left(\deg_{\mathsf{Ch}}(a'_l) \cdot \left(\sum_{i=1}^{n} \deg_{\mathsf{Ch}}(a_i)\right)\right) \\ & + (\deg_{\mathsf{Ch}}(a_r) \cdot \deg_{\mathsf{Ch}}(a'_l)) \\ & + \left(\sum_{i=1}^{n+m} \deg_{\mathsf{Ch}}(a_i) \cdot \left(\sum_{i<j,\sigma(j)<\sigma(i)} \deg_{\mathsf{Ch}}(a_j)\right)\right) \end{aligned}$$

To make the formula more intuitive, let us provide the following interpretation. The summand indexed by σ should be thought of as moving a_i, which previously occupied what we might describe as "slot i" in the tensor product to "slot $\sigma(i)$" – this explains why σ^{-1} rather than σ occurs in the indices. Moving the a_i past each other then incurs signs coming from the symmetry isomorphism in $\mathsf{Ch}(k)$ (see Definition 4.1.2.1), and this is how the last summand of s arises. The other three summands of s involving chain degrees arise from moving a_r and a'_l to their correct positions. Finally, $\operatorname{sgn}(\sigma)$ is needed for compatibility with the part of the boundary operator coming from $\partial^{\mathrm{Bar}_A(A,A)_\bullet}$ – see Construction 6.3.2.1.

A tedious, but straightforward, calculation shows that the above multiplication is compatible with the boundary operator as well as associative and commutative, and with unit $1 \otimes 1$, making $\mathrm{C}^{\mathrm{Bar}}(A)$ into an object of $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$ (see Proposition 6.3.2.3 for cofibrancy of the underlying chain complex). Let us just mention one aspect of the required calculations when checking that the multiplication is compatible with the boundary operator. The boundary operator has two summands, with one arising from $\partial^{\mathrm{Bar}_A(A,A)_\bullet}$. With regards to that summand, multiplying first and then applying the boundary operator results in (a priori) extra summands (compared to applying the boundary operator first and then multiplying), where originally non-neighboring elements have been multiplied together. However, these summands always arise in pairs from two elements of $B_{n,m}$ that only differ by a transposition, and using that A is commutative one can see that they always cancel each other out. The rest of the needed calculations are mostly checking that the signs match.

With respect to this commutative algebra structure on $\mathrm{C}^{\mathrm{Bar}}(A)$, it is easy to check that the morphism $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ from Construction 6.3.2.1 becomes a morphism in $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$.

Furthermore, the inclusion of $A \otimes A \cong \mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A)$ (see the proof of Proposition 6.3.2.3 for this notation) into $\mathrm{C}^{\mathrm{Bar}}(A)$ becomes a morphism of commutative algebras as well, and the left-$A \otimes A$-module structure[74] on $\mathrm{C}^{\mathrm{Bar}}(A)$ that was discussed in Construction 6.3.2.1 can be identified with the one induced from this morphism of commutative algebras.

[74] As A is commutative we have $A = A^{\mathrm{op}}$.

The left-$A \otimes A$-module structure on A considered in Construction 6.3.2.1 can similarly be identified with the one arising from the morphism of commutative algebras $A \otimes A \to A$ that is given by the multiplication morphism. That the morphism $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ is a morphism of left-$A \otimes A$-modules is then reflected in the commutativity of the diagram

$$\begin{tikzcd} \mathrm{C}^{\mathrm{Bar}}(A) \arrow[rr] & & A \\ & A \otimes A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

of commutative algebras in $\mathsf{Ch}(k)$.

We can thus summarize these constructions in the commutative diagram

$$\begin{tikzcd} \mathrm{C}^{\mathrm{Bar}} \arrow[rr] & & \mathrm{id}_{\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})} \\ & A \mapsto A \otimes A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

of natural transformations between endofunctors of $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$.

Finally, note that the *right*-$A \otimes A$-module structure on A considered in Proposition 6.3.2.4 can also be identified with the one arising from the morphism of commutative algebras $A \otimes A \to A$ considered above. ♢

6.3.2.4 C for commutative algebras

Combining the description of the standard Hochschild complex as a relative tensor product with the bar resolution in Section 6.3.2.2 and the commutative algebra structure on the bar resolution constructed in Construction 6.3.2.6, we can now upgrade the standard Hochschild complex for commutative algebras to an object of $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$.

Proposition 6.3.2.7. *The composition of the forgetful functor*

$$\mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$$

with the standard Hochschild complex functor

$$\mathrm{C}\colon\ \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathsf{Mixed}_{\mathrm{cof}}$$

from Definition 6.3.1.6 and the forgetful functor $\mathsf{Mixed}_{\mathrm{cof}} \to \mathsf{Ch}(k)^{\mathrm{cof}}$ *factors* *through* $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$*, so that we obtain a commutative diagram*

$$\begin{tikzcd}
\mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr, "\mathrm{C}"] \arrow[d] & & \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[d] \\
\mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"'] & \mathsf{Mixed}_{\mathrm{cof}} \arrow[r] & \mathsf{Ch}(k)^{\mathrm{cof}}
\end{tikzcd}$$

where we denote the lift by C *as well, and all the unlabeled functors are the respective forgetful functors.*

Furthermore, the functor

$$\mathrm{C}\colon \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$$

is given by $A \mapsto A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)$*, where the* $A\otimes A$*-module structures on* A *and* $\mathrm{C}^{\mathrm{Bar}}(A)$ *arise via the natural transformations of functors to* $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$ *discussed in Construction 6.3.2.6 and the relative tensor product is taken in* $\mathrm{CAlg}(\mathsf{Ch}(k))$*.* ♡

Proof. Follows immediately from Proposition 6.3.2.4 in combination with Construction 6.3.2.6 using that the symmetric monoidal forgetful functor from $\mathrm{CAlg}(\mathsf{Ch}(k))$ to $\mathsf{Ch}(k)$ preserves relative tensor products[75]. □

Remark 6.3.2.8. Going through the definition, it is straightforward to check that the natural isomorphisms encoding functoriality in k of C of associative algebras as described in Remark 6.3.1.7 are multiplicative after restricting to commutative differential graded algebras. So concretely, if $\varphi\colon k \to k'$ is a morphism of commutative rings, then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"] \arrow[d, "k'\otimes_k -"'] & \mathrm{CAlg}(\mathsf{Mixed}_{k,\mathrm{cof}}) \arrow[d, "k'\otimes_k -"] \\
\mathrm{CAlg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"'] & \mathrm{CAlg}(\mathsf{Mixed}_{k',\mathrm{cof}})
\end{tikzcd}$$

lifting the commutative diagram from Remark 6.3.1.7. ♢

6.3.2.5 $\overline{\mathrm{C}}$ for commutative algebras concentrated in degree 0

Like in Section 6.3.1.4, we unpack the commutative algebra structure on the standard Hochschild complex $\mathrm{C}(R)$ in the case that R is concentrated in degree 0.

[75] See Proposition E.8.0.1 and [HA, 3.2.3.1 (4)]. Note that in 1-categories, geometric realizations – i. e. colimits over $\mathbf{\Delta}^{\mathrm{op}}$ – are calculated as coequalizers (see [Rie14, 8.3.8]), so that relative tensor products are the “classical” ones.

Definition 6.3.2.9 ([Lod98, 1.3.4]). Let R be a commutative k-algebra and $n \geq 0$ an integer. Then we define an action of the symmetric group Σ_n on $\mathrm{C}_n(R)$ as follows. For $r_0, \ldots, r_n$ elements of R, we define the action of σ on $r_0 \otimes r_1 \otimes \cdots \otimes r_n$ as

$$\sigma \cdot (r_0 \otimes r_1 \otimes \cdots \otimes r_n) \coloneqq r_0 \otimes r_{\sigma^{-1}(1)} \otimes \cdots \otimes r_{\sigma^{-1}(n)}$$

and extend this k-linearly to an action of Σ_n on $\mathrm{C}_n(R)$. An action of Σ_n on $\overline{\mathrm{C}}_n(R)$ is defined analogously. ♢

Proposition 6.3.2.10. *Let R be a commutative k-algebra with projective underlying k-module. Then the unit 1 of R, considered as an element of $\mathrm{C}(R)_0$, is the unit of the commutative algebra structure on $\mathrm{C}(R)$, and the following formula holds for the multiplication*[76].

$$\begin{aligned}&(r_0 \otimes r_1 \otimes \cdots \otimes r_n) \cdot (r_0' \otimes r_{n+1} \otimes \cdots \otimes r_{n+m})\\ &= \sum_{\sigma \in B_{n,m}} \operatorname{sgn}(\sigma) \cdot \sigma \cdot (r_0 \cdot r_0' \otimes r_1 \otimes \cdots \otimes r_{n+m})\end{aligned}$$ ♡

Proof. Follows directly from Construction 6.3.2.6 and Proposition 6.3.2.7. □

We also obtain an induced multiplication on the normalized standard Hochschild complex.

Proposition 6.3.2.11. *Let R be a commutative k-algebra with projective underlying k-module. Then the commutative algebra structure on $\mathrm{C}(R)$ induces a commutative algebra structure on $\overline{\mathrm{C}}(R)$ that makes the quotient morphism*

$$\mathrm{C}(R) \to \overline{\mathrm{C}}(R)$$

into a morphism in $\mathrm{CAlg}(\mathsf{Ch}(k))$. ♡

Proof. Follows immediately from Propositions 6.3.1.10 and 6.3.2.10. □

Remark 6.3.2.12. Given a morphism of commutative rings $\varphi\colon k \to k'$, the diagram of natural transformations

$$\begin{tikzcd} \mathrm{C}(k' \otimes_k -) \arrow[r] \arrow[d, "\cong"'] & \overline{\mathrm{C}}(k' \otimes_k -) \arrow[d, "\cong"] \\ k' \otimes_k \mathrm{C}(-) \arrow[r] & k' \otimes_k \overline{\mathrm{C}}(-) \end{tikzcd}$$

discussed in Remark 6.3.1.11 can be lifted to a commutative diagram of natural transformations from the category of commutative k-algebras with projective underlying k-module to the category $\mathrm{CAlg}(\mathsf{Mixed}_{k'})$, such that the left natural isomorphism is the one from Remark 6.3.2.8 and the top and bottom natural transformations are the ones from Proposition 6.3.2.11. ♢

[76] We identify $\mathrm{C}(R)_n$ for $n \geq 0$ with the tensor product $R^{\otimes(n+1)}$ for these formulas.

Warning 6.3.2.13. Let R be a commutative k-algebra with projective underlying k-module. While $\mathrm{C}(R)$ has both a strict mixed complex structure as well as the structure of a differential graded algebra, it is *not* in general an algebra in Mixed. To see this, let r and r' be elements of R Then, using the formulas from Propositions 6.3.1.9 and 6.3.2.10 we obtain

$$\begin{aligned} \mathrm{d}(r \cdot r') &= 1 \otimes r \cdot r' + r \cdot r' \otimes 1 \\ \mathrm{d}(r) \cdot r' + r \cdot \mathrm{d}(r') &= ((1 \otimes r) + (r \otimes 1)) \cdot r' + r \cdot ((1 \otimes r') + (r' \otimes 1)) \\ &= (r' \otimes r) + (r \cdot r' \otimes 1) + (r \otimes r') + (r \cdot r' \otimes 1) \end{aligned}$$

which shows that, in general, d does not satisfy the Leibniz rule and hence $\mathrm{C}(R)$ does not form an algebra in Mixed – see Remark 4.2.1.12.

The formulas simplify slightly for $\overline{\mathrm{C}}(R)$ so that we get

$$\begin{aligned} \mathrm{d}(r \cdot r') &= 1 \otimes \overline{r \cdot r'} \\ \mathrm{d}(r) \cdot r' + r \cdot \mathrm{d}(r') &= (r' \otimes \overline{r}) + \big(r \otimes \overline{r'}\big) \end{aligned}$$

which is however nevertheless not in general equal.

We can note though that

$$\partial\big(1 \otimes \overline{r} \otimes \overline{r'}\big) = r \otimes \overline{r'} - 1 \otimes \overline{r \cdot r'} + r' \otimes \overline{r}$$

so that the Leibniz rule is at least satisfied up to homotopy for elements of degree 0 – which is to be expected, as $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ has the structure of an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, and we will see in Section 6.3.4 that $\mathrm{C}(R)$ represents the underlying mixed complex of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ if we consider it as an object of $\mathsf{Mixed}_{\mathrm{cof}}$, and the underlying algebra in $\mathcal{D}(k)$ of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ if we consider it as an object of $\mathrm{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})$. ♢

Despite Warning 6.3.2.13, we can show instances of the Leibniz rule for the normalized standard Hochschild complex under additional assumptions, as we show next.

Proposition 6.3.2.14. *Let R be a commutative k-algebra with projective underlying k-module. Let $n \geq 1$ and $r, s_1, \ldots, s_n$ elements of $\overline{\mathrm{C}}(R)$ (of arbitrary degree). Then the following partial Leibniz rule identity holds.*

$$\mathrm{d}(r \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_n)) = \mathrm{d}(r) \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_n) \qquad \heartsuit$$

Proof. We first note that it suffices to prove the case $n = 1$. For suppose we have already proved the statement for all $1 \leq n \leq m$, and that $r, s_1, \ldots, s_{m+1}$ are elements of $\overline{\mathrm{C}}(R)$. Then the following calculation shows how we can deduce the claim for $n = m + 1$.

$$\begin{aligned} &\mathrm{d}(r \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_{m+1})) \\ &= \mathrm{d}((r \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_m)) \cdot \mathrm{d}(s_{m+1})) \end{aligned}$$

Applying the claim for $n = 1$.

$$= \mathrm{d}(r \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_m)) \cdot \mathrm{d}(s_{m+1})$$

Applying the claim for $n = m$.

$$= \mathrm{d}(r) \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_m) \cdot \mathrm{d}(s_{m+1})$$

So now assume that $n, m \geq 0$, that r is an element of $\overline{\mathrm{C}}_n(R)$ and s is an element of $\overline{\mathrm{C}}_m(R)$. We have to show that $\mathrm{d}(r \cdot \mathrm{d}(s)) = \mathrm{d}(r) \cdot \mathrm{d}(s)$.

Using notation from Section 2.3 (34), the formula from Proposition 6.3.1.10 for the differential $\mathrm{d}(r)$ of an element r in degree n of $\overline{\mathrm{C}}(R)$ can be written in a more concise way as

$$\mathrm{d}(r) = \sum_{\tau \in C_{n+1}} \mathrm{sgn}(\tau) \cdot \tau \cdot (1 \otimes r)$$

where $1 \otimes r$ is to interpreted as notation for $1 \otimes \overline{r_0} \otimes \cdots \otimes \overline{r_n}$ if $r = r_0 \otimes \overline{r_1} \otimes \cdots \otimes \overline{r_n}$ for $r_0, \ldots, r_n$ elements of R, and k-linearly extended for other elements. We now begin by unpacking the definition of $\mathrm{d}(r \cdot \mathrm{d}(s))$.

$$\begin{aligned}
&\mathrm{d}(r \cdot \mathrm{d}(s)) \\
&= \mathrm{d}\left(r \cdot \sum_{\tau_r \in C_{m+1}} \mathrm{sgn}(\tau_r) \cdot \tau_r \cdot (1 \otimes s) \right) \\
&= \mathrm{d}\left(\sum_{\substack{\tau_r \in C_{m+1}, \\ \sigma \in B_{n,m+1}}} \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau_r) \cdot \sigma \cdot \left(\mathrm{id}_{\{1,\ldots,n\}} \amalg \tau_r\right) \cdot (r \otimes s) \right) \\
&= \sum_{\substack{\tau_r \in C_{m+1}, \\ \sigma \in B_{n,m+1}, \\ \tau \in C_{n+m+2}}} \Big(\mathrm{sgn}(\tau) \cdot \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau_r) \\
&\qquad\qquad \cdot \tau \cdot \left(\mathrm{id}_{\{1\}} \amalg \sigma\right) \cdot \left(\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r\right) \cdot (1 \otimes r \otimes s) \Big) \\
&= \sum_{\substack{\tau_r \in C_{m+1}, \\ \sigma \in B_{n,m+1}, \\ \tau \in C_{n+m+2}}} \Big(\mathrm{sgn}\big(\tau \circ \left(\mathrm{id}_{\{1\}} \amalg \sigma\right) \circ \left(\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r\right)\big) \\
&\qquad\qquad \cdot \big(\tau \circ \left(\mathrm{id}_{\{1\}} \amalg \sigma\right) \circ \left(\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r\right)\big) \cdot (1 \otimes r \otimes s) \Big)
\end{aligned}$$

Next we unpack the definition of $\mathrm{d}(r) \cdot \mathrm{d}(s)$.

$$\begin{aligned}
&\mathrm{d}(r) \cdot \mathrm{d}(s) \\
&= \left(\sum_{\tau_l \in C_{n+1}} \mathrm{sgn}(\tau_l) \cdot \tau_l \cdot (1 \otimes r) \right) \cdot \left(\sum_{\tau_r \in C_{m+1}} \mathrm{sgn}(\tau_r) \cdot \tau_r \cdot (1 \otimes s) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{\tau_l \in C_{n+1} \\ \tau_r \in C_{m+1}}} \operatorname{sgn}(\tau_l \amalg \tau_r) \cdot (\tau_l \cdot (1 \otimes r)) \cdot (\tau_r \cdot (1 \otimes s)) \\
&= \sum_{\substack{\tau_l \in C_{n+1} \\ \tau_r \in C_{m+1} \\ \sigma \in B_{n+1,m+1}}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau_l \amalg \tau_r) \cdot \sigma \cdot (\tau_l \amalg \tau_r) \cdot (1 \otimes r \otimes s) \\
&= \sum_{\substack{\tau_l \in C_{n+1} \\ \tau_r \in C_{m+1} \\ \sigma \in B_{n+1,m+1}}} \operatorname{sgn}(\sigma \circ (\tau_l \amalg \tau_r)) \cdot (\sigma \circ (\tau_l \amalg \tau_r)) \cdot (1 \otimes r \otimes s)
\end{aligned}
$$

The claim thus boils down to a statement about different decompositions of elements of Σ_{n+m+2} that we now make concrete. We define two maps of sets as follows.

$$
\begin{aligned}
f \colon C_{m+1} \times B_{n,m+1} \times C_{n+1+m+1} &\to \Sigma_{n+1+m+1} \\
(\tau_r, \sigma, \tau) &\mapsto \tau \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big) \circ \big(\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r\big) \\
g \colon C_{n+1} \times C_{m+1} \times B_{n+1,m+1} &\to \Sigma_{n+1+m+1} \\
(\tau_l, \tau_r, \sigma) &\mapsto \sigma \circ (\tau_l \amalg \tau_r)
\end{aligned}
$$

To show $\mathrm{d}(r \cdot \mathrm{d}(s))$ is equal $\mathrm{d}(r) \cdot \mathrm{d}(s)$ it then suffices to show that for every element ρ of $\Sigma_{n+1+m+1}$ the preimages of ρ under f and g satisfy $|f^{-1}(\rho)| = |g^{-1}(\rho)|$. We will show this by going through the following steps.

(1) Proof that f is injective.

(2) Proof that g is injective

(3) Definition of a subset $C_{n+1,m+1}$ of $\Sigma_{n+1+m+1}$.

(4) Proof that $\mathrm{Im}(g) = C_{n+1,m+1}$.

(5) Proof that $\mathrm{Im}(f) \subseteq C_{n+1,m+1}$.

(6) Proof that $\mathrm{Im}(f) = C_{n+1,m+1}$.

Step (1): Let (τ_r, σ, τ) be an element of $C_{m+1} \times B_{n,m+1} \times C_{n+1+m+1}$, and let ρ be the composition $\rho = \tau \circ (\mathrm{id}_{\{1\}} \amalg \sigma) \circ (\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r)$. What we have to show is that τ_r, σ, and τ are uniquely determined by ρ. First note that $\rho(1) = \tau(1)$. As elements of $C_{n+1+m+1}$ are determined uniquely by their value on a single element, this means that τ is uniquely determined by ρ. As $\mathrm{id}_{\{1\}} \amalg \sigma$ preserves the order of the elements of the subset $\{n+1+1, \ldots, n+1+m+1\}$, we obtain

$$
r_{\{n+1+1,\ldots,n+1+m+1\}}\Big(\big(\mathrm{id}_{\{1\}} \amalg \sigma\big) \circ \big(\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r\big)\Big) = \tau_r
$$

which shows the claim.

Step (2): Let σ be an element of $B_{n+1,m+1}$, τ_l an element of C_{n+1} and τ_r an element of C_{m+1}. As σ preserves the order of the elements of the subsets $\{1, \ldots, n+1\}$ as well as $\{n+1+1, \ldots, n+1+m+1\}$, we obtain

$$r_{\{1,\ldots,n+1\}}(\sigma \circ (\tau_l \amalg \tau_r)) = \tau_l$$

and similarly

$$r_{\{n+1+1,\ldots,n+1+m+1\}}(\sigma \circ (\tau_l \amalg \tau_r)) = \tau_r$$

which implies the claim.

Step (3): We let $C_{n+1,m+1}$ be the subset of $\Sigma_{n+1+m+1}$ consisting of those permutations ρ for which $r_{\{1,\ldots,n+1\}}(\rho)$ is an element of C_{n+1} and where $r_{\{n+1+1,\ldots,n+1+m+1\}}(\rho)$ is an element of C_{m+1}. One should think of $C_{n+1,m+1}$ as a variant of $B_{n+1,m+1}$; The permutations in $B_{n+1,m+1}$ are those that preserve the order of the elements of the two subsets $\{1, \ldots, n+1\}$ and $\{n+1, \ldots, n+1+m+1\}$,[77] and the permutations in $C_{n+1,m+1}$ are those which *cyclically* preserve the order of the elements of those subsets.

Step (4): The argument used in step (2) shows that $\mathrm{Im}(g) \subseteq C_{n+1,m+1}$. For the other direction, suppose that ρ is an element of $C_{n+1,m+1}$. Then let $\tau_l = r_{\{1,\ldots,n+1\}}(\rho)$ and $\tau_r = r_{\{n+1,\ldots,n+1+m+1\}}(\rho)$, and furthermore define $\sigma \coloneqq \rho \circ \left(\tau_l^{-1} \amalg \tau_r^{-1}\right)$. Then we obtain

$$r_{\{1,\ldots,n+1\}}(\sigma) = \tau_l \circ \tau_l^{-1} = \mathrm{id} \quad \text{and} \quad r_{\{n+1+1,\ldots,n+1+m+1\}}(\sigma) = \tau_r \circ \tau_r^{-1} = \mathrm{id}$$

so that σ is an element of $B_{n+1,m+1}$. This shows that $C_{n+1,m+1} \subseteq \mathrm{Im}(g)$.

Step (5): It follows from the previous step that permutations of the form

$$(\mathrm{id}_{\{1\}} \amalg \sigma) \circ (\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r)$$

for σ an element of $B_{n,m+1}$ and τ_r an element of C_{m+1} lie in $C_{n+1,m+1}$. It thus suffices to show that $C_{n+1,m+1}$ is closed under postcomposition with elements of $C_{n+1+m+1}$. This follows from the fact that if X is a subset of $\{1, \ldots, n+1+m+1\}$ and τ an element of $C_{n+1+m+1}$, then $r_X(\tau)$ is an element of $C_{|X|}$.

Step (6): By the previous two steps it suffices to show that

$$|\mathrm{Im}(f)| = |\mathrm{Im}(g)|$$

and as both f and g are injective, it suffices to show that

$$|C_{m+1}| \cdot |B_{n,m+1}| \cdot |C_{n+1+m+1}| = |C_{n+1}| \cdot |C_{m+1}| \cdot |B_{n+1,m+1}|$$

which is verified by the following calculation.

$$\begin{aligned} &|C_{m+1}| \cdot |B_{n,m+1}| \cdot |C_{n+1+m+1}| \\ &= (m+1) \cdot \left(\frac{(n+m+1)!}{n! \cdot (m+1)!}\right) \cdot (n+1+m+1) \end{aligned}$$

[77] So the respective restrictions yield the elements $\mathrm{id}_{\{1,\ldots,n+1\}}$ of Σ_{n+1} and $\mathrm{id}_{\{1,\ldots,m+1\}}$ of Σ_{m+1}.

$$= (m+1) \cdot \left(\frac{(n+1) \cdot (n+1+m+1)!}{(n+1)! \cdot (m+1)! \cdot (n+1+m+1)} \right) \cdot (n+1+m+1)$$
$$= (n+1) \cdot (m+1) \cdot \left(\frac{(n+1+m+1)!}{(n+1)! \cdot (m+1)!} \right)$$
$$= |C_{n+1}| \cdot |C_{m+1}| \cdot |B_{n+1,m+1}| \qquad \square$$

6.3.3 Relative tensor products in $\mathsf{Ch}(k)$ and $\mathcal{D}(k)$

The canonical functor $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ is symmetric monoidal – see Proposition 4.3.2.1 – and thus preserves tensor products. In this section we discuss how γ interacts with *relative* tensor products. There is no reason to expect that γ preserves $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits in general, so we can not just apply Proposition E.8.0.1. Instead, we will show that γ preserves relative tensor products if one of the two modules is cofibrant as a module. Cofibrancy is here taken to be with respect to the model structure on $\mathrm{RMod}_R(\mathsf{Ch}(k))$ and $\mathrm{LMod}_R(\mathsf{Ch}(k))$ for an algebra R in $\mathsf{Ch}(k)$ from Theorem 4.2.2.1[78]. Note that as $\mathsf{Ch}(k)$ is a 1-category, geometric realizations – i. e. colimits over $\mathbf{\Delta}^{\mathrm{op}}$ – are calculated as coequalizers[79], so that the relative tensor product in $\mathsf{Ch}(k)$ is the "classical" one.

We begin by noting in Remark 6.3.3.1 below that there is a canonical comparison map from $\gamma(X) \otimes_{\gamma(R)} \gamma(Y)$ to $\gamma(X \otimes_R Y)$.

Remark 6.3.3.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor of monoidal ∞-categories, and assume that the monoidal structures on $\mathcal{C}$ and $\mathcal{D}$ are compatible with $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits in the sense of [HA, 3.1.1.18]. The relative tensor product induces a functor

$$\mathrm{RMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{LMod}(\mathcal{C}) \xrightarrow{-\otimes_{-}-} \mathcal{C}$$

and similarly for $\mathcal{D}$, see [HA, 4.4.2.10 and 4.4.2.11].

By [HA, 4.4.2.8] this functor can be identified as the functor mapping a triple (M, R, N) to $|\mathrm{Bar}_R(M,N)_\bullet|$, the geometric realization of the simplicial object $\mathrm{Bar}_R(M,N)_\bullet$ which can be described as $M \otimes R^{\otimes \bullet} \otimes N$, see also Section E.8.

As F is monoidal, it follows from the definition of the bar construction [HA, 4.4.2.7] that there is a natural equivalence as follows.

$$\mathrm{Bar}_{F(R)}(F(M), F(N))_\bullet \simeq F \circ \mathrm{Bar}_R(M,N)_\bullet$$

As there is a natural transformation

$$|F \circ X_\bullet| = \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(F \circ X_\bullet) \to F\left(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet\right) = F(|X_\bullet|)$$

[78] $\mathsf{Ch}(k)$ satisfies the assumptions by Fact 4.1.3.1.
[79] See [Rie14, 8.3.8].

for simplicial objects $X_\bullet$ in $\mathcal{C}$, we thus obtain a canonical natural transformation comparing first applying F, and then taking the relative tensor product with first taking the relative tensor product and then applying F.

$$\begin{tikzcd}
\mathrm{RMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{LMod}(\mathcal{C}) \arrow[r, "-\otimes_{-}-"] \arrow[d, "\mathrm{RMod}(F)\otimes_{\mathrm{Alg}(F)}\mathrm{LMod}(F)"'] \arrow[dr, phantom, "\Nearrow"] & \mathcal{C} \arrow[d, "F"] \\
\mathrm{RMod}(\mathcal{D}) \times_{\mathrm{Alg}(\mathcal{D})} \mathrm{LMod}(\mathcal{D}) \arrow[r, "-\otimes_{-}-"'] & \mathcal{D}
\end{tikzcd}$$

♢

Remark 6.3.3.2. We would like to compare relative tensor products of chain complexes with relative tensor products in $\mathcal{D}(k)$. There is a slight issue here that Remark 6.3.3.1 does not directly apply to give us what we want: We can not apply it to $\gamma\colon \mathsf{Ch}(k) \to \mathcal{D}(k)$ as this functor is not monoidal, so the monoidal functor we would want to consider is $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$, but there is no reason for the full subcategory $\mathsf{Ch}(k)^{\mathrm{cof}}$ of $\mathsf{Ch}(k)$ to be closed under $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits.

However, this is not actually a problem. If R is an algebra in $\mathsf{Ch}(k)$, and M and N are right and left modules over R, and such that the underlying chain complexes of R, M, and N are cofibrant, then, because γ is monoidal on $\mathsf{Ch}(k)^{\mathrm{cof}}$, we obtain an equivalence

$$\mathrm{Bar}_{\gamma(R)}(\gamma(M), \gamma(N))_\bullet \simeq \gamma \circ \mathrm{Bar}_R(M, N)_\bullet$$

just like in Remark 6.3.3.1.

We also still obtain a canonical morphism

$$\begin{aligned}\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}\big(\mathrm{Bar}_{\gamma(R)}(\gamma(M), \gamma(N))_\bullet\big) &\simeq \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(\gamma \circ \mathrm{Bar}_R(M, N)_\bullet) \\ &\to \gamma\Big(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Bar}_R(M, N)_\bullet)\Big)\end{aligned}$$

where on the right the colimit is taken in $\mathsf{Ch}(k)$ rather than $\mathsf{Ch}(k)^{\mathrm{cof}}$, and the γ is the functor

$$\gamma\colon \mathsf{Ch}(k) \to \mathcal{D}(k)$$

that is given by postcomposing the other functor called γ with the cofibrant replacement functor.

The upshot is that we still have a canonical comparison transformation as in Remark 6.3.3.1, even if it doesn't *quite* fit into the setup of Remark 6.3.3.1.

♢

Proposition 6.3.3.3. *Let* (M, R, N) *be an object of*

$$\mathrm{RMod}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \times_{\mathrm{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})} \mathrm{LMod}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$$

i. e. R is a differential graded algebra, M is a right module over R, N is a left module over R, and all three have cofibrant underlying chain complex.

Assume that one of M and N is cofibrant as a module over R with respect to the model structure of Theorem 4.2.2.1. Then the relative tensor product $M \otimes_R N$, calculated in $\mathsf{Ch}(k)$, is again cofibrant and the canonical comparison morphism (see Remark 6.3.3.2)

$$\gamma(M) \otimes_{\gamma(R)} \gamma(N) \to \gamma(M \otimes_R N)$$

is an equivalence. ♡

Proof. Let R be an object of $\mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$. We will use the notation

$$\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{RMod}_R} \colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathrm{RMod}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$$

as well as $\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}$, $\mathrm{Free}_{\mathcal{D}}^{\mathrm{RMod}_R}$, and $\mathrm{Free}_{\mathcal{D}}^{\mathrm{LMod}_R}$ for the left adjoints to the respective forgetful functors ev_{m}. We also let C be the collection of objects (M, R, N) of

$$\mathrm{RMod}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \times_{\mathrm{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})} \mathrm{LMod}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$$

and $C^{\simeq}$ the subcollection of those tuples (M, R, N) for which the canonical comparison morphism

$$\gamma(M) \otimes_{\gamma(R)} \gamma(N) \to \gamma(M \otimes_R N)$$

is an equivalence. When we refer to colimits below while talking about objects and morphisms in $\mathsf{Ch}(k)^{\mathrm{cof}}$, those colimits are always to be taken in the category $\mathsf{Ch}(k)$.

We first show the claim regarding cofibrancy of the relative tensor product, and will do the case where N is cofibrant as a module – the other case is analogous. Fix R and M as in the statement. Then it suffices to show that the functor

$$M \otimes_R - \colon \mathrm{LMod}_{\mathsf{Ch}}(k) \to \mathsf{Ch}(k)$$

maps generating cofibrations to cofibrations and preserves colimits. That the functor preserves colimits follows from [HA, 4.4.2.15]. Let $i \colon X \to Y$ be a cofibration in $\mathsf{Ch}(k)$. Then it remains to show that

$$M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(i) \colon M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \to M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y)$$

is again a cofibration. But this morphism can be identified with the morphism

$$M \otimes i \colon M \otimes X \to M \otimes Y$$

which is a cofibration as M is cofibrant and i a cofibration.

Let us now turn towards the claim that $\gamma(M) \otimes_{\gamma(R)} \gamma(N) \to \gamma(M \otimes_R N)$ is an equivalence if one of M and N is cofibrant as a module. By the definition of the model structure on modules[80] and [Hov99, 2.1.18 (b) and 2.1.9] it suffices to show the following.

[80] Theorem 4.2.2.1

(1) Let (M, R, N) be in C. Then $(M, R, 0)$ and $(0, R, N)$ are in $C^{\simeq}$.

(2) Let R be an object of the category $\mathrm{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})$, let M be an object of $\mathrm{RMod}(\mathsf{Ch}(k)^{\mathrm{cof}})$, and let X be an object in $\mathsf{Ch}(k)^{\mathrm{cof}}$.

Then $(M, R, \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X))$ is in $C^{\simeq}$.

(3) Let (M, R, N) be in $C^{\simeq}$ with N cofibrant as a module, let $i\colon X \to Y$ be a cofibration between cofibrant objects of $\mathsf{Ch}(k)$, and let

$$f\colon \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \to N$$

be a morphism in $\mathrm{LMod}_R\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$. Then

$$\Big(M, R, N \amalg_{\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)} \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y)\Big)$$

is again in $C^{\simeq}$, where the pushouts are formed with respect to the morphisms f and $\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(i)$.

The analogous statement holds for pushouts of this form in the first component.

(4) Let R be an object of $\mathrm{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})$ and M an object of $\mathrm{RMod}(\mathsf{Ch}(k)^{\mathrm{cof}})$.

Let λ be an ordinal and let $F\colon \lambda \to \mathrm{LMod}(\mathsf{Ch}(k)^{\mathrm{cof}})$ be a λ-sequence[81]. Assume that for every morphism $\alpha \to \alpha+1$ in λ the induced morphism $F(\alpha) \to F(\alpha+1)$ is a cofibration in $\mathrm{LMod}(\mathsf{Ch}(k)^{\mathrm{cof}})$, and that for every object α of λ the left-R-module $F(\alpha)$ is cofibrant and the triple $(M, R, F(\alpha))$ is in $C^{\simeq}$.

Then $(M, R, \mathrm{colim}_\lambda F)$ is also in $C^{\simeq}$. The analogous statement holds for transfinite compositions in the first component as well.

As all statements are symmetrical, we will only show the statements with regards to the *last* component.

Proof of claim (1): As both $\gamma(M) \otimes_{\gamma(R)} 0 \simeq 0$ and $M \otimes_R 0 \cong 0$, this follows from γ preserving the zero object by Proposition 4.3.2.1 (3).

Proof of claim (2): Consider the following commutative diagram

$$\begin{tikzcd}
\gamma(M) \otimes \gamma(X) \arrow[r] \arrow[d] & \gamma(M \otimes X) \arrow[d] \\
\gamma(M) \otimes_{\gamma(R)} \gamma\Big(\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)\Big) \arrow[r] & \gamma\Big(M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)\Big)
\end{tikzcd}$$

where the horizontal morphisms are the canonical comparison morphisms, and the vertical morphisms are induced by the morphism[82]

$$(M, k, X) \to (M, R, \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X))$$

[81] See for example [Hov99, 2.1.1] for a definition.

[82] See [HA, 4.4.2.9] for the identification of the relative tensor product over the unit k with the (non-relative) tensor product.

in

$$\mathrm{RMod}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \times_{\mathrm{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})} \mathrm{LMod}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$$

that is given by the identity of M, the unit morphism $k \to R$, and the morphism from X to the underlying object of $\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)$ that exhibits the latter as a free left-R-module generated by X.

It follows from Proposition E.7.4.1 that the induced morphism

$$\gamma(X) \to \gamma\Big(\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)\Big)$$

exhibits the codomain as the free left-$\gamma(R)$-module generated by $\gamma(X)$, so it follows from associativity [HA, 4.4.3.14] and unitality [HA, 4.4.3.16] of the relative tensor product that both the left and right vertical morphisms in the above diagram are equivalences[83]. As the top horizontal morphism is an equivalence as well, so must be the bottom horizontal morphism.

Proof of claim (3): Applying the canonical comparison transformation for the relative tensor products to the commutative square

$$\begin{tikzcd}
\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \arrow[r, "f"] \arrow[d, "\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(i)"'] & N \arrow[d] \\
\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y) \arrow[r] & P
\end{tikzcd} \tag{$*$}$$

where we write P for the pushout, we obtain the commuting cube

$$\begin{tikzcd}[column sep=small]
& \gamma(M) \otimes_{\gamma(R)} \gamma\Big(\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)\Big) \arrow[rr] \arrow[dl] \arrow[dd] & & \gamma\Big(M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)\Big) \arrow[dl] \arrow[dd] \\
\gamma(M) \otimes_{\gamma(R)} \gamma(N) \arrow[rr] \arrow[dd] & & \gamma(M \otimes_R N) \arrow[dd] & \\
& \gamma(M) \otimes_{\gamma(R)} \gamma\Big(\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y)\Big) \arrow[rr] \arrow[dl] & & \gamma\Big(M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y)\Big) \arrow[dl] \\
\gamma(M) \otimes_{\gamma(R)} \gamma(P) \arrow[rr] & & \gamma(M \otimes_R P) &
\end{tikzcd} \tag{$**$}$$

in $\mathcal{D}(k)$. We need to show that the bottom front horizontal morphism is an equivalence. For this it suffices to show the following.

[83] One can easily see from the definition of free modules that $\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(k) \simeq R$, and that $\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \simeq \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(k) \otimes X$. One thus obtains equivalences

$$M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \simeq M \otimes_R (R \otimes X) \simeq (M \otimes_R R) \otimes X \simeq M \otimes X$$

and similarly for the other relevant relative tensor product in $\mathcal{D}(k)$.

(a) The right side in diagram (∗∗) is a pushout square.

(b) The left side in diagram (∗∗) is a pushout square.

(c) The horizontal morphism in diagram (∗∗) other than the bottom front one are equivalences.

Proof of claim (a): In the commutative square

$$\begin{tikzcd} M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \arrow[r, "M\otimes_R f"] \arrow[d, "M\otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(i)"'] & M \otimes_R N \arrow[d] \\ M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y) \arrow[r] & M \otimes_R P \end{tikzcd}$$

the chain complex $M \otimes_R N$ is cofibrant and $M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(i)$ is a cofibration by what we already showed at the beginning of the proof. As (∗) is a pushout square, and $M \otimes_R -$ preserves colimits by [HA, 4.4.2.15], this is again a pushout square, and by [HTT, A.2.4.4] even a homotopy pushout square. The claim thus follows by applying [HA, 1.3.4.24].

Proof of claim (b): Follows from [HA, 1.3.4.24] using that (∗) is a homotopy pushout by [HTT, A.2.4.4].

Proof of claim (c): For the two back horizontal morphisms this follows from claim (2), and for the top front horizontal morphism this is by assumption.

Proof of claim (4): Analogous to (3), this time using that transfinite compositions are already homotopy colimits if all morphisms of the form $F(\alpha) \to F(\alpha+1)$ are cofibrations, which follows from [HTT, A.2.9.24 (1)], which shows that such diagrams are cofibrant in the projective model structure on λ-diagrams. □

6.3.4 The standard Hochschild complex as a model for HH

In this section we compare the Hochschild homology functors defined in Section 6.2 with the standard Hochschild complex functors as defined in Sections 6.3.1 and 6.3.2, showing that the latter represent the former.

We first discuss the case where we take into account the mixed complex structure, but not multiplicative structure, in Section 6.3.4.1, and then the case of commutative algebras, where we take into account the commutative algebra structure on Hochschild homology, but not the mixed structure, in Section 6.3.4.2.

6.3.4.1 The mixed case

The following comparison result by Hoyois[Hoy18] shows that the standard Hochschild complex of A, considered as a strict mixed complex, is a model for the mixed complex $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\gamma(A))$.

Proposition 6.3.4.1 ([Hoy18, 2.3])**.** *There is a commuting diagram*[84]

$$\begin{tikzcd}
\mathrm{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \arrow[r, "\mathrm{B}^{\mathrm{cyc}}_{\bullet}"] \arrow[d, "\mathrm{Alg}(\gamma)"'] \arrow[rrr, bend left, "\mathrm{C}"] & \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr, "|-|_{\mathsf{Mixed}}"] \arrow[d, "\gamma_*"] & & \mathsf{Mixed}_{\mathrm{cof}} \arrow[d, "\gamma_{\mathsf{Mixed}}"] \\
\mathrm{Alg}(\mathcal{D}(k)) \arrow[r, "\mathrm{B}^{\mathrm{cyc}}_{\bullet}"'] \arrow[rrr, bend right, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \arrow[r, "|-|"'] & \mathcal{D}(k)^{\mathrm{B}\mathbb{T}} \arrow[r, "\simeq"'] & \mathcal{M}\mathrm{ixed}
\end{tikzcd}$$

where the horizontal equivalence at the bottom left is the monoidal equivalence from Construction 5.4.0.1. ♡

Proof. The top and bottom rectangles commute by the definitions of C and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$, see Definition 6.3.1.6 and Definition 6.2.1.2. For the left square in the middle see Remark 6.1.2.17.

For $X_\bullet$ a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathsf{Ch}(k)$, the underlying chain complex of $|X_\bullet|_{\mathsf{Mixed}}$ is defined in Construction 6.3.1.2 as the total complex of a certain double complex, which is an upper[85] half plane complex. If $X_{[n]_{\mathbf{\Lambda}}}$ is acyclic for every $n \geq 0$, then it follows that the rows of the corresponding double complex are all acyclic, so that we can apply the acyclic assembly lemma [Wei94, 2.7.3] to conclude that the total complex $|X_\bullet|_{\mathsf{Mixed}}$ is acyclic. As colimits of (double) complexes as well as functor categories are calculated degreewise, and the construction of the total complex from a double complex preserves colimits, it follows by using the long exact sequence of homology that every morphism $X_\bullet \to Y_\bullet$ in $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}})$ that is pointwise a quasiisomorphism is mapped under $|-|_{\mathsf{Mixed}}$ to a quasiisomorphism.

The upshot is that $|-|_{\mathsf{Mixed}}$ induces a functor

$$K\colon\ \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \to \mathcal{M}\mathrm{ixed}$$

[84]Here, γ refers to the symmetric monoidal functor $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$. The construction $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ is defined in Definition 6.1.2.16, $|-|$ is defined in Definition 6.1.3.4, $|-|_{\mathsf{Mixed}}$ is defined in Construction 6.3.1.2, C is defined in Definition 6.3.1.6, and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is defined in Definition 6.2.1.2.

[85]Or right, depending on the convention. We will assume in this proof that we convert a complex of complexes to a double complex such that $X_{i,j} = (X_j)_i$. If $X_\bullet = \mathrm{B}^{\mathrm{cyc}}_{\bullet}(A)$, then the row indexed by $n \geq 0$ contains $A^{\otimes(n+1)}$, and the rows indexed by $n < 0$ are empty.

of ∞-categories that fits into a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}) \arrow[r, "|-|_{\mathsf{Mixed}}"] \arrow[d, "\gamma_*"'] & \mathsf{Mixed}_{\mathrm{cof}} \arrow[d, "\gamma_{\mathsf{Mixed}}"] \\
\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \arrow[r, "K"'] & \mathcal{M}\mathrm{ixed}
\end{tikzcd}$$

This is the functor also called K that is defined in [Hoy18, Right before 2.2].

We are thus left to construct a commuting triangle

$$\begin{tikzcd}
\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \arrow[r, "|-|"] \arrow[dr, "K"'] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[d, "\simeq"] \\
& \mathcal{M}\mathrm{ixed}
\end{tikzcd}$$

where the vertical equivalence is the one from Construction 5.4.0.1. This is exactly what [Hoy18, 2.3] provides – as long as we chose the correct vertical equivalence. However, the vertical equivalence has been chosen in Construction 5.4.0.1 and Convention 5.2.4.3 in reference to [Hoy18, 2.3] as exactly the one that is required to obtain the above commuting triangle. $\square$

Remark 6.3.4.2. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the symmetric monoidal functor $k' \otimes_k -\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ch}(k')^{\mathrm{cof}}$ (see Fact 4.1.5.1) induces a natural transformation from the the commutative diagram from Proposition 6.3.4.1 for k to the one for k'.

To be more precise, functoriality of the cyclic bar construction (see Remark 6.1.2.17) with respect to the commutative diagram

$$\begin{tikzcd}
\mathsf{Ch}(k)^{\mathrm{cof}} \arrow[r, "k'\otimes_k -"] \arrow[d, "\gamma"'] & \mathsf{Ch}(k')^{\mathrm{cof}} \arrow[d, "\gamma"] \\
\mathcal{D}(k) \arrow[r, "k'\otimes_k -"'] & \mathcal{D}(k')
\end{tikzcd} \tag{$*$}$$

of symmetric monoidal functors from Remark 4.3.2.2 yields a commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[dl] \arrow[dd] \\
\mathrm{Alg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[rr, crossing over] \arrow[dd] & & \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k')^{\mathrm{cof}}\big) & \\
& \mathrm{Alg}(\mathcal{D}(k)) \arrow[rr] \arrow[dl] & & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \arrow[dl] \\
\mathrm{Alg}(\mathcal{D}(k')) \arrow[rr] & & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k')) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the horizontal functors are all $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$, the vertical functors are induced by γ, and the functors from the back to the front are induced by $k' \otimes_k -$. Existence of a commutative cube

$$\begin{tikzcd}
& \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathsf{Mixed}_{k,\mathrm{cof}} \arrow[dl] \arrow[dd] \\
\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[rr, crossing over] \arrow[dd] & & \mathsf{Mixed}_{k',\mathrm{cof}} & \\
& \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)) \arrow[rr] \arrow[dl] & & \mathcal{M}\mathrm{ixed}_k \arrow[dl] \\
\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k')) \arrow[rr] & & \mathcal{M}\mathrm{ixed}_{k'} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the horizontal functors are $|-|$ and $|-|_{\mathsf{Mixed}}$, and the left and right sides are induced by diagram $(*)$ is implicit in the proof of [Hoy18, 2.3], though unfortunately not explicitly stated[86]. Combining the two commutative cubes we obtain a commutative cube

$$\begin{tikzcd}
& \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr, "\mathrm{C}"] \arrow[dl] \arrow[dd] & & \mathsf{Mixed}_{k,\mathrm{cof}} \arrow[dl] \arrow[dd] \\
\mathrm{Alg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[rr, crossing over, "\mathrm{C}"] \arrow[dd] & & \mathsf{Mixed}_{k',\mathrm{cof}} & \\
& \mathrm{Alg}(\mathcal{D}(k)) \arrow[rr, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}"] \arrow[dl] & & \mathcal{M}\mathrm{ixed}_k \arrow[dl] \\
\mathrm{Alg}(\mathcal{D}(k')) \arrow[rr, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}"'] & & \mathcal{M}\mathrm{ixed}_{k'} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$, where the front and back sides are the big outer squares in Proposition 6.3.4.1, the left and right sides are induced by diagram $(*)$, the top is the diagram from Remark 6.3.1.7 and the bottom is the diagram from Remark 6.2.1.6. $\diamondsuit$

6.3.4.2 The commutative case

We now compare the standard Hochschild complex C in the commutative case to $\mathrm{HH}\colon \mathrm{CAlg}(\mathcal{D}(k)) \to \mathrm{CAlg}(\mathcal{D}(k))$ from Definition 6.2.2.1, which will be possible because we can write both as a relative tensor product according to Corollary 6.2.2.6 and Proposition 6.3.2.7, and discussed how to compare relative tensor products in $\mathsf{Ch}(k)$ with relative tensor products in $\mathcal{D}(k)$ in Section 6.3.3.

[86] See also Remark 5.4.0.3

Proposition 6.3.4.3. *There is a commuting diagram*

$$\begin{tikzcd} \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"] \arrow[d, "\mathrm{CAlg}(\gamma)"'] & \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[d, "\mathrm{CAlg}(\gamma)"] \\ \mathrm{CAlg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}"'] & \mathrm{CAlg}(\mathcal{D}(k)) \end{tikzcd} \tag{6.16}$$

where C *is the functor from Proposition 6.3.2.7 and* HH *is the functor from Definition 6.2.2.1 and* γ *is the symmetric monoidal functor* $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$*.* ♡

Proof. By Proposition 6.3.2.7 we know that $\mathrm{C}(A)$ is given as the relative tensor product $A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)$ in $\mathrm{CAlg}(\mathsf{Ch}(k))$ – see Construction 6.3.2.6 and Construction E.8.0.4 for a definition of the relevant $A \otimes A$-module structures.

Like in Remark 6.3.3.1 and Remark 6.3.3.2 we obtain a natural comparison transformation

$$\mathrm{CAlg}(\gamma)(A) \otimes_{\mathrm{CAlg}(\gamma)(A\otimes A)} \mathrm{CAlg}(\gamma)\Big(\mathrm{C}^{\mathrm{Bar}}(A)\Big) \to \mathrm{CAlg}(\gamma)\Big(A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)\Big) \tag{$*$}$$

where we use that we already know that the relative tensor product will have cofibrant underlying chain complex[87]. We want to show that this morphism is an equivalence. As the forgetful functor

$$\mathrm{ev}_{\mathfrak{a}} \colon \mathrm{CAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$$

[87]Here are some more details. $\mathrm{CAlg}(\gamma)\colon \mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}}) \to \mathrm{CAlg}(\mathcal{D}(k))$ is symmetric monoidal, and thus induces a natural equivalence of bar constructions as follows.

$$\mathrm{Bar}_{\mathrm{CAlg}(\gamma)(A\otimes A)}\Big(\mathrm{CAlg}(\gamma)(A), \mathrm{CAlg}(\gamma)(\mathrm{C}^{\mathrm{Bar}}(A))\Big)_{\bullet} \simeq \mathrm{CAlg}(\gamma) \circ \mathrm{Bar}_{A\otimes A}\Big(A, \mathrm{C}^{\mathrm{Bar}}(A)\Big)_{\bullet}$$

The relative tensor product $A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)$ is given by the colimit

$$\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Bar}_{A\otimes A}\Big(A, \mathrm{C}^{\mathrm{Bar}}(A)\Big)_{\bullet}$$

calculated in $\mathrm{CAlg}(\mathsf{Ch}(k))$ (see the introduction to Section 6.3.3), so comes with a cocone diagram

$$(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright} \to \mathrm{CAlg}(\mathsf{Ch}(k))$$

but as we know that the relative tensor product has cofibrant underlying chain complex in this instance, this functor actually factors over $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$. Postcomposing this cocone diagram (as a diagram in $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$) with $\mathrm{CAlg}(\gamma)$, we obtain a cocone diagram from $\mathrm{CAlg}(\gamma)\circ\mathrm{Bar}_{A\otimes A}(A, \mathrm{C}^{\mathrm{Bar}}(A))_{\bullet}$ to $\mathrm{CAlg}(\gamma)(A\otimes_{A\otimes A}\mathrm{C}^{\mathrm{Bar}}(A))$, and hence by the universal property of colim a morphism as follows.

$$\begin{aligned} &\mathrm{CAlg}(\gamma)(A) \otimes_{\mathrm{CAlg}(\gamma)(A\otimes A)} \mathrm{CAlg}(\gamma)\Big(\mathrm{C}^{\mathrm{Bar}}(A)\Big) \\ &\simeq \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Bar}_{\mathrm{CAlg}(\gamma)(A\otimes A)}\Big(\mathrm{CAlg}(\gamma)(A), \mathrm{CAlg}(\gamma)(\mathrm{C}^{\mathrm{Bar}}(A))\Big)_{\bullet} \\ &\simeq \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{CAlg}(\gamma) \circ \mathrm{Bar}_{A\otimes A}(A, \mathrm{C}^{\mathrm{Bar}}(A))_{\bullet} \\ &\to \mathrm{CAlg}(\gamma)(A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)) \end{aligned}$$

detects equivalences by [HA, 3.2.2.6], it suffices to show that the underlying morphism in $\mathcal{D}(k)$ is an equivalence. By [HA, 3.2.3.1 (4)] and Proposition E.4.2.3 (5) in combination with Proposition E.8.0.1, both forgetful functors

$$\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{CAlg}(\mathsf{Ch}(k)) \to \mathsf{Ch}(k) \qquad \text{and} \qquad \mathrm{ev}_{\mathfrak{a}}\colon \mathrm{CAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$$

preserve relative tensor products, so that we can identify the composition of the natural transformation $(*)$ of functors $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}}) \to \mathrm{CAlg}(\mathcal{D}(k))$ with $\mathrm{ev}_{\mathfrak{a}}$ with the natural comparison transformation

$$\gamma(A) \otimes_{\gamma(A\otimes A)} \gamma\Big(\mathrm{C}^{\mathrm{Bar}}(A)\Big) \to \gamma\Big(A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)\Big)$$

from Remark 6.3.3.2. As $\mathrm{C}^{\mathrm{Bar}}(A)$ is cofibrant as a left-$A \otimes A$-module by Proposition 6.3.2.3, we can apply Proposition 6.3.3.3 to conclude that this is an equivalence.

We have now seen that the composition $\mathrm{CAlg}(\gamma) \circ \mathrm{C}$ in (6.16) is homotopic to the functor that is described by

$$A \mapsto \mathrm{CAlg}(\gamma)(A) \otimes_{\mathrm{CAlg}(\gamma)(A\otimes A)} \mathrm{CAlg}(\gamma)\Big(\mathrm{C}^{\mathrm{Bar}}(A)\Big)$$

where the the $A \otimes A$-module structures are as in Construction 6.3.2.6 and Construction E.8.0.4. The natural morphism $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ of left-$A \otimes A$-modules from Construction 6.3.2.6 provides a natural transformation

$$\begin{aligned} &\mathrm{CAlg}(\gamma)(A) \otimes_{\mathrm{CAlg}(\gamma)(A\otimes A)} \mathrm{CAlg}(\gamma)\Big(\mathrm{C}^{\mathrm{Bar}}(A)\Big) \\ &\to \mathrm{CAlg}(\gamma)(A) \otimes_{\mathrm{CAlg}(\gamma)(A\otimes A)} \mathrm{CAlg}(\gamma)(A) \end{aligned}$$

that is an equivalence by Proposition 6.3.2.2[88]. As γ and $\mathrm{CAlg}(\gamma)$ are symmetric monoidal, we can further identify $\gamma(A \otimes A)$ with $\gamma(A) \otimes \gamma(A)$ and the left and right module structures of $\gamma(A)$ over $\gamma(A \otimes A)$ (which arise from the morphism of commutative algebras $A \otimes A \to A$ given by the multiplication morphism) with the module structures arising from the multiplication morphism $\gamma(A) \otimes \gamma(A) \to \gamma(A)$.

We have thus identified the composition $\mathrm{CAlg}(\gamma) \circ \mathrm{C}$ in (6.16) with the functor described by

$$A \mapsto \gamma(A) \otimes_{\gamma(A)\otimes\gamma(A)} \gamma(A)$$

which is precisely the description of $\mathrm{HH} \circ \mathrm{CAlg}(\gamma)$ one obtains from Corollary 6.2.2.6. □

[88] Using that equivalences of left-$\mathrm{CAlg}(\gamma)(A \otimes A)$-modules are detected by the composition of the forgetful functors $\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{LMod}_{\mathrm{CAlg}(\gamma)(A\otimes A)}(\mathrm{CAlg}(\mathcal{D}(k))) \to \mathrm{CAlg}(\mathcal{D}(k))$ and $\mathrm{ev}_{\mathfrak{a}}\, \mathrm{CAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$ by [HA, 3.2.3.1 (4)] and [HA, 4.2.3.3 (2)].

Remark 6.3.4.4. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then there is a commutative cube

$$\begin{tikzcd}
 & \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr, "\mathrm{C}"] \arrow[dl] \arrow[dd] & & \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[dl] \arrow[dd] \\
\mathrm{CAlg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[rr, "\mathrm{C}"] \arrow[dd] & & \mathrm{CAlg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[dd] & \\
 & \mathrm{CAlg}(\mathcal{D}(k)) \arrow[rr, "\mathrm{HH}"] \arrow[dl] & & \mathrm{CAlg}(\mathcal{D}(k)) \arrow[dl] \\
\mathrm{CAlg}(\mathcal{D}(k')) \arrow[rr, "\mathrm{HH}"'] & & \mathrm{CAlg}(\mathcal{D}(k')) &
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$, where the top square is the one from Remark 6.3.2.8, the bottom square is induced by the one from Remark 6.2.1.6, the left and right squares are induced by the one from Remark 4.3.2.2, and the front and back squares are the ones from Proposition 6.3.4.3. To see this, one goes through the construction of the fillers for the different sides, which are ultimately constructed from symmetric monoidality of different functors and the universal property of colimits – see Remark 6.3.3.1. Using the universal property of colimits, one is left to check commutativity of a diagram of equivalences of simplicial objects that looks in level n like the outer diagram of equivalences depicted below.

$$\begin{tikzcd}
 & (k' \otimes_k \gamma(R))^{\otimes_{k'}(n+1)} \arrow[r, dash] \arrow[dl, dash] \arrow[dd, dash, dashed] & (\gamma(k' \otimes_k R))^{\otimes_{k'}(n+1)} \arrow[dr, dash] \arrow[dd, dash, dashed] & \\
k' \otimes_k \big(\gamma(R)^{\otimes_k(n+1)}\big) \arrow[dr, dash] & & & \gamma\Big((k' \otimes_k R)^{\otimes_{k'}(n+1)}\Big) \arrow[dl, dash] \\
 & k' \otimes_k \gamma\big(R^{\otimes_k(n+1)}\big) \arrow[r, dash] & \gamma\big(k' \otimes_k \big(R^{\otimes_k(n+1)}\big)\big) &
\end{tikzcd}$$

The two diagonal equivalences on the left and right arise from γ and $k' \otimes_k -$ being symmetric monoidal, and the two horizontal equivalences arise from the commutative diagram

$$\begin{tikzcd}
\mathsf{Ch}(k)^{\mathrm{cof}} \arrow[r, "k' \otimes_k -"] \arrow[d, "\gamma"'] & \mathsf{Ch}(k')^{\mathrm{cof}} \arrow[d, "\gamma"] \\
\mathcal{D}(k) \arrow[r, "k' \otimes_k -"'] & \mathcal{D}(k')
\end{tikzcd}$$

from Remark 4.3.2.2. This latter commutative square is actually a commutative square of symmetric monoidal functors, which is how we obtain the filler for the above diagram: The dashed equivalences (defined so as to make the left and right triangle commute) are precisely the equivalences exhibiting the compositions $k' \otimes_k \gamma(-)$ and $\gamma(k' \otimes_k -)$ as symmetric monoidal functors, and the filler for the square in the middle is the one exhibiting the homotopy between those two compositions being an homotopy of symmetric monoidal functors. $\diamondsuit$

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