On the Hochschild homology of hypersurfaces as a mixed complex

Volume 2

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This thesis has been submitted to the PhD School of
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Date of submission:	2022-04-12
Date of defense:	2022-07-05
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ISBN 978-87-7125-055-8

Contents

Volume 1

1. Introduction **1**
1.1. Motivation 3
1.2. Hochschild homology as a mixed complex 4
1.2.1. Hochschild homology as an object with circle action . 4
1.2.2. Mixed complexes 5
1.3. The first step in the proof of the main result 7
1.4. The second step in the proof of the main result 10
1.5. Overview over the chapters of this thesis 12
1.6. Future directions 14
1.7. Acknowledgments 16

2. Notation and conventions **19**
2.1. Prerequisites 19
2.2. On how this thesis is structured 19
2.3. Various notations and conventions 20
2.4. Size issues 29

3. Bialgebras and modules over them **31**
3.1. Modules over algebras 33
3.1.1. Cocartesian families of monoidal ∞-categories 34
3.1.1.1. Definition 34
3.1.1.2. The universal family 36
3.1.1.3. Compatibility of fibers with products 37
3.1.2. Algebras in cocartesian families 38
3.1.2.1. Definition 39
3.1.2.2. Comparison with Lurie's definition 40
3.1.2.3. Functoriality when varying families 42
3.1.2.4. Functoriality when varying the operad 45
3.1.2.5. Functoriality encoded by families 47
3.1.2.6. Algebras in cocartesian families and products 50
3.1.3. Functorial construction of ∞-categories of left modules 52
3.1.3.1. Review of the relevant operads 54
3.1.3.2. Construction of LMod 55
3.1.3.3. LMod and colimits 63
3.2. LMod and monoidality 65
3.2.1. LMod and products 66

3.2.2. $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ as a symmetric monoidal ∞-category 67
3.2.2.1. The sym. monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ 67
3.2.2.2. The sym. monoidal structure on $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_{\infty})$ 69
3.2.2.3. The sym. monoidal structure on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ . . 83
3.2.3. LMod as a symmetric monoidal functor 86
3.3. Bialgebras 94
3.3.1. Bialgebras in (co)cartesian sym. monoidal ∞-categories 96
3.4. Modules over bialgebras 97
3.4.1. Algebras in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ 98
3.4.2. LMod as a functor from $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ 113

4. Mixed complexes **117**
4.1. Chain complexes 117
4.1.1. $\mathsf{Ch}(k)$ as a 1-category 118
4.1.2. $\mathsf{Ch}(k)$ as a closed symmetric monoidal 1-category . . . 118
4.1.3. $\mathsf{Ch}(k)$ as a model category 119
4.1.4. Homotopies in $\mathsf{Ch}(k)$ 120
4.1.5. Extension of scalars 122
4.2. Strict mixed complexes 122
4.2.1. Mixed as a closed symmetric monoidal 1-category . . . 123
4.2.1.1. The bialgebra D 123
4.2.1.2. Definition of Mixed 124
4.2.1.3. Diagrams depicting strict mixed complexes . 125
4.2.1.4. The symmetric monoidal structure on Mixed 126
4.2.1.5. Algebras in Mixed 128
4.2.1.6. The closed sym. monoidal structure on Mixed 128
4.2.2. Mixed and $\mathrm{Alg}(\mathsf{Mixed})$ as model categories 129
4.2.2.1. Model categories of algebras and modules . . 130
4.2.2.2. The model structure on Mixed 131
4.2.2.3. The model structure on $\mathrm{Alg}(\mathsf{Mixed})$ 135
4.2.2.4. Homotopies in Mixed 139
4.2.2.5. Homotopies in $\mathrm{Alg}(\mathsf{Ch}(k))$ 140
4.2.2.6. Homotopies in $\mathrm{Alg}(\mathsf{Mixed})$ 145
4.2.3. Strongly homotopy linear morphisms of strict mixed complexes 148
4.2.3.1. Definition 149
4.2.3.2. Interpretation as zigzags 150
4.3. The derived category of k 156
4.3.1. Semiadditive ∞-categories 156
4.3.2. Properties of $\mathcal{D}(k)$ 159
4.3.3. Homology 164
4.3.4. Properties of the truncation functors 169
4.4. The ∞-category of mixed complexes 171
4.4.1. Construction of comparison functors 173
4.4.2. The comparison functors are equivalences 177

4.4.3. $\mathcal{M}$ixed is stable . . . 184
4.4.4. Strongly homotopy linear morphisms . . . 184

5. Mixed complexes and circle actions **187**
5.1. Formality of certain $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebras . . . 188
5.1.1. Duality . . . 190
5.1.2. Construction of a resolution . . . 197
5.1.2.1. Notation for freely generated dgas . . . 197
5.1.2.2. Construction of A as a directed colimit . . . 198
5.1.2.3. Homology of A . . . 200
5.1.2.4. Construction of A_{n+1} from A_n . . . 202
5.1.2.5. Identification of B_n up to quasiisomorphism 205
5.1.2.6. The resolution in $\mathcal{D}(k)$ . . . 206
5.1.2.7. Free generation of certain associative algebras 207
5.1.3. Formality of certain associative algebras . . . 210
5.1.4. Group homology . . . 212
5.1.5. Formality of certain commutative algebras . . . 213
5.1.6. Identification of some mapping spaces . . . 219
5.1.6.1. Identification of a mapping space in an overcategory . . . 220
5.1.6.2. The homology of a pushout of commutative algebras . . . 221
5.1.6.3. On a mapping space of commutative algebras 226
5.1.7. Formality of certain $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebras . . . 228
5.2. The k-linear circle as an $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebra . . . 232
5.2.1. The circle group . . . 232
5.2.2. The linearization functor . . . 233
5.2.3. Definition of the k-linear circle . . . 237
5.2.4. Formality of the k-linear circle as an $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebra 237
5.3. Group actions and modules over group rings . . . 240
5.4. The monoidal equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}$ixed . . . 257

6. Hochschild homology **261**
6.1. The cyclic bar construction and geometric realization of cyclic objects . . . 261
6.1.1. Connes' cyclic category $\mathbf{\Lambda}$ . . . 262
6.1.1.1. The simplex category $\mathbf{\Delta}$ . . . 262
6.1.1.2. Definition of $\mathbf{\Lambda}$ via posets . . . 263
6.1.1.3. Definition of $\mathbf{\Lambda}$ via generators and relations . 265
6.1.1.4. Comparison of the two definitions of $\mathbf{\Lambda}$ . . . 266
6.1.1.5. Cyclic objects . . . 267
6.1.1.6. Self-duality of $\mathbf{\Lambda}$ . . . 267
6.1.2. The cyclic bar construction as a cyclic object . . . 268
6.1.2.1. The symmetric monoidal envelope . . . 269
6.1.2.2. From associative algebras to active diagrams 272

6.1.2.3. Tensoring active diagrams together 281
6.1.2.4. The functor $V\colon \mathbf{\Lambda} \to \mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ 287
6.1.2.5. The definition of $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ 289
6.1.2.6. $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ for cocartesian sym. mon. ∞-categories 289
6.1.2.7. $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ and sifted colimits 294
6.1.3. Geometric realization of cyclic objects 295
6.1.3.1. The ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$ 295
6.1.3.2. Definition of the geometric realization 296
6.1.3.3. Monoidality 298
6.2. Hochschild homology 301
6.2.1. Definition of Hochschild homology 301
6.2.2. Hochschild homology and commutative algebras . . . 303
6.2.2.1. HH for commutative algebras 303
6.2.2.2. Circle actions on tensor products with $\mathbb{T}$ . . 305
6.2.2.3. HH on CAlg as a tensor product with $\mathbb{T}$ . . . 306
6.2.2.4. HH on CAlg as a pushout 308
6.2.2.5. HH on CAlg as a relative tensor product . . 309
6.2.3. Hochschild homology and relative tensor products . . 310
6.3. The standard Hochschild complex 311
6.3.1. The std. Hochschild complex for associative algebras . 311
6.3.1.1. The cyclic bar constr. for chain complexes . . 312
6.3.1.2. Geometric realization of cyclic chain complexes 313
6.3.1.3. The standard Hochschild complex 316
6.3.1.4. C for algebras concentrated in degree 0 . . . 316
6.3.1.5. The normalized standard Hochschild complex 318
6.3.2. The std. Hochschild complex for commutative algebras 319
6.3.2.1. The bar resolution 320
6.3.2.2. C as a relative tensor product 324
6.3.2.3. The shuffle product 325
6.3.2.4. C for commutative algebras 327
6.3.2.5. $\overline{\mathrm{C}}$ for commutative algebras concentrated in degree 0 328
6.3.3. Relative tensor products in $\mathsf{Ch}(k)$ and $\mathcal{D}(k)$ 334
6.3.4. The standard Hochschild complex as a model for HH . 339
6.3.4.1. The mixed case 339
6.3.4.2. The commutative case 342

Bibliography for volume 1 **347**

Volume 2

7. Hochschild homology of polynomial algebras 351

7.1. The mixed complex of de Rham forms 353

7.2. De Rham forms as a strict model in CAlg($\mathsf{Ch}(k)$) 354

7.2.1. A smaller replacement for the bar complex 355

7.2.2. Construction of the quasiisomorphism 365

7.3. De Rham forms as a strict model in Mixed 372

7.3.1. Definition of the higher homotopies 373

7.3.2. Simplification of the boundary 374

7.3.3. Identification of summands of $\epsilon_X^{(l-1)} \circ \mathrm{d}$ of a first type 389

7.3.4. Reindexing of summands of $\epsilon_X^{(l-1)} \circ \mathrm{d}$ of a second type 398

7.3.5. A first look at $\mathrm{d} \circ \epsilon_X^{(l-1)}$ 406

7.3.6. Progress so far 410

7.3.7. Reindexing remaining summands from the boundary . 411

7.3.8. Subdivisions of the remaining indexing sets 420

7.3.9. Canceling of some summands of $\epsilon_X^{(l-1)} \circ \mathrm{d}$ 424

7.3.10. Matching up of the remaining summands 428

7.3.11. Conclusion 432

7.4. De Rham forms as a strict model in Alg(Mixed) 435

7.4.1. A first cofibrant model 438

7.4.2. An improved cofibrant model 440

7.4.3. Comparing the algebra and mixed structure separately 454

7.4.4. Compatibility of Φ with d in the case of a single variable 464

7.4.5. A free resolution for de Rham forms 468

7.4.5.1. The general construction 469

7.4.5.2. Specific choices for Y_0, Y_1, and Y_2 471

7.4.5.3. Proof that the construction is a cofibrant resolution 485

7.4.6. Naturality of ϵ 486

7.4.7. Naturality of Φ 493

7.4.8. Compatibility of Φ with d in degree 0 495

7.4.9. Proof of Conjecture B for sets of cardinality at most 2 498

7.5. De Rham forms as a strict model in Alg(Mixed) and morphisms 502

7.5.1. Conjecture C for zero variables in the domain 504

7.5.2. Conjecture C for one variable in the domain 504

7.5.2.1. Morphisms and homotopies out of $\Omega'^{\bullet}_{k[t]/k}$ . . 507

7.5.2.2. Construction of $\Omega'^{\bullet}_{F/k}$ 509

7.5.2.3. Conclusion 518

7.5.3. Conjecture D 527

7.5.4. Conjecture C for two variables in the domain 527

8. Hochschild homology of certain quotients of commutative algebras 531

8.1. Hochschild homology of certain quotients as relative tensor products . . . 532
8.1.1. Certain quotients as relative tensor products . . . 532
8.1.2. Application to Hochschild homology . . . 534
8.2. A sufficiently cofibrant strict model of k . . . 536
8.2.1. Construction of the strict model . . . 536
8.2.2. Cofibrancy . . . 539
8.2.3. Quasiisomorphism . . . 541
8.3. A formula for Hochschild homology of certain quotients . . . 542

9. Hochschild homology of certain quotients of polynomial algebras 547

9.1. Prerequisites on polynomials and dividing with remainder . . . 550
9.1.1. Monomial orders . . . 551
9.1.1.1. Partial, total, and well-orders . . . 551
9.1.1.2. The standard partial order on $\mathbb{Z}^n_{\geq 0}$ . . . 552
9.1.1.3. Definition of monomial orders . . . 552
9.1.1.4. Properties of monomial orders . . . 553
9.1.2. Degrees for multivariable polynomials . . . 554
9.1.3. Dividing multivariable polynomials with remainder . . . 557
9.1.3.1. Monic polynomials . . . 557
9.1.3.2. Reduced polynomials . . . 558
9.1.3.3. Division with remainder . . . 559
9.1.3.4. Full sum decomposition . . . 561
9.1.3.5. Properties of remainders . . . 563
9.2. A strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of medium size . . . 564
9.2.1. $k[x_1, \ldots, x_n]$ as a module over $k[t]$ . . . 565
9.2.2. A strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ . . . 566
9.2.3. A basis for the strict model . . . 570
9.2.3.1. Interaction of q^1_f with d and multiplication . 570
9.2.3.2. The basis . . . 571
9.2.3.3. Description of boundary and differential . . . 572
9.3. A smaller strict model for the underlying complex . . . 575
9.3.1. The smaller strict model as a subcomplex . . . 576
9.3.2. A different description of the smaller model . . . 578
9.4. Logarithmic dimension of polynomials . . . 580
9.4.1. Logarithm for polynomials . . . 582
9.4.2. Logarithmic dimension for polynomials . . . 582
9.5. A smaller strict model for the mixed complex . . . 587
9.5.1. Restricting the strict mixed structure . . . 588
9.5.2. An alternative description of the smaller strict mixed model . . . 588
9.6. On the quasiisomorphisms constructed by BACH . . . 590

9.7. On a question of Larsen . . . 594

10. Example: $x_1^2 - x_2x_3$ **599**

10.1. Applying Theorem A . . . 599
10.2. Comparison with the mixed complex of de Rham forms . . . 601
10.3. Grading . . . 602
10.4. Non-diagonal pieces . . . 603
10.4.1. A first look at $Y((6,4))$ and $Y((7,5))$ . . . 603
10.4.2. A new basis . . . 605
10.4.3. Non-diagonal pieces of K are acyclic . . . 606
10.5. Diagonal pieces . . . 607
10.5.1. A first look at $Y((5,5))$ and $Y((6,6))$ . . . 607
10.5.2. A new basis . . . 609
10.5.3. Another look at $Y((5,5))$ . . . 611
10.5.4. Another look at $Y((6,6))$ . . . 612
10.5.5. A basis for $K((j,j))$ . . . 613
10.5.6. $K((j,j))$ for $j < 5$ . . . 615
10.5.7. Splitting an acyclic summand off of $K((j,j))$ for $j > 5$ 615
10.5.8. Description of the strict mixed structure . . . 616
10.5.9. A smaller model for $K((j,j))$ for $j > 5$ . . . 616
10.6. $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)\big)$ as a non-split extension . 617
10.7. Non-formality of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)\big)$ . . . 620

A. ∞-category theory **625**

A.1. Homotopy categories of model categories . . . 625
A.2. Mapping spaces . . . 626
A.3. The $(\infty,2)$-category of ∞-categories . . . 628
A.3.1. Pullbacks in $\mathcal{C}\mathrm{at}_\infty$ . . . 628
A.3.2. Natural transformations . . . 629

B. (Fully) faithful functors and monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ **631**

B.1. Monomorphisms . . . 631
B.1.1. Monomorphisms in the ∞-category $\mathcal{S}$ . . . 632
B.1.2. Monomorphisms and composition . . . 634
B.1.3. Monomorphisms and limits . . . 634
B.2. (Fully) faithful functors . . . 635
B.3. (Fully) Faithful functors and Fun . . . 636
B.4. Monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ . . . 637
B.4.1. Equivalent characterizations of monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ 638
B.4.2. Pseudomonic functors and replete images . . . 640
B.4.3. Lifting along monomorphisms . . . 641
B.4.4. (Fully) faithful functors are monomorphisms . . . 642
B.5. Stability of (fully) faithful functors and monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ . . . 642
B.5.1. Functor ∞-categories . . . 643

B.5.2. Pullbacks along another functor 644
B.5.3. Pullbacks 645
B.6. Subcategories 646

C. (Co)Cartesian Fibrations 649
C.1. Stability properties of (co)cartesian fibrations 649
C.1.1. Pullbacks 650
C.1.2. Restriction along fully faithful functors 650
C.1.3. Morphisms of cartesian fibrations 652
C.2. Cocartesian fibrations and products 652

D. More ∞-category theory 661
D.1. Undercategories 661
D.1.1. Model independent construction 661
D.1.2. Undercategories, faithful functors, fully faithful functors, and monomorphisms 662
D.1.3. Mapping spaces in undercategories 662
D.2. Adjunctions 664
D.2.1. Equivalent characterizations of adjoints 664
D.2.2. Adjunctions and Fun 667

E. ∞-operads and algebras 669
E.1. ∞-operads 669
E.1.1. Symmetric monoidal functors 670
E.1.2. Conservative morphisms of ∞-operads 672
E.1.3. Base changes of cocartesian fibrations of ∞-operads . 673
E.2. Alg and base change 674
E.3. Properties preserved by Alg 678
E.3.1. Pullbacks 678
E.3.2. Cocartesian fibrations 679
E.3.3. Adjoints 681
E.3.4. Reflecting equivalences 682
E.3.5. Fully faithfulness 682
E.4. Induced ∞-operad structures on Alg 683
E.4.1. The quasicategorical model 684
E.4.2. Properties of the induced ∞-operad structure 688
E.5. Iterating Alg 692
E.6. The commutative ∞-operad 695
E.7. Colimits and free algebras 698
E.7.1. Operadic colimits 698
E.7.1.1. A criterion for operadic colimits 698
E.7.1.2. Symmetric monoidal functors and operadic colimits 700
E.7.2. Free algebras 700
E.7.2.1. Detection of free algebras 701

E.7.2.2. Symmetric monoidal functors and free algebras 702
E.7.3. Induced functors on Alg and colimits 704
E.7.4. Free modules 705
E.8. Relative tensor products 707

F. Cartesian symmetric monoidal ∞-categories 713
F.1. Cocartesian fibrations and cartesian sym. monoidal structures 713
F.2. Monoids and limits 717
F.3. Cartesian sym. monoidal ∞-categories and iterating Mon and Alg 718

Bibliography for volume 2 727

Chapter 7.

Hochschild homology of polynomial algebras

In Definition 6.2.1.2 we defined a monoidal functor

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{M}\mathrm{ixed}$$

that thus induces a functor

$$\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{D}(k)) \simeq \mathrm{Alg}(\mathrm{Alg}(\mathcal{D}(k))) \to \mathrm{Alg}(\mathcal{M}\mathrm{ixed})$$

that we will also denote by $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$.

An important collection of examples of commutative (so in particular $\mathbb{E}_2$-) algebras in $\mathcal{D}(k)$ is given by polynomial algebras, i. e. algebras of the form $k[X]$ for X a set[1], and the goal of this chapter is to describe $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of polynomial algebras as algebras in $\mathcal{M}\mathrm{ixed}$. Concretely, given a set X, we would like to obtain a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$, as an object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, i. e. an object A in $\mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})$ such that there is an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \simeq \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(A)$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. We would also like A to be as efficient (i. e. small) as possible.

By the results of Section 6.3.4 we know that the standard Hochschild complex $\mathrm{C}(k[X])$ of a polynomial k-algebra $k[X]$, considered as either a commutative differential graded algebra, or a strict mixed complex, represents HH and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of $k[X]$, respectively. However, we have no comparison result available that compares $\mathrm{C}(k[X])$ and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as associative algebras of mixed complexes – while the standard Hochschild complex is a strict mixed complex as well as a differential graded algebra, it satisfies the Leibniz rule only up to homotopy, so we can not even consider it as a strict algebra in strict mixed complexes[2]! Even without this obstacle, $\mathrm{C}(k[X])$ would not be the kind of strict model we hope for, as it is not very efficient.

The first step on the road to finding a small strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ thus needs to be to define an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$

[1] See Definition 7.0.0.1 for a definition.
[2] See Warning 6.3.2.13

that we later hope to prove is such a strict model. For R a commutative k-algebra we will thus in Section 7.1 review the definition of the strict mixed complex of de Rham forms on R, denoted by $\Omega^\bullet_{R/k}$, which has a very concise description. Indeed, as the underlying complex has no non-zero boundary operators, so it is not possible to find a "smaller" quasiisomorphic chain complex.

Our goal, which we will only be able to prove if $|X| \leq 2$, and which is formulated as Conjecture B, is then to produce an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \simeq \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[X]/k}\right)$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, i. e. to show that $\Omega^\bullet_{k[X]/k}$ is a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$.

In Section 7.2 we will begin comparing $\Omega^\bullet_{k[X]/k}$ with Hochschild homology of $k[X]$ by constructing a quasiisomorphism ϵ_X from $\Omega^\bullet_{k[X]/k}$ to the normalized standard Hochschild complex $\overline{\mathrm{C}}(k[X])$. This quasiisomorphism is multiplicative, so as we already know that $\mathrm{C}(k[X])$, and hence also $\overline{\mathrm{C}}(k[X])$, is a strict model for $\mathrm{HH}(k[X])$ as an object of $\mathrm{Alg}(\mathcal{D}(k))$, we can conclude that $\Omega^\bullet_{k[X]/k}$ is so as well.

To show that $\Omega^\bullet_{k[X]/k}$ is also a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object of $\mathcal{M}\mathrm{ixed}$ it would suffice to show that ϵ_X is even a morphism of strict mixed complexes. This is unfortunately not the case, but we can instead upgrade ϵ_X to a strongly homotopy linear quasiisomorphism[3], and will do so in Section 7.3.

The partial results regarding only the algebra and only the mixed structure from Sections 7.2 and 7.3 will then be used as input in Section 7.4, where we will show that $\Omega^\bullet_{k[X]/k}$ is even a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ as long as $|X| \leq 2$.

Suppose now that X is a set with $|X| \leq 2$ and f an element of $k[X]$. Denote the morphism of commutative k-algebras $k[t] \to k[X]$ that maps t to f by F. Now that we know that $\Omega^\bullet_{k[t]/k}$ represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t])$ and $\Omega^\bullet_{k[X]/k}$ represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ we can ask whether the induced morphism $\Omega^\bullet_{F/k}$ also represents the morphism $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)$ in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. We are thus asking for a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[X]/k}\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{F/k}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[Y]) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[Y]/k}\right)
\end{tikzcd}$$

[3] See Section 4.2.3 for this notion.

in Alg($\mathcal{M}$ixed) such that the two horizontal morphisms are equivalences. We will formulate the claim that such a square exists for F as Conjecture C, and prove this conjecture for $|X| \leq 1$, as well as for $|X| = 2$ as long as 2 is invertible in k, in Section 7.5. We will also discuss Conjecture D, which is very closely related to Conjecture C and will be an essential ingredient in the results of later chapters.

We end the introduction to this chapter by fixing some notation concerning polynomial algebras.

Definition 7.0.0.1. Let X be a set. Then $k[X]$ denotes the *polynomial k-algebra generated by X*, i. e. the free commutative k-algebra generated by X. Its underlying k-module is free, and a basis is given by elements of the form[4] $x^{\vec{i}}$ with $\vec{i}$ an element of $\mathbb{Z}_{\geq 0}^{\times X}$ such that all but finitely many components are zero. We also use notation such as $k[x_1, \ldots, x_n]$ for the polynomial k-algebra that is generated by n formal variables $x_1, \ldots, x_n$, and trust that this will not lead to confusion.

Note that as the underlying k-module of a polynomial k-algebra is free, a polynomial k-algebra is cofibrant when considered as a chain complex concentrated in degree 0.[5] ♢

7.1. The mixed complex of de Rham forms

Given a commutative k-algebra R, we denote by $\Omega^1_{R/k}$ the k-module of *Kähler differentials* – for a definition see [Lod98, 1.1.9 and 1.3.7 to 1.3.9]. One then defines [Lod98, 1.3.11] $\Omega^n_{R/k}$ for $n \geq 0$ to be the exterior product $\Lambda^n_R \Omega^1_{R/k}$. Equipping $\Omega^\bullet_{R/k}$ with the zero boundary operator we obtain a commutative differential graded algebra. $\Omega^1_{R/k}$ also comes with a derivation [Lod98, 1.3.8] $\mathrm{d}\colon \Omega^0_{R/k} = R \to \Omega^1_{R/k}$, and the unique extension of d to an operator of degree 1 on $\Omega^\bullet_{R/k}$ that satisfies $\mathrm{d} \circ \mathrm{d} = 0$ and the Leibniz rule makes $\Omega^\bullet_{R/k}$ into an object of CAlg(Mixed)[6], called the *mixed complex of de Rham forms of R*. Elements of $\Omega^n_{R/k}$ are of the form $r_0 \,\mathrm{d}\, r_1 \cdots \mathrm{d}\, r_n$, with

$$\mathrm{d}(r_0 \,\mathrm{d}\, r_1 \cdots \mathrm{d}\, r_n) = \mathrm{d}\, r_0 \,\mathrm{d}\, r_1 \cdots \mathrm{d}\, r_n$$

and

$$(r_0 \,\mathrm{d}\, r_1 \cdots \mathrm{d}\, r_n) \cdot (r'_0 \,\mathrm{d}\, r'_1 \cdots \mathrm{d}\, r'_m) = r_0 r'_0 \,\mathrm{d}\, r_1 \cdots \mathrm{d}\, r_n \,\mathrm{d}\, r'_1 \cdots \mathrm{d}\, r'_m$$

describing the differential and multiplication [Lod98, 1.3.11 and 2.3.1]. This construction is functorial in morphisms of commutative k-algebras $f\colon R \to R'$ – there is a unique morphism in CAlg(Mixed) from $\Omega^\bullet_{R/k}$ to $\Omega^\bullet_{R'/k}$ that is given by f in degree 0.

[4] See Section 2.3 (32) for this notation.
[5] See [Hov99, 2.3.6]
[6] See Remark 4.2.1.12.

For $R = k[X]$ for some set X, the $k[X]$-module $\Omega^1_{k[X]/k}$ is free with basis given by $\{ \mathrm{d}\,x \mid x \in X \}$ – see [Lod98, 1.3.10 and 1.3.11]. It follows that we can identify $\Omega^\bullet_{k[X]/k}$ with $k[X] \otimes \Lambda_k(k \cdot \{ \mathrm{d}\,x \mid x \in X \})$, where $k \cdot \{ \mathrm{d}\,x \mid x \in X \}$ is the chain complex that is freely generated by $\{ \mathrm{d}\,x \mid x \in X \}$, where we give the elements $\mathrm{d}\,x$ chain degree 1. In particular, $\Omega^\bullet_{k[X]/k}$ is levelwise free as a k-module, and hence cofibrant by [Hov99, 2.3.6]. We can thus make the following definition.

Definition 7.1.0.1. We denote by

$$\Omega^\bullet_{-/k} \colon \mathrm{CAlg}(\mathrm{LMod}_k(\mathsf{Ab})) \to \mathrm{CAlg}(\mathsf{Mixed})$$

the functor sending a k-algebra R to the commutative algebra in strict mixed complexes $\Omega^\bullet_{R/k}$ discussed above. We also denote by[7]

$$\Omega^\bullet_{k[-]/k} \colon \mathsf{Set} \to \mathrm{CAlg}(\mathsf{Mixed}_{\mathrm{cof}})$$

the functor sending a set X to $\Omega^\bullet_{k[X]/k}$. ♢

Remark 7.1.0.2. $\Omega^\bullet_{-/k}$ is also functorial in k: For $\varphi\colon k \to k'$ a morphism of commutative rings and R a k-algebra, there is an evident isomorphism

$$\begin{aligned} k' \otimes_k \Omega^\bullet_{R/k} &\cong \Omega^\bullet_{k' \otimes_k R/k'} \\ a \otimes (r_0 \,\mathrm{d}\,r_1 \cdots \mathrm{d}\,r_n) &\mapsto (a \otimes r_0)\,\mathrm{d}(1 \otimes r_1) \cdots \mathrm{d}(1 \otimes r_n) \end{aligned}$$

in $\mathrm{CAlg}(\mathsf{Mixed}_{k'})$ that is natural in R and exhibits

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{LMod}_k(\mathsf{Ab})) \arrow[r, "\Omega^\bullet_{-/k}"] \arrow[d, "k' \otimes_k -"'] & \mathrm{CAlg}(\mathsf{Mixed}_k) \arrow[d, "k' \otimes_k -"] \\ \mathrm{CAlg}(\mathrm{LMod}_{k'}(\mathsf{Ab})) \arrow[r, "\Omega^\bullet_{-/k'}"'] & \mathrm{CAlg}(\mathsf{Mixed}_{k'}) \end{tikzcd}$$

as a commutative diagram in Cat. ♢

7.2. De Rham forms as a strict model in $\mathrm{CAlg}(\mathsf{Ch}(k))$

The reason the mixed complex of de Rham forms is relevant for us is the close relationship with the (normalized) standard Hochschild complex that we will discuss in this section.

In Section 6.3.2.1 we discussed the bar resolution $\mathrm{C}^{\mathrm{Bar}}(A)$ of an associative algebra A and saw in Proposition 6.3.2.4 that the standard Hochschild

[7] See Definition 4.2.1.2 for a definition of $\mathsf{Mixed}_{\mathrm{cof}}$.

complex of A is given by the relative tensor product $A \otimes_{A \otimes A^{\mathrm{op}}} \mathrm{C}^{\mathrm{Bar}}(A)$. In Section 7.2.1 we will, for a set X, construct a morphism $\widetilde{\epsilon}_X$ of left-$k[X] \otimes k[X]$-modules (in chain complexes) $\mathrm{C}^{\mathrm{sm}}(X) \to \mathrm{C}^{\mathrm{Bar}}(k[X])$. Tensoring with $k[X]$ over $k[X] \otimes k[X]$ we then obtain a morphism of chain complexes that we will be able to identify with a morphism $\Omega^{\bullet}_{k[X]/k} \to \mathrm{C}(k[X])$. In this manner we will obtain a natural transformation

$$\epsilon \colon \Omega^{\bullet}_{k[-]/k} \to \overline{\mathrm{C}}(k[-])$$

of functors $\mathsf{Set} \to \mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$ that will turn out to be a pointwise quasi-isomorphism, thereby providing a convenient multiplicative model $\Omega^{\bullet}_{k[X]/k}$ for $\mathrm{HH}(k[X])$. This will be discussed in Section 7.2.2.

While ϵ_X (for a set X) is a morphism of differential graded algebras, it is not a morphism of strict mixed complexes. However ϵ_X can be upgraded to a strongly homotopy linear morphism in the sense of Section 4.2.3. This will be shown in the next section, Section 7.3.

7.2.1. A smaller replacement for the bar complex

In this section we will in Construction 7.2.1.1 first construct $\mathrm{C}^{\mathrm{sm}}(X)$ and $\widetilde{\epsilon}_X$, before showing in Proposition 7.2.1.2 that they have good homotopical properties.

Construction 7.2.1.1 ([Lod98, 3.2.2])**.** Let X be a set. We will construct a commutative triangle of left-$k[X] \otimes k[X]$-modules in $\mathsf{Ch}(k)$

$$\begin{tikzcd} \mathrm{C}^{\mathrm{sm}}(X) \arrow[rr, "\widetilde{\epsilon}_X"] \arrow[dr] & & \mathrm{C}^{\mathrm{Bar}}(k[X]) \arrow[dl] \\ & k[X] & \end{tikzcd} \tag{7.1}$$

where $\mathrm{C}^{\mathrm{Bar}}(k[X])$ refers to the bar resolution as constructed in Construction 6.3.2.1, and the right diagonal morphism is the one also defined in Construction 6.3.2.1. We will use notation from Section 2.3 (34).

Definition of $\mathrm{C}^{\mathrm{sm}}(X)$ *as a graded left-*$k[X] \otimes k[X]$*-module*: We define

$$\mathrm{C}^{\mathrm{sm}}(X)_n := k[X] \otimes \Lambda^n(k \cdot X) \otimes k[X]$$

and the action of $k[X] \otimes k[X]$ as follows, with l', r', l, r elements of $k[X]$ and $x_1, \dots, x_n$ elements of X.

$$(l' \otimes r') \cdot (l \otimes x_1 \cdots x_n \otimes r) := l'l \otimes x_1 \cdots x_n \otimes rr'$$

Note that if there exist $i \neq j$ with $x_i = x_j$, then the right hand side is also 0, so the action is well-defined[8].

[8]See (29) in Section 2.3 for a definition of the exterior algebra $\Lambda(k \cdot X)$.

Definition of the boundary operator on $\mathrm{C}^{\mathrm{sm}}(X)$: We make the following definition for l, r elements of $k[X]$ and $x_1, \ldots, x_n$ elements of X.

$$\begin{aligned}\partial(l \otimes x_1 \cdots x_n \otimes r) \coloneqq \sum_{i=1}^{n} (-1)^{i-1} \Big(& (lx_i \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes r) \\ & - (l \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_i r)\Big)\end{aligned}$$

For well-definedness, assume that $1 \leq j < j' \leq n$ such that $x_j = x_{j'}$. We then have to check that the formula just given for $\partial(l \otimes x_1 \cdots x_n \otimes r)$ is zero. One can immediately see that the summands for $i \notin \{j, j'\}$ vanish, as the middle tensor factor $x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n$ then contains both x_j and $x_{j'}$ as factors. Thus we are left with the following sum.

$$\begin{aligned} & (-1)^{j-1}(lx_j \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_n \otimes r) \\ - & (-1)^{j-1}(l \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_n \otimes x_j r) \\ + & (-1)^{j'-1}(lx_{j'} \otimes x_1 \cdots x_{j'-1} \cdot x_{j'+1} \cdots x_n \otimes r) \\ - & (-1)^{j'-1}(l \otimes x_1 \cdots x_{j'-1} \cdot x_{j'+1} \cdots x_n \otimes x_{j'} r)\end{aligned}$$

To see that this is zero, we will argue that the first and third terms cancel, the argument for the second and fourth term canceling is completely analogous. For this, we carry out the following calculation.

$$\begin{aligned} & (-1)^{j-1}(lx_j \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_n \otimes r) \\ = & (-1)^{j-1}(lx_j \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{j'-1} \cdot x_{j'} \cdot x_{j'+1} \cdots x_n \otimes r)\end{aligned}$$

Using that $x_{j'} = x_j$.

$$= (-1)^{j-1}(lx_{j'} \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{j'-1} \cdot x_j \cdot x_{j'+1} \cdots x_n \otimes r)$$

Now we move the factor x_j in the inner tensor factor to the spot between x_{j-1} and x_{j+1}. This involves moving past $j' - j - 1$ other factors, so incurs a sign $(-1)^{j'-j-1}$.

$$\begin{aligned} & = (-1)^{j-1}(-1)^{j'-j-1}(lx_{j'} \otimes x_1 \cdots x_{j'-1} \cdot x_{j'+1} \cdots x_n \otimes r) \\ & = -(-1)^{j'-1}(lx_{j'} \otimes x_1 \cdots x_{j'-1} \cdot x_{j'+1} \cdots x_n \otimes r)\end{aligned}$$

It is clear from the definition that ∂ is compatible with the left-$k[X] \otimes k[X]$-module structure.

∂ squares to zero on $\mathrm{C}^{\mathrm{sm}}(X)$: For l, r elements of $k[X]$ and $x_1, \ldots, x_n$ elements of X we obtain the following calculation[9], where we use $1_{j>i}$ as ad

[9] $x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{i-1} \cdot x_{i+1} \cdots x_n$ is to be as interpreted as the product from x_1 to x_n while omitting x_j and x_i, also when $j > i$.

hoc notation for 0 if $j \not> i$ and 1 if $j > i$.

$$
\begin{aligned}
&\partial(\partial(l \otimes x_1 \cdots x_n \otimes r)) \\
&= \partial\left(\sum_{i=1}^{n}(-1)^{i-1}((lx_i \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes r))\right) \\
&\quad - \partial\left(\sum_{i=1}^{n}(-1)^{i-1}((l \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_i r))\right)
\end{aligned}
$$

The indices in the sums below range from 1 to n.

$$
\begin{aligned}
&= +\sum_{i \neq j}\Big((-1)^{i-1}(-1)^{j-1_{j>i}-1} \\
&\qquad\qquad \cdot (lx_i x_j \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes r)\Big) \\
&\quad -\sum_{i \neq j}\Big((-1)^{i-1}(-1)^{j-1_{j>i}-1} \\
&\qquad\qquad \cdot (lx_i \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_j r)\Big) \\
&\quad -\sum_{i \neq j}\Big((-1)^{i-1}(-1)^{j-1_{j>i}-1} \\
&\qquad\qquad \cdot (lx_j \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_i r)\Big) \\
&\quad +\sum_{i \neq j}\Big((-1)^{i-1}(-1)^{j-1_{j>i}-1} \\
&\qquad\qquad \cdot (l \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_j x_i r)\Big)
\end{aligned}
$$

The second and third line cancel by pairing the summand within the second line indexed by (i, j) with the summand within the third line indexed by (j, i), as the sign arising from the $1_{j>i}$ expression will differ between the two terms. Furthermore, the first and fourth line each already vanish individually, which one sees by pairing the summand indexed by (i, j) with the one indexed by (j, i).

Definition of $\mathrm{C}^{\mathrm{sm}}(X) \to k[X]$ *as a morphism of graded* $k[X] \otimes k[X]$*-modules*: We define this morphism to be given by

$$
(l \otimes x_1 \cdots x_n \otimes r) \mapsto \begin{cases} l \cdot r & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}
$$

for l, r elements of $k[X]$ and $x_1, \ldots, x_n$ elements of X. It is clear that this is well-defined and compatible with the $k[X] \otimes k[X]$-action.

Compatibility of $\mathrm{C}^{\mathrm{sm}}(X) \to k[X]$ *with* ∂: Let l and r be elements of $k[X]$ and x an element of X. We have to show that $\partial(l \otimes x \otimes r)$ is mapped to zero. But we have $\partial(l \otimes x \otimes r) = lx \otimes r - l \otimes xr$, which is mapped to $lxr - lxr = 0$.

Definition of $\widetilde{\epsilon}_X$ as a morphism of graded $k[X] \otimes k[X]$-modules: For l and r elements of $k[X]$ and $x_1, \dots, x_n$ elements of X, we make the following definition.

$$\widetilde{\epsilon}_X(l \otimes x_1 \cdots x_n \otimes r) \coloneqq \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r$$

To see that this is well-defined on $k[X] \otimes \Lambda^n(k \cdot X) \otimes k[X]$, we need to verify that the formula on the right hand side is 0 if $x_i = x_j$ for some $1 \le i < j \le n$. But we can split up Σ_n as the union of left cosets of the subgroup $\{\mathrm{id}, (i\ j)\}$ in Σ_n, where $(i\ j)$ denotes the transposition that exchanges i and j, and thus carry out the following calculation.

$$\begin{aligned}
&\sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\
&= \sum_{[\sigma] \in \Sigma_n/(i\ j)} \Big(\operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\
&\qquad + \operatorname{sgn}(\sigma \circ (i\ j)) l \otimes x_{(i\ j)(\sigma^{-1}(1))} \otimes \cdots \otimes x_{(i\ j)(\sigma^{-1}(n))} \otimes r \Big)
\end{aligned}$$

As $x_i = x_j$, we can simplify the indices of x in the second summand. We also use that $\operatorname{sgn}((i\ j)) = -1$.

$$\begin{aligned}
&= \sum_{[\sigma] \in \Sigma_n/(i\ j)} \big(\operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\
&\qquad - \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \big) \\
&= 0
\end{aligned}$$

That the definition of $\widetilde{\epsilon}_X$ is compatible with the left-$k[X] \otimes k[X]$-module structures is clear.

Some comments on how to relate $\widetilde{\epsilon}_X$ with actions of Σ_n: We can define an action of the symmetric group Σ_n on $\mathrm{C}^{\mathrm{Bar}}(k[X])_n$ that is given by permuting the inner n tensor factors, i. e. we make the following definition for $y_0, \dots, y_{n+1}$ elements of $k[X]$.

$$\sigma \cdot (y_0 \otimes y_1 \otimes \cdots \otimes y_n \otimes y_{n+1}) \coloneqq y_0 \otimes y_{\sigma^{-1}(1)} \otimes \cdots \otimes y_{\sigma^{-1}(n)} \otimes y_{n+1}$$

In particular we can then write $\widetilde{\epsilon}_X$ as follows, where $l, x_1, \dots, x_n, r$ are elements of $k[X]$.

$$\widetilde{\epsilon}_X(l \otimes x_1 \cdots x_n \otimes r) = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)(\sigma \cdot (l \otimes x_1 \cdots x_n \otimes r))$$

Finally, let us note that if S is a set with n elements and we write an element of $\mathrm{C}^{\mathrm{Bar}}(k[X])_n$ as $l \otimes y_{\varphi(1)} \otimes \cdots \otimes y_{\varphi(n)} \otimes r$ for $\varphi\colon \{1, \dots, n\} \to S$ a bijection and $l, y_{\varphi(1)}, \dots, y_{\varphi(n)}, r$ elements of $k[X]$, then the action of $\sigma \in \Sigma_n$ takes the following form.

$$\sigma \cdot \left(l \otimes y_{\varphi(1)} \otimes \cdots \otimes y_{\varphi(n)} \otimes r\right) = l \otimes y_{\varphi(\sigma^{-1}(1))} \otimes \cdots \otimes y_{\varphi(\sigma^{-1}(n))} \otimes r \qquad (*)$$

Compatibility of $\widetilde{\epsilon}_X$ *with* ∂: We carry out the following calculation, for l and r elements of $k[X]$ and $x_1, \dots, x_n$ elements of X.

$$\begin{aligned}
&\partial(\widetilde{\epsilon}_X(l \otimes x_1 \cdots x_n \otimes r)) \\
&= \partial\left(\sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r\right)
\end{aligned}$$

We apply the formula for the boundary operator of $\mathrm{C}^{\mathrm{Bar}}(k[X])$ as defined in Construction 6.3.2.1, writing the summands for $i = 0$ and $i = n$ as separate terms.

$$\begin{aligned}
= &\sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\
&+ \sum_{i=1}^{n-1} (-1)^i \sum_{\sigma\in\Sigma_n} \\
&\quad \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i)} x_{\sigma^{-1}(i+1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\
&+ (-1)^n \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n-1)} \otimes x_{\sigma^{-1}(n)} r
\end{aligned}$$

We now split up the set Σ_n the sum in the second line is indexed over as the union of the right cosets of the subgroup generated by the transposition $(i\ i+1)$. Note that the right cosets have the form $\{\sigma, (i\ i+1)\sigma\}$.

$$\begin{aligned}
= &\sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\
&+ \sum_{i=1}^{n-1} (-1)^i \sum_{[\sigma]\in(i\ i+1)\backslash\Sigma_n} \\
&\quad \Big(\operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i)} x_{\sigma^{-1}(i+1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\
&+ \operatorname{sgn}((i\ i+1)\circ\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i+1)} x_{\sigma^{-1}(i)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r\Big) \\
&+ (-1)^n \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n-1)} \otimes x_{\sigma^{-1}(n)} r
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\
&\quad + \sum_{i=1}^{n-1} (-1)^i \sum_{[\sigma]\in(i\ i+1)\backslash\Sigma_n} \\
&\qquad\qquad \Big(\operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i)} x_{\sigma^{-1}(i+1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\
&\quad - \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i+1)} x_{\sigma^{-1}(i)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \Big) \\
&\quad + (-1)^n \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n-1)} \otimes x_{\sigma^{-1}(n)} r
\end{aligned}
$$

The middle summands now cancel, using that $x_{\sigma^{-1}(i)}$ and $x_{\sigma^{-1}(i+1)}$ commute in $k[X]$.

$$
\begin{aligned}
&= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\
&\quad + (-1)^n \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n-1)} \otimes x_{\sigma^{-1}(n)} r
\end{aligned}
$$

Now let σ' be an element of Σ_n and assume that i is such that $\sigma'(i) = 1$. Then $\sigma = \sigma_{1\to n} \circ \sigma' \circ \sigma_{n\to i}$ fixes n, so that we can consider σ as an element of[10] Σ_{n-1}. The upshot is that if σ' maps i to 1, then we can write it uniquely as $\sigma' = \sigma_{n\to 1} \circ \sigma \circ \sigma_{i\to n}$ for σ an element of Σ_{n-1}. Analogously, if σ' maps i to n, then we can write it uniquely as $\sigma' = \sigma \circ \sigma_{i\to n}$ for σ an element of Σ_{n-1}.

Continuing the calculation from above, we can now rewrite the sums as follows.

$$
\begin{aligned}
&= \sum_{i=1}^{n} \sum_{\sigma\in\Sigma_{n-1}} \Big(\operatorname{sgn}(\sigma_{n\to 1} \circ \sigma \circ \sigma_{i\to n}) \\
&\qquad\qquad \cdot l x_{(\sigma_{n\to 1}\circ\sigma\circ\sigma_{i\to n})^{-1}(1)} \otimes x_{(\sigma_{n\to 1}\circ\sigma\circ\sigma_{i\to n})^{-1}(2)} \\
&\qquad\qquad \otimes \cdots \otimes x_{(\sigma_{n\to 1}\circ\sigma\circ\sigma_{i\to n})^{-1}(n)} \otimes r \Big) \\
&\quad + (-1)^n \sum_{i=1}^{n} \sum_{\sigma\in\Sigma_{n-1}} \Big(\operatorname{sgn}(\sigma \circ \sigma_{i\to n}) \\
&\qquad\qquad \cdot l \otimes x_{(\sigma\circ\sigma_{i\to n})^{-1}(1)} \otimes \cdots \otimes x_{(\sigma\circ\sigma_{i\to n})^{-1}(n-1)} \otimes x_{(\sigma\circ\sigma_{i\to n})^{-1}(n)} r \Big)
\end{aligned}
$$

[10] We consider Σ_{n-1} as a subset of Σ_n by extending with $n \mapsto n$.

The sign of $\sigma_{j\to j'}$ is $(-1)^{j-j'}$, as one can see by writing $\sigma_{j\to j'}$ as the composition of transpositions $((j'+1)\ j')\circ((j'+2)\ j'+1)\cdots\circ(j\ (j-1))$ if $j>j'$, and similarly if $j'>j$.

$$\begin{aligned}
&= \sum_{i=1}^{n}\sum_{\sigma\in\Sigma_{n-1}} \Big((-1)^{n-1+i-n}\operatorname{sgn}(\sigma)\\
&\qquad\qquad \cdot lx_{\sigma_{n\to i}(\sigma^{-1}(\sigma_{1\to n}(1)))}\otimes x_{\sigma_{n\to i}(\sigma^{-1}(\sigma_{1\to n}(2)))}\\
&\qquad\qquad \otimes\cdots\otimes x_{\sigma_{n\to i}(\sigma^{-1}(\sigma_{1\to n}(n)))}\otimes r\Big)\\
&\quad + (-1)^n\sum_{i=1}^{n}\sum_{\sigma\in\Sigma_{n-1}}\Big((-1)^{i-n}\operatorname{sgn}(\sigma)\cdot\\
&\qquad\qquad l\otimes x_{\sigma_{n\to i}(\sigma^{-1}(1))}\otimes\cdots\otimes x_{\sigma_{n\to i}(\sigma^{-1}(n-1))}\otimes x_{\sigma_{n\to i}(\sigma^{-1}(n))}r\Big)\\
&= \sum_{i=1}^{n}\sum_{\sigma\in\Sigma_{n-1}}\\
&\quad \Big((-1)^{i-1}\operatorname{sgn}(\sigma)lx_i\otimes x_{\sigma_{n\to i}(\sigma^{-1}(1))}\otimes\cdots\otimes x_{\sigma_{n\to i}(\sigma^{-1}(n-1))}\otimes r\Big)\\
&\quad - \sum_{i=1}^{n}\sum_{\sigma\in\Sigma_{n-1}}\\
&\quad \Big((-1)^{i-1}\operatorname{sgn}(\sigma)l\otimes x_{\sigma_{n\to i}(\sigma^{-1}(1))}\otimes\cdots\otimes x_{\sigma_{n\to i}(\sigma^{-1}(n-1))}\otimes x_ir\Big)
\end{aligned}$$

We can now apply $(*)$.

$$\begin{aligned}
&= \sum_{i=1}^{n}\sum_{\sigma\in\Sigma_{n-1}}(-1)^{i-1}\operatorname{sgn}(\sigma)\big(\sigma\cdot\big(lx_i\otimes x_{\sigma_{n\to i}(1)}\otimes\cdots\otimes x_{\sigma_{n\to i}(n-1)}\otimes r\big)\big)\\
&\quad - \sum_{i=1}^{n}\sum_{\sigma\in\Sigma_{n-1}}(-1)^{i-1}\operatorname{sgn}(\sigma)\big(\sigma\cdot\big(l\otimes x_{\sigma_{n\to i}(1)}\otimes\cdots\otimes x_{\sigma_{n\to i}(n-1)}\otimes x_ir\big)\big)
\end{aligned}$$

We now evaluate $\sigma_{n\to i}$ in the indices.

$$\begin{aligned}
&= \sum_{i=1}^{n}\sum_{\sigma\in\Sigma_{n-1}}(-1)^{i-1}\operatorname{sgn}(\sigma)\\
&\quad\cdot(\sigma\cdot(lx_i\otimes x_1\otimes\cdots\otimes x_{i-1}\otimes x_{i+1}\otimes\cdots\otimes x_n\otimes r))\\
&\quad - \sum_{i=1}^{n}\sum_{\sigma\in\Sigma_{n-1}}(-1)^{i-1}\operatorname{sgn}(\sigma)\\
&\quad\cdot(\sigma\cdot(l\otimes x_1\otimes\cdots\otimes x_{i-1}\otimes x_{i+1}\otimes\cdots\otimes x_n\otimes x_ir))
\end{aligned}$$

$$= \sum_{\sigma \in \Sigma_{n-1}} \operatorname{sgn}(\sigma)$$
$$\cdot \left(\sigma \cdot \left(\sum_{i=1}^{n} (-1)^{i-1} l x_i \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_n \otimes r \right) \right)$$
$$- \sum_{\sigma \in \Sigma_{n-1}} \operatorname{sgn}(\sigma)$$
$$\cdot \left(\sigma \cdot \left(\sum_{i=1}^{n} (-1)^{i-1} l \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_n \otimes x_i r \right) \right)$$

We can now plug in the definition of the boundary operator on $\mathrm{C}^{\mathrm{sm}}(X)$.

$$= \sum_{\sigma \in \Sigma_{n-1}} \operatorname{sgn}(\sigma)(\sigma \cdot (\partial(l \otimes x_1 \otimes \cdots \otimes x_n \otimes r)))$$

Finally, we can use the definition of $\widetilde{\epsilon}_X$.

$$= \widetilde{\epsilon}_X(\partial(l \otimes x_1 \otimes \cdots \otimes x_n \otimes r))$$

Commutativity of diagram (7.1): Clear from the definitions. ♢

We next show that $\widetilde{\epsilon}_X$ is an equivalence between cofibrant replacements of $k[X]$ in $\mathrm{LMod}_{k[X] \otimes k[X]}(\mathsf{Ch}(k))$.

Proposition 7.2.1.2. *For X a set the following hold.*

(1) $\mathrm{C}^{\mathrm{sm}}(X)$ *as defined in Construction 7.2.1.1 is cofibrant as an object in the model category* $\mathrm{LMod}_{k[X] \otimes k[X]}(\mathsf{Ch}(k))$ *with respect to the model structure of Theorem 4.2.2.1 (where* $\mathsf{Ch}(k)$ *carries the model structure of Fact 4.1.3.1).*

(2) The morphism of chain complexes $\widetilde{\epsilon}_X \colon \mathrm{C}^{\mathrm{sm}}(X) \to \mathrm{C}^{\mathrm{Bar}}(X)$ *as defined in Construction 7.2.1.1 is a quasiisomorphism.* ♡

Proof. Proof of claim (1): The category of left-$k[X] \otimes k[X]$-modules in $\mathsf{Ch}(k)$ is isomorphic to $\mathsf{Ch}(k[X] \otimes k[X])$. We can equip $\mathsf{Ch}(k[X] \otimes k[X])$ with the projective model structure from Fact 4.1.3.1, and comparing weak equivalences and fibrations we then see that the isomorphism between $\mathrm{LMod}_{k[X] \otimes k[X]}(\mathsf{Ch}(k))$ and $\mathsf{Ch}(k[X] \otimes k[X])$ is even an isomorphism of model categories. As $\mathrm{C}^{\mathrm{sm}}(X)$ is concentrated in nonnegative degrees and is levelwise free as an $k[X] \otimes k[X]$-module we can then apply [Hov99, 2.3.6], which shows the claim.

Proof of claim (2): The proof of this claim follows the ideas of [Lod98, 3.2.2]. Considering only the underlying chain complexes, it follows directly from the definitions that morphisms in diagram (7.1) are natural in the set

X. We thus obtain a commutative triangle

$$\begin{tikzcd} \mathrm{C}^{\mathrm{sm}}(-) \arrow[rr, "\widetilde{\epsilon}"] \arrow[dr, "p"'] & & \mathrm{C}^{\mathrm{Bar}}(k[-]) \arrow[dl] \\ & k[-] & \end{tikzcd}$$

of natural transformations of functors $\mathsf{Set} \to \mathsf{Ch}(k)$. That the right diagonal morphism is a quasiisomorphism has been shown in Proposition 6.3.2.2, so it suffices to show that for any set X the left diagonal morphism

$$p_X\colon\ \mathrm{C}^{\mathrm{sm}}(X) \to k[X]$$

is a quasiisomorphism.

Both $k[-]$ as well as $\Lambda^n(k \cdot -)$, considered as functors

$$\mathsf{Set} \to \mathrm{LMod}_k(\mathsf{Ab})$$

preserve filtered colimits.[11] Colimits of chain complexes are detected levelwise, the tensor product commutes with colimits in each variable separately, and if J is a filtered category and $n \geq 0$ an integer, then the diagonal functor $\mathsf{J} \to \mathsf{J}^n$ is cofinal [HTT, 5.3.1.22 and 4.1.1.8]. This implies that $\mathrm{C}^{\mathrm{sm}}(-)$ and $k[-]$ preserve filtered colimits as functors $\mathsf{Set} \to \mathsf{Ch}(k)$. Homology preserves filtered colimits as well [Wei94, 2.6.15], so quasiisomorphisms are closed under filtered colimits. As any set can be written as the filtered colimit of its finite subsets, this implies that it suffices to show that $p\colon\ \mathrm{C}^{\mathrm{sm}}(-) \to k[-]$ is a quasiisomorphism on finite sets.

[11] One can prove this by directly checking the universal property. We sketch this for $\Lambda^n(k \cdot -)$. So let J be a filtered category, $F\colon \mathsf{J} \to \mathsf{Set}$ a functor, Y a k-module, and $g_i\colon \Lambda^n(k \cdot F(i)) \to Y$ a morphism of k-modules for each object i of J such that $g_i \circ (\Lambda^n(k \cdot F(f))) = g_j$ for every morphism $f\colon j \to i$ in J. Then we have to check that there exists a unique morphism of k-modules $g\colon \Lambda^n(k \cdot (\operatorname{colim} F)) \to Y$ such that $g \circ (\Lambda^n(k \cdot \iota_i)) = g_i$ for every object i in J, where $\iota_i\colon F(i) \to \operatorname{colim} F$ is the morphism that exhibits $\operatorname{colim} F$ as a colimit. The k-module $\Lambda^n(k \cdot (\operatorname{colim} F))$ is free, with basis given by elements of the form $x_1 \cdots x_n$ with $x_1, \ldots, x_n$ elements of $\operatorname{colim} F$ such that $x_a \neq x_b$ for $a \neq b$. For such $x_1, \ldots, x_n$, there must be (as J is filtered) an object i of J and elements $x'_1, \ldots, x'_n$ of $F(i)$ such that $x_a = \iota_i(x'_a)$ for $1 \leq a \leq n$ (filteredness was used to find a single such i that works for all n elements at once). But then we must have $g(x_1 \cdots x_n) = (g \circ (\Lambda^n(k \cdot \iota_i)))(x'_1 \cdots x'_n) = g_i(x'_1 \cdots x'_n)$. This shows uniqueness. If i' is a different object of J and $x''_1, \ldots, x''_n$ elements of $F(i')$ such that $x_a = \iota_{i'}(x''_a)$ for $1 \leq a \leq n$, then, as J is filtered, there must exist morphisms $f\colon i \to j$ and $f'\colon i' \to j$ in J such that $F(f)(x'_a) = F(f')(x''_a)$ for $1 \leq a \leq n$. We thus obtain

$$\begin{aligned} g_i\big(x'_1 \cdots x'_n\big) &= (g_j \circ (\Lambda^n(k \cdot F(f))))\big(x'_1 \cdots x'_n\big) = g_j\big(F(f)(x'_1) \cdots F(f)(x'_n)\big) \\ &= g_j\big(F(f')(x''_1) \cdots F(f')(x''_n)\big) = \big(g_j \circ (\Lambda^n(k \cdot F(f')))\big)\big(x''_1 \cdots x''_n\big) = g_{i'}\big(x''_1 \cdots x''_n\big) \end{aligned}$$

so that the above formula for $g(x_1 \cdots x_n)$ is independent of the choice of $x'_1, \ldots x'_n$, which implies that this defines a morphism g that is compatible with the g_i as required.

Now suppose that the set X is the disjoint union of Y and Y', with $\iota\colon Y \to X$ and $\iota'\colon Y' \to X$ the inclusions. We obtain a commutative diagram of chain complexes as follows, to be explained below.

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{sm}}(Y) \otimes \mathrm{C}^{\mathrm{sm}}(Y') \arrow[r] \arrow[d, "p_Y \otimes p_{Y'}"'] & \mathrm{C}^{\mathrm{sm}}(X) \arrow[d, "p_X"] \\
k[Y] \otimes k[Y'] \arrow[r] & k[X]
\end{tikzcd}$$

The top horizontal morphism is defined by k-linearly extending the assignment

$$\begin{aligned}
&(l \otimes y_1 \cdots y_n \otimes r) \otimes (l' \otimes y'_1 \cdots y'_n \otimes r') \\
\mapsto\ & k[\iota](l) \cdot k[\iota'](l') \otimes \iota(y_1) \cdots \iota(y_n) \cdot \iota'(y'_1) \cdots \iota'(y'_n) \otimes k[\iota](r) \cdot k[\iota'](r')
\end{aligned}$$

where l, r are elements of $k[Y]$, $y_1, \ldots y_n$ are elements of Y, l', r' are elements of $k[Y']$, and $y'_1, \ldots, y'_n$ are elements of Y'. It is immediate that this is well-defined, and checking compatibility with the boundary operator requires only unpacking the definitions and using that $k[X]$ is commutative. The bottom horizontal morphism is given by composing $k[\iota] \otimes k[\iota']$ with the multiplication $k[X] \otimes k[X] \to k[X]$.

Both the horizontal morphisms in the above diagram are isomorphisms, as one can easily see by considering the respective bases consisting of tensor products of monomials. To show that p_X is a quasiisomorphism, it thus suffices to show that $p_Y \otimes p_{Y'}$ is a quasiisomorphism.

Assume for the moment that p_Y and $p_{Y'}$ are quasiisomorphisms. As $k[Y]$ and $k[Y']$ are concentrated in degree 0, we can read off their homology and can thus conclude that $\mathrm{C}^{\mathrm{sm}}(Y)$, $\mathrm{C}^{\mathrm{sm}}(Y')$, $k[Y]$, and $k[Y']$ are all chain complexes that have free homology. The Künneth spectral sequences[12] that converge to the homology of the tensor products $\mathrm{C}^{\mathrm{sm}}(Y) \otimes \mathrm{C}^{\mathrm{sm}}(Y')$ and $k[Y] \otimes k[Y']$ thus collapse already on the second page, from which we can deduce that $p_Y \otimes p_{Y'}$ is also a quasiisomorphism.

It thus suffices to show that p_Y and $p_{Y'}$ are quasiisomorphisms in order to conclude that p_X is a quasiisomorphism as well, if X is the disjoint union of Y and Y'. As every finite set can be written as the disjoint union of sets that have exactly one element, we have thus reduced the claim to showing that $p_{\{x\}}$ is a quasiisomorphism.

We now show that $p_{\{x\}}$ is a chain homotopy equivalence. Note that the chain complex $\Lambda(k \cdot \{x\})$ is free with basis 1 in degree 0, free with basis x in degree 1, and zero in other degrees. We can define a section s of $p_{\{x\}}$ by $s(r) = 1 \otimes r$, so it suffices to construct a morphism of k-modules $h\colon k[x] \otimes k[x] \to k[x] \otimes k \cdot \{x\} \otimes k[x]$ that satisfies $\partial \circ h = \mathrm{id} - s \circ p_{\{x\}}$ on

[12] See for example [Rot08, 10.90].

elements of degree 0 and $h \circ \partial = \mathrm{id}$ on elements of degree 1. For this we define h as follows on basis elements, where $n, m \geq 0$.

$$h(x^n \otimes x^m) := \sum_{i=0}^{n-1} x^i \otimes x \otimes x^{n+m-i-1}$$

Then we obtain the following calculation for the first identity.

$$\begin{aligned}
&\partial(h(x^n \otimes x^m)) \\
&= \sum_{i=0}^{n-1} \partial\big(x^i \otimes x \otimes x^{n+m-i-1}\big) \\
&= \sum_{i=0}^{n-1} \big(x^{i+1} \otimes x^{n+m-i-1} - x^i \otimes x^{n+m-i}\big) \\
&= \sum_{i=1}^{n} x^i \otimes x^{n+m-i} - \sum_{i=0}^{n-1} x^i \otimes x^{n+m-i} \\
&= x^n \otimes x^m - 1 \otimes x^{n+m} \\
&= \big(\mathrm{id} - s \circ p_{\{x\}}\big)(x^n \otimes x^m)
\end{aligned}$$

The following calculation shows the second identity.

$$\begin{aligned}
&h(\partial(x^n \otimes x \otimes x^m)) \\
&= h\big(x^{n+1} \otimes x^m\big) - h\big(x^n \otimes x^{m+1}\big) \\
&= \sum_{i=0}^{n} x^i \otimes x \otimes x^{n+m-i} - \sum_{i=0}^{n-1} x^i \otimes x \otimes x^{n+m-i} \\
&= x^n \otimes x \otimes x^m \\
&= \mathrm{id}(x^n \otimes x \otimes x^m)
\end{aligned}$$

This proves the claim. □

7.2.2. A quasiisomorphism between de Rham forms and the standard Hochschild complex

In this section we define and discuss ϵ, a natural quasiisomorphism from $\Omega^\bullet_{k[-]/k}$ to $\overline{\mathrm{C}}(k[-])$.

Construction 7.2.2.1. For every set X we are going to construct a morphism of chain complexes

$$\epsilon_X : \Omega^\bullet_{k[X]/k} \to \overline{\mathrm{C}}(k[X])$$

where $\overline{\mathrm{C}}$ refers to the normalized standard Hochschild complex defined in Proposition 6.3.1.10.

So let X be a set. We define ϵ_X as a composition as follows, where we will explain the individual morphisms below.

$$\begin{tikzcd}
\Omega^{\bullet}_{k[X]/k} \arrow[rr, "\epsilon_X"] \arrow[d, "\epsilon'_X"', "\cong"] & & \overline{\mathrm{C}}(k[X]) \\
k[X] \otimes \Lambda(k \cdot X) \arrow[d, "\epsilon''_X"', "\cong"] & & \mathrm{C}(k[X]) \arrow[u] \\
k[X] \otimes_{k[X]\otimes k[X]} \mathrm{C}^{\mathrm{sm}}(X) \arrow[rr, "k[X]\otimes_{k[X]\otimes k[X]}\widetilde{\epsilon}_X"'] & & k[X] \otimes_{k[X]\otimes k[X]} \mathrm{C}^{\mathrm{Bar}}(k[X]) \arrow[u, "\cong", "\epsilon'''_X"']
\end{tikzcd}$$

In $k[X] \otimes \Lambda(k \cdot X)$ the elements of X in the exterior product are to have degree 1, and we make the resulting graded k-module into a chain complex by equipping it with the zero boundary operator. The isomorphism ϵ'_X is then the one suggested in Section 7.1, its inverse is defined by

$$l \otimes x_1 \cdots x_n \mapsto l \cdot \mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n$$

where l is an element of $k[X]$ and $x_1, \ldots, x_n$ are elements of X.

$\mathrm{C}^{\mathrm{sm}}(X)$ is as in Construction 7.2.1.1, so is given by $k[X] \otimes \Lambda(k \cdot X) \otimes k[X]$ as a graded k-module. We can thus define ϵ''_X as

$$l \otimes x_1 \cdots x_n \mapsto l \otimes (1 \otimes x_1 \cdots x_n \otimes 1)$$

where l is an element of $k[X]$ and $x_1, \ldots, x_n$ are elements of X, and it is clear that this is an isomorphism of graded k-modules. We still have to check that ϵ''_X is a morphism of chain complexes, i. e. is compatible with the boundary operators, which the following calculations shows it is.

$$\begin{aligned}
&\partial(l \otimes (1 \otimes x_1 \cdots x_n \otimes 1)) \\
&= \sum_{i=1}^{n} (-1)^{i-1} l \otimes \Big(x_i \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes 1 \\
&\qquad\qquad - 1 \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_i \Big) \\
&= \sum_{i=1}^{n} (-1)^{i-1} \big(x_i l \otimes (1 \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes 1) \\
&\qquad\qquad - l x_i \otimes (1 \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes 1) \big) \\
&= \sum_{i=1}^{n} (-1)^{i-1} 0 = 0
\end{aligned}$$

$\widetilde{\epsilon}_X$ was defined in Construction 7.2.1.1, and the lower horizontal morphism is just the induced one. The isomorphism ϵ'''_X is to be the isomorphism from Proposition 6.3.2.4, given by

$$a \otimes (a_0 \otimes \cdots \otimes a_{n+1}) \mapsto (a_{n+1} \cdot a \cdot a_0) \otimes a_1 \otimes \cdots \otimes a_n$$

with $a, a_0, \ldots, a_n$ elements of $k[X]$. Finally, the morphism from the standard Hochschild complex to the normalized standard Hochschild complex is the quotient morphism from Proposition 6.3.1.10.

Going through all the definitions, ϵ_X is described by the following formula[13]

$$\begin{aligned}\epsilon_X(r \cdot \mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n) &= \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) r \otimes \overline{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes \overline{x_{\sigma^{-1}(n)}} \\ &= \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma)\sigma \cdot (r \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_n})\end{aligned}$$

where r is an element of $k[X]$ and $x_1, \ldots, x_n$ are elements of X. ♢

Proposition 7.2.2.2. *The following statements hold regarding the morphisms constructed in Construction 7.2.2.1.*

(1) Let X be a set, $x_1, \ldots, x_n$ elements of X, and r an element of $k[X]$. Then ϵ_X maps the element $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$ of $\Omega^n_{k[X]/k}$ to the element $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$ of $\overline{\mathrm{C}}_n(k[X])$.

(2) Let X be a set. Then ϵ_X is a morphism of commutative differential graded algebras, with respect to the commutative algebra structure on the normalized standard Hochschild complex from Proposition 6.3.2.11.

(3) The morphisms ϵ_X assemble to a natural transformation

$$\epsilon\colon \Omega^\bullet_{k[-]/k} \to \overline{\mathrm{C}}(k[-])$$

of functors $\mathsf{Set} \to \mathrm{CAlg}(\mathsf{Ch}(k))$.

(4) For every set X the chain complexes $\Omega^\bullet_{k[X]/k}$ and $\overline{\mathrm{C}}(k[X])$ are cofibrant, so the natural transformation $\epsilon\colon \Omega^\bullet_{k[-]/k} \to \overline{\mathrm{C}}(k[-])$ from claim (3) can be lifted to a natural transformation of functors $\mathsf{Set} \to \mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$.

(5) Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the diagram

$$\begin{tikzcd} k' \otimes_k \Omega^\bullet_{k[-]/k} \arrow[r] \arrow[d, "\cong"'] & k' \otimes_k \overline{\mathrm{C}}(k[-]) \arrow[d, "\cong"] \\ \Omega^\bullet_{k'[-]/k'} \arrow[r] & \overline{\mathrm{C}}(k'[-]) \end{tikzcd} \tag{7.2}$$

[13] For the action of Σ_n on $\overline{\mathrm{C}}(k[X])$, see Definition 6.3.2.9.

of natural transformations of functors $\mathsf{Set} \to \mathrm{CAlg}(\mathsf{Ch}(k')^{\mathrm{cof}})$ *commutes, where the horizontal functors are induced by* ϵ*, the left natural isomorphism is the one from Remark 7.1.0.2*[14]*, and the right natural isomorphism is the one from Remark 6.3.1.11.*

(6) For every set X*, the morphism* ϵ_X *is a quasiisomorphism.* ♡

Proof. Proof of claim (1): If x is an element of X, then we can consider x as an element of $k[X]$ and thus of $\overline{\mathrm{C}}_0(k[X])$. By Proposition 6.3.1.10 we then have $\mathrm{d}\,x = 1 \otimes \overline{x}$ in $\overline{\mathrm{C}}_1(k[X])$, and using Proposition 6.3.2.10 we obtain that for $x_1, \ldots, x_n$ and r as in the claim the equation

$$r \cdot \mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n = \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) r \otimes \overline{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes \overline{x_{\sigma^{-1}(n)}}$$

holds in $\overline{\mathrm{C}}_n(k[X])$, which shows the claim, as the right hand side is the formula for $\epsilon_X(r \cdot \mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n)$ given in Construction 7.2.2.1.

Proof of claim (2): Follows immediately from claim (1).

Proof of claim (3): Let $f\colon X \to Y$ be a map of sets, and denote by $F = k[f]$ the induced morphism of commutative k-algebras $k[X] \to k[Y]$. We have to show that $\overline{\mathrm{C}}(F) \circ \epsilon_X = \epsilon_Y \circ \Omega^\bullet_{F/k}$. So let $x_1, \ldots, x_n$ be elements of X and r an element of $k[X]$. We first evaluate the left hand side on $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$. By (1), ϵ_X maps $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$ to $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$. As $\overline{\mathrm{C}}(F)$ is compatible with the strict mixed structure as well as multiplication, and given by F on degree 0 (see Propositions 6.3.1.10, 6.3.2.7 and 6.3.2.11) we obtain the following.

$$(\overline{\mathrm{C}}(F) \circ \epsilon_X)(r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n) = F(r)\,\mathrm{d}\,f(x_1) \cdots \mathrm{d}\,f(x_n)$$

We now evaluate $\epsilon_Y \circ \Omega^\bullet_{F/k}$ on $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$. The morphism $\Omega^\bullet_{F/k}$ maps this element to $F(r)\,\mathrm{d}\,f(x_1) \cdots \mathrm{d}\,f(x_n)$. It is crucial to note at this point that this description of this element is again of the form that allows us to apply (1), i. e. $f(x_i)$ is an element of the set Y, not merely an element of $k[Y]$, see also Warning 7.2.2.5. We can thus apply (1) to conclude that

$$(\epsilon_Y \circ \Omega^\bullet_{F/k})(r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n) = F(r)\,\mathrm{d}\,f(x_1) \cdots \mathrm{d}\,f(x_n)$$

which shows the claim.

Proof of claim (4): For $\Omega^\bullet_{k[X]/k}$ this is discussed before Definition 7.1.0.1. For $\overline{\mathrm{C}}(k[X])$, note that $k[X]$ and $\overline{(k[X])} = k[X]/(k \cdot 1)$ are free k-modules with bases $\left\{ x^{\vec{j}} \;\middle|\; \vec{j} \in \mathbb{Z}^{\times X} \right\}$ and $\left\{ x^{\vec{j}} \;\middle|\; \vec{j} \in \mathbb{Z}^{\times X}, \vec{j} \neq \vec{0} \right\}$, respectively, and thus $\overline{\mathrm{C}}(k[X])$ is cofibrant by Proposition 6.3.1.10 and [Hov99, 2.3.6].

Proof of claim (5): It suffices to check that the square commutes when evaluated at a set X, which can be checked by writing a generic element of

[14]Composed with the natural isomorphism $\Omega^\bullet_{k' \otimes_k k[-]/k} \cong \Omega^\bullet_{k'[-]/k}$ that is induced by the natural isomorphism $k' \otimes_k k[-] \cong k'[-]$ that is given by $l \otimes r \mapsto l \cdot \varphi[-](r)$.

the upper left chain complex as $r' \otimes (r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n)$ for $x_1, \ldots, x_n$ elements of X, r an element of $k[X]$, and r' an element of k', and verifying that the images in the lower right along the two compositions agree, by applying claim (1) in a manner similar to the proof of claim (3).

Proof of claim (6): ϵ_X is defined as the composite of five morphisms in Construction 7.2.2.1. Three of those were already remarked to be isomorphisms in Construction 7.2.2.1, and a fourth morphism is the quotient morphism $\mathrm{C}(k[X]) \to \overline{\mathrm{C}}(k[X])$, which was shown in Proposition 6.3.1.10 to be a quasiisomorphism. It thus remains to show that the fifth involved morphism, $k[X] \otimes_{k[X]\otimes k[X]} \widetilde{\epsilon}_X$, is a quasiisomorphism as well.

For this, we note as in the proof of claim (1) of Proposition 7.2.1.2 that the model categories $\mathrm{LMod}_{k[X]}(\mathsf{Ch}(k))$ and $\mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k))$ are isomorphic to $\mathsf{Ch}(k[X])$ and $\mathsf{Ch}(k[X] \otimes k[X])$, respectively. The functor

$$k[X] \otimes_{k[X]\otimes k[X]} -\colon\ \mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k)) \to \mathrm{LMod}_{k[X]}(\mathsf{Ch}(k))$$

can be identified with the extension of scalars functor along the multiplication morphism $k[X] \otimes k[X] \to k[X]$ and is thus by Fact 4.1.5.1 a left Quillen functor and hence preserves weak equivalences between cofibrant objects by [Hov99, 1.1.12]. But $\widetilde{\epsilon}_X$ is a quasiisomorphism by claim (2) of Proposition 7.2.1.2, $\mathrm{C}^{\mathrm{sm}}(X)$ is cofibrant as an object of $\mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k))$ by claim (1) of Proposition 7.2.1.2, and $\mathrm{C}^{\mathrm{Bar}}(k[X])$ is cofibrant as an object of $\mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k))$ by Proposition 6.3.2.3. □

As an immediate conclusion of Proposition 7.2.2.2 we obtain the following result showing that $\Omega^{\bullet}_{k[X]/k}$ is a strict multiplicative (but not mixed) model for $\mathrm{HH}(k[X])$.

Corollary 7.2.2.3. *Let X be a set. Then there is an equivalence*

$$\mathrm{HH}(k[X]) \simeq \mathrm{CAlg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big)$$

in $\mathrm{CAlg}(\mathcal{D}(k))$*. Concretely, such an equivalence is given by the composition*[15]

$$\mathrm{HH}(k[X]) \xrightarrow{\simeq} \mathrm{CAlg}(\gamma)(\mathrm{C}(k[X])) \xrightarrow{\simeq} \mathrm{CAlg}(\gamma)\big(\overline{\mathrm{C}}(k[X])\big) \xleftarrow{\simeq} \mathrm{CAlg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big)$$

where the left equivalence is the one from Proposition 6.3.4.3, the middle one is induced by the quotient morphism from Propositions 6.3.1.10 and 6.3.2.11, and the right equivalence is induced from ϵ_X as constructed in Construction 7.2.2.1. ♡

Proof. Follows directly from Propositions 6.3.4.3, 6.3.1.10 and 6.3.2.11 in combination with Proposition 7.2.2.2 (2), (4), and (6). □

[15] If we later refer to “the equivalence from Corollary 7.2.2.3” we mean this specific one.

Proposition 7.2.2.4. *Let $\varphi\colon k \to k'$ be a morphism of commutative rings and X a set. Then there is a commutative square*

$$\begin{tikzcd}
k' \otimes_k \mathrm{HH}(k[X]) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & k' \otimes_k \mathrm{CAlg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[d, "\simeq"] \\
\mathrm{HH}(k' \otimes_k k[X]) \arrow[d, "\simeq"'] & \mathrm{CAlg}(\gamma)\Big(k' \otimes_k \Omega^{\bullet}_{k[X]/k}\Big) \arrow[d, "\simeq"] \\
\mathrm{HH}(k'[X]) \arrow[r, "\simeq"'] & \mathrm{CAlg}(\gamma)\Big(\Omega^{\bullet}_{k'[X]/k'}\Big)
\end{tikzcd} \tag{7.3}$$

in $\mathrm{CAlg}(\mathcal{D}(k'))$*, where the two horizontal equivalences are (induced from) those from Corollary 7.2.2.3, the top left vertical equivalence is the one from Remark 6.2.1.6, the bottom left vertical equivalence is induced from the isomorphism $k' \otimes_k k[X] \cong k'[X]$ that is given by including both tensor factors in $k'[X]$ and then multiplying, the top right vertical equivalence is the one from Remark 4.4.1.3, and the bottom right equivalence is induced by the isomorphism that is given by applying the unit in the first tensor factor and $\Omega^{\bullet}_{\iota_k[X]/k}$ in the second, and then multiplying.* ♡

Proof. Consider the following diagram in $\mathrm{CAlg}(\mathcal{D}(k'))$ that will be explained below. To save space we write γ for $\mathrm{CAlg}(\gamma)$ in this diagram.

$$\begin{tikzcd}
k' \otimes_k \mathrm{HH}(k[X]) \arrow[r, "\simeq"] \arrow[d, dash, "\simeq"] & \mathrm{HH}(k' \otimes_k k[X]) \arrow[r, "\simeq"] & \mathrm{HH}(k'[X]) \arrow[d, dash, "\simeq"] \\
k' \otimes_k \gamma(\mathrm{C}(k[X])) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \gamma(k' \otimes_k \mathrm{C}(k[X])) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \gamma(\mathrm{C}(k'[X])) \arrow[d, "\simeq"] \\
k' \otimes_k \gamma\big(\overline{\mathrm{C}}(k[X])\big) \arrow[r, "\simeq"] & \gamma\big(k' \otimes_k \overline{\mathrm{C}}(k[X])\big) \arrow[r, "\simeq"] & \gamma\big(\overline{\mathrm{C}}(k'[X])\big) \\
k' \otimes_k \gamma\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[r, "\simeq"] \arrow[u, "k' \otimes_k \mathrm{CAlg}(\gamma)(\epsilon_X)", "\simeq"'] & \gamma\Big(k' \otimes_k \Omega^{\bullet}_{k'[X]/k}\Big) \arrow[r, "\simeq"] \arrow[u, "\mathrm{CAlg}(\gamma)\big(k' \otimes_k \epsilon_X\big)", "\simeq"'] & \gamma\Big(\Omega^{\bullet}_{k'[X]/k}\Big) \arrow[u, "\mathrm{CAlg}(\gamma)(\epsilon_X)", "\simeq"']
\end{tikzcd}$$

The big outer rectangle is exactly given by the transpose of diagram (without a filler so far) (7.3), after replacing the horizontal equivalences by their definition in Corollary 7.2.2.3. The middle vertical morphisms are all induced by the quotient morphism from the standard Hochschild complex to the normalized standard Hochschild complex, see Propositions 6.3.1.10 and 6.3.2.11. The two middle left horizontal equivalences are the ones from Remark 4.4.1.3, the middle right horizontal equivalences are the ones from Remarks 6.3.1.7 and 6.3.1.11, combined with the equivalence $k' \otimes_k k[X] \cong k'[X]$ that was already mentioned in the statement.

It now suffices to give a filler for all the small squares and rectangles in the above diagram. The top rectangle has a filler by Remark 6.3.4.4 and minor considerations regarding the isomorphism $k' \otimes_k k[X] \cong k'[X]$ using naturality of the equivalence Proposition 6.3.4.3. The middle left and bottom left squares have fillers by naturality of the equivalences from Remark 4.4.1.3. The middle right square has a filler by Remark 6.3.1.11. The bottom right square has a filler by Proposition 7.2.2.2 (5). $\square$

Warning 7.2.2.5. Let X be a nonempty set. Then ϵ_X is not *not* strictly compatible with the strict mixed structures on domain and codomain. Indeed, if x is an element of X, then we have

$$\mathrm{d}\big(\epsilon_X\big(x^2\big)\big) = \mathrm{d}\big(x^2\big) = 1 \otimes \overline{x^2}$$

which is not equal (though homologous) to the following.

$$\epsilon_X\big(\mathrm{d}\big(x^2\big)\big) = \epsilon_X(2x\,\mathrm{d}\,x) = 2x \otimes \overline{x}$$

In Section 7.3 we will however see that ϵ can be upgraded to a strongly homotopy linear morphism. $\diamondsuit$

Warning 7.2.2.6. A previous version of this text claimed that ϵ as defined in Construction 7.2.2.1 can even be considered as a natural transformation $\Omega^\bullet_{-/k} \to \overline{\mathrm{C}}(-)$ of functors from the full subcategory of the category of k-algebras spanned by the polynomial algebras, to $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$, a claim that fed into the eventual proof of the main result Theorem A.

That claim is however incorrect, as was pointed out by Thomas Nikolaus. Indeed, if we consider the morphism of commutative rings $\varphi\colon \mathbb{Z}[x] \to \mathbb{Z}[y]$ that maps x to y^2, then the diagram

$$\begin{tikzcd}
\Omega^\bullet_{\mathbb{Z}[x]/\mathbb{Z}} \arrow[r, "\epsilon_{\{x\}}"] \arrow[d, "\Omega^\bullet_{\varphi/\mathbb{Z}}"'] & \overline{\mathrm{C}}(\mathbb{Z}[x]) \arrow[d, "\overline{\mathrm{C}}(\varphi)"] \\
\Omega^\bullet_{\mathbb{Z}[y]/\mathbb{Z}} \arrow[r, "\epsilon_{\{y\}}"'] & \overline{\mathrm{C}}(\mathbb{Z}[y])
\end{tikzcd}$$

does not commute, as one can check using the element $\mathrm{d}\,x$ of the top left; The composition along the top right maps this element to $1 \otimes \overline{y^2}$ in the bottom right, whereas the composition along the bottom left maps this element to $2y \otimes \overline{y}$. This phenomenon is closely related to ϵ failing to preserve the differential, see Warning 7.2.2.5. $\diamondsuit$

7.3. De Rham forms as a strict model in Mixed

Let X be a set. As a conclusion to Section 7.2 we showed in Corollary 7.2.2.3 that $\Omega^\bullet_{k[X]/k}$ is a strict model for $\mathrm{HH}(k[X])$ as an object in $\mathrm{CAlg}(\mathcal{D}(k))$. In this section we show that it is also a model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object in $\mathcal{M}\mathrm{ixed}$.

To do so we show that ϵ_X can be upgraded to a strongly homotopy linear morphism in the sense of Section 4.2.3. We will define the data necessary for this, i. e. morphisms $\epsilon_X^{(l)}$ for $l \geq 0$ (where $\epsilon_X^{(0)} = \epsilon_X$), in Section 7.3.1, and the rest of the section will then be devoted to proving that this makes ϵ_X into a strongly homotopy linear morphism.

As $\Omega^\bullet_{k[X]/k}$ has zero boundary operator, this amounts to

$$\partial \circ \epsilon_X^{(l)} = \epsilon_X^{(l-1)} \circ \mathrm{d} - \mathrm{d} \circ \epsilon_X^{(l-1)} \tag{7.4}$$

holding for $l \geq 1$. We will be able to use the partial Leibniz rule for d on the normalized standard Hochschild complex that we proved in Proposition 6.3.2.14 to reduce to only needing to show the above identity for elements of degree 0. This will make up the bulk of this section.

A general pattern that will occur many times in this verification will be that we are given a sum of two sums, each of which are indexed over somewhat complicated indexing sets. We then produce a bijection between those two sets and show that the summands that correspond along this bijection agree, perhaps up to sign. The strategy to show that (7.4) holds will thus be to write both sides as sums over some indexing set, then to subdivide the respective indexing sets sufficiently to be able to pairwise match up the subsets; some will match up on the same side of (7.4) and cancel, others from one side will match with the other side. As the indexing sets we consider will often involve permutations, we will make heavy use of notation and definitions from Section 2.3 (34).

We now give a short overview over the main steps of the proof.

In Section 7.3.2 we will begin by writing the left hand side of (7.4) as a sum indexed by a set I. We then write I as a disjoint union of various subsets, some of which have "cancel" in their notation, and show that the sums over those subsets vanish.

In Section 7.3.3 we begin by considering $\epsilon_X^{(l-1)} \circ \mathrm{d}$, and immediately subdivide the resulting summands into two types. We will also match up the summands of the first type with sums over some subsets of I, i. e. with summands from the left hand side of (7.4). In Section 7.3.4 we will then turn towards the summands of the second type, and rewrite them as a sum over a new indexing set I^{d} that is better suited for later simplifications. In Section 7.3.5 we consider $\mathrm{d} \circ \epsilon_X^{(l-1)}$ and write this as a sum over a indexing set I^1. We then sum up the progress made so far in showing (7.4) in Section 7.3.6.

While I^{d} and I^1 are defined using similar notions, this does not hold for I, so in Section 7.3.7 we replace the remaining subsets of I (those over which the sums have not been matched up yet) by sets $I^{\partial}_{\mathrm{even}}$ and $I^{\partial}_{\mathrm{odd}}$ that are defined in a way similar to I^{d} and I^{1}.

In Section 7.3.8 we then write $I^{\partial}_{\mathrm{even}}$, $I^{\partial}_{\mathrm{odd}}$, I^{d}, and I^{1} as disjoint unions of various subsets. In Section 7.3.9 we show how the sums over some of the subsets of I^{d} cancel with each other, and in Section 7.3.10 we show how the remaining sums match up with each other.

Finally, we put everything together in Section 7.3.11 to prove that $\epsilon_X^{(\bullet)}$ indeed upgrades ϵ_X to a strongly homotopy linear morphism.

7.3.1. Definition of the higher homotopies

Construction 7.3.1.1. Let X be a totally ordered set. We will construct morphisms of $\mathbb{Z}$-graded k-modules

$$\epsilon_X^{(l)} \colon \Omega^{\bullet}_{k[X]/k} \to \overline{\mathrm{C}}(k[X])$$

of degree $2l$ for every $l \geq 0$, such that $\epsilon_X^{(0)} = \epsilon_X$, where ϵ_X is as defined in Construction 7.2.2.1.

The construction and later verifications that we will need to do to show that $\epsilon_X^{(\bullet)}$ forms a strongly homotopy linear morphism are somewhat involved, so we begin by introducing some auxiliary notation and definitions.

First let $l \geq 1$ be an integer. Then we let E_l be the following subset of the symmetric group Σ_{2l}[16], where we consider σ to be extended by $\sigma(0) = 0$.

$$\begin{aligned} E_l \coloneqq \Big\{ \sigma \in \Sigma_{2l} \;\Big|\; & \forall\, 0 \leq i \leq l-1 \colon \sigma \text{ cyclically preserves the} \\ & \qquad\qquad \text{ordering of } \{2i, 2i+1, 2i+2\} \Big\} \end{aligned}$$

Note that as $\sigma(0)$ was defined to be 0 the condition in particular implies that $\sigma(1) < \sigma(2)$.

Next, if $l, m \geq 0$ are integers, then we first define a set $C(l,m)$ as follows.

$$\begin{aligned} C(l,m) \coloneqq \Big\{ (c_1, \ldots, c_{l+1}) \in \{1, \ldots, m+1\}^{l+1} \;\Big|\; & c_{l+1} = m+1 \text{ and} \\ & c_i + 1 \leq c_{i+1} - 1 \text{ for } 1 \leq i \leq l \Big\} \end{aligned}$$

Let $l, m \geq 0$ be integers, $y_1, \ldots, y_m$ elements of $k[X]$, and $(c_1, \ldots, c_{l+1})$ an element of $C(l,m)$. Then we define an element $T((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1}))$ in $\overline{\mathrm{C}}_{2l}(k[X])$ as follows.

$$T((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1})) \coloneqq \prod_{j=1}^{c_1 - 1} y_j \otimes \overline{y_{c_1}} \otimes \overline{\prod_{j=c_1+1}^{c_2-1} y_j} \otimes \cdots \otimes \overline{y_{c_l}} \otimes \overline{\prod_{j=c_l+1}^{c_{l+1}-1} y_j}$$

[16] The symmetric group Σ_{2l} is the group of bijections of the set $\{1, \ldots, 2l\}$.

Note that as $c_{l+1} - 1 = m + 1 - 1 = m$, the last tensor factor does not contain undefined factors. The condition $c_i + 1 \leq c_{i+1} - 1$ in the definition of $C(l, m)$ is made precisely to ensure that the products $\prod_{j=c_i+1}^{c_{i+1}-1} y_j$ are not 1 and thus that $T((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1}))$ is not zero. We will furthermore use the notation $T((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1}))_i$, where $0 \leq i \leq 2l$, for the i-th tensor factor of $T((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1}))$.

We can now define $\epsilon_X^{(l)}$ on degree 0, where we can prescribe the value on monomials in X and then extend k-linearly. Every monomial in X can be written uniquely as $\prod_{j=1}^{m} y_j$ where $m \geq 0$, each y_j is an element of X, and such that $j < j'$ implies $y_j < y_{j'}$. For example if $X = \{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$, then the monomial $x_1^2 x_2 x_3^3$ would be written as the product $x_1 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_3 \cdot x_3$. On elements of this form we define $\epsilon_X^{(l)}$ as

$$\epsilon_X^{(l)}\left(\prod_{j=1}^{m} y_j\right) = \sum_{\sigma \in E_l} \mathrm{sgn}(\sigma) \cdot \sigma \cdot \left(\sum_{\substack{(c_1, \ldots, c_{l+1}) \\ \in C(l,m)}} T((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1}))\right)$$

Note that in the case $l = 0$ the set E_l consists only of the identity, $C(l, m)$ only of the 1-tuple $(m+1)$, and that $T((y_1, \ldots, y_m), (m+1)) = \prod_{j=1}^{m} y_j$. The above definition of $\epsilon_X^{(0)}$ thus recovers the definition of ϵ_X from Construction 7.2.2.1 on elements of degree 0.

To define $\epsilon_X^{(l)}$ in degrees other than 0, we set

$$\epsilon_X^{(l)}(f \,\mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n) := \epsilon_X^{(l)}(f) \cdot \epsilon_X(\mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n)$$

for f an element of $k[X]$ and $x_1, \ldots, x_n$ elements of X, and extend k-linearly. Note that Proposition 7.2.2.2 (2) implies that $\epsilon_X^{(0)} = \epsilon_X$. ♢

7.3.2. Simplification of the boundary

We begin the verification that

$$\partial \circ \epsilon_X^{(l)} = \epsilon_X^{(l-1)} \circ \mathrm{d} - \mathrm{d} \circ \epsilon_X^{(l-1)}$$

holds for $l \geq 1$ by subdividing the left side, and showing that some parts cancel directly.

Definition 7.3.2.1. Let X be a set. Then we define for integers $0 \leq i \leq n$ a morphism of k-modules

$$\partial_i \colon \overline{\mathrm{C}}_n(k[X]) \to \overline{\mathrm{C}}_{n-1}(k[X])$$

as the k-linear extension of

$$\partial_i \colon x^{\vec{v_0}} \otimes \overline{x^{\vec{v_1}}} \otimes \cdots \otimes \overline{x^{\vec{v_n}}} \mapsto x^{\vec{v_0}} \otimes \cdots \otimes \overline{x^{\vec{v_{i-1}}}} \otimes \overline{x^{\vec{v_i} + \vec{v_{i+1}}}} \otimes \overline{x^{\vec{v_{i+2}}}} \otimes \cdots \otimes \overline{x^{\vec{v_n}}}$$

for $0 \leq i \leq n-1$ and

$\partial_n \colon x^{\vec{v_0}} \otimes \overline{x^{\vec{v_1}}} \otimes \cdots \otimes \overline{x^{\vec{v_n}}} \mapsto x^{\vec{v_n}+\vec{v_0}} \otimes \overline{x^{\vec{v_1}}} \otimes \cdots \otimes \overline{x^{\vec{v_{n-1}}}}$

for $i = n$, with $\vec{v_0}, \ldots, \vec{v_n}$ elements of $\mathbb{Z}_{\geq 0}^X$ (with all but finitely many components zero) such that $\vec{v_1}, \ldots, \vec{v_n}$ are non-zero. $\diamondsuit$

Remark 7.3.2.2. Let X be a totally ordered set. Then it follows directly from the definition of the boundary operator on the normalized standard Hochschild complex of $k[X]$ in Propositions 6.3.1.9 and 6.3.1.10 that for $n \geq 1$

$$\partial \colon \overline{\mathrm{C}}_n(k[X]) \to \overline{\mathrm{C}}_{n-1}(k[X])$$

is given by the following sum.

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i$$

This implies in particular the following formula, where $l \geq 1$, and $y_1, \ldots, y_m$ and other notation is as in Construction 7.3.1.1.

$$\partial\left(\epsilon_X^{(l)}\left(\prod_{j=1}^{m} y_j\right)\right) = \sum_{\substack{0 \leq i \leq 2l,\\ \sigma \in E_l\\ \vec{c} \in C(l,m)}} (-1)^i \cdot \operatorname{sgn}(\sigma) \cdot \partial_i(\sigma \cdot T((y_1, \ldots, y_m), \vec{c}))$$

$\diamondsuit$

Definition 7.3.2.3. In this definition we will use notation from Construction 7.3.1.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. We will define several subsets of the set

$$I \coloneqq \{0, \ldots, 2l\} \times E_l \times C(l, m)$$

that by Remark 7.3.2.2 is the indexing set of a sum that $\partial(\epsilon_X^{(l)}(\prod_{j=1}^m y_j))$ can be expressed as.

For $1 \leq i \leq 2l-1$ we define the following set.

$$I_i^{\text{cancel}} \coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = i \text{ and for all } 0 \leq p \leq l-1 \text{ it holds that}$$
$$\{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \not\subseteq \{2p, 2p+1, 2p+2\}\Big\}$$

For $i = 0$ and $i = 2l$ we make the following definitions.

$$I_0^{\text{cancel}} \coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = 0 \text{ and } \sigma^{-1}(1) \neq 1\Big\}$$
$$I_{2l}^{\text{cancel}} \coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = 2l \text{ and } \sigma^{-1}(2l) \neq 2\Big\}$$

The above three subsets of I cover the large part of I where $\sigma^{-1}(i)$ and $\sigma^{-1}(i+1)$ do not take certain special values. We now define a number of additional subsets to deal with the remaining elements. We begin with the case in which i is neither 0 nor $2l$, and where $2p+1$ is involved. So we make the following definitions for $1 \leq i \leq 2l-1$ and $1 \leq p \leq l-1$.

$$\begin{aligned} I^{\text{cancel}}_{i,2p,2p+1} \coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 2p \\ \text{and } \sigma^{-1}(i+1) = 2p+1 \\ \text{and } c_{p+1} + 1 < c_{p+2} - 1\Big\} \\ I^{\text{cancel}}_{i,2p+1,2p+2} \coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 2p+1 \\ \text{and } \sigma^{-1}(i+1) = 2p+2 \\ \text{and } c_p + 1 < c_{p+1} - 1\Big\} \end{aligned}$$

While $p = 0$ would be impossible in the definition of $I^{\text{cancel}}_{i,2p,2p+1}$, it is possible for $I^{\text{cancel}}_{i,2p+1,2p+2}$, though we need a slightly different definition, as there is no c_0. So we make the following definition for $1 \leq i \leq 2l-1$.

$$\begin{aligned} I^{\text{cancel}}_{i,1,2} \coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 1 \text{ and } \sigma^{-1}(i+1) = 2 \\ \text{and } 0 < c_1 - 1\Big\} \end{aligned}$$

Now we consider the case where $2p+1$ is not involved. We make the following definition for $1 \leq i \leq 2l-1$ and $1 \leq p \leq l-1$.

$$\begin{aligned} I_{i,2p+2,2p} \coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 2p+2 \\ \text{and } \sigma^{-1}(i+1) = 2p\Big\} \end{aligned}$$

We next consider the cases $i = 0$ and $i = 2l$.[17]

$$\begin{aligned} I^{\text{cancel}}_{0,0,1} &\coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = 0 \text{ and } \sigma^{-1}(1) = 1 \text{ and } c_1 + 1 < c_2 - 1\Big\} \\ I_{0,0,1} &\coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = 0 \text{ and } \sigma^{-1}(1) = 1 \text{ and } c_1 + 1 = c_2 - 1\Big\} \\ I_{2l,2,0} &\coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = 2l \text{ and } \sigma^{-1}(2l) = 2\Big\} \end{aligned}$$

We now need to cover the left over complement. So we make the following definition for $1 \leq i \leq 2l-1$ and $1 \leq p \leq l-1$.

$$\begin{aligned} I_{i,2p,2p+1} \coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 2p \\ \text{and } \sigma^{-1}(i+1) = 2p+1 \\ \text{and } c_{p+1} + 1 = c_{p+2} - 1\Big\} \\ I_{i,2p+1,2p+2} \coloneqq \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 2p+1 \\ \text{and } \sigma^{-1}(i+1) = 2p+2 \end{aligned}$$

$$\text{and } c_p + 1 = c_{p+1} - 1\Big\}$$

Finally, we define the following for $1 \leq i \leq 2l - 1$.

$$I_{i,1,2} := \Big\{(i', \sigma, \vec{c}) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 1 \text{ and } \sigma^{-1}(i+1) = 2$$
$$\text{and } c_1 = 1\Big\}$$

Still with l, m, and $y_1, \ldots, y_m$ as above, we also introduce the following shorthand notation. For $(i, \sigma, \vec{c})$ an element of I we define

$$B((i, \sigma, \vec{c})) := (-1)^i \cdot \operatorname{sgn}(\sigma) \cdot \partial_i(\sigma \cdot T((y_1, \ldots, y_m), \vec{c}))$$

so that we with Remark 7.3.2.2 have the following concise formula for the boundary of $\epsilon_X^{(l)}(\prod_{j=1}^m y_j)$.

$$\partial\left(\epsilon_X^{(l)}\left(\prod_{j=1}^m y_j\right)\right) = \sum_{v \in I} B(v)$$

$\diamondsuit$

Proposition 7.3.2.4. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then I is the disjoint union of the following subsets.*

I_i^{cancel}	*for* $0 \leq i \leq 2l$
$I_{i,2p,2p+1}^{\text{cancel}}$	*for* $1 \leq i \leq 2l-1$ *and* $1 \leq p \leq l-1$
$I_{i,2p+1,2p+2}^{\text{cancel}}$	*for* $1 \leq i \leq 2l-1$ *and* $0 \leq p \leq l-1$
$I_{i,2p+2,2p}$	*for* $1 \leq i \leq 2l-1$ *and* $1 \leq p \leq l-1$
$I_{i,2p,2p+1}$	*for* $1 \leq i \leq 2l-1$ *and* $1 \leq p \leq l-1$
$I_{i,2p+1,2p+2}$	*for* $1 \leq i \leq 2l-1$ *and* $0 \leq p \leq l-1$
$I_{0,0,1}^{\text{cancel}}$	
$I_{0,0,1}$	
$I_{2l,2,0}$	

$\heartsuit$

Proof. We provide a proof here, but even the very diligent reader that otherwise reads all proofs might prefer to go through the case distinctions for themselves rather than reading the proof. The only arguments appearing apart from nested case distinctions is to look into the definitions of E_l and $C(l, m)$ to see how they exclude certain values, e. g. $\sigma^{-1}(i)$ can not be 0 if $i > 0$ or $\sigma(2p) = \sigma(2p+1) + 1$ is not possible.

[17] Note that $l \geq 1$ implies that c_2 is well-defined.

In all listed subsets there is a unique integer occurring as the first component of the elements. We can thus consider the possible values for the first component separately.

We begin with the value 0. So let $(0, \sigma, \overrightarrow{c})$ be an element of I. We have to show that $(0, \sigma, \overrightarrow{c})$ is an element of exactly one of the subsets I_0^{cancel}, $I_{0,0,1}^{\text{cancel}}$, and $I_{0,0,1}$. If $\sigma^{-1}(1) \neq 1$, then the element lies in I_0^{cancel} but not in the other two subsets. If instead $\sigma^{-1}(1) = 1$, then the element lies in $I_{0,0,1}^{\text{cancel}}$ if and only if $c_1 + 1 < c_2 - 1$ and in $I_{0,0,1}$ if and only if $c_1 + 1 = c_2 - 1$. As $c_1 + 1 \leq c_2 - 1$ by the definition of $C(l, m)$, this covers all cases.

We next consider elements for which the first component is $2l$. So let $(2l, \sigma, \overrightarrow{c})$ be an element of I. We have to show that $(0, \sigma, \overrightarrow{c})$ is an element of exactly one of the subsets I_{2l}^{cancel} and $I_{2l,2,0}$. But the element is in I_{2l}^{cancel} if and only if $\sigma^{-1}(2l) \neq 2$, and in $I_{2l,2,0}$ otherwise.

Now let $1 \leq i \leq 2l - 1$ and $(i, \sigma, \overrightarrow{c})$ an element of I. We have to show that this element lies in precisely one of the following subsets of I.

$$\begin{array}{rl}
I_i^{\text{cancel}} & \\
I_{i,2p,2p+1}^{\text{cancel}} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p+1,2p+2}^{\text{cancel}} & \text{for } 0 \leq p \leq l-1 \\
I_{i,2p+2,2p} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p,2p+1} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p+1,2p+2} & \text{for } 0 \leq p \leq l-1
\end{array}$$

We first note that $(i, \sigma, \overrightarrow{c})$ is an element of I_i^{cancel} if and only if for all $0 \leq p \leq l - 1$ it holds that $\{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \nsubseteq \{2p, 2p+1, 2p+2\}$. It thus remains to show that $(i, \sigma, \overrightarrow{c})$ is an element of one of the other subsets listed above if and only if there exists a $0 \leq p \leq l - 1$ such that $\{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \subseteq \{2p, 2p+1, 2p+2\}$. It follows directly from the definitions that if $(i, \sigma, \overrightarrow{c})$ is an element of one of those subsets, then there exists such a $0 \leq p \leq l - 1$.

We thus assume that $0 \leq p \leq l - 1$ is such that

$$\{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \subseteq \{2p, 2p+1, 2p+2\}$$

and what we need to show is that $(i, \sigma, \overrightarrow{c})$ is an element of exactly one of the subsets of I listed below.

$$\begin{array}{rl}
I_{i,2p+2,2p} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p,2p+1}^{\text{cancel}} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p,2p+1} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p+1,2p+2}^{\text{cancel}} & \text{for } 0 \leq p \leq l-1 \\
I_{i,2p+1,2p+2} & \text{for } 0 \leq p \leq l-1
\end{array}$$

By definition of E_l it must hold either that

$$\sigma(2p) < \sigma(2p+1) < \sigma(2p+2)$$

or

$$\sigma(2p+2) < \sigma(2p) \qquad < \sigma(2p+1)$$

or

$$\sigma(2p+1) < \sigma(2p+2) < \sigma(2p)$$

which implies that it is not possible to have one of the following three equalities.

$$\begin{aligned} \sigma(2p+1) &= \sigma(2p+2)+1 \\ \sigma(2p+2) &= \sigma(2p)+1 \\ \sigma(2p) &= \sigma(2p+1)+1 \end{aligned}$$

This means that we must be in precisely one of the following three cases.

(a) $\sigma^{-1}(i) = 2p+2$ and $\sigma^{-1}(i+1) = 2p$.

(b) $\sigma^{-1}(i) = 2p$ and $\sigma^{-1}(i+1) = 2p+1$.

(c) $\sigma^{-1}(i) = 2p+1$ and $\sigma^{-1}(i+1) = 2p+2$.

We now go through these cases individually.

In case (a), we first note that $(i, \sigma, \vec{c})$ can only possibly be an element of a subset of the first type listed above. Furthermore, note that p can not be 0, because $\sigma(0) = 0 \neq i+1$. Thus we must have $1 \leq p \leq l-1$, and so $(i, \sigma, \vec{c})$ is indeed an element of $I_{i,2p+2,2p}$.

In case (b), the element $(i, \sigma, \vec{c})$ can only possibly be an element of the second or third type of subset listed above, i. e. $I^{\mathrm{cancel}}_{i,2q,2q+1}$ and $I_{i,2q,2q+1}$ for $1 \leq q \leq l-1$. Again p can not be 0, as $\sigma(0) = 0 \neq i$. By definition of $C(l,m)$ we must have $c_{p+1}+1 \leq c_{p+2}-1$, so we have either $c_{p+1}+1 < c_{p+2}-1$ or $c_{p+1}+1 = c_{p+2}-1$. The element $(i, \sigma, \vec{c})$ is an element of $I^{\mathrm{cancel}}_{i,2p,2p+1}$ precisely in the first case and of $I_{i,2p,2p+1}$ precisely in the second case.

Finally, in the case (c), the element $(i, \sigma, \vec{c})$ can only possibly be an element of the fourth or fifth type of subset listed above, i. e. $I^{\mathrm{cancel}}_{i,2q+1,2q+2}$ and $I_{i,2q+1,2q+2}$ for $0 \leq q \leq l-1$. If $p > 0$, then the argument is analogous to the case (b), but it remains to show that if $p = 0$, then $(i, \sigma, \vec{c})$ is an element of precisely one of $I^{\mathrm{cancel}}_{i,1,2}$ and $I_{i,1,2}$. It is an element of the first precisely if $c_1 > 1$ and of the second precisely if $c_1 = 1$. As $c_1 \geq 1$ by the definition of $C(l,m)$, this finishes the proof. □

Proposition 7.3.2.5. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and*

$m \geq 0$ integers, and $y_1, \dots, y_m$ as in Construction 7.3.1.1. Then the following holds for every $1 \leq i \leq 2l - 1$.

$$\sum_{v \in I_i^{\mathrm{cancel}}} B(v) = 0$$

♡

Proof. Let $(i, \sigma, \overrightarrow{c})$ be an element of I_i^{cancel}. Then we claim that the tuple $(i, (i\ i+1) \circ \sigma, \overrightarrow{c})$ is also an element of I_i^{cancel}. For this we need to show that $(i\ i+1) \circ \sigma$ is again an element of E_l, and that for all $0 \leq p \leq l-1$ the following holds.

$$\left\{\ \sigma^{-1}\big((i\ i+1)^{-1}(i)\big), \sigma^{-1}\big((i\ i+1)^{-1}(i+1)\big)\ \right\} \nsubseteq \{2p, 2p+1, 2p+2\}$$

This latter condition follows directly from $(i, \sigma, \overrightarrow{c})$ being an element of I_i^{cancel} given the following short calculation.

$$\begin{aligned} &\left\{\ \sigma^{-1}\big((i\ i+1)^{-1}(i)\big), \sigma^{-1}\big((i\ i+1)^{-1}(i+1)\big)\ \right\} \\ =& \left\{\ \sigma^{-1}(i+1), \sigma^{-1}(i)\ \right\} = \left\{\ \sigma^{-1}(i), \sigma^{-1}(i+1)\ \right\} \end{aligned}$$

We still have to show that $(i\ i+1) \circ \sigma$ is an element of E_l. So let $0 \leq p \leq l-1$. Then there is a condition on the ordering of the three integers obtained by applying $(i\ i+1) \circ \sigma$ to $2p$, $2p+1$, and $2p+2$. Applying σ to those three elements, the condition is satisfied as σ is in E_l. As postcomposing with $(i\ i+1)$ only swaps i and $i+1$, the condition will thus also be satisfied for $(i\ i+1) \circ \sigma$ as long as at most one of i and $i+1$ occurs as a value of $2p$, $2p+1$, and $2p+2$ under σ. But this is ensured by the condition that

$$\left\{\ \sigma^{-1}(i), \sigma^{-1}(i+1)\ \right\} \nsubseteq \{2p, 2p+1, 2p+2\}$$

that holds due to $(i', \sigma, \overrightarrow{c})$ being an element of I_i^{cancel}.

Now let S be a subset of Σ_{2l} containing exactly one representative of each right coset of $\{\mathrm{id}, (i\ i+1)\}$. We then obtain

$$\sum_{v \in I_i^{\mathrm{cancel}}} B(v) = \sum_{\substack{(i,\sigma,\overrightarrow{c}) \in I_i^{\mathrm{cancel}} \\ \text{such that} \\ \sigma \in S}} \left(B((i, \sigma, \overrightarrow{c})) + B((i, (i\ i+1) \circ \sigma, \overrightarrow{c}))\right)$$

so that it suffices to show that if $(i, \sigma, \overrightarrow{c})$ is an element of I_i^{cancel}, then the following holds.

$$B((i, \sigma, \overrightarrow{c})) + B((i, (i\ i+1) \circ \sigma, \overrightarrow{c})) = 0$$

But as ∂_i multiplies together the i-th and $i+1$-th tensor factor we have

$$\partial_i((i\ i+1) \cdot (\sigma \cdot T((y_1, \dots, y_m), \overrightarrow{c}))) = \partial_i(\sigma \cdot T((y_1, \dots, y_m), \overrightarrow{c}))$$

which together with $\mathrm{sgn}((i\ i+1)) = -1$ finishes the proof. □

Proposition 7.3.2.6. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I_0^{\text{cancel}}} B(v) + \sum_{v \in I_{2l}^{\text{cancel}}} B(v) = 0$$

♡

Proof. We prove this by constructing a bijection

$$\varphi\colon I_0^{\text{cancel}} \to I_{2l}^{\text{cancel}}$$

such that for every element v of I_0^{cancel} we have $B(\varphi(v)) = -B(v)$.

We define φ as follows.

$$\varphi\colon (0, \sigma, \overrightarrow{c}) \mapsto (2l, \sigma_{1\to 2l} \circ \sigma, \overrightarrow{c})$$

We also directly define the candidate inverse map as follows.

$$\psi\colon (2l, \sigma, \overrightarrow{c}) \mapsto (0, \sigma_{2l\to 1} \circ \sigma, \overrightarrow{c})$$

It is clear that φ and ψ will be mutually inverse bijections as long as both are well-defined.

Before showing well-definedness we begin with a small observation. Let $(0, \sigma, \overrightarrow{c})$ be an element of I_0^{cancel}. Then the definition of I_0^{cancel} rules out that $\sigma^{-1}(1) = 1$, and we claim that the requirement that σ is an element of E_l also rules out $\sigma^{-1}(1) = 2$. Indeed, if we had $\sigma(2) = 1$, then, as $\sigma(0) = 0$, we would have $\sigma(0) < \sigma(2)$, which due to $\sigma \in E_l$ requires that $\sigma(1)$ is an integer bigger than $\sigma(0)$ and smaller than $\sigma(2)$, which would be impossible. In a completely analogous way one can see that if $(2l, \sigma, \overrightarrow{c})$ is an element of I_{2l}^{cancel}, then $\sigma^{-1}(2l)$ can be neither 1 nor 2.

Now we turn to showing that φ is well-defined. So let $(0, \sigma, \overrightarrow{c})$ be an element of I_0^{cancel}. We have to show that $(2l, \sigma_{1\to 2l} \circ \sigma, \overrightarrow{c})$ is an element of I_{2l}^{cancel}.

We first show that $\sigma_{1\to 2l} \circ \sigma$ is an element of E_l. So let $0 \leq p \leq l-1$. As $\sigma_{1\to 2l}$ preserves the ordering of the subset $\{2, \ldots 2l\}$ it is immediate that $\sigma_{1\to 2l} \circ \sigma$ cyclically preserves the ordering of $\{2p, 2p+1, 2p+2\}$ as long as none of the three values $\sigma(2p)$, $\sigma(2p+1)$, and $\sigma(2p+2)$ is 1. So assume that $0 \leq p \leq l-1$ is such that one of these three values is 1. Our previous observation rules out that this can happen when $p = 0$, so we may assume that $1 \leq p \leq l-1$, which implies that $2p$, $2p+1$, and $2p+2$ are all at least 1 and hence their images under σ will also be at least 1, which implies that the one that is 1 will be the minimum, and σ being in E_l will then imply which of the other two values must be bigger. We now consider the three possible cases separately. So assume first that $\sigma(2p) = 1$. We then obtain that

$$\sigma(2p) < \sigma(2p+1) < \sigma(2p+2)$$

which implies the following.

$$(\sigma_{1\to 2l} \circ \sigma)(2p+1) < (\sigma_{1\to 2l} \circ \sigma)(2p+2) < (\sigma_{1\to 2l} \circ \sigma)(2p)$$

Next, assume that $\sigma(2p+1) = 1$. In this case we must have

$$\sigma(2p+1) < \sigma(2p+2) < \sigma(2p)$$

which implies the following.

$$(\sigma_{1\to 2l} \circ \sigma)(2p+2) < (\sigma_{1\to 2l} \circ \sigma)(2p) < (\sigma_{1\to 2l} \circ \sigma)(2p+1)$$

Finally, assume that $\sigma(2p+2) = 1$. Then we must have

$$\sigma(2p+2) < \sigma(2p) < \sigma(2p+1)$$

which implies the following.

$$(\sigma_{1\to 2l} \circ \sigma)(2p) < (\sigma_{1\to 2l} \circ \sigma)(2p+1) < (\sigma_{1\to 2l} \circ \sigma)(2p+2)$$

This shows that $\sigma_{1\to 2l} \circ \sigma$ is an element of E_l. To show that φ is well-defined we still need to show that

$$(\sigma_{1\to 2l} \circ \sigma)^{-1}(2l) = \sigma^{-1}\big(\sigma_{1\to 2l}^{-1}(2l)\big) = \sigma^{-1}(1)$$

is not 2. But this has been shown in the observation we made above.

We have now shown that φ is well-defined. That ψ is well-defined can be shown in a completely analogous way.

It remains to show that $B(\varphi(v)) = -B(v)$ holds for every element v of I_0^{cancel}. So let $(0, \sigma, \vec{c})$ be an element of I_0^{cancel}. Then we have the following calculation.

$$\begin{aligned}
&B(\varphi((0, \sigma, \vec{c})))\\
&= B((2l, \sigma_{1\to 2l} \circ \sigma, \vec{c}))\\
&= (-1)^{2l} \cdot \operatorname{sgn}(\sigma_{1\to 2l} \circ \sigma) \cdot \partial_{2l}(\sigma_{1\to 2l} \cdot (\sigma \cdot T((y_1, \ldots, y_m), \vec{c})))\\
&= \operatorname{sgn}(\sigma_{1\to 2l}) \cdot \operatorname{sgn}(\sigma) \cdot \partial_0(\sigma \cdot T((y_1, \ldots, y_m), \vec{c}))\\
&= (-1) \cdot \operatorname{sgn}(\sigma) \cdot \partial_0(\sigma \cdot T((y_1, \ldots, y_m), \vec{c}))\\
&= -B((0, \sigma, \vec{c}))
\end{aligned}$$

□

Proposition 7.3.2.7. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Let $1 \leq p \leq l-1$ be an integer. Then the following holds.*

$$\sum_{\substack{1 \leq i \leq 2l-1 \\ v \in I_{i,2p,2p+1}^{\text{cancel}}}} B(v) + \sum_{\substack{1 \leq i \leq 2l-1 \\ v \in I_{i,2p+1,2p+2}^{\text{cancel}}}} B(v) = 0$$

♡

Proof. We use the following notation.

$$J := \left\{ (i, v) \in \{1, \ldots, 2l-1\} \times I \;\middle|\; v \in I^{\mathrm{cancel}}_{i,2p,2p+1} \right\}$$
$$J' := \left\{ (i, v) \in \{1, \ldots, 2l-1\} \times I \;\middle|\; v \in I^{\mathrm{cancel}}_{i,2p+1,2p+2} \right\}$$

To prove this proposition it then suffices to construct maps

$$\varphi\colon J \to J' \qquad \text{and} \qquad \psi\colon J' \to J$$

that are mutually inverse bijections such that for every element (i, v) of J we have $B(w) = -B(v)$ if w is the second component of $\varphi((i, v))$.

So let $(i, (i, \sigma, \vec{c}))$ be an element of J. By definition of $I^{\mathrm{cancel}}_{i,2p,2p+1}$ we have $\sigma(2p) = i$ and $\sigma(2p+1) = i+1$ so that $\sigma(2p) < \sigma(2p+1)$. The definition of E_l then implies that we are in one of the following two cases.

(a) $\sigma(2p) < \sigma(2p+1) < \sigma(2p+2)$

(b) $\sigma(2p+2) < \sigma(2p) < \sigma(2p+1)$

If we are in case (a) we let $\tau = \sigma_{i+1 \to \sigma(2p+2)-1}$[18], and if we are instead in case (b) we let $\tau = \sigma_{i+1 \to \sigma(2p+2)}$. In both cases we define φ as follows.

$$\varphi((i, (i, \sigma, \vec{c}))) := (\tau(i+1), (\tau(i+1), \tau \circ \sigma, \vec{c} + \overrightarrow{e_{p+1}}))$$

We will show later that φ is actually well-defined, but will first define ψ. So let $(i, (i, \sigma, \vec{c}))$ be an element of J'. By definition of $I^{\mathrm{cancel}}_{i,2p+1,2p+2}$ we have $\sigma(2p+1) = i$ and $\sigma(2p+2) = i+1$ so that $\sigma(2p+1) < \sigma(2p+2)$. The definition of E_l then implies that we are in one of the following two cases.

(a) $\sigma(2p) < \sigma(2p+1) < \sigma(2p+2)$

(b) $\sigma(2p+1) < \sigma(2p+2) < \sigma(2p)$

If we are in case (a) we let $\tau' = \sigma_{i \to \sigma(2p)+1}$[19] and if we are instead in case (b) we let $\tau' = \sigma_{i \to \sigma(2p)}$. In both cases we define ψ as follows.

$$\psi((i, (i, \sigma, \vec{c}))) := (\tau'(i) - 1, (\tau'(i) - 1, \tau' \circ \sigma, \vec{c} - \overrightarrow{e_{p+1}}))$$

We next show that φ is well-defined. So let $(i, (i, \sigma, \vec{c}))$ be an element of J. We first show that $1 \leq \tau(i+1) \leq 2l-1$. That $1 \leq \tau(i+1)$ is clear. In case (a) we have that $\tau(i+1)$ is by definition strictly smaller than $\sigma(2p+2)$, which can be at most $2l$, and in case (b) we can use that $\sigma(2p+2)$ is strictly smaller than $\sigma(2p)$ by virtue of us being in case (b), and $\sigma(2p)$ is at most $2l$. This show that $\tau(i+1) \leq 2l-1$.

Next we need to show that $\tau \circ \sigma$ is an element of E_l. As τ preserves the ordering of the complement of $\{\sigma(2p+1)\}$ it immediately follows from σ

[18] Note that $\sigma(2p+1) < \sigma(2p+2)$ implies $\sigma(2p+2) - 1 \geq \sigma(2p+1) \geq 1$, so τ is well-defined.
[19] Note that $\sigma(2p) < \sigma(2p+1)$ implies $\sigma(2p) + 1 \leq \sigma(2p+1) \leq 2l$, so τ' is well-defined.

cyclically preserving the ordering of $\{2q, 2q+1, 2q+2\}$ that $\tau \circ \sigma$ does so as well, as long as $0 \leq q \leq l-1$ with $q \neq p$. But if we are in case (a) then we have

$$(\tau \circ \sigma)(2p) < (\tau \circ \sigma)(2p+1) < (\tau \circ \sigma)(2p+2)$$

and in case (b) we have

$$(\tau \circ \sigma)(2p+1) < (\tau \circ \sigma)(2p+2) < (\tau \circ \sigma)(2p)$$

so that $\tau \circ \sigma$ cyclically preserves the ordering of $\{2p, 2p+1, 2p+2\}$ as well.

To finish showing that $(\tau(i+1), \tau \circ \sigma, \vec{c} + \overrightarrow{e_{p+1}})$ is an element of I we need to show that $\vec{c'} = \vec{c} + \overrightarrow{e_{p+1}}$ is an element of $C(l,m)$. Most of the (in)equalities that need to be satisfied for this are inherited from $\vec{c}$, as $\vec{c'}$ has all components except the $p+1$-th component in common with $\vec{c}$, so we are left to show that $c_p + 1 \leq (c_{p+1}+1) - 1$ and $(c_{p+1}+1) + 1 \leq c_{p+2} - 1$. The former follows directly from $c_p + 1 \leq c_{p+1} - 1$, and the latter follows from $c_{p+1} + 1 < c_{p+2} - 1$, which is part of the definition of $I^{\text{cancel}}_{i,2p,2p+1}$.

We have now shown that $(\tau(i+1), \tau \circ \sigma, \vec{c} + \overrightarrow{e_{p+1}})$ is an element of I, and we need to show that it is even an element of $I^{\text{cancel}}_{\tau(p+1),2p+1,2p+2}$. The condition on $\tau \circ \sigma$ holds as

$$\tau(\sigma(2p+1)) = \tau(i+1)$$

and τ is defined exactly so that $\tau(i+1) + 1 = \tau(\sigma(2p+2))$. The condition on $\vec{c} + \overrightarrow{e_{p+1}}$ requires that

$$(c_p) + 1 < (c_{p+1} + 1) - 1$$

which holds as $c_p + 1 \leq c_{p+1} - 1$ due to $\vec{c}$ being in $C(l,m)$.

This finishes the proof that φ is well-defined. That ψ is well-defined can be shown completely analogously.

We next show that $\psi \circ \varphi = \mathrm{id}$. So let $(i, (i, \sigma, \vec{c}))$ be an element of J and τ as in the definition of φ so that the following holds.

$$\varphi((i,(i,\sigma,\vec{c}))) \coloneqq (\tau(i+1), (\tau(i+1), \tau \circ \sigma, \vec{c} + \overrightarrow{e_{p+1}}))$$

Then let τ' be as in the definition of ψ such that we have the following.

$$\begin{aligned}&\psi(\varphi((i,(i,\sigma,\vec{c}))))\\ &\coloneqq (\tau'(\tau(i+1)) - 1, (\tau'(\tau(i+1)) - 1, \tau' \circ \tau \circ \sigma, \vec{c} + \overrightarrow{e_{p+1}} - \overrightarrow{e_{p+1}}))\end{aligned}$$

Inspecting this it is clear that it suffices to show that $\tau' \circ \tau$ is the identity. Note that τ maps $i+1$ to some element but preserves the ordering of the complement, whereas τ' preserves the ordering of the complement of $\{\tau(i+1)\}$. The composition thus also preserves the ordering of the complement of $\{i+1\}$, so that it suffices to show that $\tau' \circ \tau$ maps $i+1$ to $i+1$.

For this we distinguish between the two cases. Let us first assume case (a). Then τ maps $i+1$ to $\sigma(2p+2) - 1$. In showing that φ is well-defined we already saw that $\varphi((i,(i,\sigma,\overrightarrow{c})))$ will be as in case (a) for ψ. Thus τ' is defined by mapping $\tau(i+1)$ to $(\tau\circ\sigma)(2p)+1$. As $\sigma(2p)$ is smaller than both $\sigma(2p+1)$ and $\sigma(2p+2)$, we have $(\tau\circ\sigma)(2p)=\sigma(2p)$ so that we obtain the following calculation, where the second equality comes from the definition of $I^{\text{cancel}}_{i,2p,2p+1}$.

$$(\tau\circ\sigma)(2p)+1=\sigma(2p)+1=i+1$$

Let us now assume case (b). Then τ maps $i+1$ to $\sigma(2p+2)$. In showing that φ is well-defined we already saw that $\varphi((i,(i,\sigma,\overrightarrow{c})))$ will be as in case (b) for ψ. Thus τ' is defined by mapping $\tau(i+1)$ to $(\tau\circ\sigma)(2p)$. As $\sigma(2p)$ is smaller than $\sigma(2p+1)$ but bigger than $\sigma(2p+2)$, we have $\tau(\sigma(2p))=\sigma(2p)+1$ so that we obtain the following calculation, where the second equality comes from the definition of $I^{\text{cancel}}_{i,2p,2p+1}$.

$$(\tau\circ\sigma)(2p)=\sigma(2p)+1=i+1$$

We have now shown that $\psi\circ\varphi=\mathrm{id}$. That $\varphi\circ\psi=\mathrm{id}$ can be proven in an analogous way.

It remains to show that for every element $(i,(i,\sigma,\overrightarrow{c}))$ of J

$$B(w)=-B((i,\sigma,\overrightarrow{c}))$$

holds if w is the second component of $\varphi((i,(i,\sigma,\overrightarrow{c})))$. Let τ again be like in the definition of $\varphi((i,(i,\sigma,\overrightarrow{c})))$, so that

$$\varphi((i,(i,\sigma,\overrightarrow{c})))=(\tau(i+1),(\tau(i+1),\tau\circ\sigma,\overrightarrow{c}+\overrightarrow{e_{p+1}}))$$

holds. We can then carry out the following calculation.

$$\begin{aligned}
&B((\tau(i+1),\tau\circ\sigma,\overrightarrow{c}+\overrightarrow{e_{p+1}}))\\
&=(-1)^{\tau(i+1)}\cdot\operatorname{sgn}(\tau\circ\sigma)\cdot\partial_{\tau(i+1)}((\tau\circ\sigma)\cdot T((y_1,\ldots,y_m),\overrightarrow{c}+\overrightarrow{e_{p+1}}))\\
&=(-1)^{\tau(i+1)}\cdot\operatorname{sgn}(\tau)\cdot\operatorname{sgn}(\sigma)\\
&\quad\cdot\partial_{\tau(i+1)}((\tau\circ\sigma)\cdot T((y_1,\ldots,y_m),\overrightarrow{c}+\overrightarrow{e_{p+1}}))\\
&=(-1)^{\tau(i+1)}\cdot(-1)^{\tau(i+1)-(i+1)}\cdot\operatorname{sgn}(\sigma)\\
&\quad\cdot\partial_{\tau(i+1)}((\tau\circ\sigma)\cdot T((y_1,\ldots,y_m),\overrightarrow{c}+\overrightarrow{e_{p+1}}))\\
&=-(-1)^{i}\cdot\operatorname{sgn}(\sigma)\partial_{\tau(i+1)}((\tau\circ\sigma)\cdot T((y_1,\ldots,y_m),\overrightarrow{c}+\overrightarrow{e_{p+1}}))
\end{aligned}$$

It now remains to show that

$$\partial_{\tau(i+1)}((\tau\circ\sigma)\cdot T((y_1,\ldots,y_m),\overrightarrow{c}+\overrightarrow{e_{p+1}}))=\partial_i(\sigma\cdot T((y_1,\ldots,y_m),\overrightarrow{c}))\quad(*)$$

On the left hand side we start with $T((y_1,\ldots,y_m),\overrightarrow{c}+\overrightarrow{e_{p+1}})$, permute the tensor factors with $\tau\circ\sigma$, and then multiply the $\tau(i+1)$-th and $\tau(i+1)+1$-th tensor factor together. Note that $(\tau\circ\sigma)^{-1}(\tau(i+1))=\sigma^{-1}(i+1)=2p+1$, and in both cases we distinguished one can furthermore check that it holds as well that $\tau^{-1}(\tau(i+1)+1) = \sigma(2p+2)$. As τ preserves the ordering of the complement of $\{i+1\}$, we can thus describe the process of obtaining the left hand side of $(*)$ from $T((y_1,\ldots,y_m),\overrightarrow{c}+\overrightarrow{e_{p+1}})$ also as follows: First we permute the tensor factors using σ, then we remove the $\sigma(2p+1) = i+1$-th tensor factor and replace the $\sigma(2p+2)$-th tensor factor by its product with the $\sigma(2p+1)$-th tensor factor.

The $\sigma(2p+2)$-th tensor factor is given by

$$T((y_1,\ldots,y_m),\overrightarrow{c}+\overrightarrow{e_{p+1}})_{2p+2} = \prod_{j=c'_{p+1}+1}^{c'_{p+2}-1} y_j$$

where we define $\overrightarrow{c'} = \overrightarrow{c}+\overrightarrow{e_{p+1}}$ for ease of notation, and the $\sigma(2p+1)$-th tensor factor is given by

$$T((y_1,\ldots,y_m),\overrightarrow{c}+\overrightarrow{e_{p+1}})_{2p+1} = y_{c'_{p+1}}$$

so that, using that $c'_{p+1} = c_{p+1}+1$ and that the other components of c' equal those of c, we obtain that the product is

$$y_{c_{p+1}+1}\cdot\left(\prod_{j=c_{p+1}+2}^{c_{p+2}-1} y_j\right) = \prod_{j=c_{p+1}+1}^{c_{p+2}-1} y_j$$

which is exactly the $2p+2$-th tensor factor of $T((y_1,\ldots,y_m),\overrightarrow{c})$. As the tensor factors of $T\Big((y_1,\ldots,y_m),\overrightarrow{c'}\Big)$ and $T((y_1,\ldots,y_m),\overrightarrow{c})$ are equal except the $2p$-th, $2p+1$-th, and $2p+2$-th, we can thus describe the process of obtaining the left hand side of $(*)$ from $T((y_1,\ldots,y_m),\overrightarrow{c})$ as follows (note that the second argument of T is now $\overrightarrow{c}$, not $\overrightarrow{c'}$): First we permute the tensor factors using σ, then we remove the $\sigma(2p+1)$-th tensor factor and replace the $\sigma(2p)$-th tensor factor by the $2p$-th tensor factor of $T\Big((y_1,\ldots,y_m),\overrightarrow{c'}\Big)$.

We have

$$\begin{aligned}
T\Big((y_1,\ldots,y_m),\overrightarrow{c'}\Big)_{2p} &= \prod_{j=c_p+1}^{(c_{p+1}+1)-1} y_j \\
&= \left(\prod_{j=c_p+1}^{c_{p+1}-1} y_j\right)\cdot y_{c_{p+1}} \\
&= T((y_1,\ldots,y_m),\overrightarrow{c})_{2p}\cdot T((y_1,\ldots,y_m),\overrightarrow{c})_{2p+1}
\end{aligned}$$

so that we can also describe the process of obtaining the left hand side of $(*)$ from $T((y_1,\ldots,y_m),\overrightarrow{c})$ as follows: First we permute the tensor factors using σ, then we remove the $\sigma(2p+1)$-th tensor factor and replace the $\sigma(2p)$-th tensor factor by the product of the $\sigma(2p)$-th tensor factor with the $\sigma(2p+1)$-th tensor factor. But this is exactly the definition of the right hand side, as $\sigma(2p) = i$. $\square$

Proposition 7.3.2.8. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I_{0,0,1}^{\text{cancel}}} B(v) + \sum_{\substack{1 \leq i \leq 2l-1 \\ v \in I_{i,1,2}^{\text{cancel}}}} B(v) = 0$$

♡

Proof. This proposition and proof is very similar to Proposition 7.3.2.7, but a easier, as we are always in case (a).[20] We will thus refer to the proof of Proposition 7.3.2.7 for more details of the proof. We use the following notation.

$$J \coloneqq \left\{ (i, v) \in \{1, \ldots, 2l-1\} \times I \;\middle|\; v \in I_{i,1,2}^{\text{cancel}} \right\}$$

To prove this proposition it then suffices to construct maps

$$\varphi\colon I_{0,0,1}^{\text{cancel}} \to J \qquad \text{and} \qquad \psi\colon J \to I_{0,0,1}^{\text{cancel}}$$

that are mutually inverse bijections such that for every element v of J the identity $B(w) = -B(v)$ holds if w is the second component of $\varphi(v)$.

We begin by defining φ, which we do as follows.[21]

$$\varphi((0, \sigma, \vec{c})) = \big(\sigma(2) - 1, \big(\sigma(2) - 1, \sigma_{1 \to \sigma(2)-1} \circ \sigma, \vec{c} + \vec{e_1}\big)\big)$$

Now let $(i, (i, \sigma, \vec{c}))$ be an element of J. Then we define ψ as follows.

$$\psi((i, (i, \sigma, \vec{c}))) = (0, \sigma_{i \to 1} \circ \sigma, \vec{c} - \vec{e_1})$$

We next show that φ is well-defined. So let $(0, \sigma, \vec{c})$ be an element of $I_{0,0,1}^{\text{cancel}}$. Then $1 \leq \sigma(2) - 1 \leq 2l - 1$ and $\sigma_{1 \to \sigma(2)-1} \circ \sigma \in E_l$ can be shown exactly as in the proof of Proposition 7.3.2.7. To see that $\vec{c} + \vec{e_1}$ is an element of $C(l, m)$ we need to show that $(c_1 + 1) + 1 \leq c_2 - 1$, which follows from the condition $c_1 + 1 < c_2 - 1$ that is part of the definition of $I_{0,0,1}^{\text{cancel}}$. To see that $(\sigma(2) - 1, \sigma_{1 \to \sigma(2)-1} \circ \sigma, \vec{c} + \vec{e_1})$ is even an element of $I_{i,1,2}^{\text{cancel}}$ we need to show a condition on the values of 1 and 2 under $\sigma_{1 \to \sigma(2)-1} \circ \sigma$, which can be done exactly as in Proposition 7.3.2.7, and that $0 < (c_1 + 1) - 1$, which follows from $c_1 \geq 1$.

The proof that ψ is well-defined is very similar. That φ and ψ are mutually inverse can be shown just as in Proposition 7.3.2.7 (though the proof is easier, as only one case needs to be considered). Finally, that $B(w) = -B(v)$ for every element v of J with w the second component of $\varphi(v)$ can also be shown in exactly the same way as in the proof of Proposition 7.3.2.7. □

[20] The reason this is a separate proposition is the fact that the condition that c_1 needs to satisfy for $\vec{c} \in C(l, m)$ is not precisely of the same form as for c_i with $i > 1$, which makes the definitions a little different, and that $\sigma(2p)$ is always 0 if $p = 0$. Those differences don't add any complications to the proof and instead make it simpler however.

[21] As $\sigma(1) = 1$ by definition of $I_{0,0,1}^{\text{cancel}}$ we must have $\sigma(2) \geq 2$, so $\sigma(2) - 1 \geq 1$.

We sum up the progress made in this section with the following proposition.

Proposition 7.3.2.9. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Let $1 \leq p \leq l-1$ be an integer. Then the following holds.*

$$\begin{aligned}\partial\Big(\epsilon_X^{(l)}(y_1 \cdots y_m)\Big) = &\sum_{\substack{1\leq i\leq 2l-1,\\ 1\leq q\leq l-1,\\ v\in I_{i,2q+2,2q}}} B(v) + \sum_{\substack{1\leq i\leq 2l-1,\\ 1\leq q\leq l-1,\\ v\in I_{i,2q,2q+1}}} B(v) + \sum_{\substack{1\leq i\leq 2l-1,\\ 0\leq q\leq l-1,\\ v\in I_{i,2q+1,2q+2}}} B(v)\\ &+ \sum_{v\in I_{0,0,1}} B(v) + \sum_{v\in I_{2l,2,0}} B(v)\end{aligned}$$

♡

Proof. This follows by combining the previous results as follows. We start by applying Remark 7.3.2.2 and Definition 7.3.2.3.

$$\begin{aligned}&\partial\Big(\epsilon_X^{(l)}(y_1 \cdots y_m)\Big)\\ = &\sum_{v\in I} B(v)\end{aligned}$$

Now we apply the decomposition of I from Proposition 7.3.2.4.

$$\begin{aligned}= &\sum_{\substack{1\leq i\leq 2l-1,\\ v\in I_i^{\text{cancel}}}} B(v)\\ &+ \sum_{v\in I_0^{\text{cancel}}} B(v) + \sum_{v\in I_{2l}^{\text{cancel}}} B(v)\\ &+ \sum_{1\leq q\leq l-1}\left(\sum_{\substack{1\leq i\leq 2l-1,\\ v\in I_{i,2q,2q+1}^{\text{cancel}}}} B(v) + \sum_{\substack{1\leq i\leq 2l-1,\\ v\in I_{i,2q+1,2q+2}^{\text{cancel}}}} B(v)\right)\\ &+ \sum_{\substack{1\leq i\leq 2l-1,\\ v\in I_{i,1,2}^{\text{cancel}}}} B(v) + \sum_{v\in I_{0,0,1}^{\text{cancel}}} B(v)\\ &+ \sum_{\substack{1\leq i\leq 2l-1,\\ 1\leq q\leq l-1,\\ v\in I_{i,2q+2,2q}}} B(v) + \sum_{\substack{1\leq i\leq 2l-1,\\ 1\leq q\leq l-1,\\ v\in I_{i,2q,2q+1}}} B(v) + \sum_{\substack{1\leq i\leq 2l-1,\\ 0\leq q\leq l-1,\\ v\in I_{i,2q+1,2q+2}}} B(v)\\ &+ \sum_{v\in I_{0,0,1}} B(v) + \sum_{v\in I_{2l,2,0}} B(v)\end{aligned}$$

The first line is zero by Proposition 7.3.2.5, the second line by Proposition 7.3.2.6, the third line by Proposition 7.3.2.7, and the fourth line by Proposition 7.3.2.8, which shows the claim. □

7.3.3. Identification of summands of $\epsilon_X^{(l-1)} \circ \mathrm{d}$ of a first type

We now begin looking into the term $\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m))$. We can write this as a sum of terms of two types, and one one type can immediately be identified with summands from $\partial(\epsilon_X^{(l)}(y_1 \cdots y_m))$.

Remark 7.3.3.1. In this remark we use notation from Construction 7.3.1.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1.

We consider $\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m))$. Unpacking the definition, we obtain the following.

$$
\begin{aligned}
&\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m)) \\
&= \epsilon_X^{(l-1)}\left(\sum_{s=1}^{m} y_1 \cdots y_{s-1} \cdot y_{s+1} \cdots y_m \cdot \mathrm{d}\, y_s\right) \\
&= \sum_{s=1}^{m} \epsilon_X^{(l-1)}(y_1 \cdots y_{s-1} \cdot y_{s+1} \cdots y_m) \cdot (1 \otimes \overline{y_s}) \\
&= \sum_{\substack{1 \leq s \leq m \\ \sigma \in E_{l-1} \\ \vec{c} \in C(l-1, m-1)}} (\mathrm{sgn}(\sigma) \cdot \sigma \cdot T((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c})) \cdot (1 \otimes \overline{y_s})
\end{aligned}
$$

We can distinguish two types of summands: Those in which y_{s-1} and y_{s+1} appear together in a tensor factor, and those in which they don't. The former happens precisely if there exists an integer $1 \leq p \leq l-1$ with $c_p < s-1$ and $c_{p+1} > s$[22], or if $c_1 > s$. Note that these possibilities exclude each other, i. e. if we count $c_1 > s$ as being the condition for $p = 0$, then if there exists a $0 \leq p \leq l-1$ satisfying the condition, then it is unique. ♢

We begin by identifying the summands of $\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m))$ in which y_{s-1} and y_{s+1} occur in the same tensor factor.

Proposition 7.3.3.2. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$
\sum_{\substack{1 \leq i \leq 2l-1, \\ 1 \leq p \leq l-1}} \sum_{v \in I_{i,2p+2,2p}} B(v)
$$

[22] Note that we "jump over" y_s, so y_{s+1} has index s rather than $s+1$.

$$= \sum_{\substack{1\leq s\leq m,\\ \sigma\in E_{l-1},\\ \vec{c}\in C(l-1,m-1),\\ 1\leq p\leq l-1\\ \text{such that}\\ c_p<s-1<s<c_{p+1}}} (\operatorname{sgn}(\sigma)\cdot\sigma\cdot T((y_1,\ldots,y_{s-1},y_{s+1},\ldots,y_m),\vec{c}))\cdot(1\otimes\overline{y_s})$$

♡

Proof. We first evaluate the product occurring in the summands on the right hand side of the equation, which by Propositions 6.3.2.10 and 6.3.2.11 yields the following for s, σ, $\vec{c}$, and p as in the sum in the statement.

$$\begin{aligned}
&(\operatorname{sgn}(\sigma)\cdot\sigma\cdot T((y_1,\ldots,y_{s-1},y_{s+1},\ldots,y_m),\vec{c}))\cdot(1\otimes\overline{y_s})\\
&=\sum_{1\leq t\leq 2l-1}\operatorname{sgn}\big(\sigma_{2l-1\to t}\circ\big(\sigma\amalg\operatorname{id}_{\{2l-1\}}\big)\big)\cdot\\
&\qquad\big(\sigma_{2l-1\to t}\circ\big(\sigma\amalg\operatorname{id}_{\{2l-1\}}\big)\big)\cdot(T((y_1,\ldots,y_{s-1},y_{s+1},\ldots,y_m),\vec{c})\otimes\overline{y_s})
\end{aligned}$$

We now make the following definitions.

$$J := \{\,(i,p,v)\in\{1,\ldots,2l-1\}\times\{1,\ldots,l-1\}\times I \mid v\in I_{i,2p+2,2p}\,\}$$

$$\begin{aligned}
J' := \Big\{\Big(s,\sigma'',\vec{c'},p,t\Big)&\in\{1,\ldots,m\}\times E_{l-1}\times C(l-1,m-1)\\
&\quad\times\{1,\ldots,l-1\}\times\{1,\ldots,2l-1\}\\
&\Big|\; c'_p<s-1<s<c'_{p+1}\Big\}
\end{aligned}$$

Furthermore, for $(s,\sigma'',\vec{c'},p,\tau)$ an element of J' we will use the following notation.

$$\begin{aligned}
B'\Big(\Big(s,\sigma'',\vec{c'},p,t\Big)\Big) &:= \operatorname{sgn}\big(\sigma_{2l-1\to t}\circ\big(\sigma''\amalg\operatorname{id}_{\{2l-1\}}\big)\big)\\
&\cdot\big(\sigma_{2l-1\to t}\circ\big(\sigma''\amalg\operatorname{id}_{\{2l-1\}}\big)\big)\cdot\Big(T\Big((y_1,\ldots,y_{s-1},y_{s+1},\ldots,y_m),\vec{c'}\Big)\otimes\overline{y_s}\Big)
\end{aligned}$$

It thus suffices to construct a bijection of sets

$$\Phi\colon J\to J'$$

such that for each element (i,p,v) of J it holds that $B'(\Phi((i,p,v))) = B(v)$.

So let $(i,p,(i,\sigma,\vec{c}))$ be an element of J. Let $s := c_{p+1}$. As $1\leq p\leq l-1$ we have $2\leq p+1\leq l$, so that c_{p+1} is defined $1\leq c_{p+1}\leq(m+1)-2<m$ is satisfied. Next we define σ' as follows.

$$\sigma' = \sigma_{\sigma(2p+2)\to 2l}\circ\sigma\circ\sigma_{2l\to 2p+2}$$

Note that $\sigma'(2l) = 2l$ so that we can consider σ' as an element of Σ_{2l-1}. We let $t := \sigma'(2p+1)$. Note that $1\leq t\leq 2l-1$ and that t is $\sigma(2p+1)$ if

$\sigma(2p+1) < \sigma(2p+2)$ and $\sigma(2p+1)-1$ otherwise. We can now define another permutation σ'' to be the following composition.

$$\begin{aligned}\sigma'' &:= \sigma_{t\to 2l-1} \circ \sigma' \circ \sigma_{2l-1\to 2p+1}\\ &= \sigma_{t\to 2l-1} \circ \sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2} \circ \sigma_{2l-1\to 2p+1}\end{aligned}$$

With this definition σ'' is an element of Σ_{2l-1} that satisfies $\sigma''(2l-1) = 2l-1$, so that we can consider σ'' as an element of $\Sigma_{2(l-1)}$.

We claim that σ'' is an element of E_{l-1}. So let $0 \leq a \leq l-2$. We have to show that σ'' cyclically preserves the ordering of $\{2a, 2a+1, 2a+2\}$. We note first that as $a \leq l-2$ implies $2a+2 \leq 2l-2$, so the image of $\{2a, 2a+1, 2a+2\}$ under $\sigma\circ\sigma_{2l\to 2p+2}\circ\sigma_{2l-1\to 2p+1}$ will have image in the complement of $\{\sigma(2p+2), \sigma^{-1}_{\sigma(2p+2)\to 2l}(t)\}$, so that $\sigma_{t\to 2l-1}\circ\sigma_{\sigma(2p+2)\to 2l}$ is order-preserving on this image. This means that it suffices to show $0 \leq a \leq l-2$ that $\sigma\circ\sigma_{2l\to 2p+2}\circ\sigma_{2l-1\to 2p+1}$ cyclically preserves the ordering of $\{2a, 2a+1, 2a+2\}$.

We first consider the case of $a < p$. Then the claim follows as $\sigma_{2l-1\to 2p+1}$ and $\sigma_{2l\to 2p+2}$ are the identity on $\{2a, 2a+1, 2a+2\}$, and σ cyclically preserves the ordering of $\{2a, 2a+1, 2a+2\}$.

We next consider the case of $a > p$. In this case both $\sigma_{2l-1\to 2p+1}$ is given by addition with 1 on $\{2a, 2a+1, 2a+2\}$, and $\sigma_{2l\to 2p+2}$ is given by addition with 1 on $\{2a+1, 2a+2, 2a+3\}$. The claim thus follows from σ cyclically preserving the ordering of $\{2(a+1), 2(a+1)+1, 2(a+1)+2\}$.

It remains to consider the case $a = p$. In this case $2p$, $2p+1$, and $2p+2$ are mapped by $\sigma_{2l-1\to 2p+1}$ to $2p$, $2p+2$, and $2p+3$, which are mapped by $\sigma_{2l\to 2p+2}$ to $2p$, $2p+3$, and $2p+4$, which are mapped by σ to $\sigma(2p)$, $\sigma(2p+3)$, and $\sigma(2p+4)$, respectively. So we have to show that σ cyclically preserves the ordering of $\{2p, 2p+3, 2p+4\}$. But by assumption σ is an element of $I_{i,2p+2,2p}$, which implies that $\sigma(2p) = \sigma(2p+2)+1$. This means that σ cyclically preserves the ordering of $\{2p, 2p+3, 2p+4\}$ if and only if σ cyclically preserves the ordering of $\{2p+2, 2p+3, 2p+4\}$, which is the case, as σ is an element of E_l.

We define $\overrightarrow{c'} \in \{1, \ldots, m\}^l$ as follows.

$$c'_a := \begin{cases} c_a & \text{for } a \leq p \\ c_{a+1}-1 & \text{for } a > p \end{cases} \qquad \text{for } 1 \leq a \leq l$$

Note that as $c_a \leq m+1$ for $1 \leq a \leq l+1$ we obtain $c'_a \leq m$ for $1 \leq a \leq l$. Furthermore, as $p \geq 1$, and $1 \leq c_1 < c_2 < \ldots < c_{l+1}$ we also obtain that $c'_a \geq 1$ for $1 \leq a \leq l$, so that $\overrightarrow{c'}$ is indeed an element of $\{1, \ldots, m\}^l$. We claim that $\overrightarrow{c'}$ is in fact an element of $C(l-1, m-1)$. For this we first note that as $p \leq l-1$ we have $c'_l = c_{l+1} - 1 = m+1-1 = m$, which handles one of the conditions. That $c'_a + 1 \leq c'_{a+1} - 1$ for $1 \leq a \leq l-1$ follows directly from the corresponding property for $\vec{c}$ as long as $a \neq p$. For $a = p$ we have

$$c'_p + 1 = c_p + 1 \leq c_{p+1} - 1 \leq c_{p+2} - 3 = c'_{p+1} - 2 \leq c'_{p+1} - 1$$

which finishes the proof that $\overrightarrow{c'}$ is an element of $C(l-1, m-1)$.

We can now define Φ as follows.

$$\Phi((i, p, (i, \sigma, \overrightarrow{c}))) := \left(s, \sigma'', \overrightarrow{c'}, p, t\right) = \left(c_{p+1}, \sigma'', \overrightarrow{c'}, p, \sigma'(2p+1)\right)$$

To show that Φ is well-defined it remains to show that it holds in the above situation that

$$c'_p < s-1 < s < c'_{p+1}$$

but unpacking the definitions, this become the following.

$$c_p < c_{p+1} - 1 < c_{p+1} < c_{p+2} - 1$$

which holds as $\overrightarrow{c}$ is an element of $C(l, m)$.

We next show that $B'(\Phi((i, p, v))) = B(v)$ holds for each element (i, p, v) of J. We continue using the notation we introduced up to now for this. We first check that the signs of the two terms agree. For this we have the following calculation.

$$\begin{aligned}
&\operatorname{sgn}\left(\sigma_{2l-1\to t} \circ \left(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\right)\right) \\
&= \operatorname{sgn}(\sigma_{2l-1\to t}) \cdot \operatorname{sgn}(\sigma'') \\
&= (-1)^{2l-1-t} \cdot \operatorname{sgn}(\sigma_{t\to 2l-1} \circ \sigma' \circ \sigma_{2l-1\to 2p+1}) \\
&= (-1)^{2l-1-t} \cdot (-1)^{2l-1-t} \cdot \operatorname{sgn}(\sigma') \cdot (-1)^{2l-1-2p-1} \\
&= \operatorname{sgn}(\sigma') \\
&= \operatorname{sgn}\left(\sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2}\right) \\
&= (-1)^{2l-\sigma(2p+2)} \cdot \operatorname{sgn}(\sigma) \cdot (-1)^{2p+2-2l} \\
&= (-1)^{\sigma(2p+2)} \cdot \operatorname{sgn}(\sigma) \\
&= (-1)^{i} \cdot \operatorname{sgn}(\sigma)
\end{aligned}$$

To complete the proof of $B'(\Phi((i, p, v))) = B(v)$ it remains to show the following.

$$\begin{aligned}
&\left(\sigma_{2l-1\to t} \circ \left(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\right)\right) \cdot \left(T\left((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \overrightarrow{c'}\right) \otimes \overline{y_s}\right) \\
&= \partial_i(\sigma \cdot T((y_1, \ldots, y_m), \overrightarrow{c}))
\end{aligned}$$

We begin by considering $T((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \overrightarrow{c'})$, in the following calculation, where we let $y'_1 = y_1, \ldots, y'_{s-1} = y_{s-1}, y'_s = y_{s+1}, \ldots, y'_{m-1} = y_m$.

$$\begin{aligned}
&T\left((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \overrightarrow{c'}\right) \\
&= \prod_{j=1}^{c'_1-1} y'_j \otimes \overline{y'_{c'_1}} \otimes \overline{\prod_{j=c'_1+1}^{c'_2-1} y'_j} \otimes \cdots \otimes \overline{y'_{c'_{l-1}}} \otimes \overline{\prod_{j=c'_{l-1}+1}^{c'_l-1} y'_j}
\end{aligned}$$

$$= \prod_{j=1}^{c_1-1} y'_j \otimes \overline{y'_{c_1}} \otimes \cdots \otimes \overline{\prod_{j=c_p+1}^{c_{p+2}-1-1} y'_j \otimes \overline{y'_{c_{p+2}-1}}} \otimes \overline{\prod_{j=c_{p+2}-1+1}^{c_{p+3}-1-1} y'_j \otimes \overline{y'_{c_{p+3}-1}}} \otimes \cdots$$

Using that $s = c_{p+1}$.

$$= \prod_{j=1}^{c_1-1} y_j \otimes \overline{y_{c_1}} \otimes \cdots \otimes \overline{\prod_{j=c_p+1}^{c_{p+1}-1} y_j \cdot \prod_{j=c_{p+1}+1}^{c_{p+2}-1} y_j \otimes \overline{y_{c_{p+2}}}} \otimes \overline{\prod_{j=c_{p+2}+1}^{c_{p+3}-1} y_j} \otimes \cdots$$

Abbreviating $T((y_1, \dots, y_m), \vec{c})$ as $T = T_0 \otimes \dots T_{2l}$, we obtain the following.

$$\begin{aligned}
&\Big(\sigma_{2l-1\to t} \circ \big(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot \Big(T\Big((y_1, \dots, y_{s-1}, y_{s+1}, \dots, y_m), \vec{c'}\Big) \otimes \overline{y_s}\Big) \\
&= \Big(\sigma_{2l-1\to t} \circ \big(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \\
&\quad \cdot \big(T_0 \otimes \cdots \otimes T_{2p-1} \otimes (T_{2p} \cdot T_{2p+2}) \otimes T_{2p+3} \otimes \cdots \otimes T_{2l} \otimes T_{2p+1}\big) \\
&= \Big(\sigma_{2l-1\to t} \circ \sigma_{t\to 2l-1} \circ \sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2} \circ \sigma_{2l-1\to 2p+1}\Big)\Big|_{\{1,\dots,2l-1\}} \\
&\quad \cdot \big(T_0 \otimes \cdots \otimes T_{2p-1} \otimes (T_{2p} \cdot T_{2p+2}) \otimes T_{2p+3} \otimes \cdots \otimes T_{2l} \otimes T_{2p+1}\big) \\
&= \Big(\sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2} \circ \sigma_{2l-1\to 2p+1}\Big)\Big|_{\{1,\dots,2l-1\}} \\
&\quad \cdot \big(T_0 \otimes \cdots \otimes T_{2p-1} \otimes (T_{2p} \cdot T_{2p+2}) \otimes T_{2p+3} \otimes \cdots \otimes T_{2l} \otimes T_{2p+1}\big) \\
&= \Big(\sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2}\Big)\Big|_{\{1,\dots,2l-1\}} \\
&\quad \cdot \big(T_0 \otimes \cdots \otimes T_{2p-1} \otimes (T_{2p} \cdot T_{2p+2}) \otimes T_{2p+1} \otimes T_{2p+3} \otimes T_{2p+4} \otimes \cdots \otimes T_{2l}\big)
\end{aligned}$$

Recall that $\sigma(2p+2) = i$ and $\sigma(2p) = i+1$. We now have to distinguish several cases. We start with $1 \le j \le 2p-1$ such that $\sigma(j) < i$. Then the permutation $\sigma' = \sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2}$ maps j to $\sigma(j)$, as both $\sigma_{\sigma(2p+2)\to 2l}$ as well as $\sigma_{2l\to 2p+2}$ act as the identity on the relevant elements. Thus the $\sigma(j)$-th tensor factor in the result is given by T_j. If instead $2p+2 < j \le 2l$ and $\sigma(j) < i$, then σ' maps $j-1$ to $\sigma(j)$. As the $j-1$-th tensor factor of the unpermuted tensor product is given by T_j, we can again conclude that the $\sigma(j)$-th tensor factor of the result is given by T_j. If $j = 2p$ or $j = 2p+2$ then we can not have $\sigma(j) < i$. If $\sigma(2p+1) < i$, then we get that $\sigma'(2p+1) = \sigma(2p+1)$. The upshot is that the 0-th to $(i-1)$-th tensor factors of the result will be given by $T_0 \otimes T_{\sigma^{-1}(1)} \otimes \cdots \otimes T_{\sigma^{-1}(i-1)}$.

We have

$$\sigma'(2p) = \sigma_{i\to 2l}(\sigma(2p)) = \sigma_{i\to 2l}(i+1) = i$$

so that we can furthermore conclude that the i-th tensor factor is given by $T_{2p} \cdot T_{2p+2} = T_{2p+2} \cdot T_{2p}$.

Now let $1 \le j \le 2p-1$ with $\sigma(j) > i$. Then $\sigma'(j) = \sigma(j)-1$, so the $(\sigma(j)-1)$-th tensor factor of the result is given by T_j. If instead $2p+2 < j \le 2l$ and $\sigma(j) > i$, then $\sigma'(j-1) = \sigma(j) - 1$, so that we can again conclude that the $(\sigma(j)-1)$-th tensor factor of the result is given by T_j. Finally, if $\sigma(2p+1) > i$, then $\sigma'(2p+1) = \sigma(2p+1) - 1$ as well. As $\sigma(\{2p, 2p+2\}) = \{i, i+1\}$, the image of $\{1, \ldots, 2p-1, 2p+3, \ldots, 2l\}$ under σ contains $\{i+2, \ldots, 2l\}$. The upshot is that the $(i+1)$-th through $2l-1$-th tensor factors of the product are given by $T_{\sigma^{-1}(i+2)} \otimes \cdots \otimes T_{\sigma^{-1}(2l)}$.

Thus we obtain

$$\begin{aligned}
&\left(\sigma_{2l-1\to t} \circ \left(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\right)\right) \cdot \left(T\Big((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c'}\Big) \otimes \overline{y_s}\right) \\
&= T_0 \otimes T_{\sigma^{-1}(1)} \otimes \cdots T_{\sigma^{-1}(i-1)} \otimes T_{\sigma^{-1}(i)} \cdot T_{\sigma^{-1}(i+1)} \otimes T_{\sigma^{-1}(i+2)} \otimes \cdots \\
&= \partial_i(\sigma \cdot T((y_1, \ldots, y_m), \vec{c}))
\end{aligned}$$

To finish the proof of this proposition it remains to show that Φ is a bijection. For this we construct an inverse Ψ. So let $(s, \sigma'', \vec{c'}, p, t)$ be an element of J'. Then we define

$$\sigma' := \sigma_{2l-1\to t} \circ \sigma'' \circ \sigma_{2p+1\to 2l-1}$$

as an element of Σ_{2l-1}. We then define $i := \sigma'(2p)$ and define σ as follows, as an element of Σ_{2l}.

$$\sigma := \sigma_{2l\to i} \circ \sigma' \circ \sigma_{2p+2\to 2l}$$

Note that as σ' is an element of Σ_{2l-1} we have that $1 \le i \le 2l-1$.

We also claim that σ is an element of E_l. So let $0 \le a \le l-1$. We have to show that σ cyclically preserves the ordering of $\{2a, 2a+1, 2a+2\}$. For this we distinguish four cases. If $a < p$, then $2a$, $2a+1$, and $2a+2$ are mapped to $2a$, $2a+1$, and $2a+2$ by $\sigma_{2p+2\to 2l}$ and $\sigma_{2p+1\to 2l-1}$. The permutation σ'' cyclically preserves the ordering of $\{2a, 2a+1, 2a+2\}$, and as $a < p \le l-1$, the image under σ'' lies in $\{1, \ldots, 2l-2\}$, so that $\sigma_{2l\to i}$ and $\sigma_{2l-1\to t}$ preserve the ordering.

Next we consider the case $a = p$. In this case we have the following.

$$\begin{aligned}
\sigma(2p) &= (\sigma_{2l\to i} \circ \sigma' \circ \sigma_{2p+2\to 2l})(2p) \\
&= (\sigma_{2l\to i} \circ \sigma')(2p) = \sigma_{2l\to i}(i) = i+1 \\
\sigma(2p+2) &= (\sigma_{2l\to i} \circ \sigma' \circ \sigma_{2p+2\to 2l})(2p+2) \\
&= (\sigma_{2l\to i} \circ \sigma')(2l) = \sigma_{2l\to i}(2l) = i
\end{aligned}$$

Which shows that σ cyclically preserves the ordering of $\{2p, 2p+1, 2p+2\}$ (it does not matter where $2p+1$ is mapped to).

We now consider the case $a = p+1$.

$$\begin{aligned}
\sigma(2p+3) &= (\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma'' \circ \sigma_{2p+1\to 2l-1} \circ \sigma_{2p+2\to 2l})(2p+3) \\
&= (\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma'' \circ \sigma_{2p+1\to 2l-1})(2p+2) \\
&= (\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma'')(2p+1) \\
\sigma(2p+3) &= (\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma'' \circ \sigma_{2p+1\to 2l-1} \circ \sigma_{2p+2\to 2l})(2p+4) \\
&= (\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma'' \circ \sigma_{2p+1\to 2l-1})(2p+3)
\end{aligned}$$

$$= (\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma'')(2p+2)$$

What we thus need to show is that the three distinct integers

$$i, \quad (\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma'')(2p+1), \quad (\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma'')(2p+2)$$

are cyclically ordered. We now note that

$$\begin{aligned}(\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma'')(2p) &= (\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma'' \circ \sigma_{2p+1\to 2l-1})(2p) \\ &= (\sigma_{2l\to i} \circ \sigma')(2p) \\ &= \sigma_{2l\to i}(i) \\ &= i+1\end{aligned}$$

so as $2p+1 \neq 2p$ and $2p+2 \neq 2p$, we can replace i by $i+1$ and instead show that $\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''$ cyclically preserves the ordering of $\{2p, 2p+1, 2p+2\}$. Note that $a \leq l-1$ and we are looking at the case where $a = p+1$, which implies that $p \leq l-2$ (even though p in general can be $l-1$ as well), which implies that the set $\{2p, 2p+1, 2p+2\}$ is mapped by σ'' to the complement of $\{2l-1, 2l\}$, so that $\sigma_{2l\to i} \circ \sigma_{2l-1\to t}$ is order preserving on this image. That $\sigma_{2l\to i} \circ \sigma_{2l-1\to t} \circ \sigma''$ cyclically preserves the order of $\{2p, 2p+1, 2p+2\}$ thus follows from σ'' doing so.

Finally, we consider the case $p+1 < a \leq l-1$. Then $2a$, $2a+1$, and $2a+2$ are mapped by $\sigma_{2p+1\to 2l-1} \circ \sigma_{2p+2\to 2l}$ to $2(a-1)$, $2(a-1)+1$, and $2(a-1)+2$. As $a \leq l-1$ we have $a-1 \leq l-2$, so that σ'' maps these elements into the complement of $\{2l-1, 2l\}$, on which $\sigma_{2l\to i} \circ \sigma_{2l-1\to t}$ is order preserving. The claim thus follows from σ'' cyclically preserving the order of $\{2(a-1), 2(a-1)+1, 2(a-1)+2\}$.

To define Ψ we still need to define $\vec{c}$, which we do as follows.

$$c_a \coloneqq \begin{cases} c'_a & \text{for } 1 \leq a \leq p \\ s & \text{for } a = p+1 \\ c'_{a-1}+1 & \text{for } p+2 \leq a \leq l+1 \end{cases} \qquad \text{for } 1 \leq a \leq l+1$$

We first note that as $1 \leq s \leq m$ and $1 \leq c'_a \leq m$ for all $1 \leq a \leq l$, we have that $\vec{c}$ is an element of $\{1, \ldots, m+1\}^{l+1}$. We next need to show that $\vec{c}$ is an element of $C(l,m)$. For this we first note that $p+1 \leq l-1+1 = l$, so $c_{l+1} = c'_l + 1 = m+1$. Furthermore, that $c_a + 1 \leq c_{a+1} - 1$ for $1 \leq a \leq l$ follows directly from $\vec{c'}$ being in $C(l-1, m-1)$ as long as $a < p$ or $a \geq p+2$, so that it only remains to consider the cases $a = p$ and $a = p+1$. But we have

$$c_p = c'_p, \qquad c_{p+1} = s, \qquad c_{p+2} = c'_{p+1} + 1$$

so that the required property follows from

$$c'_p < s-1 < s < c'_{p+1}$$

which holds as $(s, \sigma'', \overrightarrow{c'}, p, t)$ is an element of J'.

We have now defined i, σ, and $\overrightarrow{c}$ and shown that $(i, \sigma, \overrightarrow{c})$ is an element of I. In the course of doing so we also already showed that $\sigma(2p) = i + 1$ and $\sigma(2p + 2) = i$, so that $(i, \sigma, \overrightarrow{c})$ is even an element of $I_{i,2p+2,2p}$. We can thus define Ψ as follows.

$$\Psi\Big(\Big(s, \sigma'', \overrightarrow{c'}, p, t\Big)\Big) := (i, p, (i, \sigma, \overrightarrow{c}))$$

It remains to show that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the respective identity maps. So let $(i, p, (i, \sigma, \overrightarrow{c}))$ be an element of J, and let s, σ', σ'', $\overrightarrow{c'}$, and t be as in the definition of $\Phi((i, p, (i, \sigma, \overrightarrow{c})))$. Then recall that σ' and σ'' were defined (in the definition of Φ) as follows.

$$\begin{aligned}
\sigma' &= \sigma_{i \to 2l} \circ \sigma \circ \sigma_{2l \to 2p+2} \\
\sigma'' &= \sigma_{t \to 2l-1} \circ \sigma' \circ \sigma_{2l-1 \to 2p+1}
\end{aligned}$$

We first note that then

$$\sigma_{2l-1 \to t} \circ \sigma'' \circ \sigma_{2p+1 \to 2l-1} = \sigma'$$

so that the σ' defined from σ'' in the definition of $\Psi((s, \sigma'', \overrightarrow{c'}, p, t))$ recovers the σ' used in the definition of $\Phi((i, p, (i, \sigma, \overrightarrow{c})))$. Next we have

$$\begin{aligned}
\sigma'(2p) &= (\sigma_{i \to 2l} \circ \sigma \circ \sigma_{2l \to 2p+2})(2p) \\
&= (\sigma_{i \to 2l} \circ \sigma)(2p) \\
&= (\sigma_{i \to 2l})(i + 1) \\
&= i
\end{aligned}$$

so that we in the definition of $\Psi((s, \sigma'', \overrightarrow{c'}, p, t))$ also recover the correct i. It then also follows immediately from the definition that the correct σ is recovered as well. Let $\underline{\overrightarrow{c}}$ be what was called $\overrightarrow{c}$ in the definition of $\Psi((s, \sigma'', \overrightarrow{c'}, p, t))$. Then we have for $1 \leq a \leq p$ that

$$\underline{c}_a = c'_a = c_a$$

while for $p + 2 \leq a \leq l + 1$ we have

$$\underline{c}_a = c'_{a-1} + 1 = c_{a-1+1} - 1 + 1 = c_a$$

and finally, we have the following.

$$\underline{c}_{p+1} = s = c_{p+1}$$

This shows that $\Psi \circ \Phi$ is the identity.

Now let $(s, \sigma'', \overrightarrow{c'}, p, t)$ be an element of J', and let σ', σ, i, and $\overrightarrow{c}$ be as in the definition of $\Psi((s, \sigma'', \overrightarrow{c'}, p, t))$. Let $\Phi(i, p, (i, \sigma, \overrightarrow{c})) = (\underline{s}, \underline{\sigma''}, \underline{\overrightarrow{c'}}, \underline{p}, \underline{t})$.

Then we directly obtain $\underline{s} = c_{p+1} = s$ and $\underline{p} = p$. It then follows from the definition that the σ' constructed in the definition of $\Phi(i, p, (i, \sigma, \overrightarrow{c}))$ recovers the σ' constructed in the definition of $\Psi((s, \sigma'', \overrightarrow{c'}, p, t))$. We then obtain that

$$\begin{aligned}\underline{t} &= \sigma'(2p+1)\\ &= (\sigma_{2l-1\to t} \circ \sigma'' \circ \sigma_{2p+1\to 2l-1})(2p+1)\\ &= t\end{aligned}$$

from which we can then also conclude that $\underline{\sigma''} = \sigma''$. It remains to show that $\underline{\overrightarrow{c'}} = \overrightarrow{c'}$. If $1 \leq a \leq p$ then we have

$$\underline{c'}_a = c_a = c'_a$$

and if instead $p < a \leq l$, then we have

$$\underline{c'}_a = c_{a+1} - 1 = c'_{a+1-1} + 1 - 1 = c'_a$$

which finishes the proof. □

The next proposition is exactly like Proposition 7.3.3.2, just for $p = 0$.

Proposition 7.3.3.3. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\begin{aligned}&\sum_{v\in I_{2l,2,0}} B(v)\\ =& \sum_{\substack{1\leq s\leq m,\\ \sigma\in E_{l-1},\\ \overrightarrow{c}\in C(l-1,m-1)\\ \text{such that}\\ s<c_1}} (\operatorname{sgn}(\sigma)\cdot\sigma\cdot T((y_1,\ldots,y_{s-1},y_{s+1},\ldots,y_m),\overrightarrow{c}))\cdot(1\otimes\overline{y_s})\end{aligned}$$

♡

Proof. The proof is very similar to the proof of Proposition 7.3.3.2, but has some differences that require some minor changes. For example there is only $I_{2l,2,0}$ rather than $I_{i,2,0}$ for various values of i, which is related to the relevant permutations σ being forced to map 0 to 0. We will point out how the main steps differ in the case at hand to the case considered in Proposition 7.3.3.2, but avoid details, for which Proposition 7.3.3.2 should be consulted.

The proof of Proposition 7.3.3.2 begins with an unpacking of the product occurring on the right hand side, which applies in the same way in our case. We then define

$$\begin{aligned}J' := \Big\{\Big(s, \sigma'', \overrightarrow{c'}, t\Big) \in&\\ &\{1,\ldots,m\}\times E_{l-1}\times C(l-1,m-1)\times\{1,\ldots,2l-1\}\\ &\Big|\ s < c'_1\Big\}\end{aligned}$$

and for an element $(s, \sigma'', \overrightarrow{c'}, t)$ of J' we define $B'((s, \sigma'', \overrightarrow{c'}, t))$ in exactly the same way as in the proof of Proposition 7.3.3.2 (note the definition of B' there does not depend on p). It thus suffices to construct a bijection of sets

$$\Phi\colon I_{2l,2,0} \to J'$$

such that for each element v of $I_{2l,2,0}$ it holds that $B'(\Phi(v)) = B(v)$.

For the construction of Φ, let $(i, \sigma, \overrightarrow{c})$ be an element of $I_{2l,2,0}$. Then we define s, σ'', $\overrightarrow{c'}$, and t in exactly the same way as in Proposition 7.3.3.2. The verification of the required property of $\overrightarrow{c'}$ differs slightly, we have to show that $s < c'_1$ which amounts to $c_1 < c_2 - 1$, which is satisfied as $\overrightarrow{c}$ is an element of $C(l, m)$.

The proof of Proposition 7.3.3.2 continues with a verification of the identity $B'(\Phi(v)) = B(v)$, which can be done in essentially the same way, only requiring very minor modification, and less cases.

The construction of Ψ requires some modifications from the way it was done in Proposition 7.3.3.2. To start with we do not have p given as part of the input, and instead set $p = 0$. The definition of i, which is defined as $\sigma'(2p) = 0$ in Proposition 7.3.3.2, needs to be changed to $i := 2l$. The definition of σ', σ, and $\overrightarrow{c}$, using these values for p and i, is then exactly as in Proposition 7.3.3.2. The verification that σ is in E_l needs to be modified when checking the cases $a = p$ and $a = p + 1$. In the case $a = p = 0$ we have $\sigma(0) = 0$ and $\sigma(2) = 2l$, so σ cyclically preserves the ordering of $\{0, 1, 2\}$ as $1 \leq \sigma(2) < 2l$. For the case $a = p + 1 = 1$ one arrives as in the proof of Proposition 7.3.3.2 to showing that $2l$, $(\sigma_{2l-1\to t} \circ \sigma'')(1)$, and $(\sigma_{2l-1\to t} \circ \sigma'')(2)$ are cyclically ordered, which is the case if and only if 0, $(\sigma_{2l-1\to t} \circ \sigma'')(1)$, and $(\sigma_{2l-1\to t} \circ \sigma'')(2)$ are cyclically ordered. One now uses that $(\sigma_{2l-1\to t} \circ \sigma'')(0) = 0$ and proceeds as in the proof of Proposition 7.3.3.2. The remaining verification steps in the construction of Ψ are exactly as in the proof of Proposition 7.3.3.2.

The verification of $\Psi \circ \Phi = \mathrm{id}$ is the same as in the proof of Proposition 7.3.3.2 except for the argument showing that i is correctly recovered, which instead in our case is a tautology. The situation for the verification for $\Phi \circ \Psi = \mathrm{id}$ is analogous. □

7.3.4. Reindexing of summands of $\epsilon_X^{(l-1)} \circ \mathrm{d}$ of a second type

We have now shown how the summands of $\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m))$ in which y_{s-1} and y_{s+1} occur together as factors of a single tensor factor match up with summands of $\partial(\epsilon_X^{(l)}(y_1 \cdots y_m))$. We now consider those summands in which y_{s-1} and y_{s+1} do *not* occur together as factors of a single tensor factor. For this it will be helpful to introduce some further notation, and while doing so we will also immediately introduce relevant analogous definitions that will be used in the next sections for $\mathrm{d}(\epsilon_X^{(l-1)}(y_1 \cdots y_m))$ and the remaining summands from $\partial(\epsilon_X^{(l)}(y_1 \cdots y_m))$.

Definition 7.3.4.1. Let $n \geq 1$ be an integer and σ an element of Σ_n. Let us for the moment denote by $P(\sigma)$ the following set.

$$P(\sigma) \coloneqq \Big\{ p \in \{1, \ldots, n-1\} \;\Big|\; \sigma \text{ cyclically preserves the ordering of } \{p-1, p, p+1\} \Big\}$$

Then we make the following definitions

$$e_{\mathrm{even}}(\sigma) \coloneqq \max(\{\, p \in \{1, \ldots, n-1\} \mid p \notin P(\sigma) \text{ and } 2 \mid p \,\})$$
$$e_{\mathrm{odd}}(\sigma) \coloneqq \min(\{\, p \in \{1, \ldots, n-1\} \mid p \notin P(\sigma) \text{ and } 2 \nmid p \,\})$$

where we set $e_{\mathrm{even}}(\sigma) = -\infty$ if the set over which the maximum is taken is empty, and $e_{\mathrm{odd}}(\sigma) = \infty$ if the set over which the minimum is taken is empty.

Now let $n, m \geq 0$ be integers. Then we define a set $C^{\mathrm{full}}(n, m)$ as follows.

$$C^{\mathrm{full}}(n, m) \coloneqq \Big\{ (c_1, \ldots, c_{n+1}) \in \{1, \ldots, m+1\}^{n+1} \;\Big|\; c_1 < c_2 < \ldots < c_n < c_{n+1} \text{ and } c_{n+1} = m+1 \Big\}$$

Now let X be a totally ordered set, $n \geq 1$ and $m \geq 0$ integers, $y_1, \ldots, y_m$ as in Construction 7.3.1.1, and $\vec{c}$ an element of $C^{\mathrm{full}}(n, m)$. Then we define an element $T^{\mathrm{full}}((y_1, \ldots, y_m), \vec{c})$ in $\overline{\mathrm{C}}_n(k[X])$ as follows.

$$T^{\mathrm{full}}((y_1, \ldots, y_m), \vec{c}) \coloneqq \prod_{j=1}^{c_1 - 1} y_j \otimes \overline{\prod_{j=c_1}^{c_2 - 1} y_j} \otimes \overline{\prod_{j=c_2}^{c_3 - 1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_n}^{c_{n+1} - 1} y_j}$$

Finally, we also make the following definition for $n, m \geq 0$ and $\vec{c}$ an element of $C^{\mathrm{full}}(n, m)$.

$$e_{\mathrm{even}}(\vec{c}) \coloneqq \max\Big(\{\, p \in \{1, \ldots, n\} \mid c_p + 1 < c_{p+1} \text{ and } 2 \mid p \,\} \cup \{\, p \in \{0\} \mid 1 < c_1 \,\} \Big)$$
$$e_{\mathrm{odd}}(\vec{c}) \coloneqq \min(\{\, p \in \{1, \ldots, n\} \mid c_p + 1 < c_{p+1} \text{ and } 2 \nmid p \,\})$$

Again, if the set over which we take the maximum is empty then we set $e_{\mathrm{even}}(\vec{c}) = -\infty$ and if the set over which we take the minimum we set $e_{\mathrm{odd}}(\vec{c}) = \infty$. ♢

Definition 7.3.4.2. In this definition will we use notation from Construction 7.3.1.1 and continue on with similar definitions as in Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. The following set I^{d} will act as an indexing set for the summands of $\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m))$ that were not yet considered, while the set I^1 will be used for $\mathrm{d}(\epsilon_X^{(l-1)}(y_1 \cdots y_m))$.

$$\begin{aligned}
I^{\mathrm{d}} \coloneqq \Big\{ &(\sigma, \vec{c}, p) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \times \{1, \dots, 2l-1\} \\
&\mid e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c}) \text{ and } e_{\mathrm{even}}(\sigma) - 2 < p < e_{\mathrm{odd}}(\sigma) + 2 \\
&\text{and } \sigma \text{ cycl. pres. the ord. of } \{p-2, p-1, p+1\} \text{ if } 2 \mid p \text{ and } p \leq 2l-2 \\
&\text{and } \sigma \text{ cycl. pres. the ord. of } \{p-1, p+1, p+2\} \text{ if } 2 \nmid p \text{ and } p \leq 2l-3 \Big\} \\
I^1 \coloneqq \Big\{ &(\sigma, \vec{c}) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \\
&\mid e_{\mathrm{even}}(\sigma) = -\infty \text{ and } e_{\mathrm{even}}(\vec{c}) = -\infty \Big\}
\end{aligned}$$

One should think of I^{d} as something like $E_l \times C(l, m)$, but where we have an extra component p that we "jump over" in the properties that E_l and $C(l, m)$ need to satisfy. We also define some new indexing sets that we will use to reindex sums appearing in $\partial(\epsilon_X^{(l)})(y_1 \cdots y_m)$.

$$\begin{aligned}
I^{\partial}_{\mathrm{even}} \coloneqq \Big\{ &(\sigma, \vec{c}) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \\
&\mid e_{\mathrm{even}}(\vec{c}) \neq -\infty \text{ and } e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{even}}(\vec{c}) \\
&\quad \text{and } e_{\mathrm{odd}}(\vec{c}) \geq e_{\mathrm{even}}(\vec{c}) + 3 \text{ and } e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{even}}(\vec{c}) + 1 \Big\} \\
I^{\partial}_{\mathrm{odd}} \coloneqq \Big\{ &(\sigma, \vec{c}) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \\
&\mid e_{\mathrm{odd}}(\vec{c}) \neq \infty \text{ and } e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{odd}}(\vec{c}) \\
&\quad \text{and } e_{\mathrm{even}}(\vec{c}) \leq e_{\mathrm{odd}}(\vec{c}) - 3 \text{ and } e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{odd}}(\vec{c}) - 1 \Big\}
\end{aligned}$$

We also define B'' and B' as follows for every element $(\sigma', \vec{c'})$ of the set $\Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m)$ and element $(\sigma, \vec{c}, p)$ of I^{d}.

$$\begin{aligned}
B''((\sigma, \vec{c})) &\coloneqq \mathrm{sgn}(\sigma) \cdot \sigma \cdot T^{\mathrm{full}}((y_1, \dots, y_m), \vec{c}) \\
B'((\sigma, \vec{c}, p)) &\coloneqq (-1)^{p+1} \cdot B''((\sigma, \vec{c}))
\end{aligned} \qquad \diamondsuit$$

Proposition 7.3.4.3. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \dots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\begin{aligned}
\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m)) = &\sum_{\substack{1 \leq i \leq 2l-1, \\ 1 \leq p \leq l-1}} \sum_{v \in I_{i, 2p+2, 2p}} B(v) + \sum_{v \in I_{2l, 2, 0}} B(v) \\
&+ \sum_{v \in I^{\mathrm{d}}} B'(v)
\end{aligned}$$

♡

Proof. Define a set J' as follows.

$$\begin{aligned} J' \coloneqq \Big\{ &(s, \sigma, \vec{c}) \in \{1, \ldots, m\} \times E_{l-1} \times C(l-1, m-1) \\ &\Big| \text{ there is no } 1 \leq p \leq l-1 \text{ such that } c_p < s-1 < s < c_{p+1}, \\ &\text{and } c_1 \not> s \Big\} \end{aligned}$$

Then Remark 7.3.3.1 together with Propositions 7.3.3.2 and 7.3.3.3 imply

$$\begin{aligned} &\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m)) \\ = &\sum_{\substack{1 \leq i \leq 2l-1, \\ 1 \leq p \leq l-1}} \sum_{v \in I_{i,2p+2,2p}} B(v) + \sum_{v \in I_{2l,2,0}} B(v) \\ &+ \sum_{(s,\sigma,\vec{c}) \in J'} (\mathrm{sgn}(\sigma) \cdot \sigma \cdot T((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c})) \cdot (1 \otimes \overline{y_s}) \end{aligned}$$

so that it suffices to show the following.

$$\begin{aligned} &\sum_{v \in I^{\mathrm{d}}} B'(v) \\ = &\sum_{(s,\sigma,\vec{c}) \in J'} (\mathrm{sgn}(\sigma) \cdot \sigma \cdot T((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c})) \cdot (1 \otimes \overline{y_s}) \end{aligned}$$

As in the proof of Proposition 7.3.3.2, we begin by evaluating the product occurring in the summands on the right hand side of the equation, which by Propositions 6.3.2.10 and 6.3.2.11 yields the following for $(s, \sigma, \vec{c})$ an element of J'.

$$\begin{aligned} &(\mathrm{sgn}(\sigma) \cdot \sigma \cdot T((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c})) \cdot (1 \otimes \overline{y_s}) \\ = &\sum_{1 \leq t \leq 2l-1} \mathrm{sgn}\big(\sigma_{2l-1 \to t} \circ \big(\sigma \amalg \mathrm{id}_{\{2l-1\}}\big)\big) \cdot \\ &\quad \big(\sigma_{2l-1 \to t} \circ \big(\sigma \amalg \mathrm{id}_{\{2l-1\}}\big)\big) \cdot (T((y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m), \vec{c}) \otimes \overline{y_s}) \end{aligned}$$

Defining a set J as follows

$$\begin{aligned} J \coloneqq \Big\{ &(s, t, \sigma, \vec{c}) \in \{1, \ldots, m\} \times \{1, \ldots, 2l-1\} \times E_{l-1} \times C(l-1, m-1) \\ &\Big| \text{ there is no } 1 \leq q \leq l-1 \text{ such that } c_q < s-1 < s < c_{q+1}, \\ &\text{and } c_1 \not> s \Big\} \end{aligned}$$

it then suffices to show that there exists a bijection

$$\Phi \colon J \to I^{\mathrm{d}}$$

such that the following holds for all elements $(s, t, \sigma, \overrightarrow{c})$ of J.

$$\begin{aligned}
&B'(\Phi((s,t,\sigma,\overrightarrow{c}))) \\
&= \operatorname{sgn}\big(\sigma_{2l-1\to t} \circ \big(\sigma \amalg \operatorname{id}_{\{2l-1\}}\big)\big) \\
&\quad \cdot \big(\sigma_{2l-1\to t} \circ \big(\sigma \amalg \operatorname{id}_{\{2l-1\}}\big)\big) \cdot (T((y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m), \overrightarrow{c}) \otimes \overline{y_s})
\end{aligned}$$

So to define Φ, let $(s, t, \sigma, \overrightarrow{c})$ be an element of J. As $c_1 \not> s$ we must have $c_1 \leq s$. We also have $s \leq m = c_l$. As $c_1 < c_2 < \cdots < c_l$ there must thus either exists a $1 \leq q \leq l$ with $c_q = s$ or with $c_q < s < c_{q+1}$. But as we ruled out $c_q < s-1 < s < c_{q+1}$, the latter implies $c_q = s-1$. The upshot is that there is a $1 \leq q \leq l$ with either $c_q = s$ or $c_q = s-1$. If $c_q = s$ then set $p := 2q-1$. If instead $c_q = s-1$, then set $p := 2q$. We then define

$$\sigma' := \sigma_{2l-1\to t} \circ \sigma \circ \sigma_{p\to 2l-1}$$

as an element of Σ_{2l-1} and $\overrightarrow{c'}$ as follows.

$$c'_a := \begin{cases} c_{a/2}+1 & \text{if } 2 \mid a \\ c_{(a+1)/2} & \text{if } a \leq p \text{ and } 2 \nmid a \\ c_{(a-1)/2}+2 & \text{if } a > p \text{ and } 2 \nmid a \end{cases} \qquad \text{for } 1 \leq a \leq 2l$$

We want to define Φ by setting

$$\Phi((s,t,\sigma,\overrightarrow{c})) := \left(\sigma', \overrightarrow{c'}, p\right)$$

and for this we need to check various things to ensure that this is well-defined.

To begin with, we have $1 \leq q \leq l$ and defined p as either $2q$ or $2q-1$. We can thus conclude that $1 \leq p \leq 2l$, and are left to exclude that $p = 2l$ can occur. This could only occur if we had $c_l = s-1$, which can not happen, as $c_l = m$ and $s-1 < m$. Thus $1 \leq p \leq 2l-1$.

We next show that $e_{\mathrm{even}}(\sigma') - 2 < p < e_{\mathrm{odd}}(\sigma') + 2$. We begin with the left inequality. To show that $e_{\mathrm{even}}(\sigma') < p+2$ we need to show that if $p+2 \leq a \leq 2l-2$ and a is even, then σ' cyclically preserves the ordering of $\{a-1, a, a+1\}$. Unpacking the definition of σ' this amounts to σ cyclically preserving the ordering of $\{a-2, a-1, a\}$, which it does as $a-1$ is odd, $1 \leq a-1 \leq 2l-3$[23], and σ is an element of E_{l-1}. Similarly, to show that $e_{\mathrm{odd}}(\sigma') > p-2$, we need to show that if $1 \leq a \leq p-2$ and a is odd, then σ' cyclically preserves $\{a-1, a, a+1\}$, which unpacking the definition of σ amounts to σ cyclically preserving the ordering of $\{a-1, a, a+1\}$, which it does as it is an element of E_{l-1}. Similarly we can show the extra condition on σ around p, where this time the elements are “split up” by $\sigma_{p\to 2l-1}$. If $p \leq 2l-2$ is even, then σ' cyclically preserving the ordering of $\{p-2, p-1, p+1\}$ amounts to σ cyclically preserving the ordering of

[23] $1 \leq a-1$ is implied by $p+2 \leq a$.

$\{p-2, p-1, p\}$, which it does as $1 \leq p-1 \leq 2l-3$ is odd[24] and σ is an element of E_{l-1}. Similarly, if $p \leq 2l-3$ is odd, then σ' cyclically preserving the ordering of $\{p-1, p+1, p+2\}$ amounts to σ cyclically preserving the ordering of $\{p-1, p, p+1\}$, which it does as $1 \leq p \leq 2l-3$ is odd.

We now show that $\overrightarrow{c'}$ is an element of $C^{\mathrm{full}}(2l-1, m)$. For this we first need to show that c'_a is a well-defined element of $\{1, \ldots, m+1\}$ for $1 \leq a \leq 2l$. If $1 \leq a \leq 2l$ is even, then $1 \leq a/2 \leq l$, so $1 \leq c_{a/2} \leq m$ is well-defined, implying that $1 \leq c'_{a/2} \leq m+1$. If a is odd and $1 \leq a \leq p \leq 2l-1$, then $2 \leq a+1 \leq 2l$, so $1 \leq (a+1)/2 \leq l$ and $1 \leq c_{(a+1)/2} \leq m$ is well-defined. If instead a is odd with $2 \leq p+1 \leq a \leq 2l$, then $1 \leq a-1 \leq 2l-1$. As $a-1$ is even this implies that $1 \leq (a-1)/2 \leq l-1$ so that $c_{(a-1)/2}$ is well-defined and $1 \leq c_{(a-1)/2} \leq m$. As $(a-1)/2 \leq l-1$ we furthermore have that $c_{(a-1)/2} \leq c_l - 2 = m-2$, so that $1 \leq c_{(a-1)/2} + 2 \leq m$. So far we showed that $\overrightarrow{c'}$ is an element of $\{1, \ldots, m+1\}^{2l}$, so we still need to verify the (in)equalities the components need to satisfy. It follows immediately from the definition that $c'_{2l} = c_l + 1 = m+1$. It remains to show that $c'_1 < \cdots < c'_{2l}$. So let $1 \leq a \leq 2l$ be even. Assume that $2 \leq a$. Then we need to show that $c'_{a-1} < c'_a$. Depending on whether $a-1 \leq p$ or not this amounts to either $c_{a/2} < c_{a/2} + 1$, which clearly true, or $c_{(a/2)-1} + 2 < c_{a/2} + 1$, which is true as $\overrightarrow{c}$ is an element of $C(l-1, m-1)$. Now assume that $a \leq 2l-2$. Then we have to show that $c'_a < c'_{a+1}$. Again we have two cases and this amounts to either $c_{a/2} + 1 < c_{(a/2)+1}$, which is true as $\overrightarrow{c}$ is an element of $C(l-1, m-1)$, or to $c_{a/2} + 1 < c_{a/2} + 2$, which is trivially true.

For Φ being well-defined it remains to show that $e_{\mathrm{even}}(\overrightarrow{c'}) < p < e_{\mathrm{odd}}(\overrightarrow{c'})$. We begin with $e_{\mathrm{even}}(\overrightarrow{c'}) < p$. So let $p \leq a \leq 2l$ be even. Then we have to show that $c'_a + 1 = c'_{a+1}$. But unpacking the definition of $\overrightarrow{c'}$ we have $c'_a = c_{a/2} + 1$ and $c'_{a+1} = c_{a/2} + 2$, so this holds. For $p < e_{\mathrm{odd}}(\overrightarrow{c'})$ let $1 \leq a \leq p$ be odd. Then we have to show that $c'_a + 1 = c'_{a+1}$. This time we have by definition $c'_a = c_{(a+1)/2}$, and $c'_{a+1} = c_{(a+1)/2} + 1$. This finishes the proof that Φ is well-defined.

Now let $(s, t, \sigma, \overrightarrow{c})$ be an element of J, and $\Phi((s, t, \sigma, \overrightarrow{c})) = (\sigma', \overrightarrow{c'}, p)$. We want to verify the identity for $B'(\Phi((s, t, \sigma, \overrightarrow{c})))$. We begin with the following calculation.

$$
\begin{aligned}
& B'(\Phi((s,t,\sigma,\overrightarrow{c}))) \\
&= (-1)^{p+1} \cdot \mathrm{sgn}(\sigma') \cdot \sigma' \cdot T^{\mathrm{full}}\Big((y_1, \ldots, y_m), \overrightarrow{c'}\Big) \\
&= (-1)^{p+1} \cdot \mathrm{sgn}(\sigma_{2l-1 \to t} \circ \sigma \circ \sigma_{p \to 2l-1}) \cdot (\sigma_{2l-1 \to t} \circ \sigma \circ \sigma_{p \to 2l-1}) \cdot \\
&\quad \left(\prod_{j=1}^{c'_1 - 1} y_j \otimes \overline{\prod_{j=c'_1}^{c'_2 - 1} y_j} \otimes \overline{\prod_{j=c'_2}^{c'_3 - 1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l} - 1} y_j} \right)
\end{aligned}
$$

[24] $1 \leq p-1$, as $p = 1$ conflicts with the assumption that p is even.

$$
\begin{aligned}
&= (-1)^{p+1} \cdot (-1)^{p-(2l-1)} \operatorname{sgn}(\sigma_{2l-1\to t} \circ \sigma) \cdot (\sigma_{2l-1\to t} \circ \sigma) \cdot \\
&\quad \left(\prod_{j=1}^{c'_1-1} y_j \otimes \cdots \otimes \overline{\prod_{j=c'_{p-1}}^{c'_p-1} y_j} \otimes \overline{\prod_{j=c'_{p+1}}^{c'_{p+2}-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \otimes \overline{\prod_{j=c'_p}^{c'_{p+1}-1} y_j} \right) \\
&= \operatorname{sgn}(\sigma_{2l-1\to t} \circ \sigma) \cdot (\sigma_{2l-1\to t} \circ \sigma) \cdot \\
&\quad \left(\prod_{j=1}^{c'_1-1} y_j \otimes \cdots \otimes \overline{\prod_{j=c'_{p-1}}^{c'_p-1} y_j} \otimes \overline{\prod_{j=c'_{p+1}}^{c'_{p+2}-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \otimes \overline{\prod_{j=c'_p}^{c'_{p+1}-1} y_j} \right)
\end{aligned}
$$

Let $y'_1 = y_1, \ldots, y'_{s-1} = y_{s-1}$, and $y'_s = y_{s+1}, \ldots, y'_{m-1} = y_m$. It then suffices to show the following.

$$
\begin{aligned}
&\prod_{j=1}^{c'_1-1} y_j \otimes \overline{\prod_{j=c'_1}^{c'_2-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{p-1}}^{c'_p-1} y_j} \otimes \overline{\prod_{j=c'_{p+1}}^{c'_{p+2}-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \otimes \overline{\prod_{j=c'_p}^{c'_{p+1}-1} y_j} \\
&= T\big((y'_1, \ldots, y'_{m-1}), \vec{c}\big) \otimes \overline{y_s}
\end{aligned}
$$

For this we distinguish two cases according to the parity of p. If p is odd, then we obtain the following by unpacking the definition of $\vec{c'}$ and p.

$$
\begin{aligned}
&\prod_{j=1}^{c'_1-1} y_j \otimes \cdots \otimes \overline{\prod_{j=c'_{p-1}}^{c'_p-1} y_j} \otimes \overline{\prod_{j=c'_{p+1}}^{c'_{p+2}-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \otimes \overline{\prod_{j=c'_p}^{c'_{p+1}-1} y_j} \\
&= \prod_{j=1}^{c_1-1} y_j \otimes \cdots \otimes \overline{\prod_{j=c_{q-1}+1}^{c_q-1} y_j} \otimes \overline{\prod_{j=c_q+1}^{c_q+1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+2}^{c_l} y_j} \otimes \overline{\prod_{j=c_q}^{c_q} y_j} \\
&= \prod_{j=1}^{c_1-1} y_j \otimes \overline{y_{c_1}} \otimes \cdots \otimes \overline{\prod_{j=c_{q-1}+1}^{c_q-1} y_j} \otimes \overline{y_{c_q+1}} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+2}^{c_l} y_j} \otimes \overline{y_{c_q}} \\
&= \prod_{j=1}^{c_1-1} y'_j \otimes \overline{y'_{c_1}} \otimes \cdots \otimes \overline{\prod_{j=c_{q-1}+1}^{c_q-1} y'_j} \otimes \overline{y'_{c_q}} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+1}^{c_l-1} y'_j} \otimes \overline{y_s} \\
&= T\big((y'_1, \ldots, y'_{m-1}), \vec{c}\big) \otimes \overline{y_s}
\end{aligned}
$$

If p is instead even, one obtains the following instead. There is only a slight difference in the middle.

$$
\begin{aligned}
&\prod_{j=1}^{c'_1-1} y_j \otimes \overline{\prod_{j=c'_1}^{c'_2-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{p-1}}^{c'_p-1} y_j} \otimes \overline{\prod_{j=c'_{p+1}}^{c'_{p+2}-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \otimes \overline{\prod_{j=c'_p}^{c'_{p+1}-1} y_j} \\
&= \prod_{j=1}^{c_1-1} y_j \otimes \overline{\prod_{j=c_1}^{c_1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_q}^{c_q} y_j} \otimes \overline{\prod_{j=c_q+2}^{c_{q+1}} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+2}^{c_l} y_j} \otimes \overline{\prod_{j=c_q}^{c_q} y_j}
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{j=1}^{c_1-1} y_j \otimes \overline{y_{c_1}} \otimes \cdots \otimes \overline{y_{c_q}} \otimes \overline{\prod_{j=c_q+2}^{c_{q+1}} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+2}^{c_l} y_j} \otimes \overline{y_{c_q}} \\
&= \prod_{j=1}^{c_1-1} y'_j \otimes \overline{y'_{c_1}} \otimes \cdots \otimes \overline{y'_{c_q}} \otimes \overline{\prod_{j=c_q+1}^{c_{q+1}-1} y'_j} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+1}^{c_l-1} y'_j} \otimes \overline{y_s} \\
&= T\big((y'_1, \ldots, y'_{m-1}), \overrightarrow{c}\big) \otimes \overline{y_s}
\end{aligned}
$$

To finish the proof of this proposition it remains to show that Φ is a bijection, for which we construct an inverse Ψ. So let $(\sigma', \overrightarrow{c'}, p)$ be an element of I^{d}. Then we define s, t, σ, and $\overrightarrow{c}$ as follows.

$$
\begin{aligned}
s &\coloneqq c'_p \\
t &\coloneqq \sigma'(p) \\
\sigma &\coloneqq \sigma_{t\to 2l-1} \circ \sigma' \circ \sigma_{2l-1\to p} \\
c_a &\coloneqq c'_{2a} - 1 \qquad \text{for } 1 \le a \le l
\end{aligned}
$$

We want to define Ψ as

$$\Psi\Big((\sigma', \overrightarrow{c'}, p)\Big) \coloneqq (s, t, \sigma, \overrightarrow{c})$$

for which we need to check various things to ensure that this is well-defined.

We first note that $\overrightarrow{c'}$ is an element of $C^{\mathrm{full}}(2l-1, m)$ and $1 \le p \le 2l-1$, so c'_p is defined and satisfies $1 \le c_p < c_{2l} = m+1$, so $1 \le s \le m$. Next, σ' is an element of Σ_{2l-1}, so $1 \le t \le 2l-1$ is also well-defined.

We next need to show that σ is an element of E_{l-1}. For this we first note that it follows from the definition of t and σ that σ is an element of Σ_{2l-2}. So now let $1 \le a \le 2l-3$ be an odd integer. We have to show that σ cyclically preserves the ordering of $\{a-1, a, a+1\}$. As $a \le 2l-3$ we have $a+1 < 2l-1$, so this amounts to showing that σ' cyclically preserves the ordering of $\{\sigma_{2l-1\to p}(a-1), \sigma_{2l-1\to p}(a), \sigma_{2l-1\to p}(a+1)\}$. For this we need to distinguish four cases. First consider the case $a < p-1$. Then we have to show that σ' cyclically preserves the ordering of $\{a-1, a, a+1\}$, which it does, as a is odd and $a \le p-2 < e_{\mathrm{odd}}(\sigma')$. Next consider the case $a > p$. Then we have to show that σ' cyclically preserves the ordering of $\{a, a+1, a+2\}$, which it does, as $a+1$ is even and $e_{\mathrm{even}}(\sigma') < p+2 \le a+1$. The cases $a = p-1$ and $a = p$ remain. So assume $a = p-1$. Then we have to show that σ' cyclically preserves the ordering of $\{p-2, p-1, p+1\}$. Now $a \le 2l-3$ being odd implies that $p \le 2l-2$ is even, so this is part of the condition for $(\sigma', \overrightarrow{c'}, p)$ being an element of I^{d}. Similarly, if we assume $a = p$, then we have to show that σ' cyclically preserves the ordering of $\{p-1, p+1, p+2\}$, which it does as $p = a \le 2l-3$ is even.

We now turn to showing that $\overrightarrow{c}$ is an element of $C(l-1, m-1)$. If $1 \le a \le l$, then $2 \le 2a \le 2l$, so it follows that c'_{2a} is defined and satisfies $1 \le c'_1 < c'_{2a} \le m+1$, so that c_a is well-defined and satisfies $1 \le c_a \le m$.

We also obtain $c_l = c'_{2l} - 1 = m + 1 - 1 = m$. So now let $1 \leq a \leq l - 1$. Then we have to show that $c_a + 1 \leq c_{a+1} - 1$. This amounts to showing that $c'_{2a} \leq c'_{2a+2} - 2$. But this follows from $c'_{2a} < c'_{2a+1} < c'_{2a+2}$.

To finish the proof that Ψ is well-defined it remains to show that $c_1 \not> s$ and that there is no $1 \leq q \leq l - 1$ such that $c_q < s - 1 < s < c_{q+1}$. Applying the definitions of s and $\vec{c}$, this means we have to show that $c'_2 - 1 \not> c'_p$ and that there is no $1 \leq q \leq l - 1$ such that $c'_{2q} - 1 < c'_p - 1 < c'_p < c'_{2q+2} - 1$. Let us first tackle the first claim. Assume that $c'_p < c'_2 - 1$, so $c'_p + 1 < c'_2$. As $c'_1 < c'_2 < c'_3 < \cdots$ this implies that $p = 1$. As $p < e_{\mathrm{odd}}(\vec{c'})$ and $p = 1$ is odd, this means that $c'_1 + 1 = c'_2$, which contradicts $c'_p + 1 < c'_2$. Next, assume $1 \leq q \leq l - 1$ such that $c'_{2q} - 1 < c'_p - 1 < c'_p < c'_{2q+2} - 1$. Again as $c'_1 < c'_2 < \cdots$ we obtain that we must have $2q < p < 2q + 2$, so $p = q + 1$. As $e_{\mathrm{even}}(\vec{c'}) < p < e_{\mathrm{odd}}(\vec{c'})$ we can then conclude that $c'_p + 1 = c'_{p+1}$, which contradicts the assumption that $c'_p < c'_{2q+2} - 1$. This finishes the proof that Ψ is well-defined.

It remains to show that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the respective identities. So let $(s, t, \sigma, \vec{c})$ be an element of J and $\Phi((s, t, \sigma, \vec{c})) = (\sigma', \vec{c'}, p)$. Let furthermore $\Psi((\sigma', \vec{c'}, p)) = (\underline{s}, \underline{t}, \underline{\sigma}, \underline{\vec{c}})$. It follows directly from the definitions that $\underline{t} = \sigma'(p) = t$, from which we can then conclude $\underline{\sigma} = \sigma$ as well. It is also immediate from the definitions that $\underline{\vec{c}} = \vec{c}$. To show that $\underline{s} = s$, one needs to distinguish by the parity of p, and then this also follows directly by unpacking the definitions.

Now let $(\sigma', \vec{c'}, p)$ be an element of I^{d} and let $\Psi((\sigma', \vec{c'}, p)) = (s, t, \sigma, \vec{c})$, as well as $\Phi((s, t, \sigma, \vec{c})) = (\underline{\sigma'}, \underline{\vec{c'}}, \underline{p})$. We again need to distinguish by the parity of p. If p is odd, then $p < e_{\mathrm{odd}}(\vec{c'})$ implies that $c'_{p+1} - 1 = c'_p$. From this we obtain $c_{(p+1)/2} = c'_{p+1} - 1 = c'_p = s$. Thus we obtain $\underline{p} = 2((p+1)/2) - 1 = p$. If instead p is even, then we directly obtain $c_{p/2} = c'_p - 1 = s - 1$, so that $\underline{p} = 2(p/2) = p$. As $\underline{p} = p$ it then follows from the definition that $\underline{\sigma'} = \sigma'$. For $\underline{\vec{c'}}$ we obtain the following for $1 \leq a \leq 2l$.

$$\underline{c'}_a \coloneqq \begin{cases} c'_a & \text{if } 2 \mid a \\ c'_{a+1} - 1 & \text{if } a \leq p \text{ and } 2 \nmid a \\ c'_{a-1} + 1 & \text{if } a > p \text{ and } 2 \nmid a \end{cases}$$

So let $a \leq p$ be odd. Then $a < e_{\mathrm{odd}}(\vec{c'})$, so that $c'_a = c'_{a+1} - 1$. Now let $a > p$ be odd. Then $a - 1 \geq p > e_{\mathrm{even}}(\vec{c'})$ is even, so $c'_{a-1} + 1 = c'_a$. This shows that $\underline{\vec{c'}} = \vec{c'}$ and thus finishes the proof that $\Phi \circ \Psi = \mathrm{id}$ and thus the proof of this proposition. □

7.3.5. A first look at $\mathrm{d} \circ \epsilon_X^{(l-1)}$

We now turn to $\mathrm{d}(\epsilon_X^{(l-1)}(y_1 \cdots y_m))$ and write it as a sum over I^1.

Proposition 7.3.5.1. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big) = \sum_{v \in I^1} B''(v)$$

♡

Proof. We begin by evaluating the left hand side using the definition of $\epsilon_X^{(l-1)}$ from Construction 7.3.1.1 and of the differential on the normalized standard Hochschild complex in Proposition 6.3.1.10.

$$\begin{aligned}
&\mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big) \\
&= \mathrm{d}\left(\sum_{\substack{\sigma \in E_{l-1},\\ \vec{c} \in C(l-1,m)}} \operatorname{sgn}(\sigma) \cdot \sigma \cdot T((y_1, \ldots, y_m), \vec{c})\right) \\
&= \sum_{0 \leq t \leq 2l-2} \sigma_{\mathrm{cyc},2l-1}^t \cdot \left(1 \otimes \left(\sum_{\substack{\sigma \in E_{l-1},\\ \vec{c} \in C(l-1,m)}} \operatorname{sgn}(\sigma) \cdot \sigma \cdot T((y_1, \ldots, y_m), \vec{c})\right)\right)
\end{aligned}$$

Note that $\operatorname{sgn}(\sigma_{\mathrm{cyc},2l-1}) = (-1)^{(2l-1)-1} = 1$.

$$\begin{aligned}
&= \sum_{0 \leq t \leq 2l-2} \sum_{\substack{\sigma \in E_{l-1},\\ \vec{c} \in C(l-1,m)}} \operatorname{sgn}\Big(\sigma_{\mathrm{cyc},2l-1}^t \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot \\
&\qquad \Big(\sigma_{\mathrm{cyc},2l-1}^t \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot (1 \otimes T((y_1, \ldots, y_m), \vec{c}))
\end{aligned}$$

Finally, we note that if we had $c_1 = 1$, then the first tensor factor of $T((y_1, \ldots, y_m), \vec{c})$ would be 1, making $1 \otimes T((y_1, \ldots, y_m), \vec{c}) = 0$. We can thus remove those summands.

$$\begin{aligned}
&= \sum_{0 \leq t \leq 2l-2} \sum_{\substack{\sigma \in E_{l-1},\\ \vec{c} \in C(l-1,m)\\ \text{such that}\\ c_1 > 1}} \operatorname{sgn}\Big(\sigma_{\mathrm{cyc},2l-1}^t \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot \\
&\qquad \Big(\sigma_{\mathrm{cyc},2l-1}^t \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot (1 \otimes T((y_1, \ldots, y_m), \vec{c}))
\end{aligned}$$

This leads us to defining a set J as follows.

$$J \coloneqq \{\, (t, \sigma, \vec{c}) \in \{0, \ldots, 2l-2\} \times E_{l-1} \times C(l-1, m) \mid c_1 > 1 \,\}$$

It then suffices to construct a bijection

$$\Phi \colon J \to I^1$$

such that for every element $(t, \sigma, \vec{c})$ of J the following holds.

$$\begin{aligned} B''(\Phi((t, \sigma, \vec{c}))) = \operatorname{sgn}\left(\sigma^t_{\mathrm{cyc},2l-1} \circ \left(\mathrm{id}_{\{1\}} \amalg \sigma\right)\right) \\ \cdot \left(\sigma^t_{\mathrm{cyc},2l-1} \circ \left(\mathrm{id}_{\{1\}} \amalg \sigma\right)\right) \cdot (1 \otimes T((y_1, \ldots, y_m), \vec{c})) \end{aligned}$$

So let $(t, \sigma, \vec{c})$ be an element of J. Then we make the following definitions.

$$\begin{aligned} \sigma' &\coloneqq \sigma^t_{\mathrm{cyc},2l-1} \circ \left(\mathrm{id}_{\{1\}} \amalg \sigma\right) \\ c'_a &\coloneqq \begin{cases} 1 & \text{if } a = 1 \\ c_{a/2} & \text{if } a \text{ is even} \\ c_{(a-1)/2} + 1 & \text{if } 1 < a \text{ is odd} \end{cases} \qquad \text{for } 1 \leq a \leq 2l \\ \Phi((t, \sigma, \vec{c})) &\coloneqq \left(\sigma', \vec{c'}\right) \end{aligned}$$

We need to show that $(\sigma', \vec{c'})$ defined like this is a well-defined element of I^1. For this note first that as σ is an element of Σ_{2l-2} the permutation σ' is indeed an element of Σ_{2l-1}. We also need $e_{\mathrm{even}}(\sigma') = -\infty$. So let $2 \leq a \leq 2l-2$ be even. Then we have to show that σ' cyclically preserves the ordering of $\{a-1, a, a+1\}$. This amounts to σ cyclically preserving the ordering of $\{a-2, a-1, a\}$, which it does as σ is an element of E_{l-1} and $1 \leq a-1 \leq 2l-3$ is odd. Next we need to show that $\vec{c'}$ is a well-defined element of $C^{\mathrm{full}}(2l-1, m)$. If $2 \leq a \leq 2l$ is even, then $1 \leq a/2 \leq l$, so c'_a is well defined and satisfies $1 \leq c'_a \leq m+1$. If $3 \leq a \leq 2l-1$ is odd, then $1 \leq (a-1)/2 \leq l-1$ so that $c_{(a-1)/2}$ is defined and satisfies $1 \leq c_{(a-1)/2} < c_l = m+1$, which implies that $1 \leq c'_a \leq m+1$. Thus $\vec{c'}$ is an element of $\{1, \ldots, m+1\}^{2l}$. We also have $c'_{2l} = c_l = m+1$. It remains to show that $c'_1 < \cdots < c'_{2l}$. This amounts to $1 < c_1 < c_1 + 1 < c_2 < \cdots < c_l$, which holds as $c_1 > 1$ by assumption on $(t, \sigma, \vec{c})$, and as $c_a + 1 \leq c_{a+1} - 1$ for $1 \leq a \leq l-1$ as $\vec{c}$ is an element of $C(l-1, m)$. To show that $(\sigma', \vec{c'})$ is an element of I^1 it still remains to show that $e_{\mathrm{even}}(\vec{c'}) = -\infty$, which amounts to showing that $c'_1 = 1$ and that $c'_{a+1} = c'_a + 1$ for $2 \leq a \leq 2l-2$ even, both of which is the case directly from the definition of $\vec{c'}$.

We now verify the identity that needs to be satisfied for $B''(\Phi((t, \sigma, \vec{c})))$.

$$\begin{aligned} & B''(\Phi((t, \sigma, \vec{c}))) \\ &= B''\left(\left(\sigma', \vec{c'}\right)\right) \\ &= \operatorname{sgn}\left(\sigma^t_{\mathrm{cyc},2l-1} \circ \left(\mathrm{id}_{\{1\}} \amalg \sigma\right)\right) \cdot \\ &\qquad \left(\sigma^t_{\mathrm{cyc},2l-1} \circ \left(\mathrm{id}_{\{1\}} \amalg \sigma\right)\right) \cdot T^{\mathrm{full}}\left((y_1, \ldots, y_m), \vec{c'}\right) \end{aligned}$$

Verification of the identity that is needed for $B''(\Phi((t, \sigma, \vec{c})))$ is now completed by the following calculation.

$$\begin{aligned}
&T^{\mathrm{full}}\Big((y_1,\ldots,y_m),\overrightarrow{c'}\Big)\\
&=\prod_{j=1}^{c'_1-1} y_j\otimes\overline{\prod_{j=c'_1}^{c'_2-1} y_j}\otimes\overline{\prod_{j=c'_2}^{c'_3-1} y_j}\otimes\overline{\prod_{j=c'_3}^{c'_4-1} y_j}\otimes\overline{\prod_{j=c'_4}^{c'_5-1} y_j}\otimes\cdots\otimes\overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j}\\
&=\prod_{j=1}^{0} y_j\otimes\overline{\prod_{j=1}^{c_1-1} y_j}\otimes\overline{\prod_{j=c_1}^{c_1} y_j}\otimes\overline{\prod_{j=c_1+1}^{c_2-1} y_j}\otimes\overline{\prod_{j=c_2}^{c_2} y_j}\otimes\cdots\otimes\overline{\prod_{j=c_{l-1}+1}^{c_l-1} y_j}\\
&=1\otimes\overline{\prod_{j=1}^{c_1-1} y_j}\otimes\overline{y_{c_1}}\otimes\overline{\prod_{j=c_1+1}^{c_2-1} y_j}\otimes\overline{y_{c_2}}\otimes\cdots\otimes\overline{\prod_{j=c_{l-1}+1}^{c_l-1} y_j}\\
&=1\otimes T((y_1,\ldots,y_m),\overrightarrow{c})
\end{aligned}$$

It remains to show that Φ is a bijection. As usual we construct an inverse Ψ. So let $(\sigma',\overrightarrow{c'})$ be an element of I^1. Then we define $\Psi((\sigma',\overrightarrow{c'}))$ as follows.

$$\begin{aligned}
t &:= \sigma'(1)-1\\
\sigma &:= r_{\{2,\ldots,2l-1\}}\Big(\sigma_{\mathrm{cyc},2l-1}^{-t}\circ\sigma'\Big)\\
c_a &:= c'_{2a} \qquad \text{for } 1\le a\le l\\
\Psi\Big(\Big(\sigma',\overrightarrow{c'}\Big)\Big) &:= (t,\sigma,\overrightarrow{c})
\end{aligned}$$

Again we have to check some things to verify that this is well-defined. First, as σ' is an element of Σ_{2l-1}, the value of t satisfies indeed $0\le t\le 2l-2$, and the above definition of σ is an element of Σ_{2l-1}. We need to show that σ is even an element of E_{l-1}. So let $1\le a\le 2l-3$ be an odd integer. We have to show that σ cyclically preserves the ordering of $\{a-1,a,a+1\}$. But as $\sigma_{\mathrm{cyc},2l-1}^{-t}$ cyclically preserves the ordering of any set, the restriction means that what we have to show amounts to showing that σ' cyclically preserves the ordering of $\{a,a+1,a+2\}$, which it does as $2\le a+1\le 2l-2$ is even and $e_{\mathrm{even}}(\sigma)=-\infty$. We also need to show that $\overrightarrow{c}$ is an element of $C(l-1,m)$ satisfying $c_1>1$. For this we note that for $1\le a\le l$ we have $2\le 2a\le 2l$. Thus $1\le c'_1<c'_{2a}\le m+1$, from which it follows that $\overrightarrow{c}$ is an element of $\{1,\ldots,m+1\}^l$ with $c_1>1$. Directly from the definition we have $c_l=c'_{2l}=m+1$, and if $a<l$, then we have $c_a=c'_{2a}<c'_{2a+1}<c'_{2a+2}=c_{a+1}$, from which $c_a+1\le c_{a+1}-1$ follows. This shows that Ψ is well-defined.

To finish the proof of this propositions we are left to show that $\Phi\circ\Psi$ and $\Psi\circ\Phi$ are the respective identity maps. So let $(t,\sigma,\overrightarrow{c})$ be an element of J, and set $\Phi((t,\sigma,\overrightarrow{c}))=(\sigma',\overrightarrow{c'})$ and $\Psi((\sigma',\overrightarrow{c'}))=(\underline{t},\underline{\sigma},\underline{\overrightarrow{c}})$. Then the following calculations show that $\Psi\circ\Phi$ is the identity.

$$\underline{t}=\sigma'(1)-1=\sigma_{\mathrm{cyc},2l-1}^{t}(1)-1=1+t-1=t$$

$$\begin{aligned}
\underline{\sigma} &= r_{\{2,\dots,2l-1\}}\Big(\sigma_{\mathrm{cyc},2l-1}^{-\underline{t}} \circ \sigma_{\mathrm{cyc},2l-1}^{t} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big)\\
&= r_{\{2,\dots,2l-1\}}\big(\mathrm{id}_{\{1\}} \amalg \sigma\big) = \sigma\\
\underline{c}_a &= c'_{2a} = c_a \qquad \text{for } 1 \leq a \leq l
\end{aligned}$$

Now let $(\sigma', \overrightarrow{c'})$ be an element of I^1. Let $\Psi((\sigma', \overrightarrow{c'})) = (t, \sigma, \overrightarrow{c})$ and let $\Phi((t, \sigma, \overrightarrow{c})) = (\underline{\sigma'}, \underline{\overrightarrow{c'}})$. We begin by the following calculation.

$$\Big(\sigma_{\mathrm{cyc},2l-1}^{-t} \circ \sigma'\Big)(1) = \sigma_{\mathrm{cyc},2l-1}^{-\left(\sigma'(1)-1\right)}(\sigma'(1)) = \sigma'(1) - (\sigma'(1) - 1) = 1$$

This implies the following calculation showing $\underline{\sigma'} = \sigma'$.

$$\begin{aligned}
\underline{\sigma'} &= \sigma_{\mathrm{cyc},2l-1}^{t} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\\
&= \sigma_{\mathrm{cyc},2l-1}^{t} \circ \Big(\mathrm{id}_{\{1\}} \amalg r_{\{2,\dots,2l-1\}}\Big(\sigma_{\mathrm{cyc},2l-1}^{-t} \circ \sigma'\Big)\Big)\\
&= \sigma_{\mathrm{cyc},2l-1}^{t} \circ \sigma_{\mathrm{cyc},2l-1}^{-t} \circ \sigma'\\
&= \sigma'
\end{aligned}$$

It remains to show that $\underline{\overrightarrow{c'}} = \overrightarrow{c'}$. So let $1 \leq a \leq 2l$. Then we have the following calculation.

$$\begin{aligned}
\underline{c'}_a &= \begin{cases} 1 & \text{if } a = 1\\ c_{a/2} & \text{if } a \text{ is even}\\ c_{(a-1)/2} + 1 & \text{if } 1 < a \text{ is odd} \end{cases}\\
&= \begin{cases} 1 & \text{if } a = 1\\ c'_a & \text{if } a \text{ is even}\\ c'_{a-1} + 1 & \text{if } 1 < a \text{ is odd} \end{cases}
\end{aligned}$$

As $e_{\mathrm{even}}(\overrightarrow{c'}) = -\infty$ by definition of I^1 we have $c'_1 = 1$. Furthermore, if $3 \leq a \leq 2l-1$ is odd, then $2 \leq a-1 \leq 2l-2$ is even, so $c'_{a-1+1} = c'_{a-1} + 1$ for the same reason. This finishes the proof of $\Phi \circ \Psi = \mathrm{id}$. □

7.3.6. Progress so far

We can sum up progress so far as in the following proposition. Our goal is to show that the left hand side of the equation is zero.

Proposition 7.3.6.1. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \dots, y_m$ as in Construction 7.3.1.1.*

Then the following holds.

$$
\begin{aligned}
&\partial\Big(\epsilon_X^{(l)}(y_1\cdots y_m)\Big) - \epsilon_X^{(l-1)}(\mathrm{d}(y_1\cdots y_m)) + \mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1\cdots y_m)\Big)\\
=& \sum_{\substack{1\le i\le 2l-1,\\ 1\le q\le l-1,\\ v\in I_{i,2q,2q+1}}} B(v) + \sum_{\substack{1\le i\le 2l-1,\\ 0\le q\le l-1,\\ v\in I_{i,2q+1,2q+2}}} B(v) + \sum_{v\in I_{0,0,1}} B(v)\\
&- \sum_{v\in I^{\mathrm{d}}} B'(v) + \sum_{v\in I^{1}} B''(v)
\end{aligned}
$$
♡

Proof. By combining Proposition 7.3.2.9 (used for the first two lines, for $\partial(\epsilon_X^{(l)}(y_1\cdots y_m)))$, Proposition 7.3.4.3 (third line, for $-\epsilon_X^{(l-1)}(\mathrm{d}(y_1\cdots y_m)))$ Proposition 7.3.5.1 (used for the fourth line, for $\mathrm{d}(\epsilon_X^{(l-1)}(y_1\cdots y_m)))$ we obtain the following.

$$
\begin{aligned}
&\partial\Big(\epsilon_X^{(l)}(y_1\cdots y_m)\Big) - \epsilon_X^{(l-1)}(\mathrm{d}(y_1\cdots y_m)) + \mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1\cdots y_m)\Big)\\
=& \sum_{\substack{1\le i\le 2l-1,\\ 1\le q\le l-1,\\ v\in I_{i,2q+2,2q}}} B(v) + \sum_{\substack{1\le i\le 2l-1,\\ 1\le q\le l-1,\\ v\in I_{i,2q,2q+1}}} B(v) + \sum_{\substack{1\le i\le 2l-1,\\ 0\le q\le l-1,\\ v\in I_{i,2q+1,2q+2}}} B(v)\\
&+ \sum_{v\in I_{0,0,1}} B(v) + \sum_{v\in I_{2l,2,0}} B(v)\\
&- \sum_{\substack{1\le i\le 2l-1,\\ 1\le p\le l-1,\\ v\in I_{i,2p+2,2p}}} B(v) - \sum_{v\in I_{2l,2,0}} B(v) - \sum_{v\in I^{\mathrm{d}}} B'(v)\\
&+ \sum_{v\in I^{1}} B''(v)
\end{aligned}
$$

Now some summands cancel and the result follows. ☐

7.3.7. Reindexing remaining summands from the boundary

We want to show that the left hand side of the equation in Proposition 7.3.6.1 is zero, doing so via the right hand side. Of the terms there, the last two terms are written as sums of summands that are obtained by applying T^{full} to an element of $C^{\mathrm{full}}(2l-1, m)$ and then permuting and perhaps adding a sign. The other terms are however given differently, so in this section we reindex those sums to bring them into a similar form.

Proposition 7.3.7.1. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1.*

Then the following holds.

$$\sum_{\substack{1\leq i\leq 2l-1,\\ 1\leq q\leq l-1,\\ v\in I_{i,2q,2q+1}}} B(v) + \sum_{v\in I_{0,0,1}} B(v) = \sum_{v\in I^{\partial}_{\mathrm{even}}} B''(v)$$

♡

Proof. Define the subset J of I as follows.[25]

$$J := I_{0,0,1} \cup \bigcup_{\substack{1\leq i\leq 2l-1,\\ 1\leq q\leq l-1}} I_{i,2q,2q+1}$$

It then suffices to produce a bijection

$$\Phi\colon J \to I^{\partial}_{\mathrm{even}}$$

such that the following holds for every element v of J.

$$B''(\Phi(v)) = B(v)$$

So let $(i, \sigma, \vec{c})$ be an element of J. Then we make the following definitions.

$$\begin{aligned}
q &:= \sigma^{-1}(i)/2\\
\sigma' &:= \sigma_{i+1\to 2l} \circ \sigma \circ \sigma_{2l\to 2q+1}\\
c'_a &:= \begin{cases} c_{(a+1)/2} & \text{if } 2\nmid a \text{ and } a\leq 2q\\ c_{a/2}+1 & \text{if } 2\mid a \text{ and } a\leq 2q\\ c_{(a+1)/2}+1 & \text{if } 2\nmid a \text{ and } a\geq 2q+1\\ c_{a/2+1} & \text{if } 2\mid a \text{ and } a\geq 2q+1\end{cases} \qquad \text{for } 1\leq a\leq 2l\\
\Phi((i,\sigma,\vec{c})) &:= \left(\sigma', \vec{c'}\right)
\end{aligned}$$

There are various things that we need to check to verify that this is well-defined. First, $\sigma^{-1}(i+1) = \sigma^{-1}(i) + 1$ holds by assumption on elements of J, so as $0 \leq i \leq 2l-1$ implies $1 \leq \sigma^{-1}(i+1) \leq 2l$ we can conclude that we must have $0 \leq \sigma^{-1}(i) \leq 2l-1$. Furthermore, the definition of J implies that $\sigma^{-1}(i)$ is even, so q is a well-defined integer satisfying $0 \leq q \leq l-1$. This makes σ' into a well-defined element of Σ_{2l}. Furthermore, we have

$$\begin{aligned}
\sigma'(2l) &= \sigma_{i+1\to 2l}(\sigma(2q+1)) = \sigma_{i+1\to 2l}\left(\sigma\left(\sigma^{-1}(i)+1\right)\right)\\
&= \sigma_{i+1\to 2l}\left(\sigma\left(\sigma^{-1}(i)\right)+1\right) = \sigma_{i+1\to 2l}(i+1)\\
&= 2l
\end{aligned}$$

[25] The definition of $I_{0,0,1}$ is really the same one as for $I_{i,2q,2q+1}$ if we set $i = 0$ and $q = 0$, so we mostly do not need to treat this as a separate case. The only difference is that $I_{i,0,1}$ is empty unless $i = 0$, as $\sigma(0) = 0$ for every element σ of Σ_{2l}.

so that we can even consider σ' as an element of Σ_{2l-1}.

We next show that $\overrightarrow{c'}$ is a well-defined element of $C^{\mathrm{full}}(2l-1, m)$. Using that $\overrightarrow{c}$ is an element of $C(l, m)$ one easily sees that in all four cases c'_a is a well-defined integer satisfying $1 \leq c'_a \leq m + 1$[26]. We also have $c'_{2l} = c_{l+1} = m + 1$. It remains to show that $c'_a < c'_{a+1}$ for $1 \leq a \leq 2l - 1$. If $a \leq 2q - 1$ is odd or $a \geq 2q+2$ even then is immediate. If $a \leq 2q-2$ is even, then $c'_a = c_{a/2}+1$ and $c'_{a+1} = c_{a/2+1}$, so $c'_a < c'_{a+1}$ follows from $c_{a/2}+1 \leq c_{a/2+1}-1$. If $a \geq 2q+1$ is odd, then $c'_a = c_{(a+1)/2} + 1$ and $c'_{a+1} = c_{(a+1)/2+1}$, so that $c'_a < c'_{a+1}$ follows analogously. It remains to consider $a = 2q$. In this case $c'_{2q} = c_q + 1$ and $c'_{2q+1} = c_{q+1} + 1$, so $c'_{2q} < c'_{2q+1}$ as $c_q < c_{q+1}$. This completes the proof that $\overrightarrow{c'}$ is a well-defined element of $C^{\mathrm{full}}(2l-1, m)$.

We now verify the conditions required for $(\sigma', \overrightarrow{c'})$ to be an element of $I^{\partial}_{\mathrm{even}}$. Concretely we make the following claims.

$$\begin{aligned} e_{\mathrm{even}}\left(\overrightarrow{c'}\right) &= 2q \\ e_{\mathrm{even}}(\sigma') &\leq 2q \\ e_{\mathrm{odd}}\left(\overrightarrow{c'}\right) &\geq 2q+3 \\ e_{\mathrm{odd}}(\sigma') &\geq 2q+1 \end{aligned}$$

To show that $e_{\mathrm{even}}(\overrightarrow{c'}) = 2q$, we begin by first noting that $c'_{2q} = c_q + 1$ and $c'_{2q+1} = c_{q+1}+1$. As $\overrightarrow{c}$ is an element of $C(l, m)$, we have $c_q+1 < c_{q+1}$, which implies that $c'_{2q} + 1 < c'_{2q+1}$, so $e_{\mathrm{even}}(\overrightarrow{c'}) \geq 2q$. Now let $2q + 2 \leq a \leq 2l - 2$ be even. Then $c'_a + 1 = c_{a/2+1} + 1 = c'_{a+1}$, which shows that $e_{\mathrm{even}}(\overrightarrow{c'}) = 2q$. Next let $1 \leq a \leq 2q - 1$ be odd. Then $c'_a + 1 = c_{(a+1)/2} + 1 = c'_{a+1}$, so $e_{\mathrm{odd}}(\overrightarrow{c'}) \geq 2q + 1$. Furthermore, we have $c'_{2q+1} = c_{q+1} + 1$ and $c'_{2q+2} = c_{q+2}$. By definition of J it holds that $c_{q+1} + 1 = c_{q+2} - 1$, which then implies $c'_{2q+1} + 1 = c'_{2q+2}$. Thus we even get $e_{\mathrm{odd}}(\overrightarrow{c'}) \geq 2q + 3$. We next show that $e_{\mathrm{even}}(\sigma') \leq 2q$. So let $2q + 2 \leq a \leq 2l - 2$ be even. Then we have to show that σ' cyclically preserves the ordering of $\{a - 1, a, a + 1\}$, which amounts to σ cyclically preserving the ordering of $\{a, a + 1, a + 2\}$, which is the case as $a+1$ is odd and satisfies $1 \leq a+1 \leq 2l-1$. To show that $e_{\mathrm{odd}}(\sigma') \geq 2q+1$ we let $1 \leq a \leq 2q - 1$ be odd, and have to show that σ' cyclically preserves the ordering of $\{a - 1, a, a + 1\}$, which it does as σ does. This completes the proof that Φ is well-defined.

Keeping the notation used so far, we now show the following.

$$B''(\Phi((i, \sigma, \overrightarrow{c}))) = B((i, \sigma, \overrightarrow{c}))$$

[26] To exclude that we get $m + 2$ in the two cases in which 1 is added to a component of $\overrightarrow{c}$, note that in those cases the index is at most l, and $c_l < c_{l+1} = m + 1$.

We first consider the signs.

$$\begin{aligned}
&\operatorname{sgn}(\sigma') \\
&= \operatorname{sgn}(\sigma_{i+1\to 2l} \circ \sigma \circ \sigma_{2l\to 2q+1}) \\
&= (-1)^{2l-(i+1)} \cdot \operatorname{sgn}(\sigma) \cdot (-1)^{2q+1-2l} \\
&= (-1)^{i} \cdot \operatorname{sgn}(\sigma)
\end{aligned}$$

It thus remains to show the following.

$$\partial_i(\sigma \cdot T((y_1, \ldots, y_m), \overrightarrow{c})) = \sigma' \cdot T^{\text{full}}\Big((y_1, \ldots, y_m), \overrightarrow{c'}\Big)$$

For this let us write $T = T((y_1, \ldots, y_m), \overrightarrow{c})$ and T_a for the a-th tensor factor of T. Then we can carry out the following calculation for the a-th tensor factor of $T^{\text{full}}((y_1, \ldots, y_m), \overrightarrow{c'})$, with $0 \le a \le 2l - 1$.

$$\begin{aligned}
&T^{\text{full}}\Big((y_1, \ldots, y_m), \overrightarrow{c'}\Big)_a \\
&= \begin{cases} \prod_{j=1}^{c'_1-1} y_j & \text{if } a = 0 \\ \overline{\prod_{j=c'_a}^{c'_{a+1}-1} y_j} & \text{if } a > 0 \end{cases} \\
&= \begin{cases} \prod_{j=1}^{c_1} y_j & \text{if } a = 0 = q \\ \prod_{j=1}^{c_1-1} y_j & \text{if } a = 0 < q \\ \overline{\prod_{j=c_{(a+1)/2}}^{c_{(a+1)/2}} y_j} & \text{if } 0 < a \le 2q-1 \text{ is odd} \\ \overline{\prod_{j=c_{a/2}+1}^{c_{a/2+1}-1} y_j} & \text{if } 0 < a \le 2q-1 \text{ is even} \\ \overline{\prod_{j=c_q+1}^{c_{q+1}} y_j} & \text{if } 0 < a = 2q \\ \overline{\prod_{j=c_{(a+1)/2}+1}^{c_{(a+1)/2+1}-1} y_j} & \text{if } a \ge 2q+1 \text{ is odd} \\ \overline{\prod_{j=c_{a/2+1}}^{c_{a/2+1}} y_j} & \text{if } a \ge 2q+1 \text{ is even} \end{cases} \\
&= \begin{cases} \prod_{j=1}^{c_1} y_j & \text{if } a = 0 = q \\ \prod_{j=1}^{c_1-1} y_j & \text{if } a = 0 < q \\ \overline{y_{c_{(a+1)/2}}} & \text{if } 0 < a \le 2q-1 \text{ is odd} \\ \overline{\prod_{j=c_{a/2}+1}^{c_{a/2+1}-1} y_j} & \text{if } 0 < a \le 2q-1 \text{ is even} \\ \overline{\prod_{j=c_q+1}^{c_{q+1}} y_j} & \text{if } 0 < a = 2q \\ \overline{\prod_{j=c_{(a+1)/2}+1}^{c_{(a+1)/2+1}-1} y_j} & \text{if } a \ge 2q+1 \text{ is odd} \\ \overline{y_{c_{a/2+1}}} & \text{if } a \ge 2q+1 \text{ is even} \end{cases}
\end{aligned}$$

$$= \begin{cases} T_0 \cdot T_1 & \text{if } a = 0 = q \\ T_a & \text{if } a = 0 < q \\ T_a & \text{if } 0 < a \leq 2q-1 \text{ is odd} \\ T_a & \text{if } 0 < a \leq 2q-1 \text{ is even} \\ T_{2q} \cdot T_{2q+1} & \text{if } 0 < a = 2q \\ T_{a+1} & \text{if } a \geq 2q+1 \text{ is odd} \\ T_{a+1} & \text{if } a \geq 2q+1 \text{ is even} \end{cases}$$

$$= \begin{cases} T_a & \text{if } a \leq 2q-1 \\ T_{2q} \cdot T_{2q+1} & \text{if } a = 2q \\ T_{a+1} & \text{if } a \geq 2q+1 \end{cases}$$

Note that the inverse of σ' is given by

$$\sigma'^{-1} = \sigma_{2q+1 \to 2l} \circ \sigma^{-1} \circ \sigma_{2l \to i+1}$$

so that we have the following values for $0 \leq a \leq 2l-1$ (note that the cases below are exhaustive, as $2q+1$ can not occur due to $a \neq 2l$).

$$\sigma'^{-1}(a) = \begin{cases} \sigma^{-1}(a) & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \leq 2q \\ \sigma^{-1}(a) - 1 & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \geq 2q+2 \\ \sigma^{-1}(a+1) & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \leq 2q \\ \sigma^{-1}(a+1) - 1 & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \geq 2q+2 \end{cases}$$

The upshot is that the a-th tensor factor of $\sigma' \cdot T^{\mathrm{full}}((y_1, \ldots, y_m), \vec{c'})$ is given by

$$\begin{cases} T^{\mathrm{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big)_{\sigma^{-1}(a)} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \leq 2q \\ T^{\mathrm{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big)_{\sigma^{-1}(a)-1} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \geq 2q+2 \\ T^{\mathrm{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big)_{\sigma^{-1}(a+1)} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \leq 2q \\ T^{\mathrm{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big)_{\sigma^{-1}(a+1)-1} & \text{if } a \geq i+1 \\ & \text{and } \sigma^{-1}(a+1) \geq 2q+2 \end{cases}$$

$$= \begin{cases} T_{\sigma^{-1}(a)} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \leq 2q-1 \\ T_{2q} \cdot T_{2q+1} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) = 2q \\ T_{\sigma^{-1}(a)-1+1} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \geq 2q+2 \\ T_{\sigma^{-1}(a+1)} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \leq 2q-1 \\ T_{2q} \cdot T_{2q+1} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) = 2q \\ T_{\sigma^{-1}(a+1)-1+1} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \geq 2q+2 \end{cases}$$

Note that $\sigma(2q) = i$ and $\sigma(2q+1) = i+1$.

$$= \begin{cases} T_{\sigma^{-1}(a)} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \leq 2q-1 \\ T_{\sigma^{-1}(i)} \cdot T_{\sigma^{-1}(i+1)} & \text{if } a = i \\ T_{\sigma^{-1}(a)} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \geq 2q+2 \\ T_{\sigma^{-1}(a+1)} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \leq 2q-1 \\ T_{\sigma^{-1}(a+1)} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \geq 2q+2 \end{cases}$$

$$= \begin{cases} T_{\sigma^{-1}(a)} & \text{if } a \leq i \\ T_{\sigma^{-1}(i)} \cdot T_{\sigma^{-1}(i+1)} & \text{if } a = i \\ T_{\sigma^{-1}(a+1)} & \text{if } a \geq i+1 \end{cases}$$

$$= \left(\partial_i(\sigma \cdot T((y_1, \ldots, y_m), \vec{c}))\right)_a$$

This finishes the proof that $B''(\Phi((i, \sigma, \vec{c}))) = B((i, \sigma, \vec{c}))$.

We still have to show that Φ is a bijection. For this we construct an inverse Ψ. So let $(\sigma', \vec{c'})$ be an element of $I^{\partial}_{\text{even}}$. Then we make the following definitions.

$$\begin{aligned} q &\coloneqq e_{\text{even}}\left(\vec{c'}\right)/2 \\ i &\coloneqq \sigma'(2q) \\ \sigma &\coloneqq \sigma_{2l \to i+1} \circ \sigma' \circ \sigma_{2q+1 \to 2l} \\ c_a &\coloneqq \begin{cases} c'_{2a-1} & \text{if } a \leq q \\ c'_{2a-1} - 1 & \text{if } q+1 \leq a \leq l \\ m+1 & \text{if } a = l+1 \end{cases} \qquad \text{for } 1 \leq a \leq l+1 \\ \Psi\left((\sigma', \vec{c'})\right) &\coloneqq (i, \sigma, \vec{c}) \end{aligned}$$

As usual various checks are needed to show that this is indeed well-defined. To begin with $e_{\text{even}}\left(\vec{c'}\right) \neq -\infty$ by definition of $I^{\partial}_{\text{even}}$, so $0 \leq e_{\text{even}}\left(\vec{c'}\right) \leq 2l-2$, implying that q is a well-defined integer satisfying $0 \leq q \leq l-1$. This makes i a well defined integer satisfying $0 \leq i \leq 2l-1$. We note here that $i = 0$ if and only if $q = 0$.

We next show that σ is an element of E_l. So let $1 \leq a \leq 2l-1$ be odd. We have to show that σ cyclically preserves the ordering of $\{a-1, a, a+1\}$. If $a \leq 2q-1$, then this amounts to showing that σ' cyclically preserves $\{a-1, a, a+1\}$, which it does as $e_{\text{odd}}(\sigma') \geq e_{\text{even}}(\vec{c'}) + 1 = 2q+1$. If instead $a \geq 2q+3$, then this amounts to showing that σ' cyclically preserves $\{a-2, a-1, a\}$, which it does as $a-1$ is even, satisfies $a-1 \geq 2q+2$, and $e_{\text{even}}(\sigma') \leq e_{\text{even}}(\vec{c'}) = 2q$. The case $a = 2q+1$ remains. For this we just evaluate σ at $2q$ and $2q+1$ as follows

$$\sigma(2q) = i \qquad \sigma(2q+1) = i+1$$

which already shows the claim, no matter what $\sigma(2q+2)$ may be. It also handles the condition on σ required for $(i, \sigma, \vec{c})$ to be an element of $I_{i,2q,2q+1}$.

Now we show that $\vec{c}$ is an element of $C(l,m)$. We have $c_{l+1} = m+1$ by definition, and for $1 \leq a \leq l$ we have $1 \leq 2a-1 \leq 2l-1$ so that c'_{2a-1} is a well-defined integer. If furthermore $a \leq q$, then, as $q \leq l-1$, we have the following chain of inequalities.

$$1 \leq c'_{2a-1} \leq c'_{2l-3} \leq c'_{2l} - 3 = m - 2$$

If instead $q+1 \leq a$ as well as $2 \leq a$, then we have the following chain of inequalities.

$$1 \leq c'_2 \leq c'_3 - 1 \leq c'_{2a-1} - 1 \leq c'_{2l-1} - 1 \leq c'_{2l} - 2 = m - 1$$

Finally, if $a = 1$ and $q = 0$, then $e_{\text{even}}(\vec{c'}) = 0$, which implies that $1 \leq c'_1 - 1$, while $c'_{2a-1} - 1 \leq m - 1$ as in the previous case. We have thus shown so far that $c_{l+1} = m+1$ while $1 \leq c_a \leq m-1$ for $1 \leq a \leq l$. So let $1 \leq a \leq l-1$. We still have to show that $c_a + 1 \leq c_{a+1} - 1$. If $a \leq q-1$ or $a \geq q+1$ this follows from $c'_{2a-1} < c'_{2a} < c'_{2a+1}$. The case $a = q$ remains, where we have $c_q = c'_{2q-1}$ and $c_{q+1} = c'_{2q+1} - 1$. But as $e_{\text{even}}(\vec{c'}) = 2q$, we obtain the last inequality in the following chain $c'_{2q-1} < c'_{2q} < c'_{2q} + 1 < c'_{2q+1}$, which shows the claim. Using that $c'_{2q+1} = c'_{2q+2} - 1 = c'_{2q+3} - 2$ due to $e_{\text{odd}}(\vec{c'}) \geq 2q+3$ and $e_{\text{even}}(\vec{c'}) = 2q$ we obtain the short calculation

$$c_{q+1} + 1 = c'_{2q+1} - 1 + 1 = c'_{2q+1} = c'_{2q+2} - 1 = c'_{2q+3} - 2 = c_{q+2} - 1$$

which finishes the proof that Ψ is well-defined as a map to J.

It remains to show that Ψ is an inverse map to Φ. So let $(i, \sigma, \vec{c})$ be an element of J, and set $\Phi((i, \sigma, \vec{c})) = (\sigma', \vec{c'})$ and $q = \sigma^{-1}(i)/2$ as in the definition of Φ. Set furthermore $\Psi((\sigma', \vec{c'})) = (\underline{i}, \underline{\sigma}, \underline{\vec{c}})$ and $\underline{q} = e_{\text{even}}(\vec{c'})/2$ as in the definition of Ψ. In the definition of Φ it was shown that $e_{\text{even}}(\vec{c'}) = 2q$, so that $\underline{q} = q$, and unpacking the definition we then have $\underline{i} = \sigma'(2q) = i$. It then follows immediately that also $\underline{\sigma} = \sigma$, and the following calculation shows that $\underline{\vec{c}} = \vec{c}$, where $1 \leq a \leq l$.

$$\begin{aligned}
\underline{c}_a &= \begin{cases} c'_{2a-1} & \text{if } a \leq q \\ c'_{2a-1} - 1 & \text{if } q+1 \leq a \leq l \end{cases} \\
&= \begin{cases} c_a & \text{if } a \leq q \\ c_a + 1 - 1 & \text{if } q+1 \leq a \leq l \end{cases} \\
&= c_a
\end{aligned}$$

This shows that $\Psi \circ \Phi = \mathrm{id}$.

Now let $(\sigma', \vec{c'})$ be an element of $I^{\partial}_{\text{even}}$. Set $\Psi((\sigma', \vec{c'})) = (i, \sigma, \vec{c})$ and let $q = e_{\text{even}}(\vec{c'})/2$ be as in the definition of Ψ. Let $\Phi((i, \sigma, \vec{c})) = (\underline{\sigma'}, \underline{\vec{c'}})$ and

$\underline{q} = \sigma^{-1}(i)/2$ as in the definition of Φ. Then we have

$$\sigma(2\underline{q}) = (\sigma_{2l\to i+1} \circ \sigma' \circ \sigma_{2q+1\to 2l})(2q) = \sigma_{2l\to i+1}(\sigma'(2q)) = \sigma_{2l\to i+1}(i) = i$$

so that $\underline{q} = q$. It then follows that $\underline{\sigma'} = \sigma'$. It remains to show that $\underline{\vec{c'}} = \vec{c'}$. So let $1 \leq a \leq 2l$. Then this is shown by the following calculation.

$$\begin{aligned}
\underline{c'}_a &= \begin{cases} c_{(a+1)/2} & \text{if } 2 \nmid a \text{ and } a \leq 2q \\ c_{a/2} + 1 & \text{if } 2 \mid a \text{ and } a \leq 2q \\ c_{(a+1)/2} + 1 & \text{if } 2 \nmid a \text{ and } a \geq 2q+1 \\ c_{a/2+1} & \text{if } 2 \mid a \text{ and } a \geq 2q+1 \end{cases} \\
&= \begin{cases} c'_a & \text{if } 2 \nmid a \text{ and } a \leq 2q \\ c'_{a-1} + 1 & \text{if } 2 \mid a \text{ and } a \leq 2q \\ c'_a - 1 + 1 & \text{if } 2 \nmid a \text{ and } a \geq 2q+1 \\ c'_{a+1} - 1 & \text{if } 2 \mid a \text{ and } 2l > a \geq 2q+1 \\ m+1 & \text{if } a = 2l \end{cases}
\end{aligned}$$

Using that $e_{\mathrm{odd}}(\vec{c'}) \geq 2q+3$ we obtain $c'_{a-1} + 1 = c'_a$ in the second case, and using $e_{\mathrm{even}}(\vec{c'}) = 2q$ we obtain $c'_a = c'_{a+1} - 1$ in the fourth case.

$$\begin{aligned}
&= \begin{cases} c'_a & \text{if } 2 \nmid a \text{ and } a \leq 2q \\ c'_a & \text{if } 2 \mid a \text{ and } a \leq 2q \\ c'_a & \text{if } 2 \nmid a \text{ and } a \geq 2q+1 \\ c'_a & \text{if } 2 \mid a \text{ and } 2l > a \geq 2q+1 \\ m+1 & \text{if } a = 2l \end{cases} \\
&= c'_a
\end{aligned}$$

□

Proposition 7.3.7.2. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{\substack{1\leq i\leq 2l-1,\\ 0\leq q\leq l-1,\\ v\in I_{i,2q+1,2q+2}}} B(v) = -\sum_{v\in I^{\partial}_{\mathrm{odd}}} B''(v)$$

♡

Proof. The proof is completely analogous to the proof of Proposition 7.3.7.1, so we omit the details. The formulas used to define Φ in this case are

$$\begin{aligned}
q &:= \big(\sigma^{-1}(i) - 1\big)/2 \\
\sigma' &:= \sigma_{i\to 2l} \circ \sigma \circ \sigma_{2l\to 2q+1}
\end{aligned}$$

$$c'_a := \begin{cases} c_{(a+1)/2} & \text{if } 2 \nmid a \text{ and } a \leq 2q+1 \\ c_{a/2}+1 & \text{if } 2 \mid a \text{ and } a \leq 2q+1 \\ c_{(a+1)/2}+1 & \text{if } 2 \nmid a \text{ and } a \geq 2q+2 \\ c_{a/2+1} & \text{if } 2 \mid a \text{ and } a \geq 2q+2 \end{cases} \qquad \text{for } 1 \leq a \leq 2l$$

$$\Phi((i, \sigma, \vec{c})) := \left(\sigma', \vec{c'}\right)$$

and in this case $e_{\mathrm{odd}}(\vec{c'}) = 2q+1$.

The special assumption on $\vec{c}$ from the definition of J has in this case, in contrast to the proof of Proposition 7.3.7.1, a different form depending on whether $q = 0$ or not, as there is no c_0. Where this property was used in the proof of Proposition 7.3.7.1 was to show that $e_{\mathrm{odd}}(\vec{c'}) \neq 2q+1$. In our case here this property is needed to show that $e_{\mathrm{even}}(\vec{c'}) \neq 2q$, and the distinction between the cases $q = 0$ and $q \neq 0$ corresponds to the analogous distinction in the definition of e_{even}.

That the definition of σ' involves i instead of $i+1$ introduces an extra minus sign in $\operatorname{sgn}(\sigma')$, which explains the minus sign in the result.

The formulas used to define Ψ are as follows.

$$\begin{aligned} q &:= \left(e_{\mathrm{even}}\left(\vec{c'}\right) - 1\right)/2 \\ i &:= \sigma'(2q+1) \\ \sigma &:= \sigma_{2l \to i} \circ \sigma' \circ \sigma_{2q+1 \to 2l} \\ c_a &:= \begin{cases} c'_{2a-1} & \text{if } a \leq q+1 \\ c'_{2a-1} - 1 & \text{if } q+2 \leq a \leq l \\ m+1 & \text{if } a = l+1 \end{cases} \qquad \text{for } 1 \leq a \leq l+1 \\ \Psi\left((\sigma', \vec{c'})\right) &:= (i, \sigma, \vec{c}) \end{aligned}$$

Again the proof that this is well-defined is analogous to the proof of Proposition 7.3.7.1 except the special treatment of $q = 0$ as discussed above. □

We sum up our current progress.

Proposition 7.3.7.3. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\begin{aligned} &\partial\left(\epsilon_X^{(l)}(y_1 \cdots y_m)\right) - \epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m)) + \mathrm{d}\left(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\right) \\ = &\sum_{v \in I^{\partial}_{\mathrm{even}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd}}} B''(v) + \sum_{v \in I^1} B''(v) - \sum_{v \in I^{\mathrm{d}}} B'(v) \end{aligned} \qquad \heartsuit$$

Proof. Combine Propositions 7.3.6.1, 7.3.7.1 and 7.3.7.2. □

7.3.8. Subdivisions of the remaining indexing sets

To continue we need to subdivide I^{d}, $I^{\partial}_{\mathrm{even}}$, $I^{\partial}_{\mathrm{odd}}$, and I^1 into a disjoint unions of subsets, which we do in this section.

Definition 7.3.8.1. In this definition we will make use of notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let $l \geq 1$ and $m \geq 0$ be integers. We define the following subsets of I^{d}.

$$\begin{aligned}
I^{\mathrm{d,cancel}}_{>} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}} \;\Big|\; e_{\mathrm{even}}(\sigma) > e_{\mathrm{odd}}(\sigma)\Big\}\\
I^{\mathrm{d}}_{<} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}} \;\Big|\; e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)\Big\}\\
I^{\mathrm{d}}_{<,\mathrm{top}} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<} \;\Big|\; \text{if } e_{\mathrm{odd}}(\sigma) \neq \infty \text{ then } p = e_{\mathrm{odd}}(\sigma),\\
&\qquad\qquad \text{else } p = 2l-1\Big\}\\
I^{\mathrm{d,cancel}}_{<,\mathrm{top}} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<,\mathrm{top}} \;\Big|\; e_{\mathrm{even}}(\sigma) \neq -\infty\\
&\qquad\qquad \text{and } e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma)\Big\}\\
I^{\mathrm{d}}_{<,\mathrm{top},\partial} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<,\mathrm{top}} \;\Big|\; e_{\mathrm{even}}(\vec{c}) \neq -\infty\\
&\qquad\qquad \text{and } e_{\mathrm{even}}(\vec{c}) \geq e_{\mathrm{even}}(\sigma)\Big\}\\
I^{\mathrm{d}}_{<,\mathrm{top},1} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<,\mathrm{top}} \;\Big|\; e_{\mathrm{even}}(\vec{c}) = -\infty \text{ and } e_{\mathrm{even}}(\sigma) = -\infty\Big\}\\
I^{\mathrm{d}}_{<,\mathrm{bottom}} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<} \;\Big|\; p = e_{\mathrm{even}}(\sigma)\Big\}\\
I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<,\mathrm{bottom}} \;\Big|\; \text{if } e_{\mathrm{odd}}(\sigma) = \infty \text{ then } e_{\mathrm{odd}}(\vec{c}) = \infty,\\
&\qquad\qquad \text{else } e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})\Big\}\\
I^{\mathrm{d}}_{<,\mathrm{bottom},\partial} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<,\mathrm{bottom}} \;\Big|\; e_{\mathrm{odd}}(\vec{c}) \neq \infty \text{ and } e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{odd}}(\vec{c})\Big\}
\end{aligned}$$

The following subset is to be defined for $1 \leq p \leq 2l-2$.

$$I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p} \coloneqq \Big\{(\sigma, \vec{c}, p') \in I^{\mathrm{d}}_{<} \;\Big|\; p' = p \text{ and } e_{\mathrm{even}}(\sigma) < p < e_{\mathrm{odd}}(\sigma)\Big\}$$

We also define the following subsets of $\Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m)$.

$$\begin{aligned}
I^{\partial}_{\mathrm{even,d}} &\coloneqq \Big\{(\sigma, \vec{c}) \in I^{\partial}_{\mathrm{even}} \;\Big|\; \text{if } e_{\mathrm{odd}}(\sigma) = \infty \text{ then } e_{\mathrm{odd}}(\vec{c}) = \infty,\\
&\qquad\qquad \text{else } e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})\Big\}\\
I^{\partial}_{\mathrm{odd,d}} &\coloneqq \Big\{(\sigma, \vec{c}) \in I^{\partial}_{\mathrm{odd}} \;\Big|\; e_{\mathrm{even}}(\sigma) \neq -\infty \text{ and } e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma)\Big\}
\end{aligned}$$

$$I^{\partial}_{\mathrm{odd-even}} \coloneqq \Big\{(\sigma, \vec{c}) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \;\Big|\; e_{\mathrm{even}}(\vec{c}) \neq -\infty \text{ and } e_{\mathrm{odd}}(\vec{c}) \neq \infty \text{ and } e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{even}}(\vec{c}) \leq e_{\mathrm{odd}}(\vec{c}) - 3 \leq e_{\mathrm{odd}}(\sigma) - 3\Big\}$$

$$I^{\partial}_{\mathrm{odd},1} \coloneqq \Big\{(\sigma, \vec{c}) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \;\Big|\; e_{\mathrm{even}}(\vec{c}) = -\infty \text{ and } e_{\mathrm{odd}}(\vec{c}) \neq \infty \text{ and } e_{\mathrm{even}}(\sigma) = -\infty \text{ and } e_{\mathrm{odd}}(\vec{c}) \leq e_{\mathrm{odd}}(\sigma)\Big\}$$

$$I^{1}_{\mathrm{d}} \coloneqq \Big\{(\sigma, \vec{c}) \in I^{1} \;\Big|\; \text{if } e_{\mathrm{odd}}(\sigma) = \infty \text{ then } e_{\mathrm{odd}}(\vec{c}) = \infty, \text{ else } e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})\Big\}$$

♢

Proposition 7.3.8.2. *In this definition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let $l \geq 1$ and $m \geq 0$ be integers. Then the set I^{d} is the disjoint union of the following subsets.*

- $I^{\mathrm{d,cancel}}_{>}$
- $I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p}$ *for* $1 \leq p \leq 2l-2$
- $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$
- $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$
- $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$
- $I^{\mathrm{d}}_{<,\mathrm{top},1}$
- $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$ ♡

Proof. As $e_{\mathrm{even}}(\sigma) = e_{\mathrm{odd}}(\sigma)$ is never possible for parity reasons, we must always either have $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$ or $e_{\mathrm{even}}(\sigma) > e_{\mathrm{odd}}(\sigma)$, showing that I^{d} is the disjoint union of $I^{\mathrm{d,cancel}}_{>}$ and $I^{\mathrm{d}}_{<}$.

Now assume that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d}}_{<}$. We will show that then

$$e_{\mathrm{even}}(\sigma) \leq p \leq e_{\mathrm{odd}}(\sigma)$$

which implies that $I^{\mathrm{d}}_{<}$ is the disjoint union of the subsets $I^{\mathrm{d,cancel}}_{<,\mathrm{mid},q}$, where q ranges over for $1 \leq q \leq 2l-2$, and the subsets $I^{\mathrm{d}}_{<,\mathrm{top}}$ and $I^{\mathrm{d}}_{<,\mathrm{bottom}}$. By definition of I^{d} we must have

$$e_{\mathrm{even}}(\sigma) - 1 \leq p \leq e_{\mathrm{odd}}(\sigma) + 1$$

so that we only must rule out that $p = e_{\mathrm{even}}(\sigma) - 1$ and $p = e_{\mathrm{odd}}(\sigma) + 1$. For this, note that by definition of $e_{\mathrm{even}}(\sigma)$ the permutation σ does *not* cyclically preserve the ordering of $\{e_{\text{even}}(\sigma) - 1, e_{\text{even}}(\sigma), e_{\text{even}}(\sigma) + 1\}$, which means that

$$e_{\text{even}}(\sigma) - 1, \qquad e_{\text{even}}(\sigma) + 1, \qquad e_{\text{even}}(\sigma)$$

will be cyclically ordered. As $e_{\text{even}}(\sigma) < e_{\text{odd}}(\sigma)$ by definition of $I^{\text{d}}_{<}$, we also know that

$$e_{\text{even}}(\sigma) - 2, \qquad e_{\text{even}}(\sigma) - 1, \qquad e_{\text{even}}(\sigma)$$

is cyclically ordered. Combining both we obtain that

$$e_{\text{even}}(\sigma) - 2, \qquad e_{\text{even}}(\sigma) - 1, \qquad e_{\text{even}}(\sigma) + 1, \qquad e_{\text{even}}(\sigma)$$

is cyclically ordered. But this means that

$$e_{\text{even}}(\sigma) - 2, \qquad e_{\text{even}}(\sigma), \qquad e_{\text{even}}(\sigma) + 1$$

is *not* cyclically ordered, which rules out $p = e_{\text{even}}(\sigma) - 1$. Analogously one can rule out $p = e_{\text{odd}}(\sigma) + 1$.

We have now shown that I^{d} is the disjoint union of the following subsets.

- $I^{\text{d,cancel}}_{>}$
- $I^{\text{d,cancel}}_{<,\text{mid},p}$ for $1 \leq p \leq 2l - 2$
- $I^{\text{d}}_{<,\text{top}}$
- $I^{\text{d}}_{<,\text{bottom}}$

It thus remains to show the following two claims. Firstly that $I^{\text{d}}_{<,\text{top}}$ is a disjoint union of the following subsets.

- $I^{\text{d,cancel}}_{<,\text{top}}$
- $I^{\text{d}}_{<,\text{top},\partial}$
- $I^{\text{d}}_{<,\text{top},1}$

And secondly that $I^{\text{d}}_{<,\text{bottom}}$ is a disjoint union of the following subsets.

- $I^{\text{d,cancel}}_{<,\text{bottom}}$
- $I^{\text{d}}_{<,\text{bottom},\partial}$

For the first claim we begin by noting that clearly the three subsets are pairwise disjoint. So now let $(\sigma, \vec{c}, p)$ be an element of $I^{\text{d}}_{<,\text{top}}$. First assume that $e_{\text{even}}(\sigma) \neq -\infty$. If $e_{\text{even}}(\vec{c}) < e_{\text{even}}(\sigma)$ then $(\sigma, \vec{c}, p)$ is an element of $I^{\text{d,cancel}}_{<,\text{top}}$. If instead $e_{\text{even}}(\vec{c}) \geq e_{\text{even}}(\sigma)$ then it follows from $e_{\text{even}}(\sigma) \neq -\infty$ that also $e_{\text{even}}(\vec{c}) \neq -\infty$ and $(\sigma, \vec{c}, p)$ is an element of $I^{\text{d}}_{<,\text{top},\partial}$. Next assume that $e_{\text{even}}(\sigma) = -\infty$. If also $e_{\text{even}}(\vec{c}) = -\infty$, then $(\sigma, \vec{c}, p)$ is an element of $I^{\text{d}}_{<,\text{top},1}$, and otherwise it will be an element of $I^{\text{d}}_{<,\text{top},\partial}$.

For the second claim we can again note immediately that the two subsets are disjoint. So now let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{bottom}}$ and assume it is not an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$. If $e_{\mathrm{odd}}(\sigma) = \infty$, then this means $e_{\mathrm{odd}}(\vec{c}) \neq \infty$, and this implies that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. If instead $e_{\mathrm{odd}}(\sigma) \neq \infty$, then this implies $e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{odd}}(\vec{c})$, so in particular $e_{\mathrm{odd}}(\vec{c}) \neq \infty$, and thus $(\sigma, \vec{c}, p)$ is again an element of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. □

Proposition 7.3.8.3. *In this definition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let $l \geq 1$ and $m \geq 0$ be integers. Then the set $I^{\partial}_{\mathrm{odd}}$ is the disjoint union of the following subsets.*

- $I^{\partial}_{\mathrm{odd,d}}$
- $I^{\partial}_{\mathrm{odd-even}}$
- $I^{\partial}_{\mathrm{odd},1}$

Furthermore the set $I^{\partial}_{\mathrm{even}}$ is the disjoint union of the following subsets.

- $I^{\partial}_{\mathrm{even,d}}$
- $I^{\partial}_{\mathrm{odd-even}}$ ♡

Proof. While $I^{\partial}_{\mathrm{odd,d}}$ was defined as a subset of $I^{\partial}_{\mathrm{odd}}$ and $I^{\partial}_{\mathrm{even,d}}$ as a subset of $I^{\partial}_{\mathrm{even}}$, the other two relevant sets have only be defined as a subset of $\Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m)$. However it follows easily from the definition that $\vec{c}$ and σ have the necessary properties for the required subset inclusions.

We first discuss $I^{\partial}_{\mathrm{odd}}$. So let $(\sigma, \vec{c})$ be an element of $I^{\partial}_{\mathrm{odd}}$. If $e_{\mathrm{even}}(\vec{c}) = -\infty$ as well as $e_{\mathrm{even}}(\sigma) = -\infty$, then $(\sigma, \vec{c})$ could (out of the three subsets in question) only possibly be an element of $I^{\partial}_{\mathrm{odd},1}$, and indeed it is, as the other two required properties are part of the definition of $I^{\partial}_{\mathrm{odd}}$. If instead $e_{\mathrm{even}}(\vec{c}) = -\infty$ and $e_{\mathrm{even}}(\sigma) > -\infty$, then $(\sigma, \vec{c})$ is an element of (only) $I^{\partial}_{\mathrm{odd,d}}$. If we have $e_{\mathrm{even}}(\vec{c}) \neq -\infty$, and $e_{\mathrm{even}}(\sigma) > e_{\mathrm{even}}(\vec{c})$, then $(\sigma, \vec{c})$ is also only element of $I^{\partial}_{\mathrm{odd,d}}$. The last case is when $e_{\mathrm{even}}(\vec{c}) \neq -\infty$, and $e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{even}}(\vec{c})$, in which case $(\sigma, \vec{c})$ is an element of precisely $I^{\partial}_{\mathrm{odd-even}}$, with the remaining inequalities arising from the definition of $I^{\partial}_{\mathrm{odd}}$.

We now discuss $I^{\partial}_{\mathrm{even}}$. It is easy to see that elements of $I^{\partial}_{\mathrm{even,d}}$ are not elements of $I^{\partial}_{\mathrm{odd-even}}$, so the two subsets are disjoint. Now let $(\sigma, \vec{c})$ be an element of $I^{\partial}_{\mathrm{even}}$ that is not in $I^{\partial}_{\mathrm{even,d}}$. If $e_{\mathrm{odd}}(\sigma) = \infty$ this means that $e_{\mathrm{odd}}(\vec{c}) \neq \infty$, and then $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{odd-even}}$, with the other inequalities being part of the definition of $I^{\partial}_{\mathrm{even}}$. If instead $e_{\mathrm{odd}}(\sigma) \neq \infty$, then $e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{odd}}(\vec{c})$, which implies $e_{\mathrm{odd}}(\vec{c}) \neq \infty$, and combined with the properties arising from the definition of $I^{\partial}_{\mathrm{even}}$ this again shows that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{odd-even}}$. □

Proposition 7.3.8.4. *In this definition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let $l \geq 1$ and $m \geq 0$ be integers. Then the set I^1 is the disjoint union of the following subsets.*

- I^1_{d}
- $I^{\partial}_{\mathrm{odd},1}$ ♡

Proof. While I^1_{d} was defined as a subset of I^1, this is not the case for $I^{\partial}_{\mathrm{odd},1}$, but that it is a subset is clear from the definition. It is also straightforward that the two subsets are disjoint. Now let $(\sigma, \overrightarrow{c})$ be an element of I^1. Assume $e_{\mathrm{odd}}(\sigma) = \infty$. Then either $e_{\mathrm{odd}}(\overrightarrow{c}) = \infty$, in which case $(\sigma, \overrightarrow{c})$ is an element of I^1_{d}, or $e_{\mathrm{odd}}(\overrightarrow{c}) \leq e_{\mathrm{odd}}(\sigma)$, in which case $(\sigma, \overrightarrow{c})$ is an element of $I^{\partial}_{\mathrm{odd},1}$. Now assume $e_{\mathrm{odd}}(\sigma) \neq \infty$. Then either $e_{\mathrm{odd}}(\overrightarrow{c}) > e_{\mathrm{odd}}(\sigma)$, in which case $(\sigma, \overrightarrow{c})$ is an element of I^1_{d}, or $e_{\mathrm{odd}}(\overrightarrow{c}) \leq e_{\mathrm{odd}}(\sigma)$, which implies $e_{\mathrm{odd}}(\overrightarrow{c}) \neq \infty$, so that $(\sigma, \overrightarrow{c})$ is an element of $I^{\partial}_{\mathrm{odd},1}$. □

7.3.9. Canceling of some summands of $\epsilon_X^{(l-1)} \circ \mathrm{d}$

Several of the subsets we defined for I^{d} are such that the relevant sums over them cancel (which we indicated by naming them $I^{\mathrm{d,cancel}}$ with some subscript). This is what we show in this subsection.

Proposition 7.3.9.1. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I^{\mathrm{d,cancel}}_{>}} B'(v) = 0$$ ♡

Proof. Let $(\sigma, \overrightarrow{c}, p)$ be an element of $I^{\mathrm{d,cancel}}_{>}$. Then

$$e_{\mathrm{odd}}(\sigma) \leq e_{\mathrm{even}}(\sigma) - 1 \leq p \leq e_{\mathrm{odd}}(\sigma) + 1 \leq e_{\mathrm{even}}(\sigma)$$

holds, where the middle inequality is from the definition of I^{d} and the other two are from the definition of $I^{\mathrm{d,cancel}}_{>}$. This implies that[27]

$$e_{\mathrm{odd}}(\sigma) + 1 = e_{\mathrm{even}}(\sigma)$$

and either $p = e_{\mathrm{odd}}(\sigma)$ or $p = e_{\mathrm{odd}}(\sigma) + 1$.

It thus suffices to show that the map

$$\begin{aligned} \Phi \colon & \left\{ (\sigma, \overrightarrow{c}, p) \in I^{\mathrm{d,cancel}}_{>} \;\middle|\; p = e_{\mathrm{odd}}(\sigma) \right\} \\ & \to \left\{ (\sigma, \overrightarrow{c}, p) \in I^{\mathrm{d,cancel}}_{>} \;\middle|\; p = e_{\mathrm{odd}}(\sigma) + 1 \right\} \\ & (\sigma, \overrightarrow{c}, p) \mapsto (\sigma, \overrightarrow{c}, p+1) \end{aligned}$$

[27] $e_{\mathrm{even}}(\sigma) - 1 \leq e_{\mathrm{odd}}(\sigma) + 1 \leq e_{\mathrm{even}}(\sigma)$ but for parity reasons $e_{\mathrm{odd}}(\sigma) + 1 = e_{\mathrm{even}}(\sigma) - 1$ is not possible.

is a well-defined bijection and that for every element $(\sigma, \overrightarrow{c}, p)$ of $I_{>}^{\mathrm{d,cancel}}$ with $p = e_{\mathrm{odd}}(\sigma)$ it holds that $B'((\sigma, \overrightarrow{c}, p+1)) = -B'((\sigma, \overrightarrow{c}, p))$. This property of B' is obvious from the definition, so it only remains to show that Φ is a well-defined bijection.

So let $(\sigma, \overrightarrow{c}, p)$ be an element of $I_{>}^{\mathrm{d,cancel}}$ with $p = e_{\mathrm{odd}}(\sigma)$. Note that this implies that p is odd with $p \leq 2l-3$. Thus $1 \leq p+1 \leq 2l-2$. We have to show that $(\sigma, \overrightarrow{c}, p+1)$ is again an element of I^{d}. It follows from

$$e_{\mathrm{even}}(\overrightarrow{c}) < p < e_{\mathrm{odd}}(\overrightarrow{c})$$

that also

$$e_{\mathrm{even}}(\overrightarrow{c}) < p+1 < e_{\mathrm{odd}}(\overrightarrow{c})$$

for parity reasons. The discussion at the start of this proof shows that

$$e_{\mathrm{even}}(\sigma) - 1 \leq p+1 \leq e_{\mathrm{odd}}(\sigma) + 1$$

holds as well. It thus remains to show that

$$\sigma(e_{\mathrm{odd}}(\sigma) - 1), \qquad \sigma(e_{\mathrm{odd}}(\sigma)), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 2)$$

is cyclically ordered. But as $(\sigma, \overrightarrow{c}, p)$ is an element of I^{d} we know that

$$\sigma(e_{\mathrm{odd}}(\sigma) - 1), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 1), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 2)$$

is cyclically ordered, and the definition of $e_{\mathrm{even}}(\sigma) = e_{\mathrm{odd}}(\sigma) + 1$ implies that

$$\sigma(e_{\mathrm{odd}}(\sigma)), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 2), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 1)$$

is cyclically ordered. Rotating the first of these two we can phrase this as the following two lines each being cyclically ordered

$$\begin{gathered}\sigma(e_{\mathrm{odd}}(\sigma) + 2), \qquad \sigma(e_{\mathrm{odd}}(\sigma) - 1), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 1)\\ \sigma(e_{\mathrm{odd}}(\sigma)), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 2), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 1)\end{gathered}$$

which combines to

$$\sigma(e_{\mathrm{odd}}(\sigma)), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 2), \qquad \sigma(e_{\mathrm{odd}}(\sigma) - 1), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 1)$$

being cyclically ordered, from which the claim follows, so Φ is well-defined.

To show that Φ is a bijection, we let $(\sigma, \overrightarrow{c}, p)$ be an element of $I_{>}^{\mathrm{d,cancel}}$ with $p = e_{\mathrm{odd}}(\sigma) + 1$. We have to show that $(\sigma, \overrightarrow{c}, p-1)$ is again an element of I^{d}. The first two properties for this are shown completely analogously to the argument above. It thus remains to show that

$$\sigma(e_{\mathrm{odd}}(\sigma) - 1), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 1), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 2)$$

is cyclically ordered. Similarly to the argument above one finds that the following two lines are each being cyclically ordered, the first arising from $(\sigma, \overrightarrow{c}, p)$ being an element of I^{d}, the second from the definition of e_{odd}.

$$\sigma(e_{\mathrm{odd}}(\sigma) - 1), \qquad \sigma(e_{\mathrm{odd}}(\sigma)), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 2)$$

$$\sigma(e_{\mathrm{odd}}(\sigma)-1), \qquad \sigma(e_{\mathrm{odd}}(\sigma)+1), \qquad \sigma(e_{\mathrm{odd}}(\sigma))$$

which combines to

$$\sigma(e_{\mathrm{odd}}(\sigma)-1), \qquad \sigma(e_{\mathrm{odd}}(\sigma)+1), \qquad \sigma(e_{\mathrm{odd}}(\sigma)), \qquad \sigma(e_{\mathrm{odd}}(\sigma)+2)$$

being cyclically ordered, from which the claim follows. □

Proposition 7.3.9.2. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Let $1 \leq p \leq 2l-3$ be odd. Then the following holds.*

$$\sum_{v \in I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p}} B'(v) + \sum_{v \in I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p+1}} B'(v) = 0 \qquad \heartsuit$$

Proof. Let σ be an element of Σ_{2l-1} and $\vec{c}$ an element of $C^{\mathrm{full}}(2l-1,m)$. It suffices to show that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p}$ if and only if $(\sigma, \vec{c}, p+1)$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p+1}$, and that in this case it holds that $B'((\sigma, \vec{c}, p+1)) = -B'((\sigma, \vec{c}, p))$. This latter property is clear from definition.

Purely for parity reasons we immediately have that

$$e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c}) \qquad \text{and} \qquad e_{\mathrm{even}}(\sigma) < p < e_{\mathrm{odd}}(\sigma)$$

if and only if

$$e_{\mathrm{even}}(\vec{c}) < p+1 < e_{\mathrm{odd}}(\vec{c}) \qquad \text{and} \qquad e_{\mathrm{even}}(\sigma) < p+1 < e_{\mathrm{odd}}(\sigma)$$

It thus remains to show that σ cyclically preserves the ordering of the set $\{p-1, p+1, p+2\}$ if and only if σ cyclically preserves the ordering of $\{p-1, p, p+2\}$. So assume first that σ cyclically preserves the ordering of $\{p-1, p+1, p+2\}$. As $p < e_{\mathrm{odd}}(\sigma)$ is odd, we know that σ cyclically preserves the ordering of $\{p-1, p, p+1\}$, which combined with the assumption yields the claim. For the other direction we combine the assumption with $p+1 > e_{\mathrm{even}}(\sigma)$ being even, which means that σ cyclically preserves the ordering of $\{p, p+1, p+2\}$. □

Proposition 7.3.9.3. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I^{\mathrm{d,cancel}}_{<,\mathrm{top}}} B'(v) + \sum_{v \in I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}} B'(v) = 0 \qquad \heartsuit$$

Proof. It suffices to show that

$$\begin{aligned}\Phi\colon I^{\mathrm{d,cancel}}_{<,\mathrm{top}} &\to I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}\\ (\sigma, \vec{c}, p) &\mapsto (\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))\end{aligned}$$

is a well-defined bijection that satisfies $B'(\Phi(v)) = -B'(v)$ for every element v of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$.

So let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$. We first handle the property for B'. We have

$$\begin{aligned}B'((\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))) &= (-1)^{e_{\mathrm{even}}(\sigma)+1} B''((\sigma, \vec{c})) = -B''((\sigma, \vec{c}))\\ &= -(-1)^{p+1} B''((\sigma, \vec{c})) = -B'((\sigma, \vec{c}, p))\end{aligned}$$

where we used that p is odd.

Next we need to show that $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$. First we show that this is an element of I^{d}. For this we first show the following inequality.

$$e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\vec{c})$$

The inequality on the left holds by definition of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$. By definition of $I^{\mathrm{d}}_{<}$ we have $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$, which together with $e_{\mathrm{odd}}(\sigma) \leq e_{\mathrm{odd}}(\vec{c})$ due to what p is (and $(\sigma, \vec{c}, p)$ being an element of I^{d}) implies the inequality on the right. Next we show the following inequality.

$$e_{\mathrm{even}}(\sigma) - 2 < e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma) + 2$$

The left inequality is clear, and the right inequality follows from the inequality $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$, which holds by definition of $I^{\mathrm{d}}_{<}$. To finish showing that $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of I^{d} it remains to show that

$$\sigma(e_{\mathrm{even}}(\sigma) - 2), \qquad \sigma(e_{\mathrm{even}}(\sigma) - 1), \qquad \sigma(e_{\mathrm{even}}(\sigma) + 1)$$

is cyclically ordered For this we use that the following two lines are cyclically ordered, where the first one arises from the definition of $e_{\mathrm{even}}(\sigma)$, and the second from $e_{\mathrm{even}}(\sigma) - 1 < e_{\mathrm{odd}}(\sigma)$ being odd.

$$\begin{array}{lll}\sigma(e_{\mathrm{even}}(\sigma) - 1), & \sigma(e_{\mathrm{even}}(\sigma) + 1), & \sigma(e_{\mathrm{even}}(\sigma))\\ \sigma(e_{\mathrm{even}}(\sigma) - 2), & \sigma(e_{\mathrm{even}}(\sigma) - 1), & \sigma(e_{\mathrm{even}}(\sigma))\end{array}$$

Combining these two we obtain that

$$\sigma(e_{\mathrm{even}}(\sigma) - 2), \qquad \sigma(e_{\mathrm{even}}(\sigma) - 1), \qquad \sigma(e_{\mathrm{even}}(\sigma) + 1), \qquad \sigma(e_{\mathrm{even}}(\sigma))$$

is cyclically ordered, from which the claim follows. We have now shown that $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of I^{d}. That $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is then an element of $I^{\mathrm{d}}_{<,\mathrm{bottom}}$ is clear. To show that it is even an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$, we have to show that either $e_{\mathrm{odd}}(\sigma) = e_{\mathrm{odd}}(\vec{c}) = \infty$ or $e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})$. But this follows from what p must be from the definition of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$ together with the inequalities p must satisfy in the definition of I^{d}.

So far we have shown that Φ is a well-defined map, and it is clearly an injection, as σ and $\vec{c}$ already determine the value of p if $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$. It remains to show that Φ is surjective. So let $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ be an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$. If $e_{\mathrm{odd}}(\sigma) = \infty$ set $p = 2l - 1$, otherwise let $p = e_{\mathrm{odd}}(\sigma)$. Then we have to show that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$. From $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ being an element of I^{d} we can immediately conclude that $e_{\mathrm{even}}(\sigma) \neq -\infty$ and that $e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma)$. It thus only remains to show that $(\sigma, \vec{c}, p)$ is an element of I^{d}. For this we first show the following inequalities.

$$e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c})$$

That $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of I^{d} implies that $e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma)$, which together with $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$ from the definition of $I^{\mathrm{d}}_{<}$ implies the left inequality. The right inequality follows instead from the definition of $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$. We next show the following inequalities.

$$e_{\mathrm{even}}(\sigma) - 2 < p < e_{\mathrm{odd}}(\sigma) + 2$$

Here the left inequality follows from $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$ from the definition of $I^{\mathrm{d}}_{<}$, and the right inequality is clear. It remains to show that σ cyclically preserves the ordering of $\{p-1, p+1, p+2\}$, as long as $p \leq 2l - 3$. So assume that $p \leq 2l - 3$, which implies that we are in the case in which $p = e_{\mathrm{odd}}(\sigma) \neq \infty$. Then we use that the following two lines are cyclically ordered, where the first one arises from the definition of $e_{\mathrm{odd}}(\sigma)$, and the second from $e_{\mathrm{odd}}(\sigma) + 1 > e_{\mathrm{even}}(\sigma)$ being odd.

$$\begin{array}{ccc} \sigma(e_{\mathrm{odd}}(\sigma) - 1), & \sigma(e_{\mathrm{odd}}(\sigma) + 1), & \sigma(e_{\mathrm{odd}}(\sigma)) \\ \sigma(e_{\mathrm{odd}}(\sigma)), & \sigma(e_{\mathrm{odd}}(\sigma) + 1), & \sigma(e_{\mathrm{odd}}(\sigma) + 2) \end{array}$$

Combining these two we obtain that

$$\sigma(e_{\mathrm{odd}}(\sigma)), \qquad \sigma(e_{\mathrm{odd}}(\sigma) - 1), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 1), \qquad \sigma(e_{\mathrm{odd}}(\sigma) + 2)$$

from which the claim follows. $\square$

7.3.10. Matching up of the remaining summands

In this section we show how the sums over various subsets of I^{d}, I^{1}, $I^{\partial}_{\mathrm{even}}$, and $I^{\partial}_{\mathrm{odd}}$ match up.

Proposition 7.3.10.1. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1.*

Then the following holds.

$$\sum_{v\in I^{\mathrm{d}}_{<,\mathrm{top},\partial}} B'(v) = \sum_{v\in I^{\partial}_{\mathrm{even},\mathrm{d}}} B''(v) \qquad \heartsuit$$

Proof. Let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$. Then p is odd, so

$$B'((\sigma, \vec{c}, p)) = B''((\sigma, \vec{c}))$$

so that it suffices to show that

$$\Phi\colon I^{\mathrm{d}}_{<,\mathrm{top},\partial} \to I^{\partial}_{\mathrm{even},\mathrm{d}}$$
$$\Phi\colon (\sigma, \vec{c}, p) \mapsto (\sigma, \vec{c})$$

is a well-defined bijection.

So let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$. We first show that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{even}}$. For this we need that $e_{\mathrm{even}}(\vec{c}) \neq \infty$ and $e_{\mathrm{even}}(\sigma) \geq e_{\mathrm{even}}(\vec{c})$, both properties that are part of the definition of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$, and we need that $e_{\mathrm{odd}}(\vec{c}) \geq e_{\mathrm{even}}(\vec{c}) + 3$, which follows from the condition

$$e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c})$$

from the definition of I^{d} together with the parities, and finally we need that $e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{even}}(\vec{c}) + 1$, which follows from left part of the inequalities just used together with the definition of p in $I^{\mathrm{d}}_{<,\mathrm{top}}$. So now we have shown that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{even}}$. The properties that σ needs to satisfy for $(\sigma, \vec{c})$ to even be an element of $I^{\partial}_{\mathrm{even},\mathrm{d}}$ follow from what p is by the definition of $I^{\mathrm{d}}_{<,\mathrm{top}}$ and that $p < e_{\mathrm{odd}}(\vec{c})$ by the definition of I^{d}. This shows that Φ is well-defined. As p is uniquely determined by σ and $\vec{c}$ in the definition of $I^{\mathrm{d}}_{<,\mathrm{top}}$, we can also conclude that Φ is injective.

It remains to show that Φ is surjective. So let $(\sigma, \vec{c})$ be an element of $I^{\partial}_{\mathrm{even},\mathrm{d}}$. If $e_{\mathrm{odd}}(\sigma) = \infty$ set $p = 2l - 1$, otherwise let $p = e_{\mathrm{odd}}(\sigma)$. Then we have to show that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$. We can first note that the two inequalities in the definition of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$ also occur in the definition of $I^{\partial}_{\mathrm{even}}$, so that it suffices to show that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d}}_{<}$. By the definition of $I^{\partial}_{\mathrm{even}}$ we have

$$e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{odd}}(\sigma)$$

so that is only remains to show that $(\sigma, \vec{c}, p)$ is an element of I^{d}. For this we note that

$$e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c})$$

follows from the definition of $I^{\partial}_{\mathrm{even}}$ for the left inequality and from the definition of $I^{\partial}_{\mathrm{even},\mathrm{d}}$ for the right inequality. Next we consider the following inequalities.

$$e_{\mathrm{even}}(\sigma) - 2 < p < e_{\mathrm{odd}}(\sigma) + 2$$

The left inequality follows from $e_{\text{even}}(\sigma) < e_{\text{odd}}(\sigma)$, which we already showed above, and the right inequality is clear. Finally, we have to show that σ cyclically preserves the ordering of $\{p-1, p+1, p+2\}$ as long as $p \leq 2l-3$, which implies that $p = e_{\text{odd}}(\sigma) \neq \infty$. The argument for this is identical to the argument used at the end of the proof of Proposition 7.3.9.3. □

Proposition 7.3.10.2. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}} B'(v) = -\sum_{v \in I^{\partial}_{\mathrm{odd},\mathrm{d}}} B''(v) \qquad \heartsuit$$

Proof. Let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. Then p is even, so

$$B'((\sigma, \vec{c}, p)) = -B''((\sigma, \vec{c}))$$

so that it suffices to show that

$$\begin{aligned} \Phi\colon I^{\mathrm{d}}_{<,\mathrm{bottom},\partial} &\to I^{\partial}_{\mathrm{odd},\mathrm{d}} \\ \Phi\colon (\sigma, \vec{c}, p) &\mapsto (\sigma, \vec{c}) \end{aligned}$$

is a well-defined bijection.

So let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. We first show that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{odd}}$. For this we need that $e_{\text{odd}}(\vec{c}) \neq \infty$ and $e_{\text{odd}}(\sigma) \geq e_{\text{odd}}(\vec{c})$, both properties that are part of the definition of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. We also need that $e_{\text{even}}(\vec{c}) \leq e_{\text{odd}}(\vec{c}) - 3$, which follows from the condition

$$e_{\text{even}}(\vec{c}) < p < e_{\text{odd}}(\vec{c})$$

from the definition of I^{d} together with parities. Finally, we need that

$$e_{\text{even}}(\sigma) \leq e_{\text{odd}}(\vec{c}) - 1$$

which follows from $p < e_{\text{odd}}(\vec{c})$ together with $p = e_{\text{even}}(\sigma)$ from the definition of $I^{\mathrm{d}}_{<,\mathrm{bottom}}$. This finishes the proof that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{odd}}$. The properties that $(\sigma, \vec{c})$ needs to satisfy to also be an element of the subset $I^{\partial}_{\mathrm{odd},\mathrm{d}}$ follow from $p = e_{\text{even}}(\sigma)$ and the definition of I^{d}. This shows that Φ is well-defined. As p is uniquely determined by σ and $\vec{c}$ in the definition of $I^{\mathrm{d}}_{<,\mathrm{bottom}}$ we can also conclude that Φ is injective.

It remains to show that Φ is surjective. So let $(\sigma, \vec{c})$ be an element of $I^{\partial}_{\mathrm{odd},\mathrm{d}}$. We have to show that $(\sigma, \vec{c}, e_{\text{even}}(\sigma))$ is an element of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. We first note that the two inequalities in the definition of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$ also occur in the definition of $I^{\partial}_{\mathrm{odd}}$, so that it suffices to show that $(\sigma, \vec{c}, e_{\text{even}}(\sigma))$ is an element of $I^{\mathrm{d}}_{<}$. By the definition of $I^{\partial}_{\mathrm{odd}}$ we have

$$e_{\text{even}}(\sigma) < e_{\text{odd}}(\vec{c}) \leq e_{\text{odd}}(\sigma)$$

so that is only remains to show that $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of I^{d}. For this we note that

$$e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\vec{c})$$

follows from the definition of $I^{\partial}_{\mathrm{odd,d}}$ for the left inequality and from the definition of $I^{\partial}_{\mathrm{odd}}$ for the right inequality. Next we consider the following inequalities.

$$e_{\mathrm{even}}(\sigma) - 2 < e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma) + 2$$

The left inequality is clear and the right inequality follows from the inequality $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$, which we already showed above. Finally, we have to show that σ cyclically preserves the ordering of the following set.

$$\{e_{\mathrm{even}}(\sigma) - 2, e_{\mathrm{even}}(\sigma) - 1, e_{\mathrm{even}}(\sigma) + 1\}$$

The argument for this is identical to the argument used at the middle of the proof of Proposition 7.3.9.3, where it is shown that the map Φ used there is well-defined. □

Proposition 7.3.10.3. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},1}} B'(v) = \sum_{v \in I^{1}_{\mathrm{d}}} B''(v) \qquad \heartsuit$$

Proof. Let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{top},1}$. Then p is odd, so

$$B'((\sigma, \vec{c}, p)) = B''((\sigma, \vec{c}))$$

so that it suffices to show that

$$\Phi \colon I^{\mathrm{d}}_{<,\mathrm{top},1} \to I^{1}_{\mathrm{d}}$$
$$\Phi \colon (\sigma, \vec{c}, p) \mapsto (\sigma, \vec{c})$$

is a well-defined bijection.

So let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{top},1}$. That $(\sigma, \vec{c})$ is an element of I^1 then follows directly from the definition of $I^{\mathrm{d}}_{<,\mathrm{top},1}$. Suppose now that $e_{\mathrm{odd}}(\sigma) = \infty$. Then we must have $p = 2l - 1$ by the definition of $I^{\mathrm{d}}_{<,\mathrm{top}}$, which by the definition of I^{d} implies that $e_{\mathrm{odd}}(\vec{c}) > 2l - 1$ so that we can conclude that $e_{\mathrm{odd}}(\vec{c}) = \infty$ as well. If instead $e_{\mathrm{odd}}(\sigma) \neq \infty$ Then we must have $p = e_{\mathrm{odd}}(\sigma)$ by the definition of $I^{\mathrm{d}}_{<,\mathrm{top}}$, which by the definition of I^{d} implies that $e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})$. This finishes the proof that Φ is well-defined. As p is uniquely determined by σ and $\vec{c}$ in the definition of $I^{\mathrm{d}}_{<,\mathrm{top}}$ we also obtain that Φ is injective.

It remains to show that Φ is surjective. So let $(\sigma, \vec{c})$ be an element of I^1_{d}. Assume first that $e_{\mathrm{odd}}(\sigma) = \infty$. Then the definition of I^1_{d} implies that $e_{\mathrm{odd}}(\vec{c}) = \infty$ as well, and by the definition of I^1 we furthermore have that $e_{\mathrm{even}}(\vec{c}) = \infty = e_{\mathrm{even}}(\sigma)$. This directly implies all the properties needed for $(\sigma, \vec{c}, 2l-1)$ to be an element of $I^{\mathrm{d}}_{<,\mathrm{top},1}$. Assume now that $e_{\mathrm{odd}}(\sigma) \neq \infty$. Then the definition of I^1_{d} implies that $e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})$. This time all properties needed for $(\sigma, \vec{c}, e_{\mathrm{odd}}(\sigma))$ to be an element of $I^{\mathrm{d}}_{<,\mathrm{top},1}$ are directly implies except that σ must cyclically preserve the ordering of

$$\{e_{\mathrm{odd}}(\sigma) - 1, e_{\mathrm{odd}}(\sigma) + 1, e_{\mathrm{odd}}(\sigma) + 2\}$$

which follows with the same argument used at the end of the proof of Proposition 7.3.9.3. □

7.3.11. Conclusion

We can now put everything together to show that $\epsilon_X^{(\bullet)}$ forms a strongly homotopy linear morphism. As an intermediate step we first show that the identity required for this holds on elements of degree 0.

Proposition 7.3.11.1. *Let X be a totally ordered set and $l \geq 1$ an integer. Then*

$$\partial \circ \epsilon_X^{(l)} = \epsilon_X^{(l-1)} \circ \mathrm{d} - \mathrm{d} \circ \epsilon_X^{(l-1)}$$

holds on elements of $\Omega^0_{k[X]/k}$, where $\epsilon_X^{(\bullet)}$ defined as in Construction 7.3.1.1. ♡

Proof. The equation we have to show is k-linear on both sides, so it suffices to show it for a set of generators. So let $m \geq 0$ be an integer and $y_1, \ldots, y_m$ be as in Construction 7.3.1.1. It suffices to show that

$$\partial\Big(\epsilon_X^{(l)}(y_1 \cdots y_m)\Big) - \epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m)) + \mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big) = 0$$

This is done by combining various previous results as follows.

$$\partial\Big(\epsilon_X^{(l)}(y_1 \cdots y_m)\Big) - \epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m)) + \mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big)$$

Applying Proposition 7.3.7.3.

$$= \sum_{v \in I^{\partial}_{\mathrm{even}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd}}} B''(v) + \sum_{v \in I^1} B''(v) - \sum_{v \in I^{\mathrm{d}}} B'(v)$$

Applying Proposition 7.3.8.3 for $I^{\partial}_{\mathrm{even}}$ (first line) and $I^{\partial}_{\mathrm{odd}}$ (second line), Proposition 7.3.8.4 for I^1 (third line), and Proposition 7.3.8.2 for I^{d} (rest).

$$\begin{aligned}
&= \sum_{v \in I^{\partial}_{\mathrm{even},\mathrm{d}}} B''(v) + \sum_{v \in I^{\partial}_{\mathrm{odd}-\mathrm{even}}} B''(v) \\
&\quad - \sum_{v \in I^{\partial}_{\mathrm{odd},\mathrm{d}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd}-\mathrm{even}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd},1}} B''(v) \\
&\quad + \sum_{v \in I^{1}_{\mathrm{d}}} B''(v) + \sum_{v \in I^{\partial}_{\mathrm{odd},1}} B''(v) \\
&\quad - \sum_{v \in I^{\mathrm{d},\mathrm{cancel}}_{>}} B'(v) - \sum_{\substack{1 \leq p \leq 2l-q \\ v \in I^{\mathrm{d},\mathrm{cancel}}_{<,\mathrm{mid},p}}} B'(v) \\
&\quad - \left(\sum_{v \in I^{\mathrm{d},\mathrm{cancel}}_{<,\mathrm{top}}} B'(v) + \sum_{v \in I^{\mathrm{d},\mathrm{cancel}}_{<,\mathrm{bottom}}} B'(v) \right) \\
&\quad - \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},\partial}} B'(v) - \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},1}} B'(v) - \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}} B'(v)
\end{aligned}$$

The terms involving $I^{\partial}_{\mathrm{odd}-\mathrm{even}}$ in the first and second line cancel. Similarly, the terms involving $I^{\partial}_{\mathrm{odd},1}$ in the second and third line cancel. Furthermore the terms in the fourth and fifth line are zero by Propositions 7.3.9.1, 7.3.9.2 and 7.3.9.3.

$$\begin{aligned}
&= \sum_{v \in I^{\partial}_{\mathrm{even},\mathrm{d}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd},\mathrm{d}}} B''(v) + \sum_{v \in I^{1}_{\mathrm{d}}} B''(v) \\
&\quad - \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},\partial}} B'(v) - \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},1}} B'(v) - \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}} B'(v)
\end{aligned}$$

Applying Proposition 7.3.10.1 for the term involving $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$, applying Proposition 7.3.10.3 for the term involving $I^{\mathrm{d}}_{<,\mathrm{top},1}$, and finally applying Proposition 7.3.10.2 for the term involving $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$.

$$\begin{aligned}
&= \sum_{v \in I^{\partial}_{\mathrm{even},\mathrm{d}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd},\mathrm{d}}} B''(v) + \sum_{v \in I^{1}_{\mathrm{d}}} B''(v) \\
&\quad - \sum_{v \in I^{\partial}_{\mathrm{even},\mathrm{d}}} B''(v) - \sum_{v \in I^{1}_{\mathrm{d}}} B''(v) + \sum_{v \in I^{\partial}_{\mathrm{odd},\mathrm{d}}} B''(v) \\
&= 0
\end{aligned}$$

□

Proposition 7.3.11.2. *Let X be a totally ordered set. Then the quasiisomorphism of chain complexes*

$$\epsilon_X \colon \Omega^{\bullet}_{k[X]/k} \to \overline{\mathrm{C}}(k[X])$$

from Construction 7.2.2.1 and Proposition 7.2.2.2 can be upgraded to a strongly homotopy linear quasiisomorphism by equipping it with $\epsilon_X^{(\bullet)}$ *as defined in Construction 7.3.1.1.* ♡

Proof. By Definition 4.2.3.1 we have to show that

$$\partial \circ \epsilon_X^{(l)} = \epsilon_X^{(l-1)} \circ \mathrm{d} - \mathrm{d} \circ \epsilon_X^{(l-1)}$$

holds for $l > 0$[28]. As both sides of the above equation are k-linear it suffices to show this on a set of generators of $\Omega^\bullet_{k[X]/k}$. So let f be an element of $k[X]$ and $y_1, \dots, y_n$ elements of X. Then the following calculation shows that the above identity is satisfied on the element $f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m$.

$$\left(\partial \circ \epsilon_X^{(l)}\right)(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)$$

$$= \partial\left(\epsilon_X^{(l)}(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\right)$$

Applying the definition of $\epsilon_X^{(l)}$ from Construction 7.3.1.1.

$$= \partial\left(\epsilon_X^{(l)}(f) \cdot \epsilon_X(\mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\right)$$

Applying Proposition 7.2.2.2 (1).

$$= \partial\left(\epsilon_X^{(l)}(f) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m\right)$$

Applying the Leibniz rule for ∂, and using that $\partial(\mathrm{d}\, x) = 0$ in $\overline{\mathrm{C}}(k[X])$ for every element x of X, which can be seen either by direct calculation or by using that $\partial(\mathrm{d}\, x) = -\,\mathrm{d}(\partial x) = 0$ for degree reasons.

$$= \partial\left(\epsilon_X^{(l)}(f)\right) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m$$

Applying Proposition 7.3.11.1.

$$= \left(\epsilon_X^{(l-1)}(\mathrm{d}(f)) - \mathrm{d}\left(\epsilon_X^{(l-1)}(f)\right)\right) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m$$

$$= \epsilon_X^{(l-1)}(\mathrm{d}(f)) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m - \mathrm{d}\left(\epsilon_X^{(l-1)}(f)\right) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m$$

Using Proposition 7.2.2.2 (1) for the first summand and Proposition 6.3.2.14 for the second summand.

$$= \epsilon_X^{(l-1)}(\mathrm{d}(f)) \cdot \epsilon_X(\mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m) - \mathrm{d}\left(\epsilon_X^{(l-1)}(f) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m\right)$$

Also using Proposition 7.2.2.2 (1) for the second summand.

$$= \epsilon_X^{(l-1)}(\mathrm{d}(f)) \cdot \epsilon_X(\mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m) - \mathrm{d}\left(\epsilon_X^{(l-1)}(f) \cdot \epsilon_X(\mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\right)$$

Using the definition of $\epsilon_X^{(l-1)}$ from Construction 7.3.1.1.

$$= \epsilon_X^{(l-1)}(\mathrm{d}(f) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m) - \mathrm{d}\left(\epsilon_X^{(l-1)}(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\right)$$

[28] The case $l = 0$ is equivalent to ϵ_X being a morphism of chain complexes, which we already know.

Using the Leibniz rule for d in $\Omega^\bullet_{k[X]/k}$ (and that $\mathrm{d} \circ \mathrm{d} = 0$).

$$= \epsilon_X^{(l-1)}(\mathrm{d}(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)) - \mathrm{d}\Big(\epsilon_X^{(l-1)}(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\Big)$$
$$= \Big(\epsilon_X^{(l-1)} \circ \mathrm{d} - \mathrm{d} \circ \epsilon_X^{(l-1)}\Big)(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)$$

This shows that ϵ_X can be upgraded to a strongly homotopy linear quasiisomorphism using $\epsilon_X^{(\bullet)}$ constructed in Construction 7.3.1.1. $\square$

As the end result of this section we can now use Proposition 7.3.11.2 to obtain an equivalence between $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ and $\gamma_{\mathsf{Mixed}}(\Omega^\bullet_{k[X]/k})$ in $\mathcal{M}$ixed, showing that $\Omega^\bullet_{k[X]/k}$ is a strict mixed model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$.

Construction 7.3.11.3. Let X be a totally ordered set. The strongly homotopy linear quasiisomorphism ϵ_X from Proposition 7.3.11.2 induces by Proposition 7.2.2.2 (4) and Construction 4.4.4.1 a morphism

$$\gamma_{\mathsf{Mixed}}\Big(\Omega^\bullet_{k[X]/k}\Big) \to \gamma_{\mathsf{Mixed}}\big(\overline{\mathrm{C}}(k[X])\big)$$

in $\mathcal{M}$ixed, which is even an equivalence by Remark 4.4.4.2. Composing this equivalence with the equivalences from Propositions 6.3.4.1 and 6.3.1.10 yields an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \simeq \gamma_{\mathsf{Mixed}}\Big(\Omega^\bullet_{k[X]/k}\Big)$$

in the ∞-category $\mathcal{M}$ixed. $\diamondsuit$

7.4. De Rham forms as a strict model in Alg(Mixed)

In Sections 7.2 and 7.3 we showed that $\Omega^\bullet_{k[X]/k}$, which is an object in $\mathrm{CAlg}(\mathsf{Mixed}_{\mathrm{cof}})$, is a model for both $\mathrm{HH}(k[X])$ considered as an object in $\mathrm{CAlg}(\mathcal{D}(k))$, by forgetting the strict mixed structure, and of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object in $\mathcal{M}$ixed, by forgetting the algebra structure. An improved version of the latter result would be to show that $\Omega^\bullet_{k[X]/k}$ is also a model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. While it seems reasonable to expect this to hold, we will unfortunately not be able to show this in general, so we first formulate this as the following conjecture.

Conjecture B. *Let X be a set. Then there exists an equivalence*

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \simeq \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^\bullet_{k[X]/k}\Big)$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$.

We will often refer to the existence of such an equivalence for a specific set X as "Conjecture B holds for X". ♧

While we will not be able to show Conjecture B in general, we will be able to show that it holds for sets X with $|X| \leq 2$, and this is the goal of this section.

Let us now give an overview of the strategy to prove Conjecture B for $|X| \leq 2$. The very rough idea is to lift $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ to some cofibrant strict model in $\mathrm{Alg}(\mathsf{Mixed})$, use the previous results to obtain two equivalences from this model to $\Omega^{\bullet}_{k[X]/k}$, one respecting the strict mixed structure and one respecting the algebra structure, and finally use this to construct an equivalence between $\Omega^{\bullet}_{k[X]/k}$ and our generic lift that respects both.

To implement this plan we begin in Section 7.4.1 by lifting $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ to a cofibrant object $\widetilde{\mathrm{C}}''(X)$ of $\mathrm{Alg}(\mathsf{Mixed})$.

As the underlying differential graded algebra of $\widetilde{\mathrm{C}}''(X)$ is also cofibrant, we could then already lift the equivalence from Corollary 7.2.2.3 to a multiplicative quasiisomorphism as follows.

$$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}''(X)\Big) \to \Omega^{\bullet}_{k[X]/k}$$

However, we can not carry out the same argument to obtain such a quasiisomorphism that is compatible with the strict mixed structure from the equivalence from Construction 7.3.11.3, as the underlying strict mixed complex $\mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}(\widetilde{\mathrm{C}}''(X))$ of $\widetilde{\mathrm{C}}''(X)$ need not be cofibrant. This problem is related to the fact that the monoidal unit k of Mixed is not cofibrant as a strict mixed complex. To deal with this issue we will thus not actually use $\widetilde{\mathrm{C}}''(X)$, but replace it along a quasiisomorphism

$$\widetilde{\mathrm{C}}(X) \to \widetilde{\mathrm{C}}''(X)$$

in $\mathrm{Alg}(\mathsf{Mixed})$ by $\widetilde{\mathrm{C}}(X)$, which is also cofibrant and constructed so as to satisfy some specific properties that we will need. In particular, $\mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}(\widetilde{\mathrm{C}}(X))$ will be given by a coproduct $k \oplus \widetilde{\mathrm{C}}'(X)$, with the inclusion of the first summand given by the unit morphism, and such that $\widetilde{\mathrm{C}}'(X)$ is cofibrant as a strict mixed complex. The construction of $\widetilde{\mathrm{C}}(X)$ will be carried out in Section 7.4.2.

Now we can lift the equivalence from Corollary 7.2.2.3 to a quasiisomorphism

$$\Phi'_X\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}(X)\Big) \to \Omega^{\bullet}_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$, and the equivalence from Construction 7.3.11.3 to a quasiisomorphism

$$k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X) \to \Omega^{\bullet}_{k[X]/k}$$

in Mixed, and we only need to verify that the restriction to k^{cof} factors over k to obtain a quasiisomorphism

$$\Psi_X\colon\ \mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}\Big(\widetilde{\mathrm{C}}(X)\Big) \to \Omega^{\bullet}_{k[X]/k}$$

in Mixed as desired. This will be done in Section 7.4.3.

So now let us get back to what we actually want to show, that $\widetilde{\mathrm{C}}(X)$ is equivalent to $\Omega^{\bullet}_{k[X]/k}$ in Alg(Mixed). As $\widetilde{\mathrm{C}}(X)$ is cofibrant such an equivalence could be realized by a quasiisomorphism

$$\widetilde{\mathrm{C}}(X) \to \Omega^{\bullet}_{k[X]/k}$$

in Alg(Mixed). However, we know little about the elements of $\widetilde{\mathrm{C}}(X)$, apart from those that must exist by virtue of the quasiisomorphisms discussed above, so it would be easier to construct morphisms *into* rather than *out of* $\widetilde{\mathrm{C}}(X)$. As $\Omega^{\bullet}_{k[X]/k}$ is not cofibrant as an object in Alg(Mixed), we can not hope for there to be an actual strict morphism

$$\Omega^{\bullet}_{k[X]/k} \to \widetilde{\mathrm{C}}(X)$$

in Alg(Mixed), so instead we will attempt to construct a morphism Ξ_X from a cofibrant replacement of $\Omega^{\bullet}_{k[X]/k}$ to $\widetilde{\mathrm{C}}(X)$.

To be able to actually construct Ξ_X will require good control over the (low-degree) generators of said cofibrant replacement, so we construct a specific cofibrant replacement $\Omega'^{\bullet}_{k[X]/k}$ of $\Omega^{\bullet}_{k[X]/k}$ in Section 7.4.5.

The set X will occur as free generators of $\Omega'^{\bullet}_{k[X]/k}$ in degree 0, so the construction of Ξ_X will begin by defining $\Xi_X(x)$ to be such that $(\Phi'_X \circ \Xi_X)(x) = x$ for elements x in X. As Φ'_X is a quasiisomorphism it suffices to check that $\Phi'_X \circ \Xi_X$ is a quasiisomorphism to conclude that Ξ_X is one. The information mentioned so far would suffice to show that Ξ_X induces an isomorphism on H_0, but to handle the other homology groups we also need control over where $\Phi'_X \circ \Xi_X$ maps $\mathrm{d}\, x$ for x an element in X.

Thus we need to study how Φ'_X interacts with d. In Section 7.4.4 we will begin with the one variable case $\Phi_{\{t\}}$. We will not quite be able to show that the $\Phi'_{\{t\}}$ is compatible with d, but we find that this holds up to sign. By postcomposing with an automorphism that tweaks signs we can thus define new morphisms Φ_X to replace the usage of Φ'_X such that $\Phi_{\{t\}}$ is compatible with d.

To deduce from this that Φ_X is also compatible with d on elements of degree 0, as long as $|X| \leq 2$, we need a naturality statement for Φ. We show the required statement in Section 7.4.7, after we showed a similar naturality statement for ϵ in Section 7.4.6. The reason we only show this naturality statement for ϵ in Section 7.4.6 rather than earlier is that the proof uses the cofibrant resolution of $\Omega^{\bullet}_{k[t]/k}$ that was constructed in Section 7.4.5. After having handled the required naturality of Φ we can then show that Φ_X is compatible with d on degree 0 elements in Section 7.4.8.

Finally, in Section 7.4.9 we will put everything together and actually construct the quasiisomorphism

$$\Xi_X \colon \Omega'^{\bullet}_{k[X]/k} \to \widetilde{\mathrm{C}}(X)$$

that is a morphism in $\mathrm{Alg}(\mathsf{Mixed})$, and thereby prove Conjecture B for $|X| \leq 2$. To do so it will be very relevant to use the comparison morphisms Φ_X as well as Ψ_X; to begin with we need to prescribe the images of the generators X as we mentioned before, which we do by lifting elements along Φ_X, and in later steps there will be obstructions in the form of cycles that need to be boundaries, which we can verify by checking that the homology class represented by the cycle maps to zero along one of the two comparison morphisms.

7.4.1. A first cofibrant model

In this section we lift $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ to a first cofibrant model $\widetilde{\mathrm{C}}''(X)$ in $\mathrm{Alg}(\mathsf{Mixed})$. We actually need slightly more and lift not only $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$, but the morphism $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \to \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ that is induced by the unit morphism. We need this relative version in order to carry out the identification of the restriction to k that is needed for the strict mixed comparison morphism, as was explained in the introduction to Section 7.4.

Proposition 7.4.1.1. *Let X be a set. Then there exists a morphism*

$$\widetilde{\iota}'' \colon \widetilde{\mathrm{C}}''(\emptyset) \to \widetilde{\mathrm{C}}''(X)$$

in $\mathrm{Alg}(\mathsf{Mixed})$*, such that $\widetilde{\mathrm{C}}''(\emptyset)$ and $\widetilde{\mathrm{C}}''(X)$ are cofibrant, together with a commutative square*

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq", no head] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(\emptyset)\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\iota}''\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, "\simeq"', no head] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(X)\right)
\end{tikzcd} \tag{7.5}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$*, where the left morphism is induced by the unit morphism $\iota_{k[X]} \colon k \to k[X]$ and the horizontal morphisms are equivalences.* ♡

Proof. By Propositions 4.4.1.7 and 4.4.2.3 the ∞-category $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ is the underlying ∞-category of the combinatorial model category $\mathrm{Alg}(\mathsf{Mixed})$, where $\mathrm{Alg}(\mathsf{Mixed})$ carries the model structure from Proposition 4.2.2.9. As the 1-category $[1]$ is small[29], we can apply [HA, 1.3.4.25] to lift functors $[1] \to \mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ to functors $[1] \to \mathrm{Alg}(\mathsf{Mixed})$ that are cofibrant with respect to the projective model structure.

Let us for the moment denote the functor $[1] \to \mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ that is encoded by the morphism $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)$ by θ. Applying [HA, 1.3.4.25] to θ we thus obtain a functor

$$\Theta \colon [1] \to \mathrm{Alg}(\mathsf{Mixed})$$

[29] By $[1]$ we mean the 1-category with two objects 0 and 1, and a unique non-identity morphism $0 \to 1$.

that is cofibrant with respect to the projective model structure on the functor category $\mathrm{Fun}([1], \mathrm{Alg}(\mathsf{Mixed}))$, and that lifts θ in the sense that there is a commutative diagram as follows.

$$\begin{tikzcd} {[1]} \arrow[r, "\Theta"] \arrow[dr, "\theta"'] & \mathrm{Alg}(\mathsf{Mixed}) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})"] \\ & \mathrm{Alg}(\mathcal{M}\mathrm{ixed}) \end{tikzcd}$$

The functor Θ corresponds to a morphism in $\mathrm{Alg}(\mathsf{Mixed})$ that we are going to denote by

$$\widetilde{\iota}'' \colon \widetilde{\mathrm{C}}''(\emptyset) \to \widetilde{\mathrm{C}}''(X)$$

so that the commutative triangle above corresponds exactly to the commuting square (7.5).

It remains to show that $\widetilde{\mathrm{C}}''(\emptyset)$ and $\widetilde{\mathrm{C}}''(X)$ are cofibrant objects. As Θ is cofibrant with respect to the projective model structure, it is also cofibrant with respect to the injective model structure by [HTT, A.2.8.5], which by definition[30] means that it is pointwise cofibrant. □

We can directly improve Proposition 7.4.1.1 by showing that we can replace $\widetilde{\mathrm{C}}''(\emptyset)$ by k, which we do in the following proposition.

Proposition 7.4.1.2. *Let X be a set. Then there exists a cofibrant object $\widetilde{\mathrm{C}}''(X)$ in* $\mathrm{Alg}(\mathsf{Mixed})$ *so that there is a commutative square*

$$\begin{tikzcd} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\iota_{k[X]})"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}''(X)}\right)"] \\ \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(X)\right) \end{tikzcd} \tag{7.6}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, *where the left morphism is induced by the unit morphism $\iota_{k[X]} \colon k \to k[X]$, the right morphism is induced by the unit morphism*

$$\iota_{\widetilde{\mathrm{C}}''(X)} \colon k \to \widetilde{\mathrm{C}}''(X)$$

and the horizontal morphisms are equivalences. ♡

Proof. Let

$$\widetilde{\iota}'' \colon \widetilde{\mathrm{C}}''(\emptyset) \to \widetilde{\mathrm{C}}''(X)$$

[30] See [HTT, A.2.8.1 and A.2.8.2].

be as in Proposition 7.4.1.1. Then $\widetilde{\mathrm{C}}''(X)$ is cofibrant and the diagram

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(\emptyset)\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\iota}''\right)"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[l, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}''(\emptyset)}\right)"'] \arrow[dl, bend left, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}''(X)}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(X)\right) &
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ commutes, where $\iota_{\widetilde{\mathrm{C}}''(\emptyset)}\colon k \to \widetilde{\mathrm{C}}''(\emptyset)$ is the unit morphism and the square is the one supplied by Proposition 7.4.1.1. It thus suffices to show that $\iota_{\widetilde{\mathrm{C}}''(\emptyset)}\colon k \to \widetilde{\mathrm{C}}''(\emptyset)$ is a quasiisomorphism.

As quasiisomorphisms are detected on underlying morphisms of chain complexes, we can forget about the strict mixed structure and only consider the unit morphism of the differential graded algebra $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}''(\emptyset))$. There is a composite equivalence

$$\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}''(\emptyset)\Big)\Big) \simeq \mathrm{HH}(k) \simeq \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k/k}\Big) \simeq \mathrm{Alg}(\gamma)(k)$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where the first equivalence is obtained by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ to the equivalence at the top left in the diagram above combined with compatibility of $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1, the second equivalence is the one from Corollary 7.2.2.3, and the third equivalence arises from the isomorphism $\Omega^{\bullet}_{k/k} \cong k$.

As initial object k is cofibrant in $\mathrm{Alg}(\mathsf{Ch}(k))$, so as every object in the model category $\mathrm{Alg}(\mathsf{Ch}(k))$ is fibrant, the above equivalence in $\mathrm{Alg}(\mathcal{D}(k))$ can be lifted to a quasiisomorphism

$$k \to \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}''(\emptyset)\Big)$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$. But as k is the initial object in this category, this morphism must be exactly $\iota_{\widetilde{\mathrm{C}}''(\emptyset)}$, which has thus been proven to be a quasiisomorphism. □

7.4.2. An improved cofibrant model

$\widetilde{\mathrm{C}}''(X)$ as in Proposition 7.4.1.1 is a cofibrant model in $\mathrm{Alg}(\mathsf{Mixed})$ for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$, but apart from that we know nothing about $\widetilde{\mathrm{C}}''(X)$. In this section we will use $\widetilde{\mathrm{C}}''(X)$ to construct a new cofibrant model $\widetilde{\mathrm{C}}(X)$ over which we will have more control.

Before we state the result of this section we begin with some notation and a remark on pushouts of certain free algebras in strict mixed complexes.

Notation 7.4.2.1. In this section we are often going to use free associative algebras in strict mixed complexes that are generated by strict mixed complexes that are themselves free. To simplify notation, we thus define

$$\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})} \coloneqq \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})} \circ \mathrm{Free}^{\mathsf{Mixed}}$$

where $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$ and $\mathrm{Free}^{\mathsf{Mixed}}$ are as in Notation 4.2.2.10. ♢

Remark 7.4.2.2. Let X be an object in Alg(Mixed), let E be a $\mathbb{Z}$-graded set, and let $i'\colon E \to X$ be a map of $\mathbb{Z}$-graded sets. Assume that the image of i' consists only of cycles in X. Define B' to be the chain complex whose underlying graded k-module is $k \cdot E$ (i. e. the free one on E), equipped with the zero boundary operator. We also define a $\mathbb{Z}$-graded k-module $\underline{B}' \coloneqq B' \oplus B'[1]$. Then $\underline{B}'$ has two generators corresponding to every element e of E; the one in the left summand is in the same degree as e, and we will also denote this generator by e, and the one in the right summand has degree one higher than e, and we will denote this generator by $\underline{e}$. We can upgrade $\underline{B}'$ to a chain complex by defining $\partial(\underline{e}) = e$ and $\partial(e) = 0$ for every element e of E. There is an obvious morphism of chain complexes $j'\colon B' \to \underline{B}'$ that maps e to e.

We will consider the pushout diagram

$$\begin{tikzcd}
\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(B') \arrow[r, "\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(j')"] \arrow[d, "i"'] & \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(\underline{B}') \arrow[d, "\underline{i}"] \\
X \arrow[r, "\iota"'] & \underline{X}
\end{tikzcd} \qquad (*)$$

in Alg(Mixed), where i is the morphism that is determined by the morphism of chain complexes $B' \to X$ that is given by mapping e to $i'(e)$ for every element e of E (this is a morphism of chain complexes by the assumption that $i'(e)$ is a cycle).

Let Y be a chain complex. Then the underlying $\mathbb{Z}$-graded k-algebra of $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(Y)$ is given by the free graded k-algebra generated by the graded k-module $\mathrm{D} \otimes Y \cong Y \oplus Y[1]$. This follows from Proposition 4.2.2.11 and the analogous statement proven with Proposition E.7.2.2 (2) in the same manner by using that the forgetful functor from $\mathsf{Ch}(k)$ to the category of $\mathbb{Z}$-graded k-modules is symmetric monoidal and preserves colimits.

As the forgetful functor from Alg(Mixed) to Alg($\mathsf{Ch}(k)$) preserves colimits by Proposition 4.2.2.12 and the forgetful functor from Alg($\mathsf{Ch}(k)$) to the category of $\mathbb{Z}$-graded k-algebras does so as well by Proposition E.7.3.1, we then obtain that diagram $(*)$ is on underlying graded k-algebras given by a pushout[31]

$$\begin{tikzcd}
\mathrm{Free}(k\cdot E \oplus k\cdot \mathrm{d}\,E) \arrow[r] \arrow[d, "i"'] & \mathrm{Free}(k\cdot E \oplus k\cdot \mathrm{d}\,E) \amalg \mathrm{Free}(k\cdot \underline{E} \oplus k\cdot \mathrm{d}\,\underline{E}) \arrow[d, "\underline{i}"] \\
X \arrow[r, "\iota"'] & \underline{X}
\end{tikzcd}$$

where Free is ad hoc notation for the free associative $\mathbb{Z}$-graded k-algebra on a $\mathbb{Z}$-graded k-module[32], $\amalg$ refers to the coproduct in the category of $\mathbb{Z}$-graded k-algebras, i. e. the free product, and the top morphism is the inclusion of the first summand. From this it follows that the underlying graded k-algebra of $\underline{X}$ is given by the coproduct (in graded k-algebras) of X and the free graded k-algebra on elements $\underline{e}$ and $\mathrm{d}\,\underline{e}$ for $e \in E$. ♢

Proposition 7.4.2.3. *Let Y be an object in* $\mathrm{Alg}(\mathsf{Mixed})$ *and Y' a sub-$\mathbb{Z}$-graded-k-module of $\mathrm{H}_*(Y)$ such that $\mathrm{H}_*(Y)$ is the direct sum of Y' with a copy of k generated by the homology class $[1]$ that is represented by the multiplicative unit 1 of Y[33]. Assume furthermore that the homology of Y is concentrated in non-negative degrees.*

Then there exists a quasiisomorphism

$$\Theta\colon X \to Y$$

in $\mathrm{Alg}(\mathsf{Mixed})$ *such that X is cofibrant, concentrated in nonnegative degrees, and satisfies the following additional property. There must exist a sub-strict-mixed-complex X' of* $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$ *that is cofibrant as an object of* Mixed *such that the morphism of strict mixed complexes*

$$k \oplus X' \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$$

that is induced by the unit $k \to X$ and the inclusion $X' \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$ is an isomorphism. Furthermore, the restriction of $\mathrm{H}_*(\Theta)$ to $\mathrm{H}_*(X')$ must corestrict to an isomorphism $\mathrm{H}_*(X') \xrightarrow{\cong} Y'$. ♡

Proof. We will inductively construct a diagram in $\mathrm{Alg}(\mathsf{Mixed})$ as indicated below, satisfying properties (a), (b), (c), (d), (e), (f) and (g) that will be

[31] We denote by $\mathrm{d}\,E$ a $\mathbb{Z}$-graded set that consists of an element that we denote by $\mathrm{d}\,e$ of degree one higher than e for each element e of E. We use a similar convention for $\underline{E}$.
[32] We also use that Free preserves coproducts to rewrite the top right object as a coproduct.
[33] This element is a cycle and satisfies $\mathrm{d}(1) = 0$ due to the Leibniz rule that is satisfied by both ∂ as well as d.

explained below.

$$
\begin{tikzcd}
 & & k & & \\
X_{-1} \arrow[r, "\iota_{-1}^{0}"] \arrow[urr, bend left, "p_{-1}"] \arrow[drr, bend right, "\Theta_{-1}"'] & X_0 \arrow[r, "\iota_{0}^{1}"] \arrow[ur, "p_0"] \arrow[dr, "\Theta_0"'] & X_1 \arrow[r, "\iota_{1}^{2}"] \arrow[u, "p_1"] \arrow[d, "\Theta_1"'] & X_2 \arrow[r] \arrow[ul, "p_2"'] \arrow[dl, "\Theta_2"] & \cdots \\
 & & Y & &
\end{tikzcd}
\tag{$*$}
$$

Beyond the notation indicated in the diagram, we will denote the morphism from X_n to X_m for $-1 \leq n \leq m$ by $\iota_n^m := \iota_{m-1}^m \circ \cdots \circ \iota_n^{n+1}$. All morphisms ι_n^m are going to be levelwise injective, so if x is an element of X_n, we will also just write x for the element $\iota_n^m(x)$ of X_m. Finally, we define $K_n := \mathrm{Ker}(p_n)$ for $n \geq -1$. Note that as p_n is a morphism of chain complexes K_n will be closed under ∂.

Now we can formulate the properties that (7.4.2.3) needs to satisfy.

(a) $X_{-1} = k$.

(b) X_n is concentrated in non-negative degrees for all $n \geq -1$.

(c) $\mathrm{H}_*(\Theta_n)$ is an isomorphism for $* < n$ if $n \geq -1$ and surjective for all $*$ if $n \geq 0$.

(d) $\mathrm{H}_*(\Theta_n)$ maps $\mathrm{H}_*(K_n)$ into Y' for all $n \geq -1$.

(e) Let $n \geq -1$. Then there is a $\mathbb{Z}$-graded set E_n and a morphism of $\mathbb{Z}$-graded sets $i'_n \colon E_n \to X_n$ satisfying the following properties. Let e be an element of E_n. Then the image $i'_n(e)$ in X_n must be a cycle as well as lie in K_n. We denote by $B'_n := k \cdot E_n$ the chain complex with zero boundary operator whose underlying $\mathbb{Z}$-graded k-module is freely generated by E_n. We furthermore denote by $\underline{B}'_n$ the $\mathbb{Z}$-graded k-module that is given by $(k \cdot E_n) \oplus (k \cdot E_n)[1]$. If e is an element of E_n, then we will also use e to refer to e as en element of the left summand, and $\underline{e}$ to refer to e as an element of the right summand. Note that $\underline{e}$ has degree 1 higher than e. We can then make $\underline{B}'_n$ into a chain complex by defining $\partial(\underline{e}) = e$ and $\partial(e) = 0$ for every element e of E_n. There is a morphism of chain complexes $j'_n \colon B'_n \to \underline{B}'_n$ that maps e to e. Now we can finally formulate the property that E_n needs to satisfy. We require that there is a pushout diagram

$$\begin{tikzcd}
\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(B'_n) \arrow[r, "\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(j'_n)"] \arrow[d, "i_n"'] & \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(\underline{B}'_n) \arrow[d, "\underline{i}_n"] \\
X_n \arrow[r, "\iota_n^{n+1}"'] & X_{n+1}
\end{tikzcd} \qquad (**)$$

in $\mathrm{Alg}(\mathsf{Mixed})$, where i_n is the morphism that is determined by the morphism of chain complexes $B'_n \to X_n$ that is given by mapping e to $i'_n(e)$ for e an element of E_n (this is a morphism of chain complexes by the assumption that every element of E_n be a cycle in X_n).

(f) ι_n^{n+1} is a cofibration in $\mathrm{Alg}(\mathsf{Mixed})$ for $n \geq -1$.

(g) $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_n^{n+1})$ is a cofibration in Mixed for $n \geq -1$.

Before we construct diagram $(*)$ with these properties, let us first explain how to deduce the claim from it. We define

$$X \coloneqq \operatorname*{colim}_{n \geq -1} X_n$$

with the colimit taken in $\mathrm{Alg}(\mathsf{Mixed})$, and let $p\colon X \to k$ and $\Theta\colon X \to Y$ be the morphisms induced by p_n and Θ_n. We furthermore define

$$X' \coloneqq \mathrm{Ker}\big(\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(p)\big)$$

which is a sub-strict-mixed-complex of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$ as $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(p)$ is a morphism of strict mixed complexes. It remains to check the properties that X and Θ need to satisfy. Before we go through the individual claims, let us first note that the forgetful functors from $\mathrm{Alg}(\mathsf{Mixed})$ to $\mathrm{Alg}(\mathsf{Ch}(k))$, Mixed, as well as $\mathsf{Ch}(k)$ all detect filtered colimits by Proposition 4.2.2.12, so in particular every element of X already occurs in X_n for some $n \geq -1$. That X is concentrated in nonnegative degrees then follows directly from (b).

We continue by showing that Θ is a quasiisomorphism. It follows immediately from (c) that $\mathrm{H}_m(\Theta)$ is surjective for any integer m. Now assume that m is an integer and z is a cycle of chain degree m in X such that $\Theta(z)$ is a boundary. There must be an $n \geq -1$ such that z is an element of X_n, and we may assume that $n > m$. Then (c) implies that $\mathrm{H}_m(\Theta_n)$ is an isomorphism, so z must be a boundary in X_n and hence in X. Thus Θ is a quasiisomorphism.

Next we need to show that X is a cofibrant object in $\mathrm{Alg}(\mathsf{Mixed})$. This means that the morphism from the initial object k must be a cofibration. By (a) we can identify this morphism with the inclusion $X_{-1} \to X$, which is a transfinite composition of

$$X_{-1} \xrightarrow{\iota_{-1}^{0}} X_0 \xrightarrow{\iota_0^{1}} X_1 \longrightarrow \cdots$$

so that the claim follows from each ι_n^{n+1} being a cofibration in Alg(Mixed) by (f), as cofibrations are closed under transfinite compositions.

We now turn towards the properties X' needs to satisfy. As p is a morphism in Alg(Mixed), it must be compatible with the respective unit morphisms, so that the composition of the unit morphism $k \to X$ with p must be the identity. The splitting lemma now implies that the morphism of strict mixed complexes $k \oplus X' \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$ that is induced by the unit $k \to X$ and the inclusion $X' \to X$ is an isomorphism. Let m be an integer. Using the just mentioned isomorphism and the one from the statement of the proposition we obtain a composition

$$\mathrm{H}_m(k) \oplus \mathrm{H}_n(X') \xrightarrow{\cong} \mathrm{H}_m(X) \xrightarrow{\mathrm{H}_m(\Theta)} \mathrm{H}_m(Y) \xrightarrow{\cong} \mathrm{H}_m(k \cdot \{[1]\}) \oplus Y'$$

that we can write as a 2×2 matrix (thinking of the direct sums as column vectors), and showing that the restriction of $\mathrm{H}_*(\Theta)$ to $\mathrm{H}_*(X')$ corestricts to an isomorphism $\mathrm{H}_*(X') \xrightarrow{\cong} Y'$ means showing the the component $\mathrm{H}_n(X') \to \mathrm{H}_m(k \cdot \{[1]\})$ is zero and the component $\mathrm{H}_n(X') \to Y'$ is an isomorphism. (d) implies that the restriction of $\mathrm{H}_m(\Theta)$ to $\mathrm{H}_m(X')$ factors over Y', which handles the former. As Θ is a morphism in Alg(Mixed) we also know that the composition of Θ with the unit morphism $k \to X$ is given by the unit morphism $k \to Y$, which shows that matrix is of the form

$$\begin{bmatrix} \cong & 0 \\ 0 & ? \end{bmatrix}$$

Combining this with the fact that $\mathrm{H}_m(\Theta)$ is an isomorphism as we already showed above we can conclude that the component $\mathrm{H}_n(X') \to Y'$ (indicated with a question mark above) must be an isomorphism as well.

It remains to show that X' is a cofibrant strict mixed complex. Using that the forgetful functor $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ from Alg(Mixed) to Mixed preserves transfinite compositions we can show, using the same argument as when we showed that X was cofibrant in Alg(Mixed), only this time using (g) instead of (f), that the unit morphism $k \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$ is a cofibration in Mixed. We can identify this unit morphism with the inclusion of the first summand $k \to k \oplus X'$. This means that the top horizontal morphism in the pushout diagram

$$\begin{tikzcd} k \arrow[r, "\mathrm{id}_k \times 0"] \arrow[d] & k \oplus X' \arrow[d, "0 \amalg \mathrm{id}_{X'}"] \\ 0 \arrow[r] & X' \end{tikzcd} \qquad (***)$$

in Mixed is a cofibration, and hence so is the bottom horizontal morphism, i. e. X' is cofibrant as an object of Mixed.

We have shown that constructing diagram ($*$) satisfying properties (a), (b), (c), (d), (e), (f) and (g) will imply the statement of the proposition, so we now turn towards actually constructing this diagram. This has two main parts. We will inductively construct X_n together with ι_{n-1}^{n}, p_n and Θ_n satisfying (a), (b), (c), (d) and (e), and separately show that this implies that (f) and (g) hold as well.

We first get this latter part out of the way. So assume that we are given a diagram ($*$) satisfying properties (a), (b), (c), (d) and (e). Then the morphisms j'_n defined in (e) for $n \geq -1$ are cofibrations of chain complexes as they are coproducts of generating cofibrations, see [Hov99, 2.3.3 and 2.3.11][34]. The functor $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ is a left Quillen functor by Theorem 4.2.2.1, so the morphisms $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(j'_n)$ in $\mathrm{Alg}(\mathsf{Mixed})$ are cofibrations as well, and hence so are the morphisms ι_n^{n+1} by the pushout diagram that is part of (e). This proves (f).

Showing (g) requires a more detailed analysis of the underlying objects of pushouts in associative algebras. Luckily, Schwede and Shipley already did most of the work for us in the proof of [SS00, 6.2], and the following argument assumes that the reader has familiarized themselves with the proof of [SS00, 6.2]. We prove (g) by induction, letting $n \geq -1$, assuming that $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_{-1}^{0}), \ldots, \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_{n-1}^{n})$ are cofibrations in Mixed, and proving that then also $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_n^{n+1})$ is a cofibration in Mixed. By (e) the morphism ι_n^{n+1} is given by a pushout in $\mathrm{Alg}(\mathsf{Mixed})$ that is the transpose of the diagram below.

$$\begin{tikzcd}
\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}\left(\mathrm{Free}^{\mathsf{Mixed}}(B'_n)\right) \arrow[r, "i_n"] \arrow[d, "\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}\left(\mathrm{Free}^{\mathsf{Mixed}}(j'_n)\right)"] & X_n \arrow[d, "\iota_n^{n+1}"] \\
\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}\left(\mathrm{Free}^{\mathsf{Mixed}}(\underline{B}'_n)\right) \arrow[r, "\underline{i}_n"'] & X_{n+1}
\end{tikzcd}$$

This is also the situation considered in the proof of [SS00, 6.2], with their functor T being given by $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$, and the the transpose of the pushout diagram above then corresponding to the pushout diagram

$$\begin{tikzcd}
T(K) \arrow[r] \arrow[d] & T(L) \arrow[d] \\
X \arrow[r] & P
\end{tikzcd}$$

that is considered at the start of the proof of [SS00, 6.2]. The proof then shows (using their notation for the intermediate steps, but ours for the end

[34] The relevant generating cofibrations are denoted by $S^{m-1} \to D^m$ in [Hov99, 2.3.3].

points) that $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_n^{n+1})$ is a transfinite composition of a sequence

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n) = P_0 \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_m \longrightarrow \cdots$$

in Mixed. As cofibrations are closed under transfinite compositions, it thus suffices to show that the morphism $P_{m-1} \to P_m$ is a cofibration for every $m \geq 1$. This morphism is defined as a pushout

$$\begin{tikzcd}
Q_m \arrow[r] \arrow[d] & \Big(\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n) \otimes \mathrm{Free}^{\mathsf{Mixed}}\big(\underline{B}'_n\big)\Big)^{\otimes m} \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n) \arrow[d] \\
P_{m-1} \arrow[r] & P_m
\end{tikzcd}$$

in Mixed, so that it suffices to show that the morphism

$$Q_m \to \Big(\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n) \otimes \mathrm{Free}^{\mathsf{Mixed}}\big(\underline{B}'_n\big)\Big)^{\otimes m} \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)$$

is a cofibration in Mixed. This morphism is in turn isomorphic to a morphism

$$\overline{Q}_m \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)} \to \mathrm{Free}^{\mathsf{Mixed}}\big(\underline{B}'_n\big)^{\otimes m} \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)}$$

that is given as a tensor product of a morphism $\overline{Q}_m \to \mathrm{Free}^{\mathsf{Mixed}}\big(\underline{B}'_n\big)^{\otimes m}$ and the identity of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)}$. Now $\overline{Q}_m$ is the colimit of a punctured hypercube built up from $\mathrm{Free}^{\mathsf{Mixed}}(j'_n)$. As j'_n is a cofibration of chain complexes[35] and $\mathrm{Free}^{\mathsf{Mixed}}$ is a left Quillen functor by Theorem 4.2.2.1, it follows that $\mathrm{Free}^{\mathsf{Mixed}}(j'_n)$ is a cofibration in Mixed. Just like in the proof of [SS00, 6.2] one can now conclude by iterated application of the pushout-product that the morphism $\overline{Q}_m \to \mathrm{Free}^{\mathsf{Mixed}}\big(\underline{B}'_n\big)^{\otimes m}$ is a cofibration in Mixed.

Where we have to deviate from the proof of [SS00, 6.2] is in how we conclude from this that the morphism

$$\overline{Q}_m \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)} \to \mathrm{Free}^{\mathsf{Mixed}}\big(\underline{B}'_n\big)^{\otimes m} \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)}$$

is a cofibration as well. While $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)$ is assumed to be cofibrant in the context of [SS00, 6.2], $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)$ actually *not* cofibrant in our situation. However, with arguments completely analogous to the proof that the statement of the proposition follows from the existence of a diagram $(*)$ satisfying properties (a), (b), (c), (d), (e), (f) and (g), we can see that $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)$ is given by the direct sum of the sub-strict-mixed-complex K_n and the image of unit morphism $k \to X_n$. That unit morphism can furthermore be identified with the morphism $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_{-1}^{n})$, which is a cofibration in Mixed by the

[35] This was shown above when we proved (f).

induction assumption. Using a pushout diagram analogous to $(***)$ we can then conclude that K_n is cofibrant as an object of Mixed. Let us now return to showing that

$$\overline{Q}_m \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)} \to \mathrm{Free}^{\mathsf{Mixed}}\left(\underline{B}'_n\right)^{\otimes m} \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)}$$

is a cofibration. The tensor product $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)} \cong (k \oplus K_n)^{\otimes(m+1)}$ is isomorphic to a direct sum of terms of the form $K_n^{\otimes i} \otimes k^{\otimes(m+1-i)} \cong K_n^{\otimes i}$. As cofibrations are closed under coproducts, it thus suffices to show that

$$\overline{Q}_m \otimes K_n^{\otimes i} \to \mathrm{Free}^{\mathsf{Mixed}}\left(\underline{B}'_n\right)^{\otimes m} \otimes K_n^{\otimes i}$$

is a cofibration in Mixed for any $i \geq 0$. Here we need to distinguish two cases. If $i > 0$, then $K_n^{\otimes i}$ is cofibrant in Mixed as K_n is cofibrant as just shown, and combining this with $\overline{Q}_m \to \mathrm{Free}^{\mathsf{Mixed}}\left(\underline{B}'_n\right)^{\otimes m}$ being a cofibration and the pushout-product axiom we obtain that the morphism above is indeed a cofibration. If instead $i = 0$, then $K_n^{\otimes i} \cong k$. This is not cofibrant as a strict mixed complex, but as it is the monoidal unit, we obtain that the above morphism in question is isomorphic to $\overline{Q}_m \to \mathrm{Free}^{\mathsf{Mixed}}\left(\underline{B}'_n\right)^{\otimes m}$ and hence nevertheless a cofibration.

We have now shown that given a diagram $(*)$ satisfying properties (a), (b), (c), (d) and (e) also properties (f) and (g) hold. So now it remains to actually construct a diagram $(*)$ satisfying properties (a), (b), (c), (d) and (e), which we do inductively.

We begin by setting $X_{-1} := k$, $p_{-1} := \mathrm{id}_k$, and $\Theta_{-1}\colon k \to Y$ the unit morphism of Y. Then (a) is handled, and (b) clearly holds for $n = -1$. As Y was assumed to have homology concentrated in non-negative degrees, and k has the same property we also have (c) for $n = -1$. Finally, $K_{-1} = 0$, so (d) is clear for $n = -1$.

Now let Z be the graded subset of Y that is given by cycles that represent a non-zero homology class in Y'. We let E_{-1} be $Z[-1]$, i. e. the $\mathbb{Z}$-graded set in which the elements of Z are all given a degree that has been lowered by 1, and define $i'_n\colon E_{-1} \to X_{-1} = k$ as the map that maps every element to 0. As the element 0 in every degree of k is an element of K_{-1} as well as a cycle we can now define X_0 via the pushout diagram $(**)$, so that (e) is satisfied for $n = -1$. We also need to define p_0 and Θ_0, which we do using the universal property of the pushout, which ultimately amounts to prescribing a cycle of the appropriate degree in k and Y to the elements $\underline{e}$ of $\underline{B}'_{-1}$ for each element e of E_{-1}. For p_0 we simply let $\underline{e}$ map to 0. For Θ_0 we note that an element e of E_{-1} corresponds to a cycle z in Y, and the degrees of $\underline{e}$ and z agree. We can thus define Θ_0 by mapping $\underline{e}$ to the corresponding cycle z.

We now need to show that (b), (c) and (d) hold for $n = 0$. By assumption Y has homology concentrated in non-negative degrees, so by construction of E_{-1} every element e of E_{-1} is of degree bigger or equal to -1, which means that the corresponding elements $\underline{e}$ are all of non-negative degrees. Applying

Remark 7.4.2.2 we can thus conclude that X_0 is concentrated in non-negative degrees, which shows (b) for $n = 0$. By construction of E_{-1} and Θ_0 it is clear that Y' is contained in the image of $\mathrm{H}_*(\Theta_0)$. As 1 must also be in the image by virtue of Θ_0 being multiplicative, we can conclude from the assumption that $\mathrm{H}_*(Y) \cong k \cdot \{[1]\} \oplus Y'$ that $\mathrm{H}_*(\Theta_0)$ is surjective. As both X_0 and Y have homology that is concentrated in non-negative degrees it is also clear that $\mathrm{H}_*(\Theta_0)$ is an isomorphism for $* < 0$. Thus (c) follows for $n = 0$. Finally, it is clear from the definitions and Remark 7.4.2.2 that a basis for K_0 is given by non-empty words in the multiplicative generators $\underline{e}$ and $\mathrm{d}\,\underline{e}$ of X_0 for e elements of E_{-1}. As Θ_0 maps every element of the form $\underline{e}$ to a cycle that represents a homology class in Y', the same is true for elements of the form $\mathrm{d}\,\underline{e}$, as Y' is closed under d for degree reasons[36]. Multiplicativity of Θ_0 now implies that $\mathrm{H}_*(\Theta_0)$ maps $\mathrm{H}_*(K_0)$ into Y', showing (d) for $n = 0$.

We now define the remainder of diagram $(*)$ by induction. So we assume that $m > 0$ such that $X_{-1}, \ldots, X_{m-1}$ as well as $p_{-1}, \ldots, p_{m-1}$ and $\Theta_{-1}, \ldots, \Theta_{m-1}$ have already been defined in such a way that (e) holds for $n = -1, \ldots, m-2$ and (b), (c) and (d) hold for $n = -1, \ldots, m-1$. We then define X_m, p_m, and Θ_m in such a way that (e) holds for $n = m-1$ and (b), (c) and (d) hold for $n = m$.

Let $L \coloneqq \mathrm{Ker}(\mathrm{H}_{m-1}(\Theta_{m-1}))$. We want to define E_{m-1} as a $\mathbb{Z}$-graded subset of K_{m-1} whose elements are cycles representing nonzero homology classes in L, and which contains at least one such cycle for each nonzero homology class in L. Note that E_{m-1} will then be concentrated in degree $m-1$. We have to show that this is in fact possible, i. e. that every homology class in L is represented by a cycle that lies in K_{m-1}. Note that, as we already mentioned before, X_{m-1} decomposes as a direct sum of $k \cdot \{1\}$ and K_{m-1}. If $m > 1$, then this immediately implies the claim, as $k \cdot \{1\}$ is then concentrated in degree $0 < m-1$ so that *every* cycle of degree $m-1$ in X_{m-1} will be in K_{m-1}. If instead $m = 1$, then a cycle representing a homology class in L is given by a sum $a \cdot 1 + l$, with a an element of k and l a cycle in K_0 of degree 0. That Θ_0 is an algebra morphism as well as (d) for $n = m-1$ imply that

$$\mathrm{H}_0(\Theta_0)([a \cdot 1] + [l]) = a \cdot [1] + \mathrm{H}_0(\Theta_0)([l])$$

with $\mathrm{H}_0(\Theta_0)([l])$ an element of Y'. The assumption that $\mathrm{H}_*(Y)$ is the direct sum of $k \cdot \{[1]\}$ and Y' then implies that we must have $a = 0$. Thus a $\mathbb{Z}$-graded subset E_{m-1} of K_{m-1} of the form described above exists.

We let $i'_{m-1} \colon E_{m-1} \to X_{m-1}$ be the inclusion map and define ι^m_{m-1} and X_m via the pushout diagram $(**)$, so that (e) is satisfied for $n = m-1$. We next define p_m and Θ_m using the universal property of the pushout. We define p_m by extending p_{m-1} by mapping $\underline{e}$ to 0 for every element e of E_{m-1}, which is compatible as $p_{m-1} \circ i_{m-1}$ maps every element of E_{m-1} to 0 as E_{m-1} is a subset of K_{m-1}.

[36] Y' is concentrated in nonnegative degrees, so the images of d applied to elements of Y' lie in degrees greater or equal to 1, and in those degrees Y' is equal to $\mathrm{H}_*(Y)$, as k is concentrated in degree 0.

We also define Θ_m as follows. Let e be an element of E_{m-1}. By definition $i'_{m-1}(e)$ is a cycle that represents a homology class that is in the kernel of $\mathrm{H}_{m-1}(\Theta_{m-1})$. There thus exists an element in degree m of Y whose boundary is $\Theta_{m-1}(i'_{m-1}(e))$, and we can thus define Θ_m as an extension of Θ_{m-1} by mapping $\underline{e}$ to one such element. It now remains to show that with these definitions (b), (c) and (d) hold for $n = m$.

Combining that E_{m-1} is concentrated in degree $m - 1 \geq 0$ with Remark 7.4.2.2 we obtain that the underlying $\mathbb{Z}$-graded k-algebra of X_m is multiplicatively generated by X_{m-1} and elements of the form $\underline{e}$ of degree m and $\mathrm{d}\,\underline{e}$ of degree $m+1$ for $e \in E_{m-1}$. Combining this with (b) for $n = m-1$ we obtain (b) for $n = m$.

This also implies that ι^m_{m-1} is an isomorphism in degrees less than or equal to $m-1$[37], and thus an isomorphism in homology in degrees less than or equal to $m-2$. Combining this with (c) for $n = m-1$ we obtain that $\mathrm{H}_*(\Theta_m)$ is an isomorphism for $* < m-1$. That $\mathrm{H}_*(\Theta_m)$ is surjective for all $*$ follows directly from $\mathrm{H}_*(\Theta_{m-1})$ being surjective for all $*$ by (c) for $n = m-1$. To show (c) for $n = m$ it thus remains to show that $\mathrm{H}_{m-1}(\Theta_m)$ is injective. As we noted that ι^m_{m-1} is an isomorphism in degrees less than or equal to $m-1$, any homology class in the kernel of $\mathrm{H}_{m-1}(\Theta_m)$ must already lie in the kernel of $\mathrm{H}_{m-1}(\Theta_{m-1})$ and hence in L. But the construction of X_m then directly implies that that homology class is zero in $\mathrm{H}_{m-1}(X_m)$. This shows (c) for $n = m$.

Finally, ι^m_{m-1} being an isomorphism in degrees less than or equal to $m-1$ implies that the restriction and corestriction of ι^m_{m-1} to a morphism of chain complexes $K_{m-1} \to K_m$ is also an isomorphism in those degrees. As $m-1 \geq 0$ this implies that the image of the restriction of $\mathrm{H}_0(\Theta_m)$ to $\mathrm{H}_0(K_m)$ is contained in the image of the restriction of $\mathrm{H}_0(\Theta_{m-1})$ to $\mathrm{H}_0(K_{m-1})$, which together with (d) for $n = m-1$ shows that $\mathrm{H}_0(\Theta_m)$ maps $\mathrm{H}_0(K_m)$ into Y'. As Y' is equal to $\mathrm{H}_*(Y)$ in degrees $* \neq 0$, this shows (d) for $n = m$. $\square$

We can now apply Proposition 7.4.2.3 to improve the cofibrant model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ from Proposition 7.4.1.2.

Proposition 7.4.2.4. *Let X be a set. Then there exists a cofibrant object $\widetilde{\mathrm{C}}(X)$ in $\mathrm{Alg}(\mathsf{Mixed})$ that is concentrated in nonnegative degrees satisfying the following properties.*

[37] This is one reason why (b) is part of the properties that we need to require of diagram $(*)$ even if we did not need this property to conclude the statement of the proposition; without assuming it in the induction each new multiplicative generator also causes new elements of potentially arbitrary low degree.

Firstly, there has to a be a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, no head, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}(X)}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, no head, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(X)\right)
\end{tikzcd} \tag{7.7}$$

in Alg($\mathcal{M}$ixed)*, where the left morphism is induced by the unit morphism* $\iota_{k[X]}\colon k \to k[X]$*, the right morphism is induced by the unit morphism*

$$\iota_{\widetilde{\mathrm{C}}(X)}\colon k \to \widetilde{\mathrm{C}}(X)$$

and the horizontal morphisms are equivalences.

Secondly, there must exists a sub-strict-mixed-complex $\widetilde{\mathrm{C}}'(X)$ *of the strict mixed complex* $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$ *that is cofibrant as an object of* Mixed *and such that the morphism of strict mixed complexes*

$$k \oplus \widetilde{\mathrm{C}}'(X) \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right)$$

that is induced by the unit morphism $k \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$ *and the inclusion* $\widetilde{\mathrm{C}}'(X) \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$ *is an isomorphism.* ♡

Proof. Let $\widetilde{\mathrm{C}}''(X)$ be as in Proposition 7.4.1.2. Then there is a composite equivalence

$$\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\left(\widetilde{\mathrm{C}}''(X)\right)\right) \simeq \mathrm{HH}(k[X]) \simeq \mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k[X]/k}\right)$$

in Alg($\mathcal{D}(k)$), where the first equivalence is obtained by applying the forgetful functor Alg($\mathrm{ev}_{\mathfrak{m}}$) to the equivalence at the bottom of diagram (7.6) supplied by Proposition 7.4.1.2 combined with compatibility of Alg($\mathrm{ev}_{\mathfrak{m}}$) with Alg($\gamma_{\mathsf{Mixed}}$) from Construction 4.4.1.1, and the second equivalence is the one from Corollary 7.2.2.3. This implies that there is an isomorphism of $\mathbb{Z}$-graded k-algebras as follows.

$$\mathrm{H}_*\left(\widetilde{\mathrm{C}}''(X)\right) \cong \mathrm{H}_*\left(\Omega^{\bullet}_{k[X]/k}\right) \cong \Omega^{\bullet}_{k[X]/k}$$

As $\Omega^{\bullet}_{k[X]/k}$ is concentrated in nonnegative degrees and can be written as a direct sum of a copy of k generated by the multiplicative unit 1 and some complement we can transfer this sum decomposition to the homology of $\widetilde{\mathrm{C}}''(X)$ and use it to apply Proposition 7.4.2.3. This yields a quasiisomorphism

$$\Theta\colon \widetilde{\mathrm{C}}(X) \to \widetilde{\mathrm{C}}''(X)$$

in $\mathrm{Alg}(\mathsf{Mixed})$ such that $\widetilde{\mathrm{C}}(X)$ is cofibrant, concentrated in nonnegative degrees, and such that there exists a cofibrant sub-strict-mixed-complex $\widetilde{\mathrm{C}}'(X)$ of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$ such that the morphism of strict mixed complexes

$$k \oplus \widetilde{\mathrm{C}}'(X) \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$$

that is induced by the unit and inclusion is an isomorphism. This already shows the second property that $\widetilde{\mathrm{C}}(X)$ needs to satisfy.

It remains to show the existence of a commutative square (7.7). This is obtained as the transpose of the outer square of the commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"] \arrow[d, dash, "\simeq"'] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}''(X)}\right)"] \arrow[dr, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}(X)}\right)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(X)\right) \\
 & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(X)\right) \arrow[u, "\simeq"', "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Theta)"]
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, with the top commutative square being the transpose of the one supplied by Proposition 7.4.1.2 and the bottom triangle commuting because k is initial in $\mathrm{Alg}(\mathsf{Mixed})$. ☐

As it will later be relevant to keep using the same equivalences as in diagram (7.7) of Proposition 7.4.2.4, we now fix $\widetilde{\mathrm{C}}$ once and for all.

Construction 7.4.2.5. Let X be set. Then we define $\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)$ to be a cofibrant object of $\mathrm{Alg}(\mathsf{Mixed}_{\mathbb{Z}})$ satisfying the conditions of Proposition 7.4.2.4. Together with $\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)$ we fix once and for all a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{\mathbb{Z}[X]}\right)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\mathbb{Z}) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}[X]) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\right)
\end{tikzcd} \tag{7.8}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed}_{\mathbb{Z}})$ and a cofibrant sub-strict-mixed-complex $\widetilde{\mathrm{C}}'_{\mathbb{Z}}(X)$ of the strict mixed complex $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X))$ as supplied by Proposition 7.4.2.4.

For other commutative rings k we then define

$$\widetilde{\mathrm{C}}_k(X) \coloneqq k \otimes_{\mathbb{Z}} \widetilde{\mathrm{C}}_{\mathbb{Z}}(X)$$

which is a cofibrant object of $\mathrm{Alg}(\mathsf{Mixed}_k)$ by Proposition 4.2.2.13. It also follows directly from $\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)$ being concentrated in nonnegative degrees that the same holds true for $\widetilde{\mathrm{C}}_k(X)$. Applying $k \otimes_{\mathbb{Z}} -$ to the inclusion of $\widetilde{\mathrm{C}}'_{\mathbb{Z}}(X)$ into $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X))$ we obtain an injection into a strict mixed complex that we can identify with $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}_k(X))$. We define $\widetilde{\mathrm{C}}'_k(X)$ to be the image of that injection, as a sub-strict-mixed-complex of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}_k(X))$. It then follows immediately from the analogous property for $\widetilde{\mathrm{C}}'_{\mathbb{Z}}$ that the morphism of strict mixed complexes

$$k \oplus \widetilde{\mathrm{C}}'_k(X) \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}_k(X)\right)$$

that is induced by the unit and inclusion is then an isomorphism. Furthermore, as the functor

$$k \otimes_{\mathbb{Z}} -\colon \mathsf{Mixed}_{\mathbb{Z}} \to \mathsf{Mixed}_k$$

preserves cofibrations by Proposition 4.2.2.3 we can also conclude that $\widetilde{\mathrm{C}}'_k(X)$ is cofibrant as an object of Mixed_k.

We also obtain the following diagram in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed}_k)$

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"] \arrow[d, dash, "\simeq"'] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[d, dash, "\simeq"] \\
k \otimes_{\mathbb{Z}} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}) \arrow[r, "k \otimes_{\mathbb{Z}} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{\mathbb{Z}[X]}\right)"] \arrow[d, dash, "\simeq"'] & k \otimes_{\mathbb{Z}} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}[X]) \arrow[d, dash, "\simeq"] \\
k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\mathbb{Z}) \arrow[r, "k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)}\right)"] \arrow[d, dash, "\simeq"'] & k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\right) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}_k(X)}\right)"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}_k(X)\right)
\end{tikzcd}$$

where the top square arises from compatibility of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ with extension of scalars as in Remark 6.2.1.6 (plus using the obvious isomorphisms $k \otimes_{\mathbb{Z}} \mathbb{Z} \cong k$ and $k \otimes_{\mathbb{Z}} \mathbb{Z}[X] \cong k[X]$ that are given by including both tensor factors into the codomain and then multiplying), the middle square is obtained by applying $k \otimes_{\mathbb{Z}} -$ to the transpose of diagram (7.8), and the bottom square arises from compatibility of $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ with extension of scalars by Remark 4.4.1.3 (together again with the isomorphism $k \otimes_{Z} \mathbb{Z} \cong k$). Transposing the outer commutative rectangle we obtain a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, no head, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\iota_{k[X]})"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}_k(X)}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, no head, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}_k(X)\right)
\end{tikzcd} \tag{7.9}$$

which we fix once and for all. With the chosen diagram (7.9) and sub-strict-mixed-complex $\widetilde{\mathrm{C}}'_k(X)$ of $\mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}(\widetilde{\mathrm{C}}_k(X))$ we have thus provided the data that shows that $\widetilde{\mathrm{C}}_k(X)$ as we defined it here satisfies the conclusion of Proposition 7.4.2.4.

If the base ring is clear from context we will as usual omit it from the notation and just write e. g. $\widetilde{\mathrm{C}}(X)$ instead of $\widetilde{\mathrm{C}}_k(X)$.

Now let X and Y be two sets and $F\colon k[X] \to k[Y]$ a morphism of commutative k-algebras. Then the composite morphism

$$\begin{aligned}
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(X)\right) &\xrightarrow{\simeq} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \\
&\xrightarrow{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[Y]) \\
&\xrightarrow{\simeq} \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(Y)\right)
\end{aligned}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, where the first and third equivalences are the ones from (7.9), can be lifted[38] to a morphism $\widetilde{\mathrm{C}}(F)$ in $\mathrm{Alg}(\mathsf{Mixed})$, which we chose once and for all. $\widetilde{\mathrm{C}}(F)$ comes together with a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, no head, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(X)\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\widetilde{\mathrm{C}}(F))"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[Y]) \arrow[r, no head, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(Y)\right)
\end{tikzcd} \tag{7.10}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, where the horizontal equivalences are those from (7.9). ♢

7.4.3. Comparing the algebra and mixed structure separately

Construction 7.4.2.5 provides a reasonably nice strict model $\widetilde{\mathrm{C}}(X)$ for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an algebra in mixed complexes. In this section we will

[38] As $\widetilde{\mathrm{C}}(X)$ is cofibrant and $\widetilde{\mathrm{C}}(Y)$ fibrant in $\mathrm{Alg}(\mathsf{Mixed})$.

construct comparison morphisms from the underlying differential graded algebra and strict mixed complex of $\widetilde{\mathrm{C}}(X)$ to $\Omega^\bullet_{k[X]/k}$.

Construction 7.4.3.1. Let X be a set. We will construct a quasiisomorphism

$$\Phi'_{k,X}\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \to \Omega^\bullet_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$. If the base ring is clear from context we will also write Φ'_X, and even Φ' if the set X is clear as well.

As in Construction 7.4.2.5 we first construct $\Phi'_{\mathbb{Z},X}$ and then extend scalars for $\Phi'_{k,X}$. There is a composite equivalence

$$\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\Big)\Big) \simeq \mathrm{HH}(\mathbb{Z}[X]) \simeq \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{\mathbb{Z}[X]/\mathbb{Z}}\Big)$$

in $\mathrm{Alg}(\mathcal{D}(\mathbb{Z}))$, where the first equivalence is obtained by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ to the equivalence at the bottom of diagram (7.8) in Construction 7.4.2.5 combined with compatibility of $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1, and the second equivalence is the one from Corollary 7.2.2.3. By Proposition 4.2.2.12 $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ preserves cofibrant objects, so $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X))$ is cofibrant as an object in $\mathrm{Alg}(\mathsf{Ch}(\mathbb{Z}))$. As $\Omega^\bullet_{\mathbb{Z}[X]/\mathbb{Z}}$ is fibrant (like every object), we can thus lift the above equivalence in $\mathrm{Alg}(\mathcal{D}(\mathbb{Z}))$ to a quasiisomorphism $\Phi'_{\mathbb{Z},X}$ (see [Hov99, 1.2.10 (ii)] and Proposition A.1.0.1) as claimed.

We now define

$$\Phi'_{k,X}\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \to \Omega^\bullet_{k[X]/k}$$

as the composition

$$\begin{aligned}
\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) &= \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(k \otimes_{\mathbb{Z}} \widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\Big)\\
&\xrightarrow{\cong} k \otimes_{\mathbb{Z}} \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\Big)\\
&\xrightarrow{k\otimes_{\mathbb{Z}}\Phi'_{\mathbb{Z},X}} k \otimes_{\mathbb{Z}} \Omega^\bullet_{\mathbb{Z}[X]/\mathbb{Z}}\\
&\xrightarrow{\cong} \Omega^\bullet_{k[X]/k}
\end{aligned}$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$, where the first equality is by definition, the first isomorphism is the one from compatibility of $\mathrm{ev}_{\mathfrak{m}}$ with extension of scalars as in Remark 4.2.1.3, and the isomorphism in the last line is given by applying the unit in the first tensor factor and $\Omega^\bullet_{\iota_k[X]/k}$ in the second, and then multiplying. To see that $\Phi'_{k,X}$ is indeed a quasiisomorphism we only need to argue that $k \otimes_{\mathbb{Z}} \Phi'_{\mathbb{Z},X}$ is a quasiisomorphism. Note that the underlying morphism of chain complexes can be identified with $k \otimes_{\mathbb{Z}} \mathrm{ev}_{\mathfrak{a}}(\Phi'_{\mathbb{Z},X})$, and the functor

$$k \otimes_{\mathbb{Z}} -\colon \mathsf{Ch}(\mathbb{Z}) \to \mathsf{Ch}(k)$$

is a left Quillen functor by Fact 4.1.5.1 and so preserves weak equivalences between cofibrant objects. By Proposition 4.2.2.12 $\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)$ has cofibrant underlying chain complex, and by the discussion surrounding Definition 7.1.0.1 $\Omega^{\bullet}_{\mathbb{Z}[X]/\mathbb{Z}}$ has cofibrant underlying chain complex as well, so as $\Phi'_{\mathbb{Z},X}$ is a quasi-isomorphism we obtain that $\Phi'_{k,X}$ is one as well. ♢

Proposition 7.4.3.2. *Let X be a set. Then there is a commutative triangle*

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big)\Big) \arrow[rr, "\mathrm{Alg}(\gamma)\left(\Phi'_{k,X}\right)"] \arrow[dr, dash, "\simeq"'] & & \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[dl, dash, "\simeq"] \\
& \mathrm{HH}(k[X]) &
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$*, where the left diagonal equivalence is obtained by applying the forgetful functor* $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ *to the equivalence at the bottom of diagram* (7.9) *in Construction 7.4.2.5 combined with compatibility of* $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ *with* $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ *from Construction 4.4.1.1, and the right diagonal equivalence is the one from Corollary 7.2.2.3.* ♡

Proof. We drop the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ from the notation in this proof to improve readability. Consider the following diagram in $\mathrm{Alg}(\mathcal{D}(k))$ that will be explained below.

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\widetilde{\mathrm{C}}_k(X)\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\Phi'_{k,X}\right)"] \arrow[d, dash, "\mathrm{id}"'] & \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(k \otimes_{\mathbb{Z}} \widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(k\otimes_{\mathbb{Z}}\Phi'_{\mathbb{Z},X}\right)"] \arrow[d, dash, "\simeq"'] & \mathrm{Alg}(\gamma)\Big(k \otimes_{\mathbb{Z}} \Omega^{\bullet}_{\mathbb{Z}[X]/\mathbb{Z}}\Big) \arrow[d, dash, "\simeq"] \\
k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma)\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\Big) \arrow[r, "k\otimes_{\mathbb{Z}}\mathrm{Alg}(\gamma)\left(\Phi'_{\mathbb{Z},X}\right)"] \arrow[d, dash, "\simeq"'] & k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{\mathbb{Z}[X]/\mathbb{Z}}\Big) \arrow[d, dash, "\mathrm{id}"] \\
k \otimes_{\mathbb{Z}} \mathrm{HH}(\mathbb{Z}[X]) \arrow[r, "\simeq"] \arrow[d, dash, "\simeq"'] & k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{\mathbb{Z}[X]/\mathbb{Z}}\Big) \arrow[d, dash, "\simeq"] \\
\mathrm{HH}(k \otimes_{\mathbb{Z}} \mathbb{Z}[X]) \arrow[d, dash, "\simeq"'] & \mathrm{Alg}(\gamma)\Big(k \otimes_{\mathbb{Z}} \Omega^{\bullet}_{\mathbb{Z}[X]/\mathbb{Z}}\Big) \arrow[d, dash, "\simeq"] \\
\mathrm{HH}(k[X]) \arrow[r, "\simeq"] & \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big)
\end{tikzcd}$$

The first square from the top is built from the composition $\Phi'_{k,X}$ is defined as in Construction 7.4.3.1. The second square is the naturality square for the equivalence in Remark 4.4.1.3. The third square is obtained from the definition of $\Phi'_{\mathbb{Z},X}$ by applying $k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma)(-)$, the left equivalence is obtained by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ to the equivalence at the bottom of diagram (7.8) in Construction 7.4.2.5 combined with compatibility of $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1 and at the end tensoring with k, and the bottom equivalence is obtained by tensoring the equivalence from Corollary 7.2.2.3 (for base ring $\mathbb{Z}$) with k. Finally, the bottom rectangle is the one from Proposition 7.2.2.4, so that in particular the bottom equivalence of the full rectangle is the one from Corollary 7.2.2.3.

Now note that on the right the top two equivalences are the same as the bottom two equivalences, so the composition of the right column is the identity. The bottom equivalence is exactly the one occurring as the right diagonal equivalence in the statement. Finally, the composition on the left is exactly the definition of the equivalence at the bottom of diagram (7.9) in Construction 7.4.2.5. □

Proposition 7.4.3.3. *Let X be a totally ordered set. Then there exists a quasiisomorphism*

$$\Psi\colon\ \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right) \to \Omega^{\bullet}_{k[X]/k}$$

in Mixed. ♡

Proof. Some parts of this proof will be analogous to Construction 7.4.3.1, but we need some additional arguments as $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right)$ is not a cofibrant object of Mixed. Proposition 7.4.2.4 and Construction 7.4.2.5 isolate this problem to the non-cofibrancy of the summand k. So let $j\colon k^{\mathrm{cof}} \to k$ be a cofibrant replacement of k in Mixed. It then follows from Construction 7.4.2.5 that $k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)$ is a cofibrant strict mixed complex and that the composition

$$k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X) \xrightarrow{j\oplus\mathrm{id}} k \oplus \widetilde{\mathrm{C}}'(X) \xrightarrow{\cong} \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right) \tag{$*$}$$

is a quasiisomorphism, where the second morphism is induced by the unit and inclusion. There is a composite equivalence

$$\begin{aligned}\gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)\right) &\simeq \gamma_{\mathsf{Mixed}}\left(\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right)\right)\\ &\simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])\\ &\simeq \gamma_{\mathsf{Mixed}}\left(\Omega^{\bullet}_{k[X]/k}\right)\end{aligned} \tag{$**$}$$

in $\mathcal{M}$ixed, where the first equivalences arises from the composite quasiisomorphism $(*)$, the second equivalence is obtained by applying the forgetful functor $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ to the equivalence at the bottom of diagram (7.9) in Construction 7.4.2.5 combined with compatibility of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1, and the third equivalence is the one from Construction 7.3.11.3.

Using that $k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)$ is a cofibrant object of Mixed and that every object, so in particular $\Omega^{\bullet}_{k[X]/k}$, is fibrant, we can now lift the composite equivalence from $(**)$ to a quasiisomorphism

$$\Psi' \colon k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X) \to \Omega^{\bullet}_{k[X]/k}$$

in Mixed.

In the following we will use the notation i_1 and i_2 for the inclusions of the first and second summands of the sums $k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)$ and $k \oplus \widetilde{\mathrm{C}}'(X)$, with the context making clear which of the two sums we are including into. We now claim the following.

Claim 1: There exist morphisms

$$\Psi'' \colon k^{\mathrm{cof}} \to k \qquad \text{and} \qquad \Psi''' \colon k \to \Omega^{\bullet}_{k[X]/k}$$

in Mixed such that Ψ'' is a quasiisomorphism and such that there exists a commutative square

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}}\right) \arrow[r, "\gamma_{\mathsf{Mixed}}(i_1)"] \arrow[d, "\gamma_{\mathsf{Mixed}}(\Psi'')"'] & \gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)\right) \arrow[d, "\gamma_{\mathsf{Mixed}}(\Psi')"] \\
\gamma_{\mathsf{Mixed}}(k) \arrow[r, "\gamma_{\mathsf{Mixed}}(\Psi''')"'] & \gamma_{\mathsf{Mixed}}\left(\Omega^{\bullet}_{k[X]/k}\right)
\end{tikzcd} \qquad (***)$$

in $\mathcal{M}\mathrm{ixed}$.

Before showing the claim we discuss how the claim implies the statement of the proposition. We define Ψ as the composition

$$\mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}\left(\widetilde{\mathrm{C}}(X)\right) \xrightarrow{\cong} k \oplus \widetilde{\mathrm{C}}'(X) \xrightarrow{\Psi''' \amalg \left(\Psi' \circ i_2\right)} \Omega^{\bullet}_{k[X]/k}$$

in Mixed, where the first morphism is the inverse isomorphism of the second morphism in $(*)$. It remains to show that the morphism

$$\Psi''' \amalg (\Psi' \circ i_2) \colon k \oplus \widetilde{\mathrm{C}}'(X) \to \Omega^{\bullet}_{k[X]/k}$$

is a quasiisomorphism. But as Ψ'' and hence $\Psi'' \oplus \mathrm{id}_{\widetilde{\mathrm{C}}'(X)}$ is a quasiisomorphism, it suffices for this to show that

$$(\Psi''' \circ \Psi'') \amalg (\Psi' \circ i_2) \colon k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X) \to \Omega^{\bullet}_{k[X]/k}$$

is a quasiisomorphism. We know that $\Psi' = (\Psi' \circ i_1) \amalg (\Psi' \circ i_2)$ is a quasiisomorphism, so it would suffice to show that $(\Psi''' \circ \Psi'') \amalg (\Psi' \circ i_2)$ is chain homotopic to $(\Psi' \circ i_1) \amalg (\Psi' \circ i_2)$, for which it in turn suffices to show that

$(\Psi''' \circ \Psi'')$ is chain homotopic to $(\Psi' \circ i_1)$. But this follows from existence of commutative diagram $(***)$, using that the underlying chain complex of k^{cof} is cofibrant by Proposition 4.2.2.12, while $\Omega^{\bullet}_{k[X]/k}$ is a fibrant chain complex, together with [Hov99, 1.2.10 (ii)] and Propositions A.1.0.1 and 4.1.4.2.

So to finish the proof it remains to show Claim 1, for which we need to unpack and rewrite the composition $\gamma_{\mathsf{Mixed}}(\Psi') \circ \gamma_{\mathsf{Mixed}}(i_1)$ that occurs in the square $(***)$ that we are to construct. Using the definition of Ψ' and $(*)$ and $(**)$ to unpack this composition we obtain that $\gamma_{\mathsf{Mixed}}(\Psi') \circ \gamma_{\mathsf{Mixed}}(i_1)$ is homotopic to the composition from the top left to the bottom right along the top row and right column of the following diagram in $\mathcal{M}$ixed, which will be explained below.

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}}\right) \arrow[r, "\gamma_{\mathsf{Mixed}}(i_1)"] \arrow[d, "\gamma_{\mathsf{Mixed}}(j)"'] & \gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)\right) \arrow[d, "\gamma_{\mathsf{Mixed}}(j \oplus \mathrm{id})"] \\
\gamma_{\mathsf{Mixed}}(k) \arrow[r, "\gamma_{\mathsf{Mixed}}(i_1)"] \arrow[d, "\mathrm{id}"'] & \gamma_{\mathsf{Mixed}}\left(k \oplus \widetilde{\mathrm{C}}'(X)\right) \arrow[d, "\simeq"] \\
\gamma_{\mathsf{Mixed}}(k) \arrow[r, "\gamma_{\mathsf{Mixed}}\left(\iota_{\widetilde{\mathrm{C}}(X)}\right)"] \arrow[d, "\simeq"'] & \gamma_{\mathsf{Mixed}}\left(\mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}\left(\widetilde{\mathrm{C}}(X)\right)\right) \arrow[d, "\simeq"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[d, "\simeq"] \\
& \gamma_{\mathsf{Mixed}}\left(\Omega^{\bullet}_{k[X]/k}\right)
\end{tikzcd}$$

The top square is obtained by applying γ_{Mixed} to a commuting square in $\mathsf{Mixed}_{\mathrm{cof}}$. In the middle square we define the vertical morphism on the right as the equivalence induced by the isomorphism occurring in $(*)$. By definition this isomorphism is given on k by the unit morphism, which implies that this square also has a filler as it is given by γ_{Mixed} applied to a commuting square in $\mathsf{Mixed}_{\mathrm{cof}}$. The bottom square is given by applying the forgetful functor $\mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}$ to diagram (7.9) in Construction 7.4.2.5. Finally, the vertical equivalence at the bottom right is the one from Construction 7.3.11.3, which also occurs in $(**)$. Commutativity of the above diagram means that $\gamma_{\mathsf{Mixed}}(\Psi') \circ \gamma_{\mathsf{Mixed}}(i_1)$ is homotopic to the composition from the top left to the bottom right along the left column.

We now consider the following commutative diagram in $\mathcal{M}\mathrm{ixed}$, which we again explain below. The composition that we just showed is homotopic to $\gamma_{\mathsf{Mixed}}(\Psi') \circ \gamma_{\mathsf{Mixed}}(i_1)$ occurs as the composition from the top left to the bottom right while staying on the top and right side.

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}}\right) \arrow[d, "\gamma_{\mathsf{Mixed}}(j)"', "\simeq"] \arrow[dddd, bend right=60, "\psi''"', "\simeq"] & \\
\gamma_{\mathsf{Mixed}}(k) \arrow[d, "\simeq"'] & \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"] \arrow[d, "\simeq"'] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[d, "\simeq"] \arrow[ddd, bend left=60, "\simeq"] \\
\gamma_{\mathsf{Mixed}}(\mathrm{C}(k)) \arrow[r, "\gamma_{\mathsf{Mixed}}\left(\mathrm{C}(\iota_{k[X]})\right)"] \arrow[d, "\simeq"'] & \gamma_{\mathsf{Mixed}}(\mathrm{C}(k[X])) \arrow[d, "\simeq"] \\
\gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(k)\right) \arrow[r, "\gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(\iota_{k[X]})\right)"] & \gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(k[X])\right) \\
 & \gamma_{\mathsf{Mixed}}\left(\Omega^{\bullet}_{k[X]/k}\right) \arrow[u, "\gamma_{\mathsf{Mixed}}\left(\epsilon_X^{(\bullet)}\right)", "\simeq"']
\end{tikzcd}$$

We start by just defining ψ'' as the composition of the equivalences in the left column (which will be explained in a moment); this shorthand will be useful to shorten notation later. The second morphism in the left column is obtained by applying the forgetful functor $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ to the top horizontal equivalence in diagram (7.9) in Construction 7.4.2.5. The top square arises from naturality of the equivalence between $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ and the standard Hochschild complex in Proposition 6.3.4.1. The bottom square arises from naturality of the quotient morphism from the standard Hochschild complex to the normalized standard Hochschild complex, see Proposition 6.3.1.10. The lower right vertical equivalence is the one induced by the strongly homotopy linear quasiisomorphism $\epsilon_X^{(\bullet)}$, see Proposition 7.3.11.2 and Construction 4.4.4.1. Finally, the long equivalence on the right is the one of Construction 7.3.11.3, which also occurs in $(**)$, and the right rectangle is obtained by unpacking its definition.

We have now shown that $\gamma_{\mathsf{Mixed}}(\Psi') \circ \gamma_{\mathsf{Mixed}}(i_1)$ is homotopic to the composition from the top left to the bottom right in the diagram above while staying to the left and bottom. Note that $\overline{\mathrm{C}}(k)$ is isomorphic to k as a strict mixed complex (as $\overline{k} = 0$), with an isomorphism given by the unit $\iota_{\overline{\mathrm{C}}(k)}$ of $\overline{\mathrm{C}}(k)$. As $\overline{\mathrm{C}}(\iota_{k[X]})$ is a morphism of differential graded algebras and equality of morphisms of strict mixed complex can be checked on the underlying morphisms of chain complexes we can conclude that

$$\overline{\mathrm{C}}\big(\iota_{k[X]}\big) \circ \iota_{\overline{\mathrm{C}}(k)} = \iota_{\overline{\mathrm{C}}(k[X])}$$

holds. We should comment here on why $\iota_{\overline{\mathrm{C}}(k)}$ and $\iota_{\overline{\mathrm{C}}(k[X])}$ are morphisms of strict mixed complexes. As $\overline{\mathrm{C}}(R)$ for a commutative k-algebra R is not in general an algebra in strict mixed complexes, it is not a purely formal fact that the unit morphism $k \to \overline{\mathrm{C}}(R)$ of the differential graded algebra structure is a morphism of strict mixed complexes rather than just a morphism of chain complexes. However, this is indeed the case, as one can check using the formula for d from Proposition 6.3.1.10.[39]

The upshot of the discussion so far is that there is a commutative diagram as follows in $\mathcal{M}$ixed.

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\big(k^{\mathrm{cof}}\big) \arrow[d, "\psi''"', "\simeq"] \arrow[rrd, "\gamma_{\mathsf{Mixed}}(\Psi')\circ\gamma_{\mathsf{Mixed}}(i_1)"] & & \\
\gamma_{\mathsf{Mixed}}\big(\overline{\mathrm{C}}(k)\big) \arrow[r, "\gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(\iota_{k[X]})\right)"] & \gamma_{\mathsf{Mixed}}\big(\overline{\mathrm{C}}(k[X])\big) & \gamma_{\mathsf{Mixed}}\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[l, "\gamma_{\mathsf{Mixed}}\left(\epsilon_X^{(\bullet)}\right)"', "\simeq"] \\
\gamma_{\mathsf{Mixed}}(k) \arrow[u, "\gamma_{\mathsf{Mixed}}\left(\iota_{\overline{\mathrm{C}}(k)}\right)", "\simeq"'] \arrow[ru, "\gamma_{\mathsf{Mixed}}\left(\iota_{\overline{\mathrm{C}}(k[X])}\right)"'] \arrow[rru, dashed] & &
\end{tikzcd}$$

As k^{cof} is a cofibrant object in Mixed we can lift the composition of the two equivalences on the left to a quasiisomorphism $\Psi''\colon k^{\mathrm{cof}} \to k$ in Mixed, and it remains to show that we can up to homotopy find a lift of the dashed composition in $\mathcal{M}$ixed to a strict morphism $\Psi'''\colon k \to \Omega^{\bullet}_{k[X]/k}$ (that such a lift exists is not automatic as k is not cofibrant in Mixed). We define Ψ''' as the unit morphism

$$\Psi''' := \iota_{\Omega^{\bullet}_{k[X]/k}} \colon k \to \Omega^{\bullet}_{k[X]/k}$$

which can be seen to be a morphism of strict mixed complexes from the

[39] That this is not automatic is underlined by the fact that the analogous property does not hold if we had used $\mathrm{C}(R)$ instead of $\overline{\mathrm{C}}(R)$ – this is one of the reasons the *normalized* standard Hochschild complex is more convenient to work with.

definition of d on $\Omega^{\bullet}_{k[X]/k}$. It then suffices to show that the triangle

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\left(\Omega^{\bullet}_{k[X]/k}\right) \arrow[rr, "\gamma_{\mathsf{Mixed}}\left(\epsilon_X^{(\bullet)}\right)", "\simeq"'] & & \gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(k[X])\right) \\
& \gamma_{\mathsf{Mixed}}(k) \arrow[ul, "\gamma_{\mathsf{Mixed}}\left(\Psi'''\right)"] \arrow[ur, "\gamma_{\mathsf{Mixed}}\left(\iota_{\overline{\mathrm{C}}(k[X])}\right)"'] &
\end{tikzcd}$$

commutes in $\mathcal{M}$ixed.

For this we first unpack the definition of the lower horizontal equivalence $\gamma_{\mathsf{Mixed}}(\Omega^{\bullet}_{k[X]/k})$ from Construction 4.4.4.1. As ad hoc notation, let us denote the natural transformation coming with the functorial cofibrant replacement on Mixed by $q\colon -^{\mathrm{cof}} \to \mathrm{id}_{\mathsf{Mixed}}$. We will also use the notation that was in use in Construction 4.4.4.1. We need to show that there is a filler for the triangle at the bottom of the following diagram, where the top is the commutative rectangle from Construction 4.4.4.1. To make the diagram a bit cleaner we abbreviate γ_{Mixed} by γ_{M}, as well as $\Omega^{\bullet}_{k[X]/k}$ and $\overline{\mathrm{C}}(k[X])$ by Ω and $\overline{\mathrm{C}}$.

$$\begin{tikzcd}
\gamma_{\mathsf{M}}\left(\Omega^{\mathrm{cof}}\right) \arrow[r, "\gamma_{\mathsf{M}}\left(\left(\epsilon_X^{\mathrm{strict}}\right)^{\mathrm{cof}}\right)", "\simeq"'] \arrow[d, "\gamma_{\mathsf{M}}(q_\Omega)"', "\simeq"] & \gamma_{\mathsf{M}}\left(\left(\overline{\mathrm{C}}^{\mathrm{shl}}\right)^{\mathrm{cof}}\right) & \gamma_{\mathsf{M}}\left(\overline{\mathrm{C}}^{\mathrm{cof}}\right) \arrow[l, "\gamma_{\mathsf{M}}\left(\left(\iota_{\overline{\mathrm{C}}}^{\mathrm{shl}}\right)^{\mathrm{cof}}\right)"', "\simeq"] \arrow[d, "\simeq"', "\gamma_{\mathsf{M}}(q_{\overline{\mathrm{C}}})"] \\
\gamma_{\mathsf{M}}(\Omega) \arrow[rr, "\gamma_{\mathsf{M}}\left(\epsilon_X^{(\bullet)}\right)", "\simeq"'] & & \gamma_{\mathsf{M}}(\overline{\mathrm{C}}) \\
& \gamma_{\mathsf{M}}(k) \arrow[ul, "\gamma_{\mathsf{M}}\left(\Psi'''\right)"] \arrow[ur, "\gamma_{\mathsf{M}}\left(\iota_{\overline{\mathrm{C}}}\right)"'] &
\end{tikzcd}$$

As all the morphism in the top rectangle are equivalences we can also partition the diagram differently and instead show that there is a morphism from $\gamma_{\mathsf{Mixed}}(k)$ to the object in the top middle such that the two shapes in the diagram below have a filler.

$$\begin{tikzcd}
\gamma_{\mathsf{M}}\left(\Omega^{\mathrm{cof}}\right) \arrow[r, "\gamma_{\mathsf{M}}\left(\left(\epsilon_X^{\mathrm{strict}}\right)^{\mathrm{cof}}\right)", "\simeq"'] \arrow[d, "\gamma_{\mathsf{M}}(q_\Omega)"', "\simeq"] & \gamma_{\mathsf{M}}\left(\left(\overline{\mathrm{C}}^{\mathrm{shl}}\right)^{\mathrm{cof}}\right) & \gamma_{\mathsf{M}}\left(\overline{\mathrm{C}}^{\mathrm{cof}}\right) \arrow[l, "\gamma_{\mathsf{M}}\left(\left(\iota_{\overline{\mathrm{C}}}^{\mathrm{shl}}\right)^{\mathrm{cof}}\right)"', "\simeq"] \arrow[d, "\simeq"', "\gamma_{\mathsf{M}}(q_{\overline{\mathrm{C}}})"] \\
\gamma_{\mathsf{M}}(\Omega) & & \gamma_{\mathsf{M}}(\overline{\mathrm{C}}) \\
& \gamma_{\mathsf{M}}(k) \arrow[ul, "\gamma_{\mathsf{M}}\left(\Psi'''\right)"] \arrow[ur, "\gamma_{\mathsf{M}}\left(\iota_{\overline{\mathrm{C}}}\right)"'] \arrow[uu, dashed] &
\end{tikzcd}$$

Next we use that $q_k\colon k^{\mathrm{cof}} \to k$ is a quasiisomorphism to reduce to showing that there exists a dashed morphism as indicated in the diagram below such that the top two triangles have a filler, with the two squares having a filler by naturality of q.

$$
\begin{tikzcd}
\gamma_{\mathsf{M}}\left(\Omega^{\mathrm{cof}}\right) \arrow[rr, "\gamma_{\mathsf{M}}\left(\left(\epsilon_X^{\mathrm{strict}}\right)^{\mathrm{cof}}\right)", "\simeq"'] \arrow[d, "\gamma_{\mathsf{M}}(q_\Omega)"', "\simeq"] & & \gamma_{\mathsf{M}}\left(\left(\overline{\mathrm{C}}^{\mathrm{shl}}\right)^{\mathrm{cof}}\right) & & \gamma_{\mathsf{M}}\left(\overline{\mathrm{C}}^{\mathrm{cof}}\right) \arrow[ll, "\gamma_{\mathsf{M}}\left(\left(\iota_{\overline{\mathrm{C}}}^{\mathrm{shl}}\right)^{\mathrm{cof}}\right)"', "\simeq"] \arrow[d, "\simeq"', "\gamma_{\mathsf{M}}\left(q_{\overline{\mathrm{C}}}\right)"] \\
\gamma_{\mathsf{M}}(\Omega) & & \gamma_{\mathsf{M}}\left(k^{\mathrm{cof}}\right) \arrow[ull, "\gamma_{\mathsf{M}}\left(\Psi'''^{\mathrm{cof}}\right)"] \arrow[u, dashed] \arrow[urr, "\gamma_{\mathsf{M}}\left(\iota_{\overline{\mathrm{C}}}^{\mathrm{cof}}\right)"'] \arrow[d, "\gamma_{\mathsf{M}}(q_k)"', "\simeq"] & & \gamma_{\mathsf{M}}\left(\overline{\mathrm{C}}\right) \\
& & \gamma_{\mathsf{M}}(k) \arrow[ull, "\gamma_{\mathsf{M}}\left(\Psi'''\right)"] \arrow[urr, "\gamma_{\mathsf{M}}\left(\iota_{\overline{\mathrm{C}}}\right)"'] & &
\end{tikzcd}
$$

To show that the square formed by the two triangles has a filler in $\mathcal{M}$ixed it suffices to show that the square

$$
\begin{tikzcd}
k^{\mathrm{cof}} \arrow[r, "\Psi'''^{\mathrm{cof}}"] \arrow[d, "\iota_{\overline{\mathrm{C}}(k[X])}^{\mathrm{cof}}"'] & \left(\Omega^{\bullet}_{k[X]/k}\right)^{\mathrm{cof}} \arrow[d, "\left(\epsilon_X^{\mathrm{strict}}\right)^{\mathrm{cof}}"] \\
\overline{\mathrm{C}}(k[X])^{\mathrm{cof}} \arrow[r, "\left(\iota_{\overline{\mathrm{C}}(k[X])}^{\mathrm{shl}}\right)^{\mathrm{cof}}"'] & \left(\overline{\mathrm{C}}(k[X])^{\mathrm{shl}}\right)^{\mathrm{cof}}
\end{tikzcd}
$$

commutes in Mixed, for which it in turn suffices to show that the diagram

$$
\begin{tikzcd}
k \arrow[r, "\Psi'''"] \arrow[d, "\iota_{\overline{\mathrm{C}}(k[X])}"'] & \Omega^{\bullet}_{k[X]/k} \arrow[d, "\epsilon_X^{\mathrm{strict}}"] \\
\overline{\mathrm{C}}(k[X]) \arrow[r, "\iota_{\overline{\mathrm{C}}(k[X])}^{\mathrm{shl}}"'] & \overline{\mathrm{C}}(k[X])^{\mathrm{shl}}
\end{tikzcd}
$$

commutes. This we can now check directly. As all morphisms are k-linear it suffices to check the image of the element 1 of k along the two compositions.

We first consider the composition along the bottom left. $\iota_{\overline{\mathrm{C}}(k[X])}$ maps 1 to 1, which is then mapped by $\iota^{\mathrm{shl}}_{\overline{\mathrm{C}}(k[X])}$ to the tuple $(1, 0, 0, \ldots)$ of $\overline{\mathrm{C}}(k[X])^{\mathrm{shl}}$, see Definition 4.2.3.3. In the composition along the top right Ψ''' maps 1 to 1, which is then mapped by $\epsilon^{\mathrm{strict}}_X$ to the tuple $\epsilon^{\mathrm{strict}}_X(1)$ that is defined as follows for $i \geq 0$, see Proposition 4.2.3.7 and Definition 4.2.3.8.

$$\epsilon^{\mathrm{strict}}_X(1)_{2i} = \epsilon^{(i)}_X(1)$$
$$\epsilon^{\mathrm{strict}}_X(1)_{2i+1} = \left(\partial\epsilon^{(i+1)}_X - \epsilon^{(i+1)}_X\partial\right)(1)$$

As $\partial(1) = 0$ we can simplify the odd case to $\epsilon^{\mathrm{strict}}_X(1)_{2i+1} = \partial(\epsilon^{(i+1)}_X(1))$. It thus suffices to show that $\epsilon^{(0)}_X(1) = 1$ and $\epsilon^{(i)}_X(1) = 0$ for $i > 0$. The former is clear as $\epsilon^{(0)}_X$ is a morphism of differential graded algebras by Proposition 7.2.2.2 (2). For the latter we check the definition of $\epsilon^{(i)}_X$ in Construction 7.3.1.1. Using the notation there, the element 1 implies that $m = 0$, and then $C(i, m)$ is empty[40], implying the claim. This finishes the proof. □

Definition 7.4.3.4. Let X be a totally ordered set. Then we choose once and for all a quasiisomorphism

$$\Psi_X\colon\ \mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}\left(\widetilde{\mathrm{C}}(X)\right) \to \Omega^{\bullet}_{k[X]/k}$$

in Mixed, as exists by Proposition 7.4.3.3. ♢

7.4.4. Compatibility of Φ with d in the case of a single variable

In Section 7.4.3 we constructed two comparison quasiisomorphisms between $\widetilde{\mathrm{C}}(X)$ and $\Omega^{\bullet}_{k[X]/k}$; one compatible with the strict mixed structure, and one compatible with the multiplicative structure. In this section we show that after possibly tweaking it slightly, the multiplicative morphism also preserves d in the special case of $X = \{t\}$.

Proposition 7.4.4.1. *There exists an element ν of $\{+1, -1\}$ such that the morphism*

$$\Phi'_{k,\{t\}}\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\left(\widetilde{\mathrm{C}}_k(\{t\})\right) \to \Omega^{\bullet}_{k[t]/k}$$

from Construction 7.4.3.1 satisfies

$$\Phi'_{k,\{t\}}(\mathrm{d}\, y) = \nu \cdot \mathrm{d}\left(\Phi'_{k,\{t\}}(y)\right) \tag{7.11}$$

for every element y of $\widetilde{\mathrm{C}}_k(\{t\})$. ♡

[40] As $i > 0$ we have that $1 \leq 1 \leq i$. Thus any element $\vec{c}$ of $C(i, m)$ must satisfy $c_1 + 1 \leq c_2 - 1$ while $1 \leq c_1, c_2 \leq 0 + 1 = 1$, which is not possible.

Proof. By definition we can identify $\Phi'_{k,\{t\}}$ with $k \otimes_{\mathbb{Z}} \Phi'_{\mathbb{Z},\{t\}}$, and as the isomorphism (actually equality) $\widetilde{\mathrm{C}}_k(\{t\}) \cong k \otimes_{\mathbb{Z}} \widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$ is compatible with the strict mixed structure by definition and the isomorphism $\Omega^{\bullet}_{k[t]/k} \cong k \otimes_{\mathbb{Z}} \Omega^{\bullet}_{\mathbb{Z}[t]/\mathbb{Z}}$ that occurs in the definition of $\Phi'_{k,\{t\}}$ is compatible with the strict mixed structure by Remark 7.1.0.2, it suffices to prove that there exists an element ν of $\{+1, -1\}$ such that (7.11) holds in the case of base ring $\mathbb{Z}$.

We next note that as $\Omega^{\bullet}_{\mathbb{Z}[t]/k}$ is concentrated in degrees 0 and 1, equation (7.11) is automatic no matter what we choose for ν if y is an element of a degree other than -1 or 0. As $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$ is concentrated in nonnegative degrees the equation also holds automatically for elements of degree -1, and every element of $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$ of degree 0 is a cycle. We are thus left showing that there exists an element ν of $\{+1, -1\}$ such that (7.11) holds for cycles y of degree 0 of $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$

As $\Phi'_{\mathbb{Z},\{t\}}$ is a quasiisomorphism and $\Omega^{\bullet}_{\mathbb{Z}[t]/\mathbb{Z}}$ has zero boundary operator, $\Phi'_{\mathbb{Z},\{t\}}$ must be surjective. We can thus lift the element t of $\Omega^{\bullet}_{\mathbb{Z}[t]/\mathbb{Z}}$ to an element $\bar{t}$ of $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$ of degree 0 such that $\Phi'_{\mathbb{Z},\{t\}}(\bar{t}) = t$. As $\Phi'_{\mathbb{Z},\{t\}}$ is multiplicative we then also have $\Phi'_{\mathbb{Z},\{t\}}(\bar{t}^n) = t^n$ for $n \geq 0$, so that we can conclude that the elements $[\bar{t}^n]$ for $n \geq 0$ form a $\mathbb{Z}$-basis for $\mathrm{H}_0(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\}))$. Let us assume for the moment that we found an element ν such that (7.11) holds for the elements $y = \bar{t}^n$ for $n \geq 0$. Then we claim (7.11) holds for all cycles y in degree 0. Indeed, any cycle y of degree 0 of $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$ must be of the form $y = \sum_{0 \leq n} c_n \cdot \bar{t}^n + \partial z$ for some element z of degree 1 and elements c_n in $\mathbb{Z}$ for $n \geq 0$, only finitely many of which are nonzero. But then we have the following calculation, using that $\Omega^{\bullet}_{\mathbb{Z}[t]/\mathbb{Z}}$ has zero boundary operator and thus $\Phi'_{\mathbb{Z},\{t\}}$ maps boundaries to zero.

$$
\begin{aligned}
\Phi'_{\mathbb{Z},\{t\}}(\mathrm{d}\, y) &= \Phi'_{\mathbb{Z},\{t\}}\left(\sum_{0 \leq n} c_n \cdot \mathrm{d}\big(\bar{t}^n\big) - \partial(\mathrm{d}\, z)\right) = \sum_{0 \leq n} c_n \cdot \Phi'_{\mathbb{Z},\{t\}}\big(\mathrm{d}\big(\bar{t}^n\big)\big) \\
&= \sum_{0 \leq n} c_n \cdot \nu \cdot \mathrm{d}\Big(\Phi'_{\mathbb{Z},\{t\}}\big(\bar{t}^n\big)\Big) = \nu \cdot \mathrm{d}\left(\Phi'_{\mathbb{Z},\{t\}}\left(\sum_{0 \leq n} c_n \cdot \bar{t}^n\right)\right) \\
&= \nu \cdot \mathrm{d}\left(\Phi'_{\mathbb{Z},\{t\}}\left(\sum_{0 \leq n} c_n \cdot \bar{t}^n + \partial z\right)\right) = \nu \cdot \mathrm{d}\Big(\Phi'_{\mathbb{Z},\{t\}}(y)\Big)
\end{aligned}
$$

It thus suffices to show that there exists an element ν of $\{+1, -1\}$ such that (7.11) holds for elements $y = \bar{t}^n$ for $n \geq 0$.

We now need some input on properties that d must satisfy on the homology of $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$. For this equip $\{t\}$ with the unique total order and let Ψ be as in Definition 7.4.3.4. Then Ψ being a quasiisomorphism as well as compatible with d, and $\Omega^{\bullet}_{\mathbb{Z}[t]/\mathbb{Z}}$ having zero boundary operator, implies that there is a commutative diagram

$$\begin{tikzcd}
\Omega^1_{\mathbb{Z}[t]/\mathbb{Z}} \arrow[r, dash, "\cong"] & \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \\
\Omega^0_{\mathbb{Z}[t]/\mathbb{Z}} \arrow[u, "\mathrm{d}"] \arrow[r, dash, "\cong"'] & \mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \arrow[u, "\mathrm{d}"']
\end{tikzcd}$$

of abelian groups where the two horizontal morphisms are isomorphisms[41]. A $\mathbb{Z}$-basis of $\Omega^0_{\mathbb{Z}[t]/\mathbb{Z}}$ is given by t^n for $n \geq 0$, and a $\mathbb{Z}$-basis of $\Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$ is given by $t^n \cdot \mathrm{d}\,t$ for $n \geq 0$. Combining this with $\mathrm{d}(t^n) = n \cdot t^{n-1} \cdot \mathrm{d}\,t$ for $n \geq 0$ one obtains the following two properties for d on $\Omega^\bullet_{\mathbb{Z}[t]/\mathbb{Z}}$.

(1) The morphism

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{d} \colon \mathbb{Q} \otimes_{\mathbb{Z}} \Omega^0_{\mathbb{Z}[t]/\mathbb{Z}} \to \mathbb{Q} \otimes_{\mathbb{Z}} \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$$

is surjective.

(2) The morphism

$$\mathrm{d} \colon \Omega^0_{\mathbb{Z}[t]/\mathbb{Z}} \to \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$$

is only divisible by units, i. e. if $\mathrm{d} = c \cdot \mathrm{d}'$ for another morphism $\mathrm{d}' \colon \Omega^0_{\mathbb{Z}[t]/\mathbb{Z}} \to \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$ and element c in $\mathbb{Z}$, then c must be a unit (so either $+1$ or -1).

Using the above commutative square we can conclude that the analogous properties hold for the homology $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$.

(1) The morphism

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{d} \colon \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big)$$

is surjective.

(2) The morphism

$$\mathrm{d} \colon \mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \to \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big)$$

is only divisible by units, i. e. if $\mathrm{d} = c \cdot \mathrm{d}'$ for another morphism

$$\mathrm{d}' \colon \mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \to \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big)$$

and element c in $\mathbb{Z}$, then c must be a unit (so either $+1$ or -1).

[41] Induced by Ψ, but we do not actually care beyond them being isomorphisms.

We now use property (1) to show that $\Phi'_{\mathbb{Z},\{t\}}(\mathrm{d}\,\overline{t}) = \nu \cdot \mathrm{d}\,t$ for a nonzero element ν in $\mathbb{Z}$. For this let a_m for $0 \leq m \leq s$ be elements of $\mathbb{Z}$ such that

$$\Phi'_{\mathbb{Z},\{t\}}(\mathrm{d}\,\overline{t}) = \sum_{0 \leq m \leq s} a_m \cdot t^m \cdot \mathrm{d}\,t$$

holds in $\Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$. We already noted that the elements $[\overline{t}^n]$ for $n \geq 0$ form a $\mathbb{Z}$-basis for $\mathrm{H}_0(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\}))$. Combining this with (1) we obtain that the elements $[\overline{t}^n \cdot \mathrm{d}\,\overline{t}]$ for $n \geq 0$ form a $\mathbb{Q}$-generating set for $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{H}_1(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\}))$. As $\Phi'_{\mathbb{Z},\{t\}}$ is a multiplicative quasiisomorphism it follows that the elements

$$\Phi'_{\mathbb{Z},\{t\}}\big(\overline{t}^n \cdot \mathrm{d}\,\overline{t}\big) = t^n \cdot \left(\sum_{0 \leq m \leq s} a_m \cdot t^m \cdot \mathrm{d}\,t\right) = \sum_{0 \leq m \leq s} a_m \cdot t^{n+m} \cdot \mathrm{d}\,t$$

for $n \geq 0$ form a $\mathbb{Q}$-linear generating set for $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$. In particular, there must exist elements b_n of $\mathbb{Q}$ for $0 \leq n \leq u$, such that

$$\mathrm{d}\,t = \sum_{0 \leq n \leq u} b_n \cdot \left(\sum_{0 \leq m \leq s} a_m \cdot t^{n+m} \cdot \mathrm{d}\,t\right)$$

holds in $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$. Note that if all a_m are zero or all b_n are zero, then the right hand side vanishes, which contradicts the equality, so we can without loss of generality assume that $0 \leq u$ and $0 \leq s$ are such that $b_u \neq 0$ and $a_s \neq 0$. But then rewriting the right hand side in terms of the $\mathbb{Q}$-basis $t^l \cdot \mathrm{d}\,t$ for $l \geq 0$ of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$ we will have a nonzero coefficient $b_u \cdot a_s$ for the summand associated to $t^{u+s} \cdot \mathrm{d}\,t$. This can only happen if $u + s = 0$, so in particular $s = 0$ so that we must have

$$\Phi'_{\mathbb{Z},\{t\}}\big(\mathrm{d}\,\overline{t}\big) = a_0 \cdot \mathrm{d}\,t$$

in $\Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$ for a_0 a nonzero element of $\mathbb{Z}$.

Set $\nu = a_0$. Then we obtain the following calculation for $n \geq 0$.

$$\begin{aligned}\Phi'_{\mathbb{Z},\{t\}}\big(\mathrm{d}(\overline{t}^n)\big) &= \Phi'_{\mathbb{Z},\{t\}}\Big(n \cdot \overline{t}^{n-1} \cdot \mathrm{d}\,\overline{t}\Big) = n \cdot t^{n-1} \cdot (\nu \cdot \mathrm{d}\,t) \\ &= \nu \cdot \big(n \cdot t^{n-1} \cdot \mathrm{d}\,t\big) = \nu \cdot \mathrm{d}(t^n) = \nu \cdot \mathrm{d}\Big(\Phi'_{\mathbb{Z},\{t\}}\big(\overline{t}^n\big)\Big)\end{aligned}$$

We have thus shown that (7.11) holds for this choice of ν for the elements $y = \overline{t}^n$ for $n \geq 0$, but we still have to show that ν is an element of $\{+1, -1\}$. But note that as $[\overline{t}^n]$ for $n \geq 0$ is a $\mathbb{Z}$-basis for $\mathrm{H}_0(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\}))$, the calculation we just made implies that the composition

$$\mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \xrightarrow{\mathrm{d}} \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \xrightarrow{\mathrm{H}_1\left(\Phi'_{\mathbb{Z},\{t\}}\right)} \mathrm{H}_1\Big(\Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}\Big)$$

is ν times the composition $\mathrm{d}\circ\mathrm{H}_0(\Phi'_{\mathbb{Z},\{t\}})$, so the above composition is divisible by ν. As $\mathrm{H}_1(\Phi'_{\mathbb{Z},\{t\}})$ is an isomorphism this implies that also the morphism

$$\mathrm{d}\colon\ \mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \to \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big)$$

is divisible by ν. Finally, (2) implies that ν must then be either $+1$ or -1. □

Definition 7.4.4.2. Let X be a set. We define a quasiisomorphism

$$\Phi_{k,X}\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \to \Omega^\bullet_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$ by

$$y \mapsto \nu^{\deg_{\mathsf{Ch}}(y)} \cdot \Phi'_{k,X}(y)$$

where $\Phi'_{k,X}$ is as in Construction 7.4.3.1 and ν as in Proposition 7.4.4.1. If k is clear from context we will also denote $\Phi_{k,X}$ by Φ_x. ♢

Proposition 7.4.4.3. *The morphism*

$$\Phi_{k,\{t\}}\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(\{t\})\Big) \to \Omega^\bullet_{k[t]/k}$$

from Definition 7.4.4.2 is compatible with d *and can thus be lifted to a morphism in* $\mathrm{Alg}(\mathsf{Mixed})$. ♡

Proof. Follows directly from the definition and Proposition 7.4.4.1. □

7.4.5. A free resolution for de Rham forms

In this section we construct a cofibrant replacement of $\Omega^\bullet_{k[X]/k}$ in the model category $\mathrm{Alg}(\mathsf{Mixed})$ for totally ordered sets X with $|X| \leq 2$, and prove some properties it satisfies. We know abstractly that a cofibrant replacement exists, but it will be crucial for applications that we have good control over the low degrees of the the cofibrant replacement that we use.

We will begin in Section 7.4.5.1 by giving a construction of a cofibrant replacement[42] that depends on the choice of certain sets $Y_0, Y_1, \ldots$. For our application we will need to make a specific choice for Y_0, Y_1, and Y_2, and we will describe those choices and show that they have the necessary properties in Section 7.4.5.2. Finally Section 7.4.5.3 will be concerned with proving that the object constructed in Section 7.4.5.1 actually is a cofibrant replacement of $\Omega^\bullet_{k[X]/k}$.

[42]We will only construct the object and morphism to $\Omega^\bullet_{k[X]/k}$, but will not yet show that it indeed is a cofibrant replacement.

7.4.5.1. The general construction

In this section we give a general construction of a morphism

$$\Omega'^{\bullet}_{k[X]/k} \to \Omega^{\bullet}_{k[X]/k}$$

in Alg(Mixed) that depends on the choice of certain sets $Y_0, Y_1, \ldots$.

Construction 7.4.5.1. Let X be a set. We will construct a commutative diagram

$$\begin{tikzcd}
A_0 \arrow[r, "\iota_0^1"] \arrow[drr, "\Theta_0"'] & A_1 \arrow[r, "\iota_1^2"] \arrow[dr, "\Theta_1"] & A_2 \arrow[r] \arrow[d, "\Theta_2"] & \ldots \arrow[r] & \Omega'^{\bullet}_{k[X]/k} \arrow[dll, "\Theta"] \\
& & \Omega^{\bullet}_{k[X]/k} & &
\end{tikzcd}$$

in Alg(Mixed), where the first line is a $\mathbb{Z}_{\geq 0}$-diagram and its colimit $\Omega'^{\bullet}_{k[X]/k}$. Beyond the notation indicated in the diagram, we will denote the morphism from A_n to $\Omega^{\bullet}_{k[X]/k}$ by ι_n, and the morphism from A_n to A_m for $m \geq n$ by ι_n^m. The objects A_n are going to be built up using free associative algebras in strict mixed complexes that are generated by strict mixed complexes that are themselves free, so to simplify notation we will use Notation 7.4.2.1. All morphisms ι_n^m are going to be levelwise injective, so if y is an element of A_n, we will also just write y for the element $\iota_n^m(y)$ of A_m.

We begin by defining

$$A_0 := \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(k \cdot X)$$

where by $k \cdot X$ we mean the chain complex that is free as a graded k-module on the set X, where we give every element of X chain degree 0.

Using the universal property of $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ and $k \cdot X$, we can now define Θ_0 as the unique morphism in Alg(Mixed) that maps an element x of X, considered as a basis element of $k \cdot X$, to the element x, considered as an element of $k[X]$ and thereby of $\Omega^0_{k[X]/k}$.

We next describe how to construct A_{n+1} from A_n, for $n \geq 0$. This will depend on the choice of a subset Y_n of $(A_n)_n$, i. e. elements of degree n in A_n. We note that we will later show that we can make some particular choices for some of these sets. The set Y_n has to satisfy the following conditions for every $n \geq 0$.

(a) Every element y of Y_n is a cycle in A_n.

(b) Every element y of Y_n is mapped to 0 by Θ_n.

(c) Let I be the graded ideal[43] in the graded k-algebra $\mathrm{H}_*(A_n)$ that is generated by the homology classes represented by elements of the following subset.

$$Y_n \cup \{\, \mathrm{d}\, y \mid y \in Y_n \,\}$$

Then we must have $I_n = \mathrm{Ker}(\mathrm{H}_n(\Theta_n))$[44].

Note that it is always possible to find a set Y_n satisfying all three requirements above, by starting with a generating set of $\mathrm{Ker}(\mathrm{H}_n(\Theta_n))$[45], and then for each of those homology classes choosing a cycle representing it. Note that as the boundary operator of $\Omega^\bullet_{k[X]/k}$ is zero, a cycle representing a homology class in the kernel of $\mathrm{H}_*(\Theta)$ must already be mapped to 0 by Θ, so (b) is then satisfied, and (a) and (c) hold by construction.

The idea behind the above requirements is that we want to divide out $\mathrm{Ker}(\mathrm{H}_n(\Theta_n))$ from A_n, but want to do so in an efficient fashion that does not create excessive new elements in homology. In particular, the assumption that the elements of Y_n all have degree n is needed to ensure that the connectivity of Θ_n increases with n.

Now let B'_n be the chain complex $B'_n \coloneqq k \cdot Y_n$, where we give elements of Y_n the the same chain degree as in A_n. If y is an element of Y_n, then we will denote the corresponding basis element of B'_n by y as well. Let $\underline{B}'_n$ the chain complex whose underlying graded k-module is given by $(k \cdot Y_n) \oplus (k \cdot Y_n)[1]$, where if y is an element of Y_n we will denote the corresponding basis element from the first summand by y again and the corresponding shifted[46] basis element of the second summand by $\underline{y}$, and where the boundary operator is determined by $\partial(\underline{y}) = y$. There is an evident morphism of chain complexes $j_n \colon B'_n \to \underline{B}'_n$ that maps y to y.

We can now define A_{n+1} and ι_n^{n+1} as in the following pushout diagram in $\mathrm{Alg}(\mathsf{Mixed})$

$$\begin{tikzcd}
B_n \coloneqq \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(B'_n) \arrow[r, "\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(j_n)"] \arrow[d, "i_n"'] & \underline{B}_n \coloneqq \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(\underline{B}'_n) \arrow[d, "\underline{i}_n"] \\
A_n \arrow[r, "\iota_n^{n+1}"'] & A_{n+1}
\end{tikzcd} \tag{7.12}$$

where i_n is the morphism in $\mathrm{Alg}(\mathsf{Mixed})$ that extends the morphism of chain complexes $B'_n \to A_n$ given by mapping y considered as an element of B'_n to y considered as an element of A_n, for every element y of Y_n. The latter is a morphism of chain complexes due to (a).

[43] That is, a subset that is closed under k-linear combinations as well as multiplication with any element of $\mathrm{H}_*(A_n)$ on either side.

[44] Note that (b) already implies that $I \subseteq \mathrm{Ker}(\mathrm{H}_*(\Theta_n))$.

[45] For example the very inefficient choice of *all* elements of $\mathrm{Ker}(\mathrm{H}_n(\Theta_n))$ works.

[46] One degree higher, see Definition 4.1.1.2.

We can define a morphism $\underline{\Theta}_n : \underline{B}_n \to \Omega^\bullet_{k[X]/k}$ in Alg(Mixed) as the one adjoint to the morphism of chain complexes $0 : \underline{B}'_n \to \Omega^\bullet_{k[X]/k}$ that maps y and $\underline{y}$ to 0 for every y in Y_n. If y is an element of Y_n, then by (b), $\Theta_n(i_n(y)) = 0$, so that $\Theta_n \circ i_n = \underline{\Theta}_n \circ \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(j_n)$, and hence, by the universal property of the pushout diagram in Alg(Mixed) above, we obtain a morphism $\Theta_{n+1} : A_{n+1} \to \Omega^\bullet_{k[X]/k}$ such that $\Theta_{n+1} \circ \iota_n^{n+1} = \Theta_n$ and $\Theta_{n+1} \circ \underline{i}_n = \underline{\Theta}_n$.

Finally, $\Omega'^\bullet_{k[X]/k}$ is defined as the colimit of the $\mathbb{Z}_{\geq 0}$-diagram

$$A_0 \xrightarrow{\iota_0^1} A_1 \xrightarrow{\iota_1^2} A_2 \xrightarrow{\iota_2^3} \ldots$$

in Alg(Mixed), and $\Theta : \Omega'^\bullet_{k[X]/k} \to \Omega^\bullet_{k[X]/k}$ is defined using the universal property of the colimit such that $\Theta \circ \iota_n = \Theta_n$ for every $n \geq 0$. ♢

Remark 7.4.5.2. This remark concerns the situation of Construction 7.4.5.1. Let $n \geq 0$ be an integer. From Remark 7.4.2.2 it follows that the underlying graded k-algebra of A_{n+1} is given by the coproduct (in graded k-algebras) of A_n and the free graded k-algebra on elements $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_n$.

Inductively we can conclude that the underlying graded k-algebra of A_n is free on the elements x and $\mathrm{d}\,x$ for $x \in X$, and $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_m$ with $m < n$. As the forgetful functor from Alg(Mixed) to Alg $\mathsf{Ch}(k)$ preserves filtered colimits by Proposition 4.2.2.12 we can also conclude that the colimit $\Omega'^\bullet_{k[X]/k}$ has an underlying graded k-algebra that is free on the elements x and $\mathrm{d}\,x$ for $x \in X$ and $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_m$ for $m \geq 0$.

Note that elements y of Y_m being of degree m implies that $\underline{y}$ is then of degree $m+1$, which is always positive. The only multiplicative generators of degree 0 are thus those of the form x for $x \in X$, and A_m is concentrated in nonnegative degrees for every $m \geq 0$. The above also implies that the morphisms $\iota_n^{n'}$ are isomorphisms in degrees smaller to or equal to n. ♢

7.4.5.2. Specific choices for Y_0, Y_1, and Y_2

In this section we discuss specific choices that we make for Y_0, Y_1, and Y_2 in Construction 7.4.5.1. We begin with a general remark explaining the maneuvers that we will make in all the proofs.

Remark 7.4.5.3. This remark concerns the situation of Construction 7.4.5.1, and we will use notation from there. In the proofs of Propositions 7.4.5.6, 7.4.5.7 and 7.4.5.8 we will for some $n \geq 0$ have defined sets $Y_0, \ldots, Y_{n-1}$ as in Construction 7.4.5.1 and shown that they satisfy (a), (b) and (c), and defined a set Y_n of elements of degree n in A_n for which we already showed that (a) and (b) holds, but we still have to show that (c) holds, i. e. that $I_n = \mathrm{Ker}(\mathrm{H}_n(\Theta_n))$, for I the graded ideal in $\mathrm{H}_*(A_n)$ that is generated by the homology classes represented by elements of $Y_n \cup \{ \mathrm{d}\,y \mid y \in Y_n \}$. In this remark we explain the general approach to proving this, in order to avoid repetition. Before we continue let us define J as the graded ideal in the graded k-algebra of cycles of A_n[47] that is generated by the elements y and $\mathrm{d}\,y$ for $y \in Y_n$[48].

Property (b) implies that $I_n \subseteq \operatorname{Ker}(\mathrm{H}_n(\Theta_n))$, so to show equality it only remains to show that every element in $\operatorname{Ker}(\mathrm{H}_n(\Theta_n))$ lies in I_n. Note that the set of homology classes represented by elements of J is exactly I. As $\Omega^\bullet_{k[X]/k}$ has zero boundary operator it also follows that a cycle in A_n represents a homology class in $\operatorname{Ker}(\mathrm{H}_n(\Theta_n))$ if and only if Θ_n maps it to 0. These two facts together imply that it suffices to show that every cycle in A_n of degree n that lies in the kernel of Θ_n is given as a sum of an element in J and a boundary.

The strategy we will employ to prove this will be by reducing step by step to the case of such cycles lying in increasingly restrictive submodules, by eliminating basis elements, as we now make more precise.

By Remark 7.4.5.2 the underlying $\mathbb{Z}$-graded k-algebra of A_n is free on the generators x and $\mathrm{d}\,x$ for $x \in X$, and $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_{n'}$ with $n' < n$. Let $\mathcal{G}$ be the set of generators just described, as a $\mathbb{Z}$-graded subset of A_n, and $\mathcal{B}$ the set of all words of degree n in $\mathcal{G}$. Then $\mathcal{B}$ is a k-basis of the underlying $\mathbb{Z}$-graded k-module of A_n. We will use a sequence of subsets

$$\mathcal{B} = \mathcal{B}_0 \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \cdots \supset \mathcal{B}_l$$

up to some subset $\mathcal{B}_l$ of $\mathcal{B}$ for $l > 0$ an integer. Suppose that we can show that one of the following two holds for every $0 \leq i < l$.

(I) For every element w of $\mathcal{B}_i \setminus \mathcal{B}_{i+1}$ there is a boundary in A_n or an element of J that, written in the basis $\mathcal{B}$, only has non-zero coefficients corresponding to the basis elements in $\mathcal{B}_{i+1}$, except for the basis element w, for which the coefficient is a unit in k. This implies that every element z of A_n of degree n that lies in the k-submodule generated by $\mathcal{B}_i$ is a sum of an element of J, a boundary, and an element z' in the k-submodule generated by $\mathcal{B}_{i+1}$. Note that every element of J and every boundary is a cycle, so z is a cycle if and only if z' is. Furthermore every element of J and every boundary is in the kernel of Θ_n, so z is in the kernel of Θ_n if and only if z' is.

(II) Every cycle z in A_n of degree n that satisfies $\Theta_n(z) = 0$ and that lies in the k-submodule generated by $\mathcal{B}_i$ already lies in the k-submodule generated by $\mathcal{B}_{i+1}$.

In both cases this implies that if we can show that every cycle in A_n of degree n that lies in the kernel of Θ_n and also lies in the k-submodule generated by $\mathcal{B}_{i+1}$ is a sum of an element of J and a boundary, then the same statement follows for such cycles that lie in the k-submodule generated by $\mathcal{B}_i$. Inductively

[47] Note that the Leibniz rule for ∂ implies that 1 is a cycle and that products of cycles are again cycles, so cycles form a sub-k-algebra of A_n.

[48] By (a) these elements are cycles.

it then thus suffices to show that cycles in A_n that lie in the k-submodule generated by $\mathcal{B}_l$ and lie in the kernel of Θ_n are a sum of an element in J and a boundary. Usually $\mathcal{B}_l$ will be of such a form that we can already show that such a cycle must be zero, and we will explain how we usually show this further below.

In the propositions below we will not usually define $\mathcal{B}_i$ explicitly. Instead we will step by step describe the difference $\mathcal{B}_i \setminus \mathcal{B}_{i+1}$ and explain how to eliminate those basis elements using an element of J or boundary in A_n in the manner described above.

Let us make one remark about the elements of $\mathcal{B}$. It follows from Remark 7.4.5.2 $\mathcal{G}$ consists only of elements of nonnegative degree, with the only elements of degree 0 being the elements of X. A concrete implication of this that we will often use is that the number of factors that are not in X that can occur in a word in $\mathcal{G}$ of specified degree is bounded. For example words in $\mathcal{G}$ of degree 1 need to consists of precisely one factor of the form $\mathrm{d}\,x$ or $\underline{y}$ with y an element of Y_0, with the other factors all from X.

We will call products of elements of X, considered as elements of A_n, *words in* X. If we are given a total order on the set X then we say that a word in X is *ordered* if it is of the form $x_1^{i_1} \cdots x_a^{i_a}$ with $a \geq 0$ an integer, $i_1, \ldots, i_a \geq 1$ integers, and $x_1 < x_2 < \cdots < x_a$ elements of X. Similarly we will call products of elements of the form x and $\mathrm{d}\,x$ for $x \in X$ *words in* X *and* $\mathrm{d}\,X$, and call such a word *ordered* if it is of the form $x_1^{i_1} \cdots x_a^{i_a} \cdot \mathrm{d}\,x'_1 \cdots \mathrm{d}\,x'_b$ with $a, b \geq 0$ an integers, $i_1, \ldots, i_a \geq 1$ integers, and $x_1 < x_2 < \cdots < x_a$ and $x'_1 < \cdots < x'_b$ elements of X. We let $\mathcal{B}_X$ be the set of words in X and $\mathcal{B}_X^{\mathrm{ord}}$ the set of ordered words in X. Analogously, we let $\mathcal{B}_{X,\mathrm{d}\,X}$ be the ($\mathbb{Z}$-graded) set of words in X and $\mathrm{d}\,X$, and $\mathcal{B}_{X,\mathrm{d}\,X}^{\mathrm{ord}}$ the ($\mathbb{Z}$-graded) set of ordered words in X and $\mathrm{d}\,X$. We will often refer to the number of factors in a word w as its *length*, and denote it by $\mathrm{len}(w)$.

Now suppose that $\mathcal{B}_l$ is a subset of $\mathcal{B}_{X,\mathrm{d}\,X}^{\mathrm{ord}}$. Then the restriction of Θ_n to the sub-k-module with basis $\mathcal{B}_l$ is injective, so any element in the kernel of that restriction is already 0. The upshot is that if we can find

$$\mathcal{B} = \mathcal{B}_0 \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \cdots \supset \mathcal{B}_l$$

such that (I) or (II) holds for every $0 \leq i < l$ and such that $\mathcal{B}_l$ is a subset of $\mathcal{B}_{X,\mathrm{d}\,X}^{\mathrm{ord}}$, then this will complete the proof that $I_n = \mathrm{Ker}(\mathrm{H}_n(\Theta_n))$. $\diamondsuit$

Remark 7.4.5.4. Let X be a totally ordered set that is either $X = \emptyset$, $X = \{x_1\}$, or $X = \{x_1, x_2\}$ with $x_1 < x_2$. For reference we provide here a table with the multiplicative generators of A_0, A_1, A_2, A_3 with Y_0, Y_1, Y_2 as defined in Propositions 7.4.5.6, 7.4.5.7 and 7.4.5.8 below. The generators are given as for the case $X = \{x_1, x_2\}$, and to read off the case $X = \{x_1\}$ (the case $X = \emptyset$) one leaves out any element that involves x_2 (that involves x_1 or x_2) The first column contains the chain degree of the elements, the second lists their names, and the third column contains the first of A_0, A_1, A_2, A_3 that contains the element.

Deg.	Elements	In
0	x_1	A_0
0	x_2	A_0
1	$\mathrm{d}\,x_1$	A_0
1	$\mathrm{d}\,x_2$	A_0
1	$\underline{x_1x_2 - x_2x_1}$	A_1
2	$\mathrm{d}\,\underline{x_1x_2 - x_2x_1}$	A_1
2	$\underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}$	A_2
2	$\underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}$	A_2
2	$\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}$	A_2
2	$\underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}$	A_2
3	$\mathrm{d}\,\underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}$	A_2
3	$\mathrm{d}\,\underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}$	A_2
3	$\mathrm{d}\,\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}$	A_2
3	$\mathrm{d}\,\underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}$	A_2
3	$\underline{\mathrm{d}\,x_1 \cdot \mathrm{d}\,x_1}$	A_3
3	$\underline{\mathrm{d}\,x_2 \cdot \mathrm{d}\,x_2}$	A_3
3	$\underline{\mathrm{d}\,\underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} \ldots}$	
	$\underline{\ldots - \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}}$	A_3
3	$\underline{\mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_2 \ldots}$	A_3
	$\underline{\ldots - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \cdot x_1 \ldots}$	
	$\underline{\ldots + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} \cdot x_2}$	
3	$\underline{\mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_1 \ldots}$	A_3
	$\underline{\ldots - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} \cdot x_1 \ldots}$	
	$\underline{\ldots + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} \cdot x_2}$	
4	$\mathrm{d}\,\underline{\mathrm{d}\,x_1 \cdot \mathrm{d}\,x_1}$	A_3
4	$\mathrm{d}\,\underline{\mathrm{d}\,x_2 \cdot \mathrm{d}\,x_2}$	A_3
4	$\mathrm{d}\,\underline{\mathrm{d}\,\underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} \ldots}$	
	$\underline{\ldots - \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}}$	A_3
4	$\mathrm{d}\,\underline{\mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_2 \ldots}$	A_3
	$\underline{\ldots - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \cdot x_1 \ldots}$	
	$\underline{\ldots + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} \cdot x_2}$	
4	$\mathrm{d}\,\underline{\mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_1 \ldots}$	A_3
	$\underline{\ldots - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} \cdot x_1 \ldots}$	
	$\underline{\ldots + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} \cdot x_2}$	

This table is intended to be used to determine what the k-basis for A_n in a specific degree is. ♢

Before we actually define Y_0, Y_1, and Y_2, we first show a helper statement.

Proposition 7.4.5.5. *This proposition concerns Construction 7.4.5.1, and we use some notation from Remark 7.4.5.3.*

Let $X = \{x_1, x_2\}$. Then the elements

$$x_1^{a_1} x_2^{a_2} \cdot (x_1 x_2 - x_2 x_1) \cdot w$$

in A_0, with $a_1, a_2 \geq 0$ and $w \in \mathcal{B}_X$, are all pairwise distinct, and the set of all such elements is k-linearly independent. ♡

Proof. Suppose that $a_1, a_2, a_1', a_2' \geq 0$ and $w, w' \in \mathcal{B}_X$ such that

$$x_1^{a_1} x_2^{a_2} \cdot (x_1 x_2 - x_2 x_1) \cdot w = x_1^{a_1'} x_2^{a_2'} \cdot (x_1 x_2 - x_2 x_1) \cdot w'$$

Then as $\mathcal{B}_X$ is k-linearly independent and the left hand side has two summands in the basis $\mathcal{B}_X$ that both begin with $x_1^{a_1} x_2^{a_2}$, but where the next factor differs, the same must be true for the two summands of the right hands side, and vice versa. This implies $a_1' = a_1$ and $a_2' = a_2$, which in turn implies that $w' = w$.

Now suppose that

$$0 = \sum_{\substack{a_1, a_2 \geq 0, \\ w \in \mathcal{B}_X}} b_{a_1, a_2, w} \cdot x_1^{a_1} x_2^{a_2} \cdot (x_1 x_2 - x_2 x_1) \cdot w$$

with $b_{a_1,a_2,w}$ elements of k, all but finitely many zero. We have to show that all coefficients $b_{a_1,a_2,w}$ are already zero. If this is already the case, then we are done. So assume that there is a coefficient $b_{a_1,a_2,w}$ that is nonzero. Then let $\widetilde{a}_1 \geq 0$ and $\widetilde{a}_2 \geq 0$ and $\widetilde{w} \in \mathcal{B}_X$ be such that $b_{\widetilde{a}_1, \widetilde{a}_2, \widetilde{w}} \neq 0$ while first minimizing $\widetilde{a}_1$ and then (for that already fixed $\widetilde{a}_1$) maximizing $\widetilde{a}_2$.

Then it suffices to show that

$$x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot (x_1 x_2 - x_2 x_1) \cdot \widetilde{w}$$

is k-linearly independent of the k-submodule spanned by elements

$$x_1^{a_1} x_2^{a_2} \cdot (x_1 x_2 - x_2 x_1) \cdot w$$

for $a_1, a_2 \geq 0$ and $w \in \mathcal{B}_X$ such that $(a_1, a_2, w) \neq (\widetilde{a}_1, \widetilde{a}_2, \widetilde{w})$ and $a_1 \geq \widetilde{a}_1$, and $a_2 \leq \widetilde{a}_2$ if $a_1 = \widetilde{a}_1$.

So assume that

$$\begin{aligned} & c \cdot \left(x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot x_1 x_2 \cdot \widetilde{w} - x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot x_2 x_1 \cdot \widetilde{w} \right) \\ = & \sum_{\substack{a_1, a_2 \geq 0, \\ w \in \mathcal{B}_X, \\ a_1 \geq \widetilde{a}_1, \\ a_2 \leq \widetilde{a}_2 \text{ if } a_1 = \widetilde{a}_1, \\ (a_1, a_2, w) \neq (\widetilde{a}_1, \widetilde{a}_2, \widetilde{w})}} c_{a_1, a_2, w} \cdot (x_1^{a_1} x_2^{a_2} \cdot x_1 x_2 \cdot w - x_1^{a_1} x_2^{a_2} \cdot x_2 x_1 \cdot w) \end{aligned} \tag{$*$}$$

for c a nonzero element of k and $c_{a_1,a_2,w}$ elements of k, only finitely many of which are nonzero. We consider for which (a_1, a_2, w) as in the indexing set we can have that one of the following two equations holds.

$$x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot x_2 x_1 \cdot \widetilde{w} = x_1^{a_1} x_2^{a_2} \cdot x_1 x_2 \cdot w \qquad \text{or} \qquad x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot x_2 x_1 \cdot \widetilde{w} = x_1^{a_1} x_2^{a_2} \cdot x_2 x_1 \cdot w$$

We first note that the on left hand side of the equations the first $\widetilde{a}_1$ factors of x_1 are always followed by at least one factor of x_2. Thus it is not possible to have $a_1 > \widetilde{a}_1$. As by assumption $a_1 \geq \widetilde{a}_1$ we can thus conclude that $a_1 = \widetilde{a}_1$. Thus we must have $a_2 \leq \widetilde{a}_2$. The factor number $a_1 + a_2 + 1$ or $a_1 + a_2 + 2$ on the right hand side of the two equations is x_1. As factors $a_1 + 1$ up to $a_1 + \widetilde{a}_2 + 1$ on the left hand side are x_2 we must thus have $a_2 \geq \widetilde{a}_2$. As factor number $a_1 + \widetilde{a}_2 + 2$ on the left hand side is x_1 on the other hand we must have $a_2 \leq \widetilde{a}_2 + 1$. We are thus left with the two options $a_2 = \widetilde{a}_2$ and $a_2 = \widetilde{a}_2 + 1$. The former would imply that $w = \widetilde{w}$, which contradicts the assumption $(a_1, a_2, w) \neq (\widetilde{a}_1, \widetilde{a}_2, \widetilde{w})$. The latter contradicts the assumptions that $a_2 \leq \widetilde{a}_2$ if $a_1 = \widetilde{a}_1$. This shows that if we write both sides of equation $(*)$ in the basis $\mathcal{B}_X$, then the left hand side has a nonzero coefficient for the basis element $x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot x_2 x_1 \cdot \widetilde{w}$ while the right hand side always has coefficient zero. This contradicts equation $(*)$, which implies all coefficients $b_{a_1,a_2,w}$ must have been zero, thereby showing the k-linear independence claim in the statement. □

Proposition 7.4.5.6. *Let X be a totally ordered set. Then the subset Y_0 of $(A_0)_0$ in Construction 7.4.5.1 can be chosen as follows.*

$$Y_0 \coloneqq \{\, x \cdot x' - x' \cdot x \mid x, x' \in X \text{ such that } x < x' \,\} \qquad \heartsuit$$

Proof. Condition (a): That the elements are cycles is clear as A_0 has zero boundary operator.

Condition (b): holds as $\Omega^{\bullet}_{k[X]/k}$ is commutative.

Condition (c): We are going to use the strategy explained in Remark 7.4.5.3 and also use notation from there. Elements of $\mathcal{B}$ are precisely words in X, and we can use elements of J to iteratively reorder the factors until we are left only with ordered words in X. □

Proposition 7.4.5.7. *Let X be a subset of the totally ordered set $\{x_1 < x_2\}$. This proposition concerns Construction 7.4.5.1, and we let Y_0 be as in Proposition 7.4.5.6.*

Then the subset Y_1 of $(A_1)_1$ in Construction 7.4.5.1 can be chosen as follows.

$$Y_1 \coloneqq \{\, x \cdot \mathrm{d}\, x' - \mathrm{d}\, x' \cdot x \mid x, x' \in X \,\} \qquad \heartsuit$$

Proof. Condition (a): All elements of Y_1 lie in A_0, which has zero boundary operator.

Condition (b): Holds as $\Omega^{\bullet}_{k[X]/k}$ is commutative.

Condition (c): We are going to use the strategy explained in Remark 7.4.5.3 and also use notation from there. The elements of $\mathcal{B}$ are words of one of the following two types, with the second only occurring if $|X| = 2$.

(1) A word in $\mathcal{G}$ with precisely one factor $\mathrm{d}\, x$ with $x \in X$ and the remaining factors in X.

(2) A word in $\mathcal{G}$ with precisely one factor $\underline{x_1x_2 - x_2x_1}$ and the remaining factors in X.

We first consider elements of type (1). We can first use elements of J to move the factor $\mathrm{d}\,x$ to the very end of the product, so that we are left with elements of the form $w \cdot \mathrm{d}\,x$ with w a word in X. If $|X| < 2$ then w will already be ordered, and if $|X| = 2$ we can then use the boundary of elements of A_1 of the form

$$w' \cdot \underline{x_1x_2 - x_2x_1} \cdot w'' \cdot \mathrm{d}\,x$$

for w' and w'' words in X to reorder w, so that we can ultimately eliminate all basis elements of type (1) except those of the form $w \cdot \mathrm{d}\,x$ with w an ordered word in X and x an element of X.

We are thus left with basis elements of the following two types, with the second only occurring if $|X| = 2$.

(1') An element of $\mathcal{B}^{\mathrm{ord}}_{X,\mathrm{d}\,X}$.

(2') A word in $\mathcal{G}$ with precisely one factor $\underline{x_1x_2 - x_2x_1}$ and the remaining factors in X.

If $|X| < 2$ we are thus done per Remark 7.4.5.3. So now assume that $|X| = 2$.

Then let $w \cdot \underline{x_1x_2 - x_2x_1} \cdot w'$ be an element of $\mathcal{B}$ of type (2'), with w and w' words in X. Assume that w is not ordered. It is then possible to order w in a finite number of steps by swapping neighboring (nonequal) factors, and there also is a minimum number of such steps required, which in this case must be positive as we assumed that w is not already ordered. Then we can write w as $w = v \cdot x_2 \cdot x_1 \cdot v'$ such that v and v' are words in X, and such that the minimum number of swappings to order $v \cdot x_1 \cdot x_2 \cdot v'$ is smaller than the minimum number of swappings to order w. Consider the following boundary.

$$\begin{aligned}
&\partial\big(v \cdot \underline{x_1x_2 - x_2x_1} \cdot v' \cdot \underline{x_1x_2 - x_2x_1} \cdot w'\big) \\
&= v \cdot (x_1x_2 - x_2x_1) \cdot v' \cdot \underline{x_1x_2 - x_2x_1} \cdot w' \\
&\quad - v \cdot \underline{x_1x_2 - x_2x_1} \cdot v' \cdot (x_1x_2 - x_2x_1) \cdot w' \\
&= v \cdot x_1 \cdot x_2 \cdot v' \cdot \underline{x_1x_2 - x_2x_1} \cdot w' - v \cdot x_2 \cdot x_1 \cdot v' \cdot \underline{x_1x_2 - x_2x_1} \cdot w' \\
&\quad - v \cdot \underline{x_1x_2 - x_2x_1} \cdot v' \cdot x_1 \cdot x_2 \cdot w' + v \cdot \underline{x_1x_2 - x_2x_1} \cdot v' \cdot x_2 \cdot x_1 \cdot w'
\end{aligned}$$

Up to sign the second summand is the element we started with, the first has a word of the same length before $\underline{x_1x_2 - x_2x_1}$, but of smaller minimum number of swappings to order it, and the last two summands have a word of smaller length before the first factor $\underline{x_1x_2 - x_2x_1}$. By induction we can thus eliminate those elements from (2') where the word in X appearing before the factor $\underline{x_1x_2 - x_2x_1}$ is not ordered.

We are thus left with basis elements of the following two types.

(1") An element of $\mathcal{B}^{\mathrm{ord}}_{X,\mathrm{d}\,X}$.

(2”) A product $x_1^{a_1} \cdot x_2^{a_2} \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w$ where w is a word in X.

To finish the proof it remains to eliminate the basis elements from (2”). We do this using method (II) from Remark 7.4.5.3. So let z' be a cycle in A_1 that is a k-linear combination of elements of type (1”) and (2”). We have to show that z' is then already a k-linear combination of elements of type (1”). For this we write $z' = z'' + z$ with z'' a k-linear combination of elements of type (1”) and z a k-linear combination of elements of type (2”). As every element of type (1”) is a cycle this implies that z is a cycle. It now suffices to show that $z = 0$.

We can write z as

$$z = \sum_{\substack{a_1, a_2 \geq 0 \\ w \in \mathcal{B}_X}} b_{a_1,a_2,w} \cdot x_1^{a_1} x_2^{a_2} \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w$$

with $b_{a_1,a_2,w}$ elements of k, all but finitely many zero. The boundary of z is then given as follows.

$$\partial(z) = \sum_{\substack{a_1, a_2 \geq 0, \\ w \in \mathcal{B}_X}} b_{a_1,a_2,w} \cdot x_1^{a_1} x_2^{a_2} \cdot x_1 x_2 \cdot w - b_{a_1,a_2,w} \cdot x_1^{a_1} x_2^{a_2} \cdot x_2 x_1 \cdot w$$

Now Proposition 7.4.5.5 directly implies that all coefficients $b_{a_1,a_2,w}$ must be zero, so $z = 0$.

We are thus now left with only basis elements of type (1”), which finishes the proof as explained in Remark 7.4.5.3. □

Proposition 7.4.5.8. *Let X be a subset of the totally ordered set $\{x_1 < x_2\}$. This proposition concerns Construction 7.4.5.1, and we let Y_0 be as in Proposition 7.4.5.6 and Y_1 as in Proposition 7.4.5.7.*

Then the subset Y_2 of $(A_2)_2$ in Construction 7.4.5.1 can be chosen as follows. If $|X| = 0$ we can let $Y_2 = \emptyset$, if $|X| = 1$ we can let $Y_2 = \{\mathrm{d}\, x_1 \cdot \mathrm{d}\, x_1\}$, and if $|x| = 2$ we can define Y_2 as follows.

$$\begin{aligned}
Y_2 \coloneqq &\left\{ \mathrm{d}\, x_1 \cdot \mathrm{d}\, x_1, \mathrm{d}\, x_2 \cdot \mathrm{d}\, x_2 \right\} \\
&\cup \left\{ \mathrm{d}\, \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} \right\} \\
&\cup \Big\{ \mathrm{d}\, x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\, x_2 \\
&\qquad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2} \cdot x_1 \\
&\qquad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} \cdot x_2 \Big\} \\
&\cup \Big\{ \mathrm{d}\, x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\, x_1 \\
&\qquad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} \cdot x_1 \\
&\qquad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} \cdot x_2 \Big\}
\end{aligned} \qquad \heartsuit$$

Proof. To keep the proof shorter as it would otherwise be we mostly will implicitly work as if we had $|X| = 2$; the proof for $|X| < 2$ can be obtained by jumping over every element or argument that involves an element of $\{x_1, x_2\} \setminus X$. To shorten notation we also make the following definitions for this proof.

$$\begin{aligned}
D &\coloneqq \mathrm{d}\,\underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} \\
C_2 &\coloneqq \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_2 \\
&\quad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \cdot x_1 \\
&\quad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} \cdot x_2 \\
C_3 &\coloneqq \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_1 \\
&\quad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} \cdot x_1 \\
&\quad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} \cdot x_2
\end{aligned}$$

Condition (a): That the elements of Y_2 are cycles can be checked by direct calculation. For this, keep in mind the signs introduced by the Leibniz rule and $\partial \circ \mathrm{d} = -\mathrm{d} \circ \partial$.

Condition (b): Θ_2 maps $\mathrm{d}\,x_1 \cdot \mathrm{d}\,x_1$ and $\mathrm{d}\,x_2 \cdot \mathrm{d}\,x_2$ (if they are defined, depending on what X is) to zero as $\mathrm{d}\,x_1$ and $\mathrm{d}\,x_2$ square to zero in $\Omega^{\bullet}_{k[X]/k}$. The elements D, C_2, and C_3 are mapped to zero because every summand has a factor of the form y or $\mathrm{d}\,y$, with y an element of Y_0 or Y_1, and those elements are already mapped to zero.

Condition (c): We are going to use the strategy explained in Remark 7.4.5.3 and also use notation from there. The elements of $\mathcal{B}$ are words of one of the following types, with types (3), (4) and (5) only occurring for $|X| = 2$.

(1) A word in $\mathcal{G}$ with precisely two factors $\mathrm{d}\,x$ and $\mathrm{d}\,x'$ with $x, x' \in X$ (the case $x = x'$ is allowed) and the remaining factors in X.

(2) A word in $\mathcal{G}$ with precisely one factor $\underline{x \cdot x' - x' \cdot x}$ with $x, x' \in X$, and the remaining factors in X.

(3) A word in $\mathcal{G}$ with precisely one factor $\underline{x_1 x_2 - x_2 x_1}$, precisely one factor $\mathrm{d}\,x$ for $x \in X$, and the remaining factors in X.

(4) A word in $\mathcal{G}$ with precisely two factors $\underline{x_1 x_2 - x_2 x_1}$, and the remaining factors in X.

(5) A word in $\mathcal{G}$ with precisely one factor $\mathrm{d}\,\underline{x_1 x_2 - x_2 x_1}$, and the remaining factors in X.

As a first step the basis elements of type (5) can be eliminated using elements of J that involve a factor of D, so that we are only left with types (1), (2), (3) and (4).

For elements of type (4) we use a similar procedure as we did for elements of type (2') in Proposition 7.4.5.7. So let $w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w''$ be an element of type (4), with w, w', and w'' elements of $\mathcal{B}_X$. Assume that w' is not ordered. Then we can write w' as $w' = v \cdot x_2 \cdot x_1 \cdot v'$ such that v and v' are elements of $\mathcal{B}_X$ and such that the minimum number of swappings to order $v \cdot x_1 \cdot x_2 \cdot v'$ is smaller than the minimum number of swappings to order w'. Consider the following boundary.

$$\begin{aligned}
&\partial\big(w \cdot \underline{x_1x_2 - x_2x_1} \cdot v \cdot \underline{x_1x_2 - x_2x_1} \cdot v' \cdot \underline{x_1x_2 - x_2x_1} \cdot w''\big)\\
= &+ w \cdot x_1x_2 \cdot v \cdot \underline{x_1x_2 - x_2x_1} \cdot v' \cdot \underline{x_1x_2 - x_2x_1} \cdot w''\\
&- w \cdot x_2x_1 \cdot v \cdot \underline{x_1x_2 - x_2x_1} \cdot v' \cdot \underline{x_1x_2 - x_2x_1} \cdot w''\\
&- w \cdot \underline{x_1x_2 - x_2x_1} \cdot v \cdot x_1x_2 \cdot v' \cdot \underline{x_1x_2 - x_2x_1} \cdot w''\\
&+ w \cdot \underline{x_1x_2 - x_2x_1} \cdot v \cdot x_2x_1 \cdot v' \cdot \underline{x_1x_2 - x_2x_1} \cdot w''\\
&+ w \cdot \underline{x_1x_2 - x_2x_1} \cdot v \cdot \underline{x_1x_2 - x_2x_1} \cdot v' \cdot x_1x_2 \cdot w''\\
&- w \cdot \underline{x_1x_2 - x_2x_1} \cdot v \cdot \underline{x_1x_2 - x_2x_1} \cdot v' \cdot x_2x_1 \cdot w''
\end{aligned}$$

Up to sign the fourth summand is the element we started with, the third has a word in X between the two factors $\underline{x_1x_2 - x_2x_1}$ of same length as w' but with smaller minimum number of swappings to order it, and the remaining four summands have a word in X of smaller length between the two factors of $\underline{x_1x_2 - x_2x_1}$. By induction we can thus eliminate elements of type (4) where the word in X between the two factors of $\underline{x_1x_2 - x_2x_1}$ are not ordered.

We are thus left with the following types of basis elements.

(1') A word in $\mathcal{G}$ with precisely two factors $\mathrm{d}\,x$ and $\mathrm{d}\,x'$ with $x, x' \in X$ (the case $x = x'$ is allowed) and the remaining factors in X.

(2') A word in $\mathcal{G}$ with precisely one factor $\underline{x \cdot x' - x' \cdot x}$ with $x, x' \in X$, and the remaining factors in X.

(3') A word in $\mathcal{G}$ with precisely one factor $\underline{x_1x_2 - x_2x_1}$, precisely one factor $\mathrm{d}\,x$ for $x \in X$, and the remaining factors in X.

(4') $w \cdot \underline{x_1x_2 - x_2x_1} \cdot x_1^{a_1}x_2^{a_2} \cdot \underline{x_1x_2 - x_2x_1} \cdot w'$ with $w, w' \in \mathcal{B}_X$.

We next show that we can also eliminate the remaining elements of type (4') using method (II) from Remark 7.4.5.3. For this we first note that words in $\mathcal{G}$ that can occur[49] in the boundaries of elements of type (1'), (2'), (3') and (4') never have a factor $\underline{x_1x_2 - x_2x_1}$, but the boundary of elements of type (4') lies in the k-submodule spanned by words in $\mathcal{G}$ that have a factor $\underline{x_1x_2 - x_2x_1}$. To eliminate (4') it thus suffices to show that if z is a k-linear combination of elements of type (4'), with $\partial(z) = 0$, then already $z = 0$.

So let z be given by

$$z = \sum_{\substack{w,w' \in \mathcal{B}_X\\ a_1,a_2 \geq 0}} b_{w,a_1,a_2,w'} \cdot w \cdot \underline{x_1x_2 - x_2x_1} \cdot x_1^{a_1}x_2^{a_2} \cdot \underline{x_1x_2 - x_2x_1} \cdot w'$$

with $b_{w,a_1,a_2,w'}$ elements of k, only finitely many of which are nonzero, and assume that $\partial(z) = 0$. If all coefficients $b_{w,a_1,a_2,w'}$ are zero, then we already

[49] By this we mean that writing the respective element in terms of the k-basis given by words in $\mathcal{G}$ the coefficient associated to that word is nonzero.

have $z = 0$ and are done, so assume that this is not the case. Then we can let $\widetilde{w} \in \mathcal{B}$ be such that there exist $a_1, a_2 \geq 0$ and $w' \in \mathcal{B}$ such that $b_{\widetilde{w},a_1,a_2,w'} \neq 0$ while minimizing $\operatorname{len}(\widetilde{w})$ with this property. The boundary $\partial(z)$ has the following form.

$$\begin{aligned}
0 &= \partial(z)\\
&= \sum_{\substack{w,w' \in \mathcal{B}_X\\ a_1,a_2 \geq 0\\ \operatorname{len}(w) \geq \operatorname{len}(\widetilde{w})}} b_{w,a_1,a_2,w'} \cdot w \cdot (x_1x_2 - x_2x_1) \cdot x_1^{a_1} x_2^{a_2} \cdot \underline{x_1x_2 - x_2x_1} \cdot w'\\
&\quad - \sum_{\substack{w,w' \in \mathcal{B}_X\\ a_1,a_2 \geq 0\\ \operatorname{len}(w) \geq \operatorname{len}(\widetilde{w})}} b_{w,a_1,a_2,w'} \cdot w \cdot \underline{x_1x_2 - x_2x_1} \cdot x_1^{a_1} x_2^{a_2} \cdot (x_1x_2 - x_2x_1) \cdot w'
\end{aligned}$$

We now apply a k-linear morphism p to this equation. p is to be a morphism from $(A_2)_1$ to the k-submodule of $(A_2)_1$ that is spanned by words in $\mathcal{G}$ of degree 1 that begin with $\widetilde{w} \cdot \underline{x_1x_2 - x_2x_1}$. We define p on the basis given by words in $\mathcal{G}$ of degree 1 by mapping words that begin with $\widetilde{w} \cdot \underline{x_1x_2 - x_2x_1}$ to themselves, and all others to 0. Then the requirement $\operatorname{len}(w) \geq \operatorname{len}(\widetilde{w})$ implies that the summands

$$b_{w,a_1,a_2,w'} \cdot w \cdot (x_1x_2 - x_2x_1) \cdot x_1^{a_1} x_2^{a_2} \cdot \underline{x_1x_2 - x_2x_1} \cdot w'$$

of the equation above are all mapped to 0 by p, and the summands

$$b_{w,a_1,a_2,w'} \cdot w \cdot \underline{x_1x_2 - x_2x_1} \cdot x_1^{a_1} x_2^{a_2} \cdot (x_1x_2 - x_2x_1) \cdot w'$$

map to 0 unless $w = \widetilde{w}$. The upshot is that we obtain the following equality[50].

$$0 = \sum_{\substack{w' \in \mathcal{B}_X\\ a_1,a_2 \geq 0}} b_{\widetilde{w},a_1,a_2,w'} \cdot \widetilde{w} \cdot \underline{x_1x_2 - x_2x_1} \cdot x_1^{a_1} x_2^{a_2} \cdot (x_1x_2 - x_2x_1) \cdot w'$$

This implies that we must also have

$$0 = \sum_{\substack{w' \in \mathcal{B}_X\\ a_1,a_2 \geq 0}} b_{\widetilde{w},a_1,a_2,w'} \cdot x_1^{a_1} x_2^{a_2} \cdot (x_1x_2 - x_2x_1) \cdot w'$$

which by Proposition 7.4.5.5 implies that $b_{\widetilde{w},a_1,a_2,w'} = 0$ for all $a_1, a_2 \geq 0$ and $w' \in \mathcal{B}_X$. This however contradicts the assumption on $\widetilde{w}$, implying that z must have been zero after all.

Thus we can eliminate elements of type (4') and are left with basis elements of the following types.

(1') A word in $\mathcal{G}$ with precisely two factors $\mathrm{d}\,x$ and $\mathrm{d}\,x'$ with $x, x' \in X$ (the case $x = x'$ is allowed) and the remaining factors in X.

[50] We also multiplied with -1.

(2') A word in $\mathcal{G}$ with precisely one factor $\underline{x \cdot x' - x' \cdot x}$ with $x, x' \in X$, and the remaining factors in X.

(3') A word in $\mathcal{G}$ with precisely one factor $\underline{x_1 x_2 - x_2 x_1}$, precisely one factor $\mathrm{d}\, x$ for $x \in X$, and the remaining factors in X.

We now consider the basis elements of type (3'). We claim that we can eliminate those elements of type (3') that do not begin with the factor $\mathrm{d}\, x$. We can show this by induction on the number of factors before the factor $\mathrm{d}\, x$. There are two main cases, depending on what the preceding factor is. We first discuss the case in which the preceding factor is an element of X, say x'. Then we can write the element as either $w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \cdot x' \cdot \mathrm{d}\, x \cdot w''$ or $w \cdot x' \cdot \mathrm{d}\, x \cdot w' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w''$ with $w, w', w'' \in \mathcal{B}_X$. We only discuss the first form, the second is completely analogous. Then consider the following boundary.

$$\begin{aligned} &\partial\big(w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \cdot \underline{x' \cdot \mathrm{d}\, x - \mathrm{d}\, x \cdot x'} \cdot w''\big) \\ = {}& w \cdot x_1 x_2 \cdot w' \cdot \underline{x' \cdot \mathrm{d}\, x - \mathrm{d}\, x \cdot x'} \cdot w'' \\ &- w \cdot x_2 x_1 \cdot w' \cdot \underline{x' \cdot \mathrm{d}\, x - \mathrm{d}\, x \cdot x'} \cdot w'' \\ &- w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \cdot x' \cdot \mathrm{d}\, x \cdot w'' \\ &+ w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \cdot \mathrm{d}\, x \cdot x' \cdot w'' \end{aligned}$$

Up to sign the third summand is the element we started with, the fourth is of type (3'), but with a smaller number of factors preceding $\mathrm{d}\, x$, and the other two are of type (2').

The other case to consider is when the factor preceding $\mathrm{d}\, x$ is $\underline{x_1 x_2 - x_2 x_1}$, so that the element is of the form $w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot \mathrm{d}\, x \cdot w'$ for $w, w' \in \mathcal{B}_X$. We assume that $x = x_1$, the case $x = x_2$ is completely analogous by using C_2 instead of C_3. Then consider the following element in J.

$$\begin{aligned} &w \cdot C_3 \cdot w' \\ = {}& w \cdot \mathrm{d}\, x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot w' + w \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\, x_1 \cdot w' \\ &- w \cdot x_1 \cdot \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} \cdot w' + w \cdot \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} \cdot x_1 \cdot w' \\ &+ w \cdot x_2 \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} \cdot w' - w \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} \cdot x_2 \cdot w' \end{aligned}$$

Up to sign the second summand is the element we started with, the first is of the (3'), but with a smaller number of factors preceding $\mathrm{d}\, x_1$, and the other four are of type (2').

We have now eliminated all elements of type (3') except those that start with $\mathrm{d}\, x$ as their first factor. Proceeding completely analogously to how we did with elements of type (2') in Proposition 7.4.5.7 we can now also eliminate those in which the word in X between $\mathrm{d}\, x$ and the factor $\underline{x_1 x_2 - x_2 x_1}$ is not ordered. We are thus left with the following basis elements.

(1") A word in $\mathcal{G}$ with precisely two factors $\mathrm{d}\, x$ and $\mathrm{d}\, x'$ with $x, x' \in X$ (the case $x = x'$ is allowed) and the remaining factors in X.

(2”) A word in $\mathcal{G}$ with precisely one factor $\underline{x \cdot x' - x' \cdot x}$ with $x, x' \in X$, and the remaining factors in X.

(3”) $\mathrm{d}\, x \cdot x_1^{a_1} x_2^{a_2} \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w$ for $x \in X$, $a_1, a_2 \geq 0$, and $w \in \mathcal{B}_X$.

We now eliminate type (2”) using method (II) from Remark 7.4.5.3. So assume that $z'' = z''' + z + z'$ is a cycle where z''' is a k-linear combination of basis elements of type (1”), z is a k-linear combination of basis elements of type (2”) and z' is a k-linear combination of basis elements of type (3”). We have to show that then $z = 0$. We first note that as every element of type (1”) is already a cycle we obtain that $z + z'$ is a cycle. We write

$$z = \sum_{\substack{w, w' \in \mathcal{B}_X, \\ x, x' \in X}} b_{w,x,x',w'} \cdot w \cdot \underline{x \cdot \mathrm{d}\, x' - \mathrm{d}\, x' \cdot x} \cdot w'$$

$$z' = \sum_{\substack{x \in X, \\ a_1, a_2 \geq 0, \\ w \in \mathcal{B}_X}} c_{x,a_1,a_2,w} \cdot \mathrm{d}\, x \cdot x_1^{a_1} x_2^{a_2} \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w$$

with $b_{w,x,x',w'}$ and $c_{x,a_1,a_2,w}$ elements of k, only finitely many of which are nonzero. If all coefficients $b_{w,x,x',w'}$ are zero, then we have $z = 0$ and are done. So assume that this is not the case. Then let $\widetilde{w}, \widetilde{w}' \in \mathcal{B}_X$ and $\widetilde{x}, \widetilde{x}' \in X$ be such that $b_{\widetilde{w},\widetilde{x},\widetilde{x}',\widetilde{w}'} \neq 0$ and choose $\widetilde{w}$ to be of maximum length with this property. From $\partial(z + z') = 0$ we then obtain the following equality.

$$\begin{aligned} & \sum_{\substack{w, w' \in \mathcal{B}_X, \\ x, x' \in X, \\ \mathrm{len}(w) \leq \mathrm{len}(\widetilde{w})}} b_{w,x,x',w'} \cdot w \cdot x \cdot \mathrm{d}\, x' \cdot w' \\ & \quad - \sum_{\substack{w, w' \in \mathcal{B}_X, \\ x, x' \in X, \\ \mathrm{len}(w) \leq \mathrm{len}(\widetilde{w})}} b_{w,x,x',w'} \cdot w \cdot \mathrm{d}\, x' \cdot x \cdot w' \\ = & \sum_{\substack{x \in X, \\ a_1, a_2 \geq 0, \\ w \in \mathcal{B}_X}} c_{x,a_1,a_2,w} \cdot \mathrm{d}\, x \cdot x_1^{a_1} x_2^{a_2} \cdot x_1 x_2 \cdot w \\ & \quad - \sum_{\substack{x \in X, \\ a_1, a_2 \geq 0, \\ w \in \mathcal{B}_X}} c_{x,a_1,a_2,w} \cdot \mathrm{d}\, x \cdot x_1^{a_1} x_2^{a_2} \cdot x_2 x_1 \cdot w \end{aligned}$$

We now apply a k-linear morphism p' to this equation. p' is to be a morphism from $(A_2)_1$ to the k-submodule of $(A_2)_1$ that is spanned by the word $\widetilde{w} \cdot \widetilde{x} \cdot \mathrm{d}\, \widetilde{x}' \cdot \widetilde{w}'$ in $\mathcal{G}$ of degree 1. We define p on the basis given by words in $\mathcal{G}$ of degree 1 by mapping the just mentioned word to itself and all others to 0. Then note that all words are mapped to zero where the length of the word preceding a factor of the form $\mathrm{d}\, x$ is unequal to $\mathrm{len}(\widetilde{w}) + 1$. The condition

$\operatorname{len}(w) \leq \operatorname{len}(\widetilde{w})$ on the left hand side of the above equation then implies that the second sum on the left hand side is mapped to zero. As all words in $\mathcal{G}$ occurring on the right hand side begin with an element of the form $\mathrm{d}\,x$ they are also all mapped to zero. We thus obtain that

$$b_{\widetilde{w},\widetilde{x},\widetilde{x}',\widetilde{w}'} \cdot \widetilde{w} \cdot \widetilde{x} \cdot \mathrm{d}\,\widetilde{x}' \cdot \widetilde{w}' = 0$$

which contradicts the assumption that $b_{\widetilde{w},\widetilde{x},\widetilde{x}',\widetilde{w}'} \neq 0$. Thus we must have $z = 0$ and can thus eliminate basis elements of type (2”).

We are thus left with the following basis elements.

(1”) A word in $\mathcal{G}$ with precisely two factors $\mathrm{d}\,x$ and $\mathrm{d}\,x'$ with $x, x' \in X$ (the case $x = x'$ is allowed) and the remaining factors in X.

(3”) $\mathrm{d}\,x \cdot x_1^{a_1} x_2^{a_2} \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w$ for $x \in X$, $a_1, a_2 \geq 0$, and $w \in \mathcal{B}_X$.

We can now eliminate type (3”) in a manner that is completely analogous to the argument as we used to eliminate (2”) in Proposition 7.4.5.7. We are thus left with only type (1”). For this we can first use boundaries of words in $\mathcal{G}$ involving two factors $\mathrm{d}\,x$ and $\mathrm{d}\,x'$ as well as a factor $\underline{x_1 x_2 - x_2 x_1}$ with the other factors in X, as well as boundaries of words in $\mathcal{G}$ involving one factor $\mathrm{d}\,x''$ and one factor $\underline{x \cdot \mathrm{d}\,x' - \mathrm{d}\,x' \cdot x}$ with the remaining factors in X, to reorder the factors so that we are left with only elements of the form $x_1^{a_1} x_2^{a_2} \cdot \mathrm{d}\,x \cdot \mathrm{d}\,x'$ with $a_1, a_2 \geq 0$ and $x, x' \in X$. As a second step we can eliminate such elements with $x = x'$ by using elements of J that involve a factor of $\mathrm{d}\,x \cdot \mathrm{d}\,x$.

We are thus left with elements of the following two types.

(1^*) $x_1^{a_1} x_2^{a_2} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2$ for $a_1, a_2 \geq 0$.

(2^*) $x_1^{a_1} x_2^{a_2} \cdot \mathrm{d}\,x_2 \cdot \mathrm{d}\,x_1$ for $a_1, a_2 \geq 0$.

We can eliminate type (2^*) using the following boundary.

$$\partial\big(x_1^{a_1} x_2^{a_2} \cdot \mathrm{d}\,\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}\big) = -x_1^{a_1} x_2^{a_2} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2 - x_1^{a_1} x_2^{a_2} \cdot \mathrm{d}\,x_2 \cdot \mathrm{d}\,x_1$$

We are thus left with only basis elements from (1^*), which form a subset of $\mathcal{B}^{\mathrm{ord}}_{X,\mathrm{d}\,X}$, so we are done. □

Definition 7.4.5.9. Let X be a totally ordered set with $|X| \leq 2$. Then we define

$$\Theta_X \colon \Omega'^{\bullet}_{k[X]/k} \to \Omega^{\bullet}_{k[X]/k}$$

to be the morphism in $\mathrm{Alg}(\mathsf{Mixed})$ constructed in Construction 7.4.5.1 where we let Y_0 be as defined in Proposition 7.4.5.6, Y_1 as defined in Proposition 7.4.5.7, Y_2 as defined in Proposition 7.4.5.8, and where for $n > 2$ we just choose some subset Y_n of $(A_n)_n$ that satisfies (a), (b) and (c) of Construction 7.4.5.1 (we argued in Construction 7.4.5.1 that it is always possible to find Y_n satisfying this). ♢

7.4.5.3. Proof that the construction is a cofibrant resolution

In this section we show that Θ_X as defined in Definition 7.4.5.9 really is a cofibrant replacement of $\Omega^\bullet_{k[X]/k}$.

Proposition 7.4.5.10. *This proposition concerns Construction 7.4.5.1. Let X be a set and $n \geq 0$ an integer. Then*

$$\mathrm{H}_m(\Theta_n)\colon\ \mathrm{H}_m(A_n) \to \mathrm{H}_m\left(\Omega^\bullet_{k[X]/k}\right)$$

is an isomorphism for $m < n$ and surjective for every m. ♡

Proof. $\Omega^\bullet_{k[X]/k}$ is generated as a graded k-algebra by the elements x and $\mathrm{d}\,x$ for $x \in X$, so as every element of X is in the image of the morphism Θ_0 in Alg(Mixed), it follows that Θ_0 is surjective. As both A_0 and $\Omega^\bullet_{k[X]/k}$ have zero boundary operator, this implies that $\mathrm{H}_*(\Theta_0)$ and hence also $\mathrm{H}_*(\Theta_n)$ is surjective as well.

Now we show that $\mathrm{H}_m(\Theta_n)$ is even an isomorphism if $m < n$. We prove this by induction. The case $n = 0$ is clear, as both A_0 and $\Omega^\bullet_{k[X]/k}$ are concentrated in nonnegative degrees, so in particular have homology concentrated in nonnegative degrees.

So now assume that $n > 0$ and we already showed that $\mathrm{H}_m(\Theta_{n-1})$ is an isomorphism for $m < n-1$. By Remark 7.4.5.2 $\iota^n_{n-1}\colon A_{n-1} \to A_n$ is an isomorphism in degrees smaller than or equal to $n-1$. This implies that in the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_m(A_{n-1}) \arrow[rr, "\mathrm{H}_m\left(\iota^n_{n-1}\right)"] \arrow[dr, "\mathrm{H}_m(\Theta_{n-1})"'] & & \mathrm{H}_m(A_n) \arrow[dl, "\mathrm{H}_m(\Theta_n)"] \\
& \mathrm{H}_m\left(\Omega^\bullet_{k[X]/k}\right) &
\end{tikzcd}$$

the top morphism is an isomorphism for $m \leq n-2$, and as the left morphism is an isomorphism in that range as well, it already follows that $\mathrm{H}_m(\Theta_n)$ is an isomorphism for $m \leq n-2$. For $m = n-1$ we still obtain that $\mathrm{H}_{n-1}(\iota^n_{n-1})$ must be surjective[51]. In order to show that $\mathrm{Ker}(\mathrm{H}_{n-1}(\Theta_n)) \cong 0$ it thus suffices to show that $\mathrm{H}_{n-1}\left(\iota^n_{n-1}\right)$ maps $\mathrm{Ker}(\mathrm{H}_{n-1}(\Theta_{n-1}))$ to zero. But is precisely what condition (c) ensures. □

[51] Given an element of $\mathrm{H}_{n-1}(A_n)$ we can represent it by a cycle of degree $n-1$ As ι^n_{n-1} is an isomorphism in degree $n-1$, there is an element z in A_{n-1} that is mapped to that cycle by ι^n_{n-1}. It thus remains to show that z is also a cycle and hence represents a homology class. But

$$\iota^n_{n-1}(\partial z) = \partial\left(\iota^n_{n-1}(z)\right) = 0$$

which implies $\partial z = 0$, as ι^n_{n-1} is also an isomorphism in degree l.

Proposition 7.4.5.11. *Let X be a totally ordered set with $|X| \leq 2$. This proposition concerns Θ_X as defined in Definition 7.4.5.9.*

The object

$$\Omega'^{\bullet}_{k[X]/k}$$

of $\mathrm{Alg}(\mathsf{Mixed})$ *is cofibrant, and the morphism*

$$\Theta_X \colon \Omega'^{\bullet}_{k[X]/k} \to \Omega^{\bullet}_{k[X]/k}$$

is a quasiisomorphism. ♡

Proof. $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ is a left Quillen functor by Definition 4.2.2.2, Proposition 4.2.2.9, and Theorem 4.2.2.1. As $k \cdot X$ is a cofibrant chain complex, this implies that A_0 is cofibrant in $\mathrm{Alg}(\mathsf{Mixed})$. Furthermore, for every $n \geq 0$, the morphism j_n is a cofibration in $\mathsf{Ch}(k)$ (it is a coproduct of generating cofibrations considered in [Hov99, 2.3.3 and 2.3.11]), so $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(j_n)$ and thus also ι_n^{n+1} are cofibrations in $\mathrm{Alg}(\mathsf{Mixed})$. As cofibrations are closed under (transfinite) compositions, this implies that $\Omega'^{\bullet}_{k[X]/k}$ is cofibrant.

We now turn to showing that Θ_X is a quasiisomorphism. Remark 7.4.5.2 implies that $\iota_n^{n'} \colon A_n \to A_{n'}$ is an isomorphism in degrees smaller to or equal to n for all $0 \leq n < n'$. Combining this with the fact that the forgetful functor from $\mathrm{Alg}(\mathsf{Mixed})$ to $\mathsf{Ch}(k)$ preserves filtered colimits by Proposition 4.2.2.12 we obtain that $\iota_n \colon A_n \to \Omega'^{\bullet}_{k[X]/k}$ is an isomorphism in degrees smaller to or equal to n as well. In particular, in the diagram

$$\begin{tikzcd}
\mathrm{H}_m(A_n) \arrow[rr, "\mathrm{H}_m(\iota_n)"] \arrow[dr, "\mathrm{H}_m(\Theta_n)"'] & & \mathrm{H}_m\left(\Omega'^{\bullet}_{k[X]/k}\right) \arrow[dl, "\mathrm{H}_m(\Theta_X)"] \\
& \mathrm{H}_m\left(\Omega^{\bullet}_{k[t]/k}\right) &
\end{tikzcd}$$

the top morphism is an isomorphisms for $m < n$. As the left morphism is as isomorphism in that range as well by Proposition 7.4.5.10 we can conclude that $\mathrm{H}_m(\Theta_X)$ is an isomorphism for $m < n$ too. It follows that $\mathrm{H}_m(\Theta_X)$ is an isomorphism for all integers m, so Θ is a quasiisomorphism. □

7.4.6. Naturality of ϵ

We explained in Warning 7.2.2.6 that the morphisms

$$\epsilon_X \colon \Omega^{\bullet}_{k[X]/k} \to \overline{\mathrm{C}}(k[X])$$

of differential graded k-algebras that were defined in Construction 7.2.2.1 and Proposition 7.2.2.2 only assemble to a natural transformation of functors from Set to $\mathrm{Alg}(\mathsf{Ch}(k))$, but not to a natural transformation of functors from CAlg(LMod$_k$(Ab)) to Alg(Ch(k)). In this section we show that a weaker statement is at least true in special cases: If X is a set with $|X| \leq 2$ and F a morphism of commutative algebras $F\colon k[t] \to k[X]$, then there is a filler for the square

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k[t]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_{\{t\}})"] \arrow[d, "\mathrm{Alg}(\gamma)(\Omega^{\bullet}_{F/k})"'] & \mathrm{Alg}(\gamma)(\overline{\mathrm{C}}(k[t])) \arrow[d, "\mathrm{Alg}(\gamma)(\overline{\mathrm{C}}(F))"] \\
\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k[X]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_X)"'] & \mathrm{Alg}(\gamma)(\overline{\mathrm{C}}(k[X]))
\end{tikzcd}$$

in Alg($\mathcal{D}(k)$).

Proposition 7.4.6.1. *Let X be a totally ordered set satisfying $|X| \leq 2$, and f an element of $k[X]$. Denote the morphism of commutative k-algebras $k[t] \to k[X]$ that maps t to f by F.*

Then there is a filler for the square

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k[t]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_{\{t\}})"] \arrow[d, "\mathrm{Alg}(\gamma)(\Omega^{\bullet}_{F/k})"'] & \mathrm{Alg}(\gamma)(\overline{\mathrm{C}}(k[t])) \arrow[d, "\mathrm{Alg}(\gamma)(\overline{\mathrm{C}}(F))"] \\
\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k[X]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_X)"'] & \mathrm{Alg}(\gamma)(\overline{\mathrm{C}}(k[X]))
\end{tikzcd}$$

in Alg($\mathcal{D}(k)$)*, where ϵ is as defined in Construction 7.2.2.1 and Proposition 7.2.2.2.* ♡

Proof. Let the morphism

$$\Theta_{\{t\}}\colon \Omega'^{\bullet}_{k[t]/k} \to \Omega^{\bullet}_{k[t]/k}$$

in Alg(Mixed) be as in Definition 7.4.5.9. By Proposition 7.4.5.11 $\Omega'^{\bullet}_{k[t]/k}$ is a cofibrant object of Alg(Mixed), and thus has cofibrant underlying chain complex by Proposition 4.2.2.12. Furthermore, $\Theta_{\{t\}}$ is a quasiisomorphism, and thus induces an equivalence

$$\mathrm{Alg}(\gamma)(\Theta_{\{t\}})\colon\ \mathrm{Alg}(\gamma)\left(\Omega'^{\bullet}_{k[X]/k}\right) \to \mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k[X]/k}\right)$$

in $\mathrm{Alg}(\mathcal{D}(k))$. It thus suffices to show that there is a filler for the square

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\left(\Omega'^{\bullet}_{k[t]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma)\left(\epsilon_{\{t\}}\circ\Theta_{\{t\}}\right)"] \arrow[d, "\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{F/k}\circ\Theta_{\{t\}}\right)"'] & \mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(k[t])\right) \arrow[d, "\mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(F)\right)"] \\
\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k[X]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_X)"'] & \mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(k[X])\right)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$.

By Proposition 4.2.2.12 the underlying differential graded algebra of cofibrant objects in $\mathrm{Alg}(\mathsf{Mixed})$ is cofibrant, so $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant as an object in $\mathrm{Alg}(\mathsf{Ch}(k))$. Like every object of $\mathrm{Alg}(\mathsf{Ch}(k))$ also $\overline{\mathrm{C}}(k[X])$ is fibrant. Combining this with Proposition A.1.0.1 and [Hov99, 1.2.10 (ii)] it suffices to show that there exists a homotopy in the model-category-theoretic sense between the two compositions in the following diagram in $\mathrm{Alg}(\mathsf{Ch}(k))$.

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[t]/k} \arrow[r, "\epsilon_{\{t\}}\circ\Theta_{\{t\}}"] \arrow[d, "\Omega^{\bullet}_{F/k}\circ\Theta_{\{t\}}"'] & \overline{\mathrm{C}}(k[t]) \arrow[d, "\overline{\mathrm{C}}(F)"] \\
\Omega^{\bullet}_{k[X]/k} \arrow[r, "\epsilon_X"'] & \overline{\mathrm{C}}(k[X])
\end{tikzcd}$$

By Propositions 4.1.4.2 and 4.2.2.17 this means that we have to define a morphism of $\mathbb{Z}$-graded k-modules

$$h\colon \Omega'^{\bullet}_{k[t]/k} \to \overline{\mathrm{C}}(k[X])$$

of degree 1 that satisfies

$$\partial(h(z)) + h(\partial(z)) = \left(\overline{\mathrm{C}}(F)\circ\epsilon_{\{t\}}\circ\Theta_{\{t\}}\right)(z) - \left(\epsilon_X\circ\Omega^{\bullet}_{F/k}\circ\Theta_{\{t\}}\right)(z)$$

for being a chain homotopy as well as the Leibniz rule for chain homotopies

$$\begin{aligned} h(z\cdot z') = h(z)\cdot\left(\epsilon_X\circ\Omega^{\bullet}_{F/k}\circ\Theta_{\{t\}}\right)(z') \\ + (-1)^{\deg_{\mathsf{Ch}}(z)}\left(\overline{\mathrm{C}}(F)\circ\epsilon_{\{t\}}\circ\Theta_{\{t\}}\right)(z)\cdot h(z') \end{aligned}$$

for all elements z and z' of $\Omega'^{\bullet}_{k[t]/k}$.

In the following we will use notation from Construction 7.4.5.1. By definition, and using that the forgetful functor from $\mathrm{Alg}(\mathsf{Mixed})$ to $\mathrm{Alg}(\mathsf{Ch}(k))$ preserves filtered colimits by Proposition 4.2.2.12, we can identify $\Omega'^{\bullet}_{k[t]/k}$ as the colimit of the diagram

$$A_0 \xrightarrow{\iota_0^1} A_1 \xrightarrow{\iota_1^2} A_2 \xrightarrow{\iota_2^3} \ldots$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$. The forgetful functor to $\mathbb{Z}$-graded k-modules also preserves filtered colimits by Proposition 4.2.2.12, and together this implies that we can define h as above by defining a compatible system of homotopies h_n of morphisms of differential graded algebras from the restriction $\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_n$ to $\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_n$. We will do this by induction.

We begin with some general remarks on how the induction step will work. So assume that $n \geq 0$ and we already have constructed a homotopy h_n of morphisms of differential graded algebras $A_n \to \overline{\mathrm{C}}(k[X])$ from $\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_n$ to $\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_n$. We wish to extend h_n to h_{n+1}. For easier notation we will use the following shorthands.

$$\begin{aligned}
\varphi' &\coloneqq \overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_n \\
\varphi &\coloneqq \overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_{n+1} \\
\psi' &\coloneqq \epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_n \\
\psi &\coloneqq \epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_{n+1}
\end{aligned}$$

By Remark 7.4.5.2 the underlying graded k-algebra of A_{n+1} is free on the elements t and $\mathrm{d}\,t$, and $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_m$ with $m \leq n$, while A_n is free on the same generators except the elements of Y_n and $\mathrm{d}\,Y_n$. Let us denote by G_{n+1} the generators for A_{n+1} that were just mentioned, and by G_n those of A_n. For compatibility with h_n we are forced to define h_{n+1} as follows on G_n.

$$h_{n+1}(g) \coloneqq h_n(g) \qquad \text{for } g \in G_n$$

On elements g in $G_{n+1} \setminus G_n$ we need to define h_{n+1} in such a way that h_{n+1} is a homotopy from φ to ψ, so we must have the following.

$$\partial(h_{n+1}(g)) = \varphi(g) - \psi(g) - h_{n+1}(\partial(g)) = -h_n(\partial(g)) \tag{$*$}$$

In the simplification we used that $\Theta_{\{t\}} \circ \iota_{n+1}$ is zero on g (and hence so are φ and ψ), and that $\partial(g)$ is an element of A_n. We claim that finding solutions to these lifting problems is the only obstacle to extending h_n to h_{n+1} as required. So assume that we can find values for $h_{n+1}(g)$ for every $g \in G_{n+1} \setminus G_n$ that satisfy $(*)$.

As we have already defined values of h_{n+1} on G_{n+1}, Proposition 4.2.2.18 implies that there is a unique way to extend this to a morphism h_{n+1} of $\mathbb{Z}$-graded k-modules from A_{n+1} to $\overline{\mathrm{C}}(k[X])$ of that increases degree by 1 and that satisfies the Leibniz rule for homotopies of differential graded algebras from φ to ψ. As h_{n+1} agrees with h_n on G_n and h_n also satisfies the analogous Leibniz rule as a homotopy of differential graded algebras from φ' to ψ', and φ and ψ restrict to φ' and ψ', the uniqueness part of Proposition 4.2.2.18 then implies that h_{n+1} extends h_n. It remains to show that h_{n+1} satisfies $\partial \circ h_{n+1} + h_{n+1} \circ \partial = \varphi - \psi$. Again by Proposition 4.2.2.18 it suffices to check this on elements of G_{n+1}. On elements of $G_{n+1} \setminus G_n$ this holds by definition, and on elements of G_n this holds because it does for h_n.

We have now shown that the only obstruction to extending h_n to h_{n+1} with all the necessary properties is finding solutions for $h_{n+1}(g)$ for elements g of $G_{n+1} \setminus G_n$ to the equation $(*)$. We claim that such a solution can always be found if $n \geq 2$. So assume that $n \geq 2$ and we have already defined h_n. Let g be an element of $G_{n+1} \setminus G_n$. Then we first claim that the right hand side of equation $(*)$ is a cycle. For this we carry out the following calculation, using that h_n is a chain homotopy from φ' to ψ'.

$$\begin{aligned}
&\partial(-h_n(\partial(g)))\\
&= h_n(\partial(\partial(g))) - \varphi'(\partial(g)) + \psi'(\partial(g))\\
&= -\varphi'(\partial(g)) + \psi'(\partial(g))\\
&= 0
\end{aligned}$$

The last step needs a comment. The element g is either of the form $\underline{y}$ or $\mathrm{d}\,\underline{y}$ for $y \in Y_n$. Thus $\partial(g)$ is either y or $-\mathrm{d}\,y$ for a $y \in Y_n$, and $\Theta_{\{t\}}$, and thus also φ' and ψ', maps every element of Y_n (and hence also $\mathrm{d}\,Y_n$) to 0.

As the right hand sides of equation $(*)$ is a cycle, it represents a homology class, and finding a solution to the equation is equivalent to the homology class being zero. As the elements of Y_n are of degree n, the element g, and hence the right hand side of $(*)$, must be of degree $n+1$ or $n+2$.[52] Thus the obstructions to extending h_n to h_{n+1} are homology classes in degree $n+1$ and $n+2$. As ϵ_X is a quasiisomorphism by Proposition 7.2.2.2 (6) and $\Omega^\bullet_{k[X]/k}$ is concentrated in degrees less than or equal to 2 (this is where we use the assumption $|X| \leq 2$), the homology of $\overline{\mathrm{C}}(k[X])$ is concentrated in degrees less than or equal to 2. Thus the homology classes obstructing extension of h_n to h_{n+1} are all trivially zero as we assumed $n \geq 2$, so that it is always possible to extend h_n to h_{n+1}.

By the above argument it thus suffices to construct h_2. Concretely, we first need to define $h_0(t)$ and $h_0(\mathrm{d}\,t)$ satisfying the following.[53]

$$\begin{aligned}
\partial(h_0(t)) &= \big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\big)(t) - \Big(\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\Big)(t) \quad (7.13)\\
\partial(h_0(\mathrm{d}\,t)) &= \big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\big)(\mathrm{d}\,t) - \Big(\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\Big)(\mathrm{d}\,t)
\end{aligned}$$

For the set $\{t\}$ the set Y_0 is empty by Proposition 7.4.5.6, so $\iota_0^1\colon A_0 \to A_1$ is an isomorphism, and hence h_0 extends to h_1 for trivial reasons. Finally, to extend h_1 to h_2 we need to define $h_2(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$ and $h_2(\mathrm{d}\,\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$ (see Proposition 7.4.5.7) satisfying the following.

$$\begin{aligned}
\partial(h_2(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})) &= -h_0(\partial(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})) \quad (7.14)\\
\partial(h_2(\mathrm{d}\,\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})) &= -h_0(\partial(\mathrm{d}\,\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}))
\end{aligned}$$

[52] Recall that if y is an element of Y_n, then $\underline{y}$ is of degree $n+1$ and $\mathrm{d}\,\underline{y}$ is then of degree $n+2$.

[53] The argument that it suffices to define h_0 on t and $\mathrm{d}\,t$ satisfying the chain homotopy identity is completely analogous to the argument we gave for extending h_n to h_{n+1}, also using Proposition 4.2.2.18. This time the analogue of $(*)$ has slightly different form as $\Theta_{\{t\}}$ does not vanish on t and $\mathrm{d}\,t$, but t and $\mathrm{d}\,t$ are cycles in A_0.

However, the obstruction for the existence of a solution for $h_2(\mathrm{d}\,\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$ is a homology class in degree 3. By the same argument as the case of extensions from A_n to A_{n+1} for $n \geq 2$ we thus already know abstractly that a solution can be found. To extend h_1 to h_2 it thus suffices to find a solution for $h_2(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$.

We begin by evaluating the right hand sides of (7.13), where we use the definitions in particular from Construction 7.4.5.1 and Construction 7.2.2.1. If $|X| = 2$ we denote the elements of X by $x_1 < x_2$, if $|X| = 1$ we denote the unique element by x_1.

$$
\begin{aligned}
&\left(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\right)(t) - \left(\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\right)(t) \\
&= \left(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}}\right)(t) - \left(\epsilon_X \circ \Omega^\bullet_{F/k}\right)(t) \\
&= \overline{\mathrm{C}}(F)(t) - \epsilon_X(f) \\
&= f - f \\
&= 0 \\
&= \partial(0)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\right)(\mathrm{d}\,t) - \left(\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\right)(\mathrm{d}\,t) \\
&= \left(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}}\right)(\mathrm{d}\,t) - \left(\epsilon_X \circ \Omega^\bullet_{F/k}\right)(\mathrm{d}\,t) \\
&= \overline{\mathrm{C}}(F)\left(1 \otimes \overline{t}\right) - \epsilon_X(\mathrm{d}\,f) \\
&= 1 \otimes \overline{f} - \epsilon_X(\mathrm{d}\,f) \\
&= \mathrm{d}(f) - \epsilon_X(\mathrm{d}\,f) \\
&= \mathrm{d}(\epsilon_X(f)) - \epsilon_X(\mathrm{d}\,f)
\end{aligned}
$$

We can now use that $\epsilon_X^{(\bullet)}$ as defined in Construction 7.3.1.1 is a strongly homotopy linear morphism, see Proposition 7.3.11.2.

$$
= -\partial\left(\epsilon_X^{(1)}(f)\right)
$$

We can thus define $h_0(t) = 0$ and $h_0(\mathrm{d}\,t) = -\epsilon_X^{(1)}(f)$.

Now we evaluate the right hand side of (7.14).

$$
\begin{aligned}
&- h_0(\partial(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})) \\
&= -h_0(t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t) \\
&= -h_0(t) \cdot \left(\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\right)(\mathrm{d}\,t) \\
&\quad - \left(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\right)(t) \cdot h_0(\mathrm{d}\,t) \\
&\quad + h_0(\mathrm{d}\,t) \cdot \left(\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\right)(t) \\
&\quad - \left(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\right)(\mathrm{d}\,t) \cdot h_0(t)
\end{aligned}
$$

$$
\begin{aligned}
&= -\big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\big)(t) \cdot h_0(\mathrm{d}\, t) \\
&\quad + h_0(\mathrm{d}\, t) \cdot \Big(\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\Big)(t) \\
&= -\Big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0 - \epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\Big)(t) \cdot h_0(\mathrm{d}\, t) \\
&= -\partial(h_0(t)) \cdot h_0(\mathrm{d}\, t) \\
&= 0 \\
&= \partial(0)
\end{aligned}
$$

Thus we can define $h_2(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}) = 0$. □

As a significantly easier variant we can also show a analogous result to Proposition 7.4.6.1 where we consider morphisms into k.

Proposition 7.4.6.2. *Let X be a set and $F\colon k[X] \to k$ a morphism of commutative k-algebras.*

Then there is a filler for the square

$$
\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_X)"] \arrow[d, "\mathrm{Alg}(\gamma)\left(\Omega^\bullet_{F/k}\right)"'] & \mathrm{Alg}(\gamma)\big(\overline{\mathrm{C}}(k[X])\big) \arrow[d, "\mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(F)\right)"] \\
\mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_\emptyset)"'] & \mathrm{Alg}(\gamma)\big(\overline{\mathrm{C}}(k)\big)
\end{tikzcd}
$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where ϵ is as defined in Construction 7.2.2.1 and Proposition 7.2.2.2. ♡

Proof. It suffices to show that the diagram

$$
\begin{tikzcd}
\Omega^\bullet_{k[X]/k} \arrow[r, "\epsilon_X"] \arrow[d, "\Omega^\bullet_{F/k}"'] & \overline{\mathrm{C}}(k[X]) \arrow[d, "\overline{\mathrm{C}}(F)"] \\
\Omega^\bullet_{k/k} \arrow[r, "\epsilon_\emptyset"'] & \overline{\mathrm{C}}(k)
\end{tikzcd}
$$

commutes strictly. For this we note that as $\overline{k} \cong 0$, the lower right chain complex is concentrated in degree 0, so it suffices to check that the two compositions agree on elements of degree 0. But on degree 0 we can identify the diagram with

$$
\begin{tikzcd}
k[X] \arrow[r, "\mathrm{id}_{k[X]}"] \arrow[d, "F"'] & k[X] \arrow[d, "F"] \\
k \arrow[r, "\mathrm{id}_k"'] & k
\end{tikzcd}
$$

which commutes. □

7.4.7. Naturality of Φ

In Definition 7.4.4.2 we defined a quasiisomorphisms

$$\Phi_X \colon \operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \to \Omega^\bullet_{k[X]/k}$$

in $\operatorname{Alg}(\mathsf{Ch}(k))$, for every set X. While the morphisms Φ_X for different sets X do not assemble to a natural transformation from the category of commutative k-algebras to $\operatorname{Alg}(\mathsf{Ch}(k))$, we show in this section that a weaker naturality property holds with respect to some specific morphisms of commutative k-algebras.

Proposition 7.4.7.1. *Let X and Y be totally ordered sets satisfying one of the following.*

(1) $|X| = 1$ *and* $|Y| \leq 2$.

(2) $|Y| = 0$.

Let F be a morphism of commutative k-algebras $k[X] \to k[Y]$.

Then there is a filler for the square

$$\begin{tikzcd}
\operatorname{Alg}(\gamma)\Big(\operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big)\Big) \arrow[r, "\operatorname{Alg}(\gamma)(\Phi_X)"] \arrow[d, "\operatorname{Alg}(\gamma)\left(\operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})\left(\widetilde{\mathrm{C}}_k(F)\right)\right)"'] & \operatorname{Alg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[d, "\operatorname{Alg}(\gamma)\left(\Omega^\bullet_{F/k}\right)"] \\
\operatorname{Alg}(\gamma)\Big(\operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(Y)\Big)\Big) \arrow[r, "\operatorname{Alg}(\gamma)(\Phi_Y)"'] & \operatorname{Alg}(\gamma)\Big(\Omega^\bullet_{k[Y]/k}\Big)
\end{tikzcd}$$

in $\operatorname{Alg}(\mathcal{D}(k))$, where $\widetilde{\mathrm{C}}$ is as in Construction 7.4.2.5 and Φ_X and Φ_Y as in Definition 7.4.4.2. ♡

Proof. In the following we will omit the forgetful functor $\operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})$ from the notation to make diagrams more compact.

By Definition 7.4.4.2 Φ_X is the composition of Φ'_X with the quasiisomorphism mapping z to $\nu^{\deg_{\mathsf{Ch}}(z)} \cdot z$ (where ν is as in Proposition 7.4.4.1), and analogously for Φ_Y. As the diagram

$$\begin{tikzcd}
\Omega^\bullet_{k[X]/k} \arrow[r, "\nu^{\deg_{\mathsf{Ch}}(-)}\cdot_", "\simeq"'] \arrow[d, "\Omega^\bullet_{F/k}"'] & \Omega^\bullet_{k[X]/k} \arrow[d, "\Omega^\bullet_{F/k}"] \\
\Omega^\bullet_{k[Y]/k} \arrow[r, "\simeq", "\nu^{\deg_{\mathsf{Ch}}(-)}\cdot_"'] & \Omega^\bullet_{k[Y]/k}
\end{tikzcd}$$

commutes there is a filler for the right square in the following (non-commuting) diagram in $\mathrm{Alg}(\mathcal{D}(k))$.

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\left(\widetilde{\mathrm{C}}_k(X)\right) \arrow[r, "\mathrm{Alg}(\gamma)(\Phi'_X)"] \arrow[d, "\mathrm{Alg}(\gamma)(\widetilde{\mathrm{C}}_k(F))"] & \mathrm{Alg}(\gamma)\left(\Omega^\bullet_{k[X]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma)(\nu^{\deg_{\mathsf{Ch}}(-)}\cdot_)", "\simeq"'] \arrow[d, "\mathrm{Alg}(\gamma)(\Omega^\bullet_{F/k})"] & \mathrm{Alg}(\gamma)\left(\Omega^\bullet_{k[X]/k}\right) \arrow[d, "\mathrm{Alg}(\gamma)(\Omega^\bullet_{F/k})"'] \\
\mathrm{Alg}(\gamma)\left(\widetilde{\mathrm{C}}_k(Y)\right) \arrow[r, "\mathrm{Alg}(\gamma)(\Phi'_Y)"'] & \mathrm{Alg}(\gamma)\left(\Omega^\bullet_{k[Y]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma)(\nu^{\deg_{\mathsf{Ch}}(-)}\cdot_)", "\simeq"'] & \mathrm{Alg}(\gamma)\left(\Omega^\bullet_{k[Y]/k}\right)
\end{tikzcd}$$

It thus suffices to find a filler for the left square.

We now unpack the definition of Φ'_X, with Φ'_Y of course being completely analogous. By Proposition 7.4.3.2 $\mathrm{Alg}(\gamma)(\Phi'_X)$ is homotopic to the composition

$$\mathrm{Alg}(\gamma)\left(\widetilde{\mathrm{C}}_k(X)\right) \simeq \mathrm{HH}(k[X]) \simeq \mathrm{Alg}(\gamma)\left(\Omega^\bullet_{k[X]/k}\right)$$

where the first equivalence is obtained by applying $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ to the equivalence at the bottom of diagram (7.9) in Construction 7.4.2.5 combined with compatibility of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1, and the second equivalence is the one from Corollary 7.2.2.3.

By definition that equivalence from Corollary 7.2.2.3 is given by the composition

$$\begin{tikzcd}
\mathrm{HH}(k[X]) \arrow[r, dash, "\simeq"] & \mathrm{Alg}(\gamma)(\mathrm{C}(k[X])) \arrow[r, "\simeq"] & \mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(k[X])\right) \\
& & \mathrm{Alg}(\gamma)\left(\Omega^\bullet_{k[X]/k}\right) \arrow[u, "\mathrm{Alg}(\gamma)(\epsilon_X)", "\simeq"']
\end{tikzcd}$$

where the first equivalence is the one from Proposition 6.3.4.3, the second one is induced by the quotient morphism from Propositions 6.3.1.10 and 6.3.2.11, and the last equivalence is induced from ϵ_X as constructed in Construction 7.2.2.1.

In the following diagram in $\mathrm{Alg}(\mathcal{D}(k))$, we let the two columns be given by the composition the equivalences Φ'_X and Φ'_Y are defined as, as we just reviewed.

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\widetilde{\mathrm{C}}_k(X)\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\widetilde{\mathrm{C}}_k(F)\right)"] \arrow[d, dash, "\simeq"'] & \mathrm{Alg}(\gamma)\Big(\widetilde{\mathrm{C}}_k(Y)\Big) \arrow[d, dash, "\simeq"] \\
\mathrm{HH}(k[X]) \arrow[r, "\mathrm{HH}(F)"] \arrow[d, dash, "\simeq"'] & \mathrm{HH}(k[Y]) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma)(\mathrm{C}(k[X])) \arrow[r, "\mathrm{Alg}(\gamma)(\mathrm{C}(F))"] \arrow[d, "\simeq"'] & \mathrm{Alg}(\gamma)(\mathrm{C}(k[Y])) \arrow[d, "\simeq"] \\
\mathrm{Alg}(\gamma)\big(\overline{\mathrm{C}}(k[X])\big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(F)\right)"] & \mathrm{Alg}(\gamma)\big(\overline{\mathrm{C}}(k[Y])\big) \\
\mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\Omega^\bullet_{F/k}\right)"] \arrow[u, "\simeq", "\mathrm{Alg}(\gamma)(\epsilon_X)"'] & \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[Y]/k}\Big) \arrow[u, "\mathrm{Alg}(\gamma)(\epsilon_Y)", "\simeq"']
\end{tikzcd}$$

There is a filler for the first square from the top by definition of $\widetilde{\mathrm{C}}_k(F)$, see Construction 7.4.2.5. The second square has a filler by naturality of the equivalence between HH and the standard Hochschild complex C in Proposition 6.3.4.3. The third square has a filler by naturality of the quotient map from the standard Hochschild complex to the normalized standard Hochschild complex, see Propositions 6.3.1.10 and 6.3.2.11. Finally, the bottom square has a filler by Proposition 7.4.6.1 (and Proposition 7.2.2.2 (3)) if we are in case (1) and by Proposition 7.4.6.2 if we are in case (2). □

7.4.8. Compatibility of Φ with d in degree 0

In Section 7.4.4 we showed that $\Phi_{\{t\}}$ is compatible with d (see Proposition 7.4.4.3). In this section we use the naturality statement from the previous Section 7.4.7 to deduce compatibility of Φ_X with d on elements of degree 0 as long as $|X| \leq 2$. Note that the following proposition still has content for $|X| = 1$. In this case it shows that the ν obtained in Proposition 7.4.4.1 is independent of the choices made along the way.

Proposition 7.4.8.1. *Let X be a totally ordered set satisfying $|X| \leq 2$. Then the quasiisomorphism*

$$\Phi_X\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \to \Omega^\bullet_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$ *from Definition 7.4.4.2 satisfies*

$$\Phi_X(\mathrm{d}\, z) = \mathrm{d}(\Phi_X(z)) \tag{7.15}$$

for every element z of degree 0 *of* $\widetilde{\mathrm{C}}_k(X)$. ♡

Proof. Let z and z' be elements of $\widetilde{\mathrm{C}}_k(X)$ of degree 0, and y an element of degree 1 such that $\partial(y) = z - z'$. Assume that

$$\Phi_X(\mathrm{d}\, z) = \mathrm{d}(\Phi_X(z))$$

holds. Then we claim that

$$\Phi_X(\mathrm{d}\, z') = \mathrm{d}(\Phi_X(z'))$$

holds as well. Indeed, this follows from the following calculation.

$$\begin{aligned}\Phi_X(\mathrm{d}\, z') &= \Phi_X(\mathrm{d}(z - \partial(y))) = \Phi_X(\mathrm{d}\, z - \mathrm{d}(\partial(y)))\\ &= \Phi_X(\mathrm{d}\, z) + \Phi_X(\partial(\mathrm{d}(y))) = \mathrm{d}(\Phi_X(z)) + \partial(\Phi_X(\mathrm{d}(y)))\\ &= \mathrm{d}(\Phi_X(z)) = \mathrm{d}(\Phi_X(z' + \partial(y))) = \mathrm{d}(\Phi_X(z') + \Phi_X(\partial(y)))\\ &= \mathrm{d}(\Phi_X(z') + \partial(\Phi_X(y))) = \mathrm{d}(\Phi_X(z'))\end{aligned}$$

As $\widetilde{\mathrm{C}}_k(X)$ is concentrated in nonnegative degrees by Construction 7.4.2.5 and Proposition 7.4.2.4, every element of degree 0 is a cycle. It thus suffices to show that for each homology class in $\mathrm{H}_0(\widetilde{\mathrm{C}}_k(X))$ there is a cycle representing it that satisfies (7.15).

As both sides of (7.15) are k-linear in z it even suffices to verify (7.15) on one cycle for each in a set of homology classes that generate $\mathrm{H}_0(\widetilde{\mathrm{C}}_k(X))$ as a k-module.

As Φ_X is a quasiisomorphism it is surjective, so that we can lift every element x of X, considered as an element of $\Omega^\bullet_{k[X]/k}$ of degree 0, to a cycle $\widetilde{x}$ in $\widetilde{\mathrm{C}}_k(X)$. Products[54] of elements of X form a k-basis for $\Omega^\bullet_{k[X]/k}$ and hence $\mathrm{H}_0(\Omega^\bullet_{k[X]/k})$. As Φ_X is a multiplicative quasiisomorphism this implies that products of elements of the form $\widetilde{x}$ for $x \in X$ are cycles representing homology classes that together form a generating set for $\mathrm{H}_0(\widetilde{\mathrm{C}}_k(X))$ as a k-module. It thus suffices to show that (7.15) is satisfied for products (with arbitrary many factors) of elements of the form $\widetilde{x}$ for $x \in X$.

Now suppose that z and z' are elements of degree 0 in $\widetilde{\mathrm{C}}_k(X)$ that both satisfy (7.15). Then we claim that the product $z \cdot z'$ satisfies (7.15) as well. This can be shown with the following simple calculation that uses that Φ_X is multiplicative and that d satisfies the Leibniz rule on both $\widetilde{\mathrm{C}}_k(X)$ and $\Omega^\bullet_{k[X]/k}$.

$$\begin{aligned}\Phi_X(\mathrm{d}(z \cdot z')) &= \Phi_X(\mathrm{d}(z) \cdot z' + z \cdot \mathrm{d}(z'))\\ &= \Phi_X(\mathrm{d}\, z) \cdot \Phi_X(z') + \Phi_X(z) \cdot \Phi_X(\mathrm{d}\, z')\\ &= \mathrm{d}(\Phi_X(z)) \cdot \Phi_X(z') + \Phi_X(z) \cdot \mathrm{d}(\Phi_X(z'))\end{aligned}$$

[54]With arbitrary (finite) number of factors, including zero factors.

$$= \mathrm{d}(\Phi_X(z) \cdot \Phi_X(z'))$$
$$= \mathrm{d}(\Phi_X(z \cdot z'))$$

Note that the element 1 satisfies (7.15) because $\mathrm{d}(1) = 0$ by the Leibniz rule in both $\widetilde{\mathrm{C}}_k(X)$ and $\Omega^\bullet_{k[X]/k}$, and $\Phi_X(1) = 1$. We are thus reduced to show that (7.15) holds for the specific elements $\widetilde{x}$ for $x \in X$.

So let x be an element of X and $F\colon k[t] \to k[X]$ the morphism of commutative k-algebras that maps t to x. By Proposition 7.4.7.1[55] there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(\{t\})\Big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Phi_{\{t\}})"] \arrow[d, "\mathrm{Alg}(\gamma)(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}(F)))"'] & \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[t]/k}\Big) \arrow[d, "\mathrm{Alg}(\gamma)(\Omega^\bullet_{F/k})"] \\
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Phi_X)"'] & \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$. As the underlying differential graded algebra of $\widetilde{\mathrm{C}}_k(\{t\})$ is cofibrant by Proposition 4.2.2.12 and every object is fibrant in $\mathrm{Alg}(\mathsf{Ch}(k))$, we obtain from [Hov99, 1.2.10 (ii)] and Proposition A.1.0.1 that the following diagram commutes up to chain homotopy of morphisms of differential graded algebras in the sense of Propositions 4.1.4.2 and 4.2.2.17.

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(\{t\})\Big) \arrow[r, "\Phi_{\{t\}}"] \arrow[d, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}(F))"'] & \Omega^\bullet_{k[t]/k} \arrow[d, "\Omega^\bullet_{F/k}"] \\
\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \arrow[r, "\Phi_X"'] & \Omega^\bullet_{k[X]/k}
\end{tikzcd}$$

So let h be such a homotopy of morphisms of differential graded algebras from $\Omega^\bullet_{F/k} \circ \Phi_{\{t\}}$ to $\Phi_X \circ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}(F))$. Lift the element t in degree 0 of $\Omega^\bullet_{k[t]/k}$ to a cycle $\widetilde{t}$ in $\widetilde{\mathrm{C}}_k(\{t\})$. Then we have the following.

$$\begin{aligned}
\Phi_X\Big(\widetilde{x} - \widetilde{\mathrm{C}}(F)\big(\widetilde{t}\big)\Big) &= \Phi_X(\widetilde{x}) - \Phi_X\Big(\widetilde{\mathrm{C}}(F)\big(\widetilde{t}\big)\Big) \\
&= x + \partial\big(h\big(\widetilde{t}\big)\big) + h\big(\partial\big(\widetilde{t}\big)\big) - \Omega^\bullet_{F/k}\big(\Phi_{\{t\}}\big(\widetilde{t}\big)\big) \\
&= x + 0 + h(0) - \Omega^\bullet_{F/k}(t) \\
&= x - x \\
&= 0
\end{aligned}$$

[55] This is the part of this proof that uses the assumption that $|X| \leq 2$.

Thus $\widetilde{x} - \widetilde{\mathrm{C}}(F)(\widetilde{t})$ is a cycle that represents a homology class that maps to 0 under $\mathrm{H}_0(\Phi_X)$. As Φ_X is a quasiisomorphism we must thus have that $\widetilde{x} - \widetilde{\mathrm{C}}(F)(\widetilde{t})$ is a boundary. By the argument we gave at the start of this proof it thus suffices to show that (7.15) holds for the element $\widetilde{\mathrm{C}}(F)(\widetilde{t})$. For this we use the following calculation, using that $\Phi_{\{t\}}$ is compatible with d by Proposition 7.4.4.3, and that $\Phi_X(\widetilde{\mathrm{C}}(F)(\widetilde{t})) = \Phi_X(\widetilde{x})$ by the above calculation.

$$\begin{aligned}\Phi_X\Big(\mathrm{d}\Big(\widetilde{\mathrm{C}}(F)\big(\widetilde{t}\big)\Big)\Big) &= \Phi_X\Big(\widetilde{\mathrm{C}}(F)\big(\mathrm{d}\,\widetilde{t}\big)\Big)\\ &= \Omega^\bullet_{F/k}\big(\Phi_{\{t\}}\big(\mathrm{d}\,\widetilde{t}\big)\big) - \partial\big(h\big(\mathrm{d}\,\widetilde{t}\big)\big) - h\big(\partial\big(\mathrm{d}\,\widetilde{t}\big)\big)\\ &= \Omega^\bullet_{F/k}\big(\mathrm{d}\big(\Phi_{\{t\}}\big(\widetilde{t}\big)\big)\big) - 0 + h\big(\mathrm{d}\big(\partial\big(\widetilde{t}\big)\big)\big)\\ &= \Omega^\bullet_{F/k}(\mathrm{d}\,t) + h(\mathrm{d}(0))\\ &= \mathrm{d}\,x\\ &= \mathrm{d}\Big(\Phi_X\Big(\widetilde{\mathrm{C}}(F)\big(\widetilde{t}\big)\Big)\Big) \qquad \square\end{aligned}$$

7.4.9. Proof of Conjecture B for sets of cardinality at most 2

The goal of Section 7.4 is to show that Conjecture B holds for $|X| \leq 2$. This is what we do in this subsection, by combining all the ingredients from the previous subsections.

Construction 7.4.9.1. Let X be a totally ordered set with $|X| \leq 2$. We will construct a morphism

$$\Xi_X \colon \Omega'^\bullet_{k[X]/k} \to \widetilde{\mathrm{C}}(X)$$

in Alg(Mixed), where $\Omega'^\bullet_{k[X]/k}$ is as defined in Definition 7.4.5.9 and Construction 7.4.5.1, and $\widetilde{\mathrm{C}}(X)$ is as defined in Construction 7.4.2.5.

In this construction we will in particular use notation from Construction 7.4.5.1, and also make use of the multiplicative quasiisomorphism Φ_X from Definition 7.4.4.2 and the strict mixed quasiisomorphism Ψ_X from Definition 7.4.3.4.

By the universal property of the colimit it suffices to construct morphisms

$$\Xi_n \colon A_n \to \widetilde{\mathrm{C}}(X)$$

in Alg(Mixed) for every $n \geq 0$ such that $\Xi_{n+1} \circ \iota_n^{n+1} = \Xi_n$. By the universal property of pushouts and $\mathrm{Free}^{\mathrm{Alg(Mixed)}}$ this amounts to the following. To define Ξ_0 we need to prescribe a cycle as the value $\Xi_0(x)$ for every element x of X. If $n \geq 0$, then to lift Ξ_n to Ξ_{n+1} amounts to prescribing a value for $\Xi_{n+1}(\underline{y})$ for every element y of Y_n, under the constraint that

$$\partial\big(\Xi_{n+1}\big(\underline{y}\big)\big) = \Xi_n(y) \qquad (*)$$

must hold. We will require one additional property that $\Xi_{n+1}(\underline{y})$ should satisfy, namely that

$$\Psi_X\big(\Xi_{n+1}(\underline{y})\big) = 0 \tag{$**$}$$

where Ψ_X is as in Definition 7.4.3.4.

Let $n \geq 0$, let y be an element of Y_n, and assume that Ξ_n has already been defined. Note that $\Xi_n(y)$ is a cycle, as y is a cycle by (a) in Construction 7.4.5.1. We claim that if the homology class represented by $\Xi_n(y)$ is zero, then a value for $\Xi_{n+1}(\underline{y})$ can be found that satisfies both $(*)$ and $(**)$. So let z be an element of $\widetilde{\mathrm{C}}(X)$ so that $\partial(z) = \Xi_n(y)$. Then $\Psi_X(z)$ is a cycle (as every element of $\Omega^\bullet_{k[X]/k}$ is), so as Ψ_X is a quasiisomorphism and $\Omega^\bullet_{k[X]/k}$ has zero boundary operator we can lift $\Psi_X(z)$ to a cycle z' in $\widetilde{\mathrm{C}}(X)$ such that $\Psi_X(z') = \Psi_X(z)$. Now set $\Xi_{n+1}(\underline{y}) \coloneqq z - z'$. Then we immediately obtain

$$\Psi_X\big(\Xi_{n+1}(\underline{y})\big) = \Psi_x(z) - \Psi_X(z') = 0$$

and, using that z' is a cycle,

$$\partial\big(\Xi_{n+1}(\underline{y})\big) = \partial(z - z') = \partial(z) = \Xi_n(y)$$

so that this definition of $\Xi_{n+1}(\underline{y})$ satisfies both $(*)$ and $(**)$.

We now define Ξ_0 and then Ξ_n for $n > 0$ by induction, in such a way that $\Psi_X \circ \Xi_n$ maps $\underline{y}$ to 0 for all elements $y \in Y_{n'}$ for $n' < n$. By the argument above it suffices for the induction step in which we extend Ξ_n to Ξ_{n+1} for $n \geq 0$ to show that the homology class represented by $\Xi_n(y)$ is zero for every element y of Y_n. As Φ_X and Ψ_X are quasiisomorphisms it in turn suffices for this to show that each of those elements is mapped to zero by $\Phi_X \circ \Xi_0$ or $\Psi_X \circ \Xi_0$.

We thus start with Ξ_0. Let x be an element of X. We need to define a cycle $\Xi_0(x)$. For this we use that as Φ_X is a quasiisomorphism and $\Omega^\bullet_{k[X]/k}$ has zero boundary operator, we can lift the element x of $\Omega^0_{k[X]/k}$ to a cycle $\Xi_0(x)$ in $\widetilde{\mathrm{C}}(X)$. This defines Ξ_0 in such a way that

$$(\Phi_X \circ \Xi_0)(x) = x \tag{7.16}$$

holds for every element x of X.

To extend Ξ_0 to Ξ_1 and then Ξ_2 we use that $\Phi_X \circ \Xi_0$ maps the elements of Y_0 and Y_1 (note that Y_1 lies already in A_0) to zero. This is the case as all elements of Y_0 and Y_1 are given by commutators, so as $\Phi_X \circ \Xi_0$ is multiplicative those elements are mapped to zero as $\Omega^\bullet_{k[X]/k}$ is commutative.

We next extend Ξ_2 to Ξ_3. In the following we will denote the element(s) of X by $x_1 < \cdots < x_{|X|}$. Then by Proposition 7.4.5.8 the elements of Y_2 are given by the full list below for $|X| = 2$, consist of the first element of the list for $|X| = 1$, and $Y_1 = \emptyset$ for $|X| = 0$.

(1) $\mathrm{d}\, x_1 \cdot \mathrm{d}\, x_1$

(2) $\mathrm{d}\,x_2 \cdot \mathrm{d}\,x_2$

(3) $\mathrm{d}\,\underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}$

(4) $$\begin{aligned} &\mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_2 \\ &- x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \cdot x_1 \\ &+ x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} \cdot x_2 \end{aligned}$$

(5) $$\begin{aligned} &\mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_1 \\ &- x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} \cdot x_1 \\ &+ x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} \cdot x_2 \end{aligned}$$

Elements (4) and (5) can be handled using Φ_X in the same way as we did with the elements of Y_0 and Y_1, as they are sums of commutators. Elements (1) and (2) can also be handled analogously with Φ_X, this time using that odd degree elements square to zero in $\Omega^{\bullet}_{k[X]/k}$. It remains to consider element (3). For this element we can use that $\Psi_X \circ \Xi_2$ maps it to 0, which is the case as by induction hypothesis $\Psi_X \circ \Xi_2$ maps every element of the form $\underline{y}$ for y an element of Y_0 or Y_1 to zero, and $\Psi_X \circ \Xi_2$ is also compatible with $\underline{\mathrm{d}}$.

Now let $n \geq 3$ and assume we have already constructed Ξ_n. To extend Ξ_n to Ξ_{n+1} it suffices to show that $\Phi_X \circ \Xi_n$ maps the elements of Y_n to zero. However the elements of Y_n are of degree n, and $\Omega^n_{k[X]/k} \cong 0$ as $|X| \leq 2 < 3 \leq n$, so this is automatically satisfied. ♢

Proposition 7.4.9.2. *Let X be a totally ordered set with $|X| \leq 2$. Then the morphism*

$$\Xi_X \colon \Omega'^{\bullet}_{k[X]/k} \to \widetilde{\mathrm{C}}(X)$$

in $\mathrm{Alg}(\mathsf{Mixed})$ *that was constructed in Construction 7.4.9.1 is a quasiisomorphism.* ♡

Proof. Let us denote the element(s) of X by $x_1 < \cdots < x_{|X|}$. The morphism

$$\Theta_X \colon \Omega'^{\bullet}_{k[X]/k} \to \Omega^{\bullet}_{k[X]/k}$$

as defined in Definition 7.4.5.9 is a quasiisomorphism by Proposition 7.4.5.11. By construction Θ_X maps the cycle $x_1^{a_1} \cdots x_{|X|}^{a_{|X|}} \cdot \mathrm{d}\,x_1^{b_1} \cdots \mathrm{d}\,x_{|X|}^{b_{|X|}}$ of $\Omega'^{\bullet}_{k[X]/k}$ with $a_1, \ldots, a_{|X|} \geq 0$ and $b_1, \ldots, b_{|X|} \in \{0,1\}$, to the element of $\Omega^{\bullet}_{k[X]/k}$ with the same name. As the homology classes of those cycles in $\Omega^{\bullet}_{k[X]/k}$ form a k-basis for the homology, the same must be true for $\Omega'^{\bullet}_{k[X]/k}$, i. e. the set

$$\left\{ \left[x_1^{a_1} \cdots x_{|X|}^{a_{|X|}} \cdot \mathrm{d}\,x_1^{b_1} \cdots \mathrm{d}\,x_{|X|}^{b_{|X|}} \right] \;\middle|\; a_1, \ldots, a_{|X|} \geq 0,\ b_1, \ldots, b_{|X|} \in \{0,1\} \right\} \qquad (*)$$

forms a k-basis of the $\mathbb{Z}$-graded k-module $\mathrm{H}_*(\Omega'^{\bullet}_{k[X]/k})$.

To show that $\mathrm{H}_*(\Xi_X)$ is an isomorphism it suffices to show that the morphism $\mathrm{H}_*(\Phi_X \circ \Xi_X)$ is an isomorphism, where Φ_X is the quasiisomorphism defined in Definition 7.4.4.2. For this it now suffices to show that the basis $(*)$ of $\mathrm{H}_*(\Omega'^{\bullet}_{k[X]/k})$ is mapped to a basis of $\mathrm{H}_*(\Omega^{\bullet}_{k[X]/k})$ under $\mathrm{H}_*(\Phi_X \circ \Xi_X)$, and for this it is in turn enough to show that $\Phi_X \circ \Xi_X$ maps the element $x_1^{a_1} \cdots x_{|X|}^{a_{|X|}} \cdot \mathrm{d}\, x_1^{b_1} \cdots \mathrm{d}\, x_{|X|}^{b_{|X|}}$ of $\Omega'^{\bullet}_{k[X]/k}$ for $a_1, \ldots, a_{|X|} \geq 0$ and $b_1, \ldots, b_{|X|} \in \{0, 1\}$ to the element of $\Omega^{\bullet}_{k[X]/k}$ with the same name. As $\Phi_X \circ \Xi_X$ is multiplicative we only need to show that $\Phi_X \circ \Xi_X$ maps elements x to x and $\mathrm{d}\, x$ to $\mathrm{d}\, x$, for each element x in X. That $\Phi_X \circ \Xi_X$ maps elements x of X to x holds by construction of Ξ_X, see (7.16). We can also deduce from this that $\mathrm{d}\, x$ is mapped to $\mathrm{d}\, x$, as Ξ_X is compatible with d, and Φ_X is compatible with d on elements of degree 0 by Proposition 7.4.8.1. $\square$

We can now sum up Section 7.4 as follows.

Corollary 7.4.9.3. *Let X be a totally ordered set with $|X| \leq 2$. Then there is a composite equivalence*

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, phantom, "\simeq"] \arrow[drr, dashed, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(X)\Big) & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big) \arrow[l, "\simeq"', "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)"] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Theta_X)"', "\simeq"] \\
 & & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[X]/k}\Big)
\end{tikzcd}$$

in Alg($\mathcal{M}$ixed)*, where the first equivalence is the one at the bottom of diagram* (7.9) *in Construction 7.4.2.5, the second equivalence is induced by Ξ_X as constructed in Construction 7.4.9.1, and which is a quasiisomorphism by Proposition 7.4.9.2, and the third equivalence is induced by Θ_X as defined in Definition 7.4.5.9, which is a quasiisomorphism by Proposition 7.4.5.11.*

In particular, Conjecture B holds for X. ♡

Remark 7.4.9.4. Usage of Ψ_X is not really necessary in Construction 7.4.9.1, as we could also have arranged for

$$\Phi_X\big(\Xi_2\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)\big) = 0$$

and

$$\Phi_X\big(\Xi_2\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\big) = \Phi_X\big(\Xi_1\big(\mathrm{d}\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big)\big)$$

instead of equation $(**)$ in Construction 7.4.9.1, and thereby also dealing with the problematic element

$$\mathrm{d}\, \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}$$

that we used Ψ_X to handle in Construction 7.4.9.1, by using Φ_X instead, having the contribution from the third summand exactly cancel out the uncontrollable (under Φ_X) first summand.

The reason Construction 7.4.9.1 was nevertheless written using Ψ_X is that it would not suffice to only use Φ_X anymore in the case $|X| = 3$, as in this case we would have to consider also obstructions to extend to generators of degree 4, and this would involve in particular an element like

$$\mathrm{d}(\underline{x \cdot \mathrm{d}\,x - \mathrm{d}\,x \cdot x}) + 2 \cdot \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x}$$

in degree 3 that can not be handled with the same idea using Φ_X only unless 2 is invertible in k. However, it is likely that the technique actually used in Construction 7.4.9.1 using Φ_X and Ψ_X extends to the three-variable case, so it would be an unnecessary assumption to assume that 2 is invertible in k.

The case $|X| = 5$ is expected to need different techniques for base rings such as $k = \mathbb{Z}$ in which 3 is not invertible, as the cofibrant resolution $\Omega'^{\bullet}_{k[X]/k}$ will have a generator in degree 6 with boundary of the form[56]

$$\begin{gathered}
x \cdot \underline{\mathrm{d}\,\underline{\mathrm{d}\,x \cdot \mathrm{d}\,x}} - \underline{\mathrm{d}\,\underline{\mathrm{d}\,x \cdot \mathrm{d}\,x}} \cdot x \\
+ \mathrm{d}\,x \cdot \underline{\mathrm{d}\,\underline{x \cdot \mathrm{d}\,x - \mathrm{d}\,x \cdot x} + 2 \cdot \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x}} - \underline{\mathrm{d}\,\underline{x \cdot \mathrm{d}\,x - \mathrm{d}\,x \cdot x} + 2 \cdot \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x}} \cdot \mathrm{d}\,x \\
+ \mathrm{d}\,\underline{x \cdot \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x} + \mathrm{d}\,x \cdot \underline{x \cdot \mathrm{d}\,x - \mathrm{d}\,x \cdot x} - \underline{x \cdot \mathrm{d}\,x - \mathrm{d}\,x \cdot x} \cdot \mathrm{d}\,x - \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x} \cdot x} \\
+3 \cdot \underline{\mathrm{d}\,x \cdot \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x} + \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x} \cdot \mathrm{d}\,x}
\end{gathered}$$

which involves interactions of the multiplicative and strict mixed structure in a way that does not seem to be handleable using only Φ_X or Ψ_X (unless 3 is invertible). $\diamondsuit$

7.5. De Rham forms as a strict model in $\mathrm{Alg}(\mathsf{Mixed})$ and morphisms

In Section 7.4 we discussed Conjecture B, which asks for showing that for polynomial k-algebras de Rham forms are a strict model for Hochschild homology as an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. The next upgrade of such an objectwise equivalence would be showing that the morphism induced on de Rham forms by a morphism of polynomial k-algebras represents the induced morphism on Hochschild homology as well. We formulate this as the following conjecture.

Conjecture C. *Let X and Y be sets and $F\colon k[X] \to k[Y]$ a morphism of*

[56] The generators of $\Omega'^{\bullet}_{k[X]/k}$, in particular including this expected generator, were found using computer calculations.

commutative k-algebras. Then there exists a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, no head, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[X]/k}\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega^{\bullet}_{F/k})"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[Y]) \arrow[r, no head, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[Y]/k}\right)
\end{tikzcd}$$

in Alg($\mathcal{M}$ixed) *such that the horizontal morphisms are equivalences.*

We will often refer to the existence of such a commutative square for a specific F as "Conjecture C holds for F". ♧

Later in this section we will show that Conjecture C holds

- if $|X| = 0$ and $|Y| \leq 2$ by Proposition 7.5.1.1 in Section 7.5.1, and
- if $|X| = 1$ and $|Y| \leq 1$ by Proposition 7.5.2.6 in Section 7.5.2, and
- if $|X| = 1$ and $|Y| = 2$ and 2 is invertible in k by Proposition 7.5.2.6 in Section 7.5.2, and
- if $|X| = 2$ and $|Y| = 0$ by Proposition 7.5.4.1 in Section 7.5.4.

For applications we will need the following variant of Conjecture C, with two squares at once, with the same equivalence in the middle (so this is stronger than just two instances of Conjecture C).

Conjecture D. *Let X be a set and f an element of $k[X]$. Denote by $F\colon k[t] \to k[X]$ the morphism of commutative k-algebras that maps t to f and by $G\colon k[t] \to k$ the morphism of commutative k-algebras that maps t to 0. Then there exists a commutative diagram*

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, no head, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k/k}\right) \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t]) \arrow[r, no head, "\simeq"] \arrow[u, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(G)"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[t]/k}\right) \arrow[u, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega^{\bullet}_{G/k})"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega^{\bullet}_{F/k})"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, no head, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[X]/k}\right)
\end{tikzcd}$$

in Alg($\mathcal{M}$ixed) *such that the horizontal morphisms are equivalences.*

We will often refer to the existence of such a commutative diagram for a specific f as "Conjecture D holds for f". ♧

In Proposition 7.5.3.1 in Section 7.5.3 we will show that Conjecture D holds if $|X| \leq 1$ or $|X| = 2$ and 2 is invertible in k.

We will discuss Conjecture C for $|X| = 0$ in Section 7.5.1, for $|X| = 1$ in Section 7.5.2, and for $|X| = 2$ in Section 7.5.4. Conjecture D will be discussed in Section 7.5.3.

7.5.1. Conjecture C for zero variables in the domain

In this short section we prove Conjecture C in the case that the domain is a polynomial ring in zero variables, in which case Conjecture C is true for formal reasons.

Proposition 7.5.1.1. *Let X be totally ordered set satisfying $|X| \leq 2$. Then there exists a filler for the square*

$$\begin{tikzcd} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k/k}\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{\iota_{k[X]}/k}\right)"] \\ \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[X]/k}\right) \end{tikzcd} \tag{7.17}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$*, where the horizontal equivalences are the ones from Corollary 7.4.9.3 (for the top horizontal equivalence applied to the empty set).*

In particular, Conjecture C holds for $F = \iota_{k[X]}$ if $|X| \leq 2$. ♡

Proof. $\Omega^{\bullet}_{k/k}$ is isomorphic to k, the monoidal unit of Mixed, considered as an object of $\mathrm{Alg}(\mathsf{Mixed})$. As γ_{Mixed} is symmetric monoidal (see Construction 4.4.1.1), k is mapped by $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ to an initial object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ by [HA, 3.2.1.8]. That there is a filler for diagram (7.17) now follows purely from the universal property of initial objects. □

7.5.2. Conjecture C for one variable in the domain

In this section we turn to the much more involved proof that Conjecture C holds for morphisms $F\colon k[t] \to k[X]$ if $|X| \leq 1$ or $|X| = 2$ and 2 is invertible in k. Using that $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant in $\mathrm{Alg}(\mathsf{Mixed})$ it will be possible to obtain a morphism

$$\Omega'^{\bullet}_{F/k} \colon \Omega'^{\bullet}_{k[t]/k} \to \Omega'^{\bullet}_{k[X]/k}$$

in Alg(Mixed) so that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t]) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega'^{\bullet}_{k[t]/k}\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega'^{\bullet}_{F/k})"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega'^{\bullet}_{k[X]/k}\right)
\end{tikzcd} \tag{7.18}$$

in Alg($\mathcal{M}$ixed). If we could then show that the square

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[t]/k} \arrow[r, "\Theta_{\{t\}}", "\simeq"'] \arrow[d, "\Omega'^{\bullet}_{F/k}"'] & \Omega^{\bullet}_{k[t]/k} \arrow[d, "\Omega^{\bullet}_{F/k}"] \\
\Omega'^{\bullet}_{k[X]/k} \arrow[r, "\simeq", "\Theta_X"'] & \Omega^{\bullet}_{k[X]/k}
\end{tikzcd} \tag{7.19}$$

in Alg(Mixed) commutes (perhaps up to homotopy of algebras in strict mixed complexes), then we would be finished. If $|X| \leq 1$, then it follows from Remark 7.4.5.2 that we only need to check that the two compositions map t to the same element (as the other generators must map to zero for degree reasons), and this is something that is actually true, both compositions mapping t to $F(t)$.

However, if $X = \{ x_1, x_2 \}$ (which we give the total order $x_1 < x_2$), then we also need to check that the two compositions agree on $\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}$. Unfortunately, this will not be the case in general. $\Omega^{\bullet}_{k[t]/k}$ is zero in degree 2, so the composition along the top right will map $\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}$ to zero, but this is not necessarily the case for the composition along the bottom left. The idea to deal with this is to replace Θ_X by a different quasiisomorphism of algebras in strict mixed complexes λ. For λ to be a quasiisomorphism and have the correct value on $\Omega'^{\bullet}_{F/k}(t)$ we will want to set $\lambda(x_i) := x_i$. We have a lot of choice in how we define λ on the higher generators $\underline{y}$ for $y \in Y_n$, which we can choose nearly arbitrarily, the only real restriction being that the following must hold.

$$\mathrm{d}\big(\lambda\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big) + \lambda\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big) - \lambda\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big) \tag{7.20}$$

So how should we choose $\lambda(\underline{y})$ for $y \in Y_n$ for $n \geq 0$ in order to ensure that we have $\lambda(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})) = 0$ so that the analogue of diagram (7.19) commutes?

The main tool available to understand $\Omega'^{\bullet}_{F/k}$ is naturality of Φ as we showed it in Proposition 7.4.7.1, and we can use this to show that

$$\Phi_X\left(\Xi_X\left(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})\right)\right) = 0 \tag{7.21}$$

holds. As $\Phi_X \circ \Xi_X$ is a quasiisomorphism and maps x_i to x_i we could thus set λ to $\Phi_X \circ \Xi_X$ if only it were a morphism in $\mathrm{Alg}(\mathsf{Mixed})$! But unfortunately, Φ_X is only multiplicative but is not in general compatible with the strict mixed structure. What we could instead do is to try to define λ in such a way that $\Phi_X \circ \Xi_X$ and λ agree on $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$. As Φ_X is multiplicative and preserves d on elements of degree 0 by Proposition 7.4.8.1, $\Phi_X \circ \Xi_X$ and λ already agree on the $\mathbb{Z}$-graded k-subalgebra generated by elements x_i and $\mathrm{d}\, x_i$. If we for example choose

$$\lambda\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) \coloneqq \Phi_X\left(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\right)$$

then it would follow that the two morphisms would also agree on elements like $x_1 \cdot \mathrm{d}\, x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1}$.

However if terms involving $\mathrm{d}(\underline{x_1 \cdot x_2 - x_2 \cdot x_1})$ appeared in the element $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$, then we would not be able to deal with this, as we have no way of accessing where $\Phi_X \circ \Xi_X$ maps such an element. So as a first simplification step we need to make a particular choice for $\Omega'^{\bullet}_{F/k}$ for which $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$ is given by a k-linear combination of products of x_1, x_2, $\mathrm{d}\, x_1$, $\mathrm{d}\, x_2$, as well as elements of the form $\underline{y}$ for $y \in Y_n$, but without factors of the form $\mathrm{d}(\underline{y})$. This can be arranged as

$$\mathrm{d}\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) + \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}$$

is a boundary in $\Omega'^{\bullet}_{k[X]/k}$.

If we now just set $\lambda(\underline{y}) \coloneqq \Phi_X(\Xi_X(\underline{y}))$, then it would follow from (7.21) that

$$\lambda\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})\Big) = 0$$

holds as well, so that the analogue of diagram (7.19) commutes. However, the next hurdle is that (7.20) needs to be satisfied. So say if $\lambda(\underline{x_1 \cdot x_2 - x_2 \cdot x_1})$ had been defined in such a way as to be 0, then we must have

$$\lambda\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big) = \lambda\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)$$

and can not choose the two values independently. This is where the assumption that 2 is divisible in k comes in, because combining this assumption with choosing $\Omega'^{\bullet}_{F/k}$ such that $\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}$ and $\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}$ always contribute to $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$ in a pairwise manner we will be able to average out $\Phi_X(\Xi_X(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}))$ and $\Phi_X(\Xi_X(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}))$ between $\lambda\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)$ and $\lambda\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)$, and similarly deal with any possible contributions from $\mathrm{d}(\underline{x_1 \cdot x_2 - x_2 \cdot x_1})$.

We will begin putting this proof strategy into practice by first unpacking the data required to construct morphisms and homotopies with domain $\Omega'^{\bullet}_{k[t]/k}$ in Section 7.5.2.1. We will then be able to show existence of an appropriate morphism $\Omega'^{\bullet}_{F/k}$ in Section 7.5.2.2. Finally, we put everything together in Section 7.5.2.3 to prove that Conjecture C holds for morphisms $F\colon k[t] \to k[X]$ if $|X| \leq 1$ or $|X| = 2$ and 2 is invertible in k.

7.5.2.1. Morphisms and homotopies out of $\Omega'^{\bullet}_{k[t]/k}$

To put this proof strategy described in the introduction to Section 7.5.2 into practice we first need to construct a morphism $\Omega'^{\bullet}_{F/k}$ with the required properties. The next two propositions are helpful for that as they simplify the amount of data we need to provide and the amount of properties we need to check in order to construct morphisms out of $\Omega'^{\bullet}_{k[t]/k}$, and homotopies of such morphisms.

Proposition 7.5.2.1. *Let X be an object of* Alg(Mixed) *such that* $\mathrm{H}_*(X) \cong 0$ *for $* > 2$ and such that elements of* $\mathrm{H}_1(X)$ *square to zero. Let $\Omega'^{\bullet}_{k[t]/k}$ be as in Definition 7.4.5.9.*

Let F' be a map of $\mathbb{Z}$-graded sets from the subset $\{\, t, \underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t} \,\}$ of $\Omega'^{\bullet}_{k[t]/k}$ to X, and assume that $F'(t)$ is a cycle and that F' satisfies the following equality.

$$\partial(F'(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})) = F'(t) \cdot \mathrm{d}(F'(t)) - \mathrm{d}(F'(t)) \cdot F'(t) \tag{7.22}$$

Then F can be extended to a morphism

$$F \colon \Omega'^{\bullet}_{k[t]/k} \to X$$

in Alg(Mixed). ♡

Proof. We are going to use notation from the construction of $\Omega'^{\bullet}_{k[t]/k}$ in Construction 7.4.5.1 in this proof.

By the universal property of $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ and $k \cdot -$ we obtain a morphism $F_0 \colon A_0 \to X$ in Alg(Mixed) that maps t to $F'(t)$, where we need to use that $F'(t)$ is a cycle. As Y_0 is empty the morphism ι_0^1 is an isomorphism, so we immediately obtain an extension of F_0 to $F_1 \colon A_1 \to X$. Again by the universal property of $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ as well as pushouts in Alg(Mixed), we can extend F_1 to a morphism $F_2 \colon A_2 \to X$ in Alg(Mixed) satisfying $F_2(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}) = F'(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$ if and only if

$$\partial(F'(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})) = F_1(t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t)$$

holds. But this es precisely ensured by (7.22).

It now suffices to assume that $n \geq 2$ and $F_n \colon A_n \to X$ is a morphism in Alg(Mixed), and then to show that F_n can be extended to a morphism $F_{n+1} \colon A_{n+1} \to X$. Again by the universal property, this requires finding a value $F_{n+1}(\underline{y})$ for every $y \in Y_n$ such that

$$\partial\big(F_{n+1}(\underline{y})\big) = F_n(y)$$

holds. But y is a cycle of degree n in A_n by Construction 7.4.5.1 (a), so $F_n(y)$ is a cycle in degree n of X, and such a solution exists if and only if the homology class represented by $F_n(y)$ is zero. If $n > 2$ then this must trivially be true as then $\mathrm{H}_n(X) \cong 0$ by assumption. If instead $n = 2$, then the only element of Y_2 is $\mathrm{d}\,t \cdot \mathrm{d}\,t$. As $\mathrm{d}\,t$ is already a cycle, the homology class $[F_n(\mathrm{d}\,t \cdot \mathrm{d}\,t)]$ is equal to the square of $[F_n(\mathrm{d}\,t)]$ and hence zero by assumption that elements of $H_1(X)$ square to zero. □

Proposition 7.5.2.2. *Let X be an object of* $\mathrm{Alg}(\mathsf{Mixed})$ *such that* $\mathrm{H}_*(X) \cong 0$ *for $* > 2$. and let $\Omega'^{\bullet}_{k[t]/k}$ be as in Definition 7.4.5.9.*

Let

$$F, G\colon \Omega'^{\bullet}_{k[t]/k} \to X$$

be two morphisms $\mathrm{Alg}(\mathsf{Mixed})$*, and assume that the elements*

$$F(t) - G(t) \qquad \text{and} \qquad F(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}) - G(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$$

are boundaries in X.

Then there exists a homotopy of algebras of strict mixed complexes in the sense of Proposition 4.2.2.20 from F to G. ♡

Proof. We are going to use notation from the construction of $\Omega'^{\bullet}_{k[t]/k}$ in Construction 7.4.5.1 in this proof.

As the forgetful functor from $\mathrm{Alg}(\mathsf{Mixed})$ to $\mathbb{Z}$-graded k-modules preserves filtered colimits by Proposition 4.2.2.12 it suffices to construct compatible homotopies of algebras of strict mixed complexes h_n from $F \circ \iota_n$ to $G \circ \iota_n$ for every $n \geq 0$.

Let us begin by constructing the homotopy h_0. By Construction 7.4.5.1 the underlying $\mathbb{Z}$-graded k-algebra of A_0 is free on $\{ t, \mathrm{d}\,t \}$. Define h_0 on $\{ t \}$ by mapping t to an element whose boundary is $F(t) - G(t)$ (such an element exists by assumption). As t is a cycle Proposition 4.2.2.21 then immediately furnishes us with an extension to a homotopy of algebras of strict mixed complexes from $F \circ \iota_0$ to $G \circ \iota_0$.

We now assume that h_n has already been defined for $n \geq 0$, and show that h_n can be extended to h_{n+1}. By Proposition 4.2.2.21 and Remark 7.4.5.2 extending h_n to h_{n+1} amounts to finding a value for $h_{n+1}(\underline{y})$ for every element y in Y_n such that

$$\partial\big(h_{n+1}(\underline{y})\big) = F\big(\underline{y}\big) - G\big(\underline{y}\big) - h_n(y) \tag{$*$}$$

holds. We now distinguish between the case $n = 0$, $n = 1$, and $n \geq 2$.

If $n = 0$, then Y_n is empty, so nothing needs to be done. If $n = 1$, then we have that $Y_n = \{ t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t \}$, so we only need to consider the element $\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}$. By assumption $F(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}) - G(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$ is a boundary, so that it suffices to show that $h_0(t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t)$ is a boundary, which the following calculation does.

$$\begin{aligned}
&h_0(t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t) \\
&= h_0(t) \cdot G(\mathrm{d}\,t) + F(t) \cdot h_0(\mathrm{d}\,t) - h_0(\mathrm{d}\,t) \cdot G(t) + F(\mathrm{d}\,t) \cdot h_0(t) \\
&= h_0(t) \cdot \mathrm{d}(G(t)) - F(t) \cdot \mathrm{d}(h_0(t)) + \mathrm{d}(h_0(t)) \cdot G(t) + \mathrm{d}(F(t)) \cdot h_0(t)
\end{aligned}$$

$$
\begin{aligned}
&= h_0(t) \cdot \mathrm{d}(G(t)) + \mathrm{d}(F(t)) \cdot h_0(t) + \mathrm{d}(h_0(t)) \cdot G(t) - F(t) \cdot \mathrm{d}(h_0(t)) \\
&= h_0(t) \cdot \mathrm{d}(G(t)) - h_0(t) \cdot \mathrm{d}(F(t)) + \mathrm{d}(h_0(t)) \cdot G(t) - \mathrm{d}(h_0(t)) \cdot F(t) \\
&= -h_0(t) \cdot \mathrm{d}(F(t) - G(t)) - \mathrm{d}(h_0(t)) \cdot (F(t) - G(t)) \\
&= -h_0(t) \cdot \mathrm{d}(\partial(h_0(t))) - \mathrm{d}(h_0(t)) \cdot \partial(h_0(t)) \\
&= h_0(t) \cdot \partial(\mathrm{d}(h_0(t))) - \partial(h_0(t)) \cdot \mathrm{d}(h_0(t)) \\
&= -\partial(h_0(t) \cdot \mathrm{d}(h_0(t)))
\end{aligned}
$$

It remains to consider the case $n \geq 2$. Note that the right hand side of equation $(*)$ is in degree $n+1 > 2$, so as $\mathrm{H}_*(X)$ is concentrated in degrees $* \leq 2$ it suffices to show that the right hand side of equation $(*)$ is a cycle. This is shown via the following calculation, with $y \in Y_n$.

$$
\begin{aligned}
&\partial\big(F(\underline{y}) - G(\underline{y}) - h_n(y)\big) \\
&= F\big(\partial(\underline{y})\big) - G\big(\partial(\underline{y})\big) - \partial(h_n(y)) \\
&= F(y) - G(y) - \partial(h_n(y)) \\
&= h_n(\partial(y)) \\
&= h_n(0) \\
&= 0
\end{aligned}
$$

□

7.5.2.2. Construction of $\Omega'^{\bullet}_{F/k}$

In this section we show the existence of a morphism $\Omega'^{\bullet}_{F/k}$ of appropriate form to put the proof strategy described in the introduction to Section 7.5.2 into practice. In order to be able to properly describe what kind of form $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$ is supposed to have we need to simplify $\Omega'^{\bullet}_{k[X]/k}$ by making it commutative. We thus introduce appropriate notation in the following definition.

Definition 7.5.2.3. Let X be a totally ordered set satisfying $|X| \leq 2$, and let $\Omega'^{\bullet}_{k[X]/k}$ be as in Definition 7.4.5.9.

Then we define

$$\xi_X \colon \Omega'^{\bullet}_{k[X]/k} \to \Omega''^{\bullet}_{k[X]/k}$$

to be the morphism of $\mathbb{Z}$-graded k-algebras that is given by quotienting out the commutator, i. e. ξ_X is initial among morphisms of $\mathbb{Z}$-graded k-algebras with commutative codomain. We will usually not use special notation to distinguish between elements of $\Omega'^{\bullet}_{k[X]/k}$ and their images under ξ_X, but make clear from context in which of the two they lie. It follows from Remark 7.4.5.2 that the $\mathbb{Z}$-graded commutative k-algebra $\Omega''^{\bullet}_{k[X]/k}$ is freely generated (as a *commutative* $\mathbb{Z}$-graded k-algebra) by the elements x and $\mathrm{d}\,x$ for $x \in X$ and $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_n$ for $n \geq 0$. ◇

We wish to show that there exists a morphism $\Omega'^{\bullet}_{F/k}$ fitting into a square (7.18) and such that $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$ has a specific form[57]. As $\Omega'^{\bullet}_{F/k}$ has to be a morphism of algebras of strict mixed complexes we already know that the boundary will have to be of the following form.

$$\partial\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})\Big) = \Omega'^{\bullet}_{F/k}(t) \cdot \mathrm{d}\Big(\Omega'^{\bullet}_{F/k}(t)\Big) - \mathrm{d}\Big(\Omega'^{\bullet}_{F/k}(t)\Big) \cdot \Omega'^{\bullet}_{F/k}(t)$$

The strategy to obtain $\Omega'^{\bullet}_{F/k}$ where $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$ is of a specified form will be to first show that every commutator as on the right hand side of the equation is the boundary of an element E of degree 2 of $\Omega'^{\bullet}_{k[X]/k}$ that is of a certain form, and then show that, up to some small adjustments, we can construct $\Omega'^{\bullet}_{k[X]/k}$ in such a way that $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$ is precisely given by E. While the following proposition does not yet refer to $\Omega'^{\bullet}_{F/k}$ it is however the crucial preparatory result in its construction, ensuring that such an E of appropriate form exists.

Proposition 7.5.2.4. *Let X be the set $X = \{ x_1, x_2 \}$ equipped with the total order $x_1 < x_2$. In this proposition we are going to use Definitions 7.4.5.9 and 7.5.2.3.*

Let J be the $\mathbb{Z}$-graded subset of $\Omega''^{\bullet}_{k[X]/k}$ that consists of elements of degree 1 of the form $g \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1}$ with g an element of $k[X]$ and of elements of degree 2 of the form

$$\begin{aligned} &g_{\mathrm{d}\, x_1} \cdot \mathrm{d}\, x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\, x_2} \cdot \mathrm{d}\, x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \\ &+ g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big) \\ &+ g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} + g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2} \end{aligned}$$

with $g_{\mathrm{d}\, x_1}$, $g_{\mathrm{d}\, x_2}$, g_{both}, g_{same,x_1}, and g_{same,x_2} elements in $k[X]$. Denote by I the $\mathbb{Z}$-graded subset of $\Omega'^{\bullet}_{k[X]/k}$ that is the preimage of J under ξ_X.

Then the following holds.

(1) Every element of the form $w \cdot w' - w' \cdot w$ for w and w' elements of degree 0 in $\Omega'^{\bullet}_{k[X]/k}$ is the boundary of an element in I.

(2) Every element of the form $w \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot w + w' \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w'$ for w and w' elements of degree 0 in $\Omega'^{\bullet}_{k[X]/k}$ is the boundary of an element of I.

(3) Every element of the form $w \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w$ for w an element of degree 0 in $\Omega'^{\bullet}_{k[X]/k}$ is the boundary of an element of I. ♡

Proof. In this proof we will make use of notation from Construction 7.4.5.1 as well as repeatedly use Remark 7.4.5.2 without further comment.

[57] For example not involving $\mathrm{d}\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)$.

Before proving the claims let us note that I is closed under k-linear combinations as well as multiplying from either side with an element of X. Furthermore, the product of an element of I of degree 1 with $\mathrm{d}\,x_1$ or $\mathrm{d}\,x_2$ is an element of I again.

If w and w' are as in (1), then we will say that E is a *lift associated to w and w' as in (1)* to mean that E is an element of I such that $\partial(E) = w \cdot w' - w' \cdot w$. We use the analogous convention for (2) and (3).

We now begin by proving (1). For this we note that as $\Omega'^{\bullet}_{k[X]/k}$ is concentrated in nonnegative degrees, the element $w \cdot w' - w' \cdot w$ of degree 0 is a cycle. As $\Omega^{\bullet}_{k[X]/k}$ is commutative, the commutator $w \cdot w' - w' \cdot w$ must be mapped to 0 by Θ_0. $\mathrm{H}_0(\Theta_1)$ is an isomorphism by Proposition 7.4.5.10, so this implies that there is an element E' in degree 1 of A_1 whose boundary is $w \cdot w' - w' \cdot w$. By Remark 7.4.5.2 E' can be written as $E' = E'' + E$ where E'' is an element of A_0 and E is in the k-submodule generated by words in X with one extra factor $\underline{x_1 \cdot x_2 - x_2 \cdot x_1}$, so that E is an element of I. As $\partial(E'') = 0$ we already have $\partial(E) = w \cdot w' - w' \cdot w$, which finishes the proof of (1).

Now we show claim (2), which we will do by reducing to more and more specific w and w', and using claim (1). First assume that w_1, w_2, w'_1 and w'_2 are elements of degree 0 of $\Omega'^{\bullet}_{k[X]/k}$ such that (2) holds for the pair (w_1, w'_1) with associated lift E_{11}, for (w_1, w'_2) with associated lift E_{12}, for (w_2, w'_1) with associated lift E_{21}, and for (w_2, w'_2) with associated lift E_{22}. Let a_1 and a_2 be elements of k. Then we claim that (2) also holds for the pair $(a_1 \cdot w_1 + a_2 \cdot w_2, a_1 \cdot w'_1 + a_2 \cdot w'_2)$, with the following associated lift.

$$E = a_1 \cdot a_1 \cdot E_{11} + a_1 \cdot a_2 \cdot E_{12} + a_1 \cdot a_2 \cdot E_{21} + a_2 \cdot a_2 \cdot E_{22}$$

That E is again an element of I follows from the argument at the start of this proof, and that the boundary is what it should be is verified by the following calculation.

$$\begin{aligned}
&(a_1 \cdot w_1 + a_2 \cdot w_2) \cdot \mathrm{d}(a_1 \cdot w'_1 + a_2 \cdot w'_2) \\
&\quad - \mathrm{d}(a_1 \cdot w'_1 + a_2 \cdot w'_2) \cdot (a_1 \cdot w_1 + a_2 \cdot w_2) \\
&\quad + (a_1 \cdot w'_1 + a_2 \cdot w'_2) \cdot \mathrm{d}(a_1 \cdot w_1 + a_2 \cdot w_2) \\
&\quad - \mathrm{d}(a_1 \cdot w_1 + a_2 \cdot w_2) \cdot (a_1 \cdot w'_1 + a_2 \cdot w'_2) \\
= {}& a_1 \cdot a_1 \cdot w_1 \cdot \mathrm{d}(w'_1) + a_1 \cdot a_2 \cdot w_1 \cdot \mathrm{d}(w'_2) \\
&\quad + a_2 \cdot a_1 \cdot w_2 \cdot \mathrm{d}(w'_1) + a_2 \cdot a_2 \cdot w_2 \cdot \mathrm{d}(w'_2) \\
&\quad - a_1 \cdot a_1 \cdot \mathrm{d}(w'_1) \cdot w_1 - a_1 \cdot a_2 \cdot \mathrm{d}(w'_1) \cdot w_2 \\
&\quad - a_2 \cdot a_1 \cdot \mathrm{d}(w'_2) \cdot w_1 - a_2 \cdot a_2 \cdot \mathrm{d}(w'_2) \cdot w_2 \\
&\quad + a_1 \cdot a_1 \cdot w'_1 \cdot \mathrm{d}(w_1) + a_1 \cdot a_2 \cdot w'_1 \cdot \mathrm{d}(w_2) \\
&\quad + a_2 \cdot a_1 \cdot w'_2 \cdot \mathrm{d}(w_1) + a_2 \cdot a_2 \cdot w'_2 \cdot \mathrm{d}(w_2) \\
&\quad - a_1 \cdot a_1 \cdot \mathrm{d}(w_1) \cdot w'_1 - a_1 \cdot a_2 \cdot \mathrm{d}(w_1) \cdot w'_2 \\
&\quad - a_2 \cdot a_1 \cdot \mathrm{d}(w_2) \cdot w'_1 - a_2 \cdot a_2 \cdot \mathrm{d}(w_2) \cdot w'_2
\end{aligned}$$

$$
\begin{aligned}
&= a_1 \cdot a_1 \cdot w_1 \cdot \mathrm{d}(w_1') - a_1 \cdot a_1 \cdot \mathrm{d}(w_1') \cdot w_1 \\
&\quad + a_1 \cdot a_1 \cdot w_1' \cdot \mathrm{d}(w_1) - a_1 \cdot a_1 \cdot \mathrm{d}(w_1) \cdot w_1' \\
&\quad + a_1 \cdot a_2 \cdot w_1 \cdot \mathrm{d}(w_2') - a_1 \cdot a_2 \cdot \mathrm{d}(w_2') \cdot w_1 \\
&\quad + a_1 \cdot a_2 \cdot w_2' \cdot \mathrm{d}(w_1) - a_1 \cdot a_2 \cdot \mathrm{d}(w_1) \cdot w_2' \\
&\quad + a_1 \cdot a_2 \cdot w_2 \cdot \mathrm{d}(w_1') - a_1 \cdot a_2 \cdot \mathrm{d}(w_1') \cdot w_2 \\
&\quad + a_1 \cdot a_2 \cdot w_1' \cdot \mathrm{d}(w_2) - a_1 \cdot a_2 \cdot \mathrm{d}(w_2) \cdot w_1' \\
&\quad + a_2 \cdot a_2 \cdot w_2 \cdot \mathrm{d}(w_2') - a_2 \cdot a_2 \cdot \mathrm{d}(w_2') \cdot w_2 \\
&\quad + a_2 \cdot a_2 \cdot w_2' \cdot \mathrm{d}(w_2) - a_2 \cdot a_2 \cdot \mathrm{d}(w_2) \cdot w_2' \\
&= \partial(a_1 \cdot a_1 \cdot E_{11} + a_1 \cdot a_2 \cdot E_{12} + a_1 \cdot a_2 \cdot E_{21} + a_2 \cdot a_2 \cdot E_{22})
\end{aligned}
$$

By the above argument it not suffices to show claim (2) for pairs (w, w') of elements of degree 0 of $\Omega'^{\bullet}_{k[X]/k}$ that are in a k-basis. By Remark 7.4.5.2 it thus suffices to consider the case in which w and w' are words in X. If w is a word of length 0 (i. e. $w = 1$), then $w \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot w + w' \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w' = 0$, so that we can use 0 as an associated lift. Now assume that we have shown (2) for pairs (w, w') where the length of w is smaller or equal to n, for $n \geq 1$, and that w is a word of length n and x an element of X. Then we claim that (2) also holds for $(x \cdot w, w')$. Indeed, let $E_{w,w'}$ be a lift associated to the pair (w, w') and $E_{x,w'}$ a lift associated to the pair (x, w') as in (2), and let $E_{w',w}$ be a lift of $w' \cdot w - w \cdot w'$ and $E_{w',x}$ be a lift of $w' \cdot x - x \cdot w'$ as in (1). Then $E = x \cdot E_{w,w'} + E_{x,w'} \cdot w + \mathrm{d}(x) \cdot E_{w',w} + E_{w',x} \cdot \mathrm{d}(w)$ is again in I and the following calculation then shows that this E is a lift associated to to the pair $(x \cdot w, w')$ as in (2).

$$
\begin{aligned}
&x \cdot w \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot x \cdot w + w' \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot w' \\
&= x \cdot w \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot x \cdot w \\
&\quad + w' \cdot \mathrm{d}(x) \cdot w + w' \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot w' - x \cdot \mathrm{d}(w) \cdot w' \\
&= x \cdot (w \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot w + w' \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w') \\
&\quad + x \cdot \mathrm{d}(w') \cdot w - x \cdot w' \cdot \mathrm{d}(w) - \mathrm{d}(w') \cdot x \cdot w \\
&\quad + w' \cdot \mathrm{d}(x) \cdot w + w' \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot w' \\
&= x \cdot (w \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot w + w' \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w') \\
&\quad + (x \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot x + w' \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot w') \cdot w \\
&\quad + \mathrm{d}(x) \cdot w' \cdot w - x \cdot w' \cdot \mathrm{d}(w) + w' \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot w' \\
&= x \cdot (w \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot w + w' \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w') \\
&\quad + (x \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot x + w' \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot w') \cdot w \\
&\quad + \mathrm{d}(x) \cdot (w' \cdot w - w \cdot w') + (w' \cdot x - x \cdot w') \cdot \mathrm{d}(w) \\
&= \partial(x \cdot E_{w,w'} + E_{x,w'} \cdot w + \mathrm{d}(x) \cdot E_{w',w} + E_{w',x} \cdot \mathrm{d}(w)) \\
&= \partial(E)
\end{aligned}
$$

It now remains to show (2) for pairs (x, w') where x is an element of X and w' is a word in X. With a completely analogous argument as the one we just carried out, this time for w' instead of w, we can even reduce to the case of pairs (x, x') with x and x' elements of X. But for such pairs

$$E = \underline{x \cdot \mathrm{d}(x') - \mathrm{d}(x') \cdot x} + \underline{x' \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x'}$$

works as an associated lift.

We now turn to showing claim (3), which we do using a similar strategy as (2). First assume that w and w' are elements of degree 0 of $\Omega'^{\bullet}_{k[X]/k}$ such that (3) holds for w with associated lift E_w, and for w' with associated lift $E_{w'}$. Let a and a' be elements of k and let $E_{w,w'}$ be a lift associated to the pair (w, w') as in (2). Then we claim that (3) also holds for $a \cdot w + a' \cdot w'$ with associated lift $E = a \cdot a \cdot E_w + a' \cdot a' \cdot E_{w'} + a \cdot a' \cdot E_{w,w'}$. That E is again an element of I is covered by the argument at the start of the proof, and the following calculation checks that the boundary is correct as well.

$$\begin{aligned}
&(a \cdot w + a' \cdot w') \cdot \mathrm{d}(a \cdot w + a' \cdot w') - \mathrm{d}(a \cdot w + a' \cdot w') \cdot (a \cdot w + a' \cdot w') \\
&= a \cdot a \cdot w \cdot \mathrm{d}(w) + a \cdot a' \cdot w \cdot \mathrm{d}(w') + a' \cdot a \cdot w' \cdot \mathrm{d}(w) + a' \cdot a' \cdot w' \cdot \mathrm{d}(w') \\
&\quad - a \cdot a \cdot \mathrm{d}(w) \cdot w - a \cdot a' \cdot \mathrm{d}(w) \cdot w' - a' \cdot a \cdot \mathrm{d}(w') \cdot w - a' \cdot a' \cdot \mathrm{d}(w') \cdot w' \\
&= a \cdot a \cdot w \cdot \mathrm{d}(w) - a \cdot a \cdot \mathrm{d}(w) \cdot w + a' \cdot a' \cdot w' \cdot \mathrm{d}(w') - a' \cdot a' \cdot \mathrm{d}(w') \cdot w' \\
&\quad + a \cdot a' \cdot w \cdot \mathrm{d}(w') - a \cdot a' \cdot \mathrm{d}(w') \cdot w + a \cdot a' \cdot w' \cdot \mathrm{d}(w) - a \cdot a' \cdot \mathrm{d}(w) \cdot w' \\
&= \partial(a \cdot a \cdot E_w + a' \cdot a' \cdot E_{w'} + a \cdot a' \cdot E_{w,w'})
\end{aligned}$$

It now suffices to show (3) for words in X. Assume that we have already shown (3) for words in X of length smaller or equal to n, and that $n \geq 1$. Let x be an element of X and w a word in X of length n. Let $E_{xw,x}$ be a lift for the pair $(x \cdot w, x)$ as in (2), E_x a lift for x as in (3), E_w a lift for w as in (3), and $E_{w,x}$ a lift for the pair (w, x) as in (1). Then $E = E_{xw,x} \cdot w - E_x \cdot w \cdot w + x \cdot x \cdot E_w + x \cdot E_{w,x} \cdot \mathrm{d}(w)$ is again in I and the following calculation shows that E is a lift for $x \cdot w$ as in (3).

$$\begin{aligned}
&x \cdot w \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x \cdot w \\
&= x \cdot w \cdot \mathrm{d}(x) \cdot w + x \cdot w \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot x \cdot w - x \cdot \mathrm{d}(w) \cdot x \cdot w \\
&= (x \cdot w \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x \cdot w + x \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x) \cdot w \\
&\quad + \mathrm{d}(x) \cdot x \cdot w \cdot w - x \cdot \mathrm{d}(x \cdot w) \cdot w + \mathrm{d}(x \cdot w) \cdot x \cdot w \\
&\quad + x \cdot w \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot x \cdot w - x \cdot \mathrm{d}(w) \cdot x \cdot w \\
&= (x \cdot w \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x \cdot w + x \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x) \cdot w \\
&\quad + \mathrm{d}(x) \cdot x \cdot w \cdot w - x \cdot \mathrm{d}(x) \cdot w \cdot w - x \cdot x \cdot \mathrm{d}(w) \cdot w \\
&\quad + \mathrm{d}(x) \cdot w \cdot x \cdot w + x \cdot \mathrm{d}(w) \cdot x \cdot w \\
&\quad + x \cdot w \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot x \cdot w - x \cdot \mathrm{d}(w) \cdot x \cdot w \\
&= (x \cdot w \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x \cdot w + x \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x) \cdot w \\
&\quad + \mathrm{d}(x) \cdot x \cdot w \cdot w - x \cdot \mathrm{d}(x) \cdot w \cdot w - x \cdot x \cdot \mathrm{d}(w) \cdot w + x \cdot w \cdot x \cdot \mathrm{d}(w)
\end{aligned}$$

$$
\begin{aligned}
&= (x \cdot w \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x \cdot w + x \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x) \cdot w \\
&\quad - (x \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x) \cdot w \cdot w + x \cdot x \cdot (w \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w) \\
&\quad - x \cdot x \cdot w \cdot \mathrm{d}(w) + x \cdot w \cdot x \cdot \mathrm{d}(w) \\
&= (x \cdot w \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x \cdot w + x \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x) \cdot w \\
&\quad - (x \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x) \cdot w \cdot w + x \cdot x \cdot (w \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w) \\
&\quad + x \cdot (w \cdot x - x \cdot w) \cdot \mathrm{d}(w) \\
&= \partial(E_{xw,x} \cdot w - E_x \cdot w \cdot w + x \cdot x \cdot E_w + x \cdot E_{w,x} \cdot \mathrm{d}(w))
\end{aligned}
$$

It thus only remains to show (3) for the elements 1, x_1, and x_2. For 1 we obtain $1 \cdot \mathrm{d}(1) - \mathrm{d}(1) \cdot 1 = 0$, so that we can use 0 as a lift. For x either x_1 or x_2 we can use $\underline{x \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x}$ as a lift. □

With the preparation of Proposition 7.5.2.4 we can now construct a morphism $\Omega'^{\bullet}_{F/k}$ with the required properties in the following proposition.

Proposition 7.5.2.5. *Let X be a totally ordered set satisfying $|X| \leq 2$, and denote the elements of X by $x_1 < \cdots < x_{|X|}$. Let f be an element of $k[X]$, and denote by $F\colon k[t] \to k[X]$ the morphism of commutative k-algebras that maps t to f.*

Then there exists a morphism

$$\Omega'^{\bullet}_{F/k} : \Omega'^{\bullet}_{k[t]/k} \to \Omega'^{\bullet}_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Mixed})$ *such that there exists a commutative diagram*

$$
\begin{tikzcd}
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_{\{t\}})", "\simeq"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega'^{\bullet}_{F/k})"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(\{t\})\Big) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\widetilde{\mathrm{C}}(F))"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big) \arrow[r, "\simeq", "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(X)\Big)
\end{tikzcd}
$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ *where $\Xi_{\{t\}}$ and Ξ_X are as in Construction 7.4.9.1, and such that $\xi_X \circ \Omega'^{\bullet}_{F/k}$ maps t to f (see Definition 7.5.2.3 for a definition of ξ_X).*

If $|X| = 2$, then $\Omega'^{\bullet}_{F/k}$ can furthermore be chosen such that there additionally exist elements $g_{\mathrm{d}\,x_1}$, $g_{\mathrm{d}\,x_2}$, g_{both}, g_{same,x_1}, g_{same,x_2}, and g_{obs} in $k[X]$ such that

$$
\begin{aligned}
&\xi_X\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})\Big) \qquad (7.23) \\
&= g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \\
&\quad + g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big) \\
&\quad + g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} + g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \\
&\quad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2
\end{aligned}
$$

holds in $\Omega''^{\bullet}_{k[X]/k}$. ♡

Proof. As $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant as an object of the model category $\mathrm{Alg}(\mathsf{Mixed})$ by Proposition 7.4.5.11, we can lift the composition

$$\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)^{-1} \circ \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(F)\Big) \circ \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(\Xi_{\{t\}}\big)$$

to a morphism

$$G\colon \Omega'^{\bullet}_{k[t]/k} \to \Omega'^{\bullet}_{k[X]/k}$$

that thus comes with a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_{\{t\}})", "\simeq"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(G)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(\{t\})\Big) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\widetilde{\mathrm{C}}(F))"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big) \arrow[r, "\simeq", "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(X)\Big)
\end{tikzcd} \tag{7.24}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. It now suffices by [Hov99, 1.2.10 (ii)] and Propositions A.1.0.1 and 4.2.2.20 to show that there is a homotopy of algebras of strict mixed complexes from G to a morphism $\Omega'^{\bullet}_{F/k}$ that takes the required form on the elements t and $\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}$.

We begin by showing that G already maps t to an acceptable value. For this we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\Omega'^{\bullet}_{k[t]/k}\Big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(G)"] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma)(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_{\{t\}}))"] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_X))"', "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}(\{t\})\Big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\widetilde{\mathrm{C}}(F))"] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma)(\Phi_{\{t\}})"] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}(X)\Big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)(\Phi_X)"', "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Omega^{\bullet}_{F/k})"] & \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where the top square is obtained from the transpose of diagram (7.24) by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ and using compatibility with γ_{Mixed} (see Construction 4.4.1.1), and the bottom square is the one from Proposition 7.4.7.1. The underlying differential graded k-algebra of $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant by Propositions 7.4.5.11 and 4.2.2.12, so we can conclude by [Hov99, 1.2.10 (ii)], Propositions A.1.0.1 and 4.2.2.17 that there exists a homotopy of differential graded k-algebras h from $\Phi_X \circ \Xi_X \circ G$ to $\Omega^\bullet_{F/k} \circ \Phi_{\{t\}} \circ \Xi_{\{t\}}$ (we omit forgetful functors in the notation here) in the sense of Proposition 4.2.2.17. We can then carry out the following calculation, where we use that $(\Phi_{\{t\}} \circ \Xi_{\{t\}})(t) = t$ by definition of $\Xi_{\{t\}}$, see around equation (7.16) of Construction 7.4.9.1.

$$\begin{aligned}(\Phi_X \circ \Xi_X \circ G)(t) &= \Big(\Omega^\bullet_{F/k} \circ \Phi_{\{t\}} \circ \Xi_{\{t\}}\Big)(t) + \partial(h(t)) + h(\partial(t))\\ &= \Omega^\bullet_{F/k}\big((\Phi_{\{t\}} \circ \Xi_{\{t\}})(t)\big) + 0 + h(0)\\ &= \Omega^\bullet_{F/k}(t)\\ &= f\end{aligned}$$

By the universal property of ξ_X there exists a commutative diagram

$$\begin{tikzcd} \Omega'^\bullet_{k[X]/k} \arrow[r, "\Xi_X"] \arrow[d, "\xi_X"'] & \widetilde{\mathrm{C}}(X) \arrow[r, "\Phi_X"] & \Omega^\bullet_{k[X]/k} \\ \Omega''^\bullet_{k[X]/k} \arrow[urr, dashed, bend right=10] & & \end{tikzcd}$$

of $\mathbb{Z}$-graded k-algebras, and as $\Phi_X \circ \Xi_X$ maps elements x_i of X to x_i by Construction 7.4.9.1, it follows from Remark 7.4.5.2 that the dashed morphism is an isomorphism in degree 0, mapping x_i to x_i. That $(\Phi_X \circ \Xi_X)(G(t)) = f$ thus implies that $\xi_X(G(t)) = f$.

If $|X| < 2$ we can now define $\Omega'^\bullet_{F/k} \coloneqq G$ and are finished. So from now on we will assume that $X = \{\, x_1, x_2 \,\}$. Unfortunately the value of G at $\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}$ is not automatically of the right form, so we will need to replace G by a homotopic morphism that takes a different value at $\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}$, but the same one at t.

By Proposition 7.5.2.4 (3) we can let E be an element of degree 2 in $\Omega'^\bullet_{k[X]/k}$ satisfying the following two properties.

(1) $\partial(E) = G(t) \cdot \mathrm{d}(G(t)) - \mathrm{d}(G(t)) \cdot G(t)$

(2) There exist elements $g_{\mathrm{d}\,x_1}$, $g_{\mathrm{d}\,x_2}$, g_{both}, g_{same,x_1}, and g_{same,x_2} in $k[X]$ such that

$$\begin{aligned}&\xi_X(E)\\ &= g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1}\\ &\quad + g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big)\\ &\quad + g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} + g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}\end{aligned}$$

holds.

We first note that

$$\partial(G(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}) - E) = G(t)\cdot \mathrm{d}(G(t)) - \mathrm{d}(G(t))\cdot G(t) - \partial(E) = 0$$

so that $G(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}) - E$ is a cycle. As Θ_0 (see the construction of $\Omega'^{\bullet}_{k[X]/k}$ in Construction 7.4.5.1) is surjective on homology by Proposition 7.4.5.10, we can find a cycle z in A_0 such that the homology classes represented by z and $G(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}) - E$ map to the same homology class in $\Omega^{\bullet}_{k[X]/k}$ under Θ_X. As Θ_X is a quasiisomorphism by Proposition 7.4.5.11 this implies that

$$G(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}) - E - z \tag{$**$}$$

must be a boundary.

We now want to apply Proposition 7.5.2.1 to obtain a morphism

$$\Omega'^{\bullet}_{F/k}\colon \Omega'^{\bullet}_{k[t]/k} \to \Omega'^{\bullet}_{k[X]/k}$$

in Alg(Mixed) with $\Omega'^{\bullet}_{F/k}(t) = G(t)$ and $\Omega'^{\bullet}_{F/k}(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}) = E + z$. We first note that as $\mathrm{H}_*(\Phi_X \circ \Xi_X)$ is a multiplicative isomorphism by Definition 7.4.4.2 and Proposition 7.4.9.2 it holds that $\mathrm{H}_*(\Omega'^{\bullet}_{k[X]/k})$ is zero above degree 2 and that odd degree elements square to zero. That $G(t)$ is a cycle is clear as G is a morphism of chain complexes and t is a cycle in $\Omega'^{\bullet}_{k[t]/k}$. Finally, (7.22) holds in this context, as this follows from (1) above combined with z being a cycle. Thus we can apply Proposition 7.5.2.1 to obtain a morphism $\Omega'^{\bullet}_{F/k}$ with the prescribed values.

We next show that $\Omega'^{\bullet}_{F/k}$ is indeed homotopic to G. For this we use Proposition 7.5.2.2, so that we have to show that

$$G(t) - \Omega'^{\bullet}_{F/k}(t) \qquad \text{and} \qquad G(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}) - \Omega'^{\bullet}_{F/k}(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t})$$

are boundaries. The first term is 0 by definition, and that the second is a boundary was ensured around ($**$) (we chose z specifically so that this would hold). Thus Proposition 7.5.2.1 applies to show that there indeed exists a homotopy of algebras in strict mixed complexes from G to $\Omega'^{\bullet}_{F/k}$.

It remains to show that the two values of $\xi_X \circ \Omega'^{\bullet}_{F/k}$ are as required. For t this is clear as

$$\xi_X\left(\Omega'^{\bullet}_{F/k}(t)\right) = \xi_X(G(t)) = f$$

holds, as we discussed at the start of this proof. For $\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}$ we note that the image of $\xi_X \circ \Theta_0$ is generated by the multiplicative generators x_1, x_2, $\mathrm{d}\,x_1$, and $\mathrm{d}\,x_2$. Thus the element z of degree 2 in A_0 must map to an element of the form $g_{\mathrm{obs}}\cdot \mathrm{d}\,x_1\cdot \mathrm{d}\,x_2$ with g_{obs} an element of $k[x_1, x_2]$. Then we obtain the following by combining the definition just made with (2).

$$\begin{aligned}&\xi_X\left(\Omega'^{\bullet}_{F/k}(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t})\right)\\ &= \xi_X(E) + \xi_X(z)\end{aligned}$$

$$\begin{aligned}
&= g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \\
&\quad + g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big) \\
&\quad + g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} + g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \\
&\quad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2
\end{aligned}$$ □

7.5.2.3. Conclusion

Having constructed $\Omega'^{\bullet}_{F/k}$ in the preceding Section 7.5.2.2 we can now use it to show Conjecture C in certain cases using the strategy sketched in the introduction to Section 7.5.2. Note that what we show is actually slightly stronger than Conjecture C, as we show that there is a specific top horizontal equivalence in diagram (7.25) that is independent of X and f. This is what allows us to even deduce Conjecture D from this, as we do in Proposition 7.5.3.1 in Section 7.5.3.

Proposition 7.5.2.6. *Let X be a set, let f be an element of $k[X]$, and denote by $F\colon k[t] \to k[X]$ the morphism of commutative k-algebras that maps t to f. Assume that one of the following holds.*

(1) $|X| \leq 1$.

(2) $|X| = 2$ *and* 2 *is invertible in* k.

Then there exists a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t]) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[t]/k}\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega'^{\bullet}_{F/k})"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[X]/k}\right)
\end{tikzcd} \tag{7.25}$$

in Alg($\mathcal{M}$ixed) *such that the top horizontal equivalence is the one from Corollary 7.4.9.3 and the bottom horizontal morphism is an equivalence*[58]*. In particular, Conjecture C holds for F.* ♡

Proof. We begin by equipping X with a total order, and will denote the elements of X by $x_1 < \cdots < x_{|X|}$. Consider the following (non-commuting)

[58] We do *not* claim that there exists a filler for such a square where also the bottom horizontal equivalence is given by the one from Corollary 7.4.9.3.

diagram in Alg($\mathcal{M}$ixed), that will be explained below.

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t]) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"] \arrow[d, dash, "\simeq"] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(\{t\})\right) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\widetilde{\mathrm{C}}(F))"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(X)\right) \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega'^{\bullet}_{k[t]/k}\right) \arrow[u, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_{\{t\}})", "\simeq"'] \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega'^{\bullet}_{F/k})"] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Theta_{\{t\}})"', "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega'^{\bullet}_{k[X]/k}\right) \arrow[u, "\simeq", "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)"'] \arrow[d, dashed, "\simeq"'] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[t]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega^{\bullet}_{F/k})"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[X]/k}\right)
\end{tikzcd} \tag{$*$}$$

The top square has a filler given by the (transpose of) commutative diagram (7.10) from the definition of $\widetilde{\mathrm{C}}(F)$ in Construction 7.4.2.5, $\Xi_{\{t\}}$, Ξ_X, and $\Theta_{\{t\}}$ are as in Construction 7.4.9.1 and Definition 7.4.5.9, and $\Omega'^{\bullet}_{F/k}$ is as in Proposition 7.5.2.5 so that the middle square has a filler as well.

By Corollary 7.4.9.3 the vertical composition on the left is the top horizontal equivalence in diagram (7.25) from the statement. As the top and middle square have fillers it thus suffices to construct a quasiisomorphism of algebras in strict mixed complexes

$$\lambda\colon \Omega'^{\bullet}_{k[X]/k} \to \Omega^{\bullet}_{k[X]/k}$$

such that the diagram

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[t]/k} \arrow[r, "\Theta_{\{t\}}", "\simeq"'] \arrow[d, "\Omega'^{\bullet}_{F/k}"'] & \Omega^{\bullet}_{k[t]/k} \arrow[d, "\Omega^{\bullet}_{F/k}"] \\
\Omega'^{\bullet}_{k[X]/k} \arrow[r, dashed, "\simeq", "\lambda"'] & \Omega^{\bullet}_{k[X]/k}
\end{tikzcd} \tag{$**$}$$

in Alg(Mixed) commutes strictly.

Suppose for the moment that we have defined a λ. Using notation from Construction 7.4.5.1, it follows from Remark 7.4.5.2 that for checking strict commutativity of $(**)$ it suffices to check that the diagram commutes on the element t as well as elements of the form $\underline{y}$ for y an element of one of the sets Y_m for $m \geq 0$ used in the definition of $\Omega'^{\bullet}_{k[t]/k}$. But elements of Y_m have degree m so that $\underline{y}$ is of degree $m+1$. As we assume $|X| \leq 2$, we have that $\Omega^{\bullet}_{k[X]/k}$ is concentrated in degrees at most 2, so diagram $(**)$ will commute on elements $\underline{y}$ for y an element of Y_m for $m \geq 2$ automatically, and if even $|X| \leq 1$ then it will commute automatically on such elements for $m \geq 1$. As Y_0 is empty by Definition 7.4.5.9 and Proposition 7.4.5.6 and Y_1 has only one element $t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t$ by Definition 7.4.5.9 and Proposition 7.4.5.7, this means that it suffices to check that the following two equations hold if $|X| = 2$, and only that the first one holds if $|X| \leq 1$.

$$\lambda\Big(\Omega'^{\bullet}_{F/k}(t)\Big) = \Omega^{\bullet}_{F/k}\big(\Theta_{\{t\}}(t)\big)$$
$$\lambda\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})\Big) = \Omega^{\bullet}_{F/k}\big(\Theta_{\{t\}}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})\big)$$

We can evaluate the right hand sides. By definition $\Theta_{\{t\}}$ maps t to t and $\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}$ to 0. Thus we need to define λ such that it is a quasiisomorphism and show that both of the following equations hold if $|X| = 2$, and that the first one holds if $|X| \leq 1$.

$$\lambda\Big(\Omega'^{\bullet}_{F/k}(t)\Big) = f \tag{$***$}$$
$$\lambda\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})\Big) = 0$$

We can now already show the statement under the assumption that $|X| \leq 1$. In that case, we let λ be the quasiisomorphism of algebras in strict mixed complexes Θ_X from Definition 7.4.5.9. We only need to verify that the first equation of $(***)$ holds for this choice of λ. As the underlying $\mathbb{Z}$-graded k-algebra of $\Omega^{\bullet}_{k[X]/k}$ is commutative, the underlying morphism of λ factors as in the following diagram of $\mathbb{Z}$-graded k-algebras.

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[X]/k} \arrow[rr, "\lambda"] \arrow[d, "\xi_X"'] & & \Omega^{\bullet}_{k[X]/k} \\
\Omega''^{\bullet}_{k[X]/k} \arrow[urr, dashed, bend right, "\lambda''"'] & &
\end{tikzcd}$$

As λ and ξ_X map the elements x_i of X to x_i (considered as elements of the respective $\mathbb{Z}$-graded k-algebras), the same holds for λ'', so that in particular $\lambda''(f) = f$. By Proposition 7.5.2.5 we know that $\xi_X\Big(\Omega'^{\bullet}_{F/k}(t)\Big) = f$, so it follows that

$$\lambda\Big(\Omega'^{\bullet}_{F/k}(t)\Big) = \lambda''\Big(\xi_X\Big(\Omega'^{\bullet}_{F/k}(t)\Big)\Big) = \lambda''(f) = f$$

holds.

We now consider the case $|X| = 2$, and thus assume that 2 is invertible in k. In this case setting λ to Θ_X will unfortunately not work in general. We will in the following use notation from the construction of $\Omega'^{\bullet}_{k[X]/k}$ in Construction 7.4.5.1, as well as the concrete choices for Y_0, Y_1 and Y_2 in Definition 7.4.5.9. We will define λ using the universal property of the definition of $\Omega'^{\bullet}_{k[X]/k}$ as a colimit by constructing a compatible system of morphisms $\lambda_n\colon A_n \to \Omega^{\bullet}_{k[X]/k}$ in $\mathrm{Alg}(\mathsf{Mixed})$ for every $n \geq 0$.

We will begin by defining λ_0 using the universal property of $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ by prescribing $\lambda_0(x_i) = x_i$. We first note that the argument that we used above in the case $|X| \leq 1$ to show that the first equation of $(***)$ holds did not use that $\lambda = \Theta_X$, but only that λ maps x_i to x_i, and hence this argument is still applicable. Thus it only remains to show that λ_0 can be extended to a morphism $\lambda\colon \Omega'^{\bullet}_{k[X]/k} \to \Omega^{\bullet}_{k[X]/k}$ in $\mathrm{Alg}(\mathsf{Mixed})$ that is a quasiisomorphism and that is such that the second equation of $(***)$ holds.

We claim that *any* extension of λ_0 to λ is automatically a quasiisomorphism. For this we note that the $\mathbb{Z}$-graded subset

$$\left\{ \left[x_1^{a_1} \cdot x_2^{a_2} \cdot \mathrm{d}(x_1)^{b_1} \cdot \mathrm{d}(x_2)^{b_2}\right] \;\middle|\; a_1, a_2 \geq 0,\; b_1, b_2 \in \{0,1\} \right\}$$

of $\mathrm{H}_*(\Omega'^{\bullet}_{k[X]/k})$ forms a k-basis of $\mathrm{H}_*(\Omega'^{\bullet}_{k[X]/k})$, as $\mathrm{H}_*(\Theta_X)$ is an isomorphism and maps this set to the set with the same description (see Construction 7.4.5.1 and Proposition 7.4.5.11). As this subset is also mapped by $\mathrm{H}_*(\lambda)$ to the same subset of $\mathrm{H}_*(\Omega^{\bullet}_{k[X]/k})$ it follows that λ is a quasiisomorphism as well.

It thus suffices to show that there is some extension of λ_0 to λ such that the second equation of $(***)$ holds. We will now inductively assume that λ_n has already been defined for $n \geq 0$ and then extend λ_n to λ_{n+1}. By construction such an extension amounts to defining a value for $\lambda_{n+1}(\underline{y})$ for every element y of Y_n, and showing that

$$\partial\big(\lambda_{n+1}(\underline{y})\big) = \lambda_n(y)$$

holds in $\Omega^{\bullet}_{k[X]/k}$. As $\Omega^{\bullet}_{k[X]/k}$ has zero boundary operator the left hand side is always zero and in particular does not depend on what we chose for $\lambda_{n+1}(\underline{y})$. So for an extension to λ_{n+1} to exist λ_n must map all elements of Y_n to zero, and then we are free to prescribe any value for $\lambda_{n+1}(\underline{y})$ for elements y of Y_n. Note that $\lambda_n(y)$ lies in $\Omega^{n}_{k[X]/k}$, so as we assumed $|X| = 2$ this is automatically zero if $n \geq 3$, and hence we can already conclude that an extension of λ_3 to λ exists.

To extend λ_0 to λ_1 we need to check that

$$\lambda_0(x_1 \cdot x_2 - x_2 \cdot x_1) = 0$$

which is clear as $\Omega^{\bullet}_{k[X]/k}$ is commutative, and can then set the following value.

$$\lambda_1\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) \coloneqq \Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big)$$

Next, to extend λ_1 to λ_2 we need to check

$$
\begin{aligned}
\lambda_1(x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1) &= 0\\
\lambda_1(x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1) &= 0\\
\lambda_1(x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2) &= 0\\
\lambda_1(x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2) &= 0
\end{aligned}
$$

all of which are clear as $\Omega^{\bullet}_{k[X]/k}$ is commutative, and can then prescribe the following values.

$$
\begin{aligned}
&\lambda_2\big(\underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1}\big) \coloneqq \Phi_X\big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1}\big)\big)\\
&\lambda_2\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big) \coloneqq\\
&\qquad\qquad \frac{1}{2}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)\big) + \Phi_X\big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\big)\big)\\
&\qquad\qquad - \frac{1}{2}\,\mathrm{d}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big)\big)\\
&\lambda_2\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big) \coloneqq\\
&\qquad\qquad \frac{1}{2}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)\big) + \Phi_X\big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\big)\big)\\
&\qquad\qquad + \frac{1}{2}\,\mathrm{d}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big)\big)\\
&\lambda_2\big(\underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2}\big) \coloneqq \Phi_X\big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2}\big)\big)
\end{aligned}
$$

Finally, we need to extend λ_2 to λ_3. For this we need to check the following.

$$
\begin{aligned}
&\lambda_2\Big(\mathrm{d}\, x_1 \cdot \mathrm{d}\, x_1\Big) = 0\\
&\lambda_2\Big(\mathrm{d}\, x_2 \cdot \mathrm{d}\, x_2\Big) = 0\\
&\lambda_2\Big(\mathrm{d}\, \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\Big) = 0\\
&\lambda_2\Big(\mathrm{d}\, x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\, x_2\\
&\qquad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2} \cdot x_1\\
&\qquad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} \cdot x_2\Big) = 0\\
&\lambda_2\Big(\mathrm{d}\, x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\, x_1\\
&\qquad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} \cdot x_1\\
&\qquad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} \cdot x_2\Big) = 0
\end{aligned}
$$

The first two equations are satisfied as odd degree elements in $\Omega^{\bullet}_{k[X]/k}$ square to zero and the last two as $\Omega^{\bullet}_{k[X]/k}$ is commutative. It remains to show that middle equation, which is shown by the following calculation. The values for

$\lambda_2(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1})$ and $\lambda_2(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2})$ were chosen precisely so as to make this work out, and this is why we needed that 2 is invertible in k.

$$\begin{aligned}
&\lambda_2\Big(\mathrm{d}\, \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\Big) \\
&= \mathrm{d}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big)\big) \\
&\quad + \frac{1}{2}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)\big) + \Phi_X\big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\big)\big) \\
&\quad - \frac{1}{2}\,\mathrm{d}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big)\big) \\
&\quad - \frac{1}{2}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)\big) + \Phi_X\big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\big)\big) \\
&\quad - \frac{1}{2}\,\mathrm{d}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big)\big) \\
&= 0
\end{aligned}$$

Thus we can extend λ_2 to λ_3 by mapping $\underline{y}$ to 0 for y an element of Y_2.

As already mentioned λ_3 can be further be extended to λ. It now only remains to show that the second equation of $(* * *)$ holds. Again as the underlying $\mathbb{Z}$-graded k-algebra of $\Omega^{\bullet}_{k[X]/k}$ is commutative, the underlying morphism of λ factors as in the following diagram of $\mathbb{Z}$-graded k-algebras.

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[X]/k} \arrow[rr, "\lambda"] \arrow[d, "\xi_X"'] & & \Omega^{\bullet}_{k[X]/k} \\
\Omega''^{\bullet}_{k[X]/k} \arrow[urr, dashed, bend right, "\lambda''"'] & &
\end{tikzcd}$$

as we already had above. Similarly we can factor $\Phi_X \circ \Xi_X$ as follows.

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[X]/k} \arrow[rr, "\Phi_X \circ \Xi_X"] \arrow[d, "\xi_X"'] & & \Omega^{\bullet}_{k[X]/k} \\
\Omega''^{\bullet}_{k[X]/k} \arrow[urr, dashed, bend right, "\Phi''_X"'] & &
\end{tikzcd}$$

We now begin with the following calculation, where we let $g_{\mathrm{d}\, x_1}$, $g_{\mathrm{d}\, x_2}$, g_{both}, g_{same,x_1}, g_{same,x_2}, and g_{obs} be elements in $k[X]$ as in Proposition 7.5.2.5 so that (7.23) holds. Note that as λ maps x_i to x_i and hence also $\mathrm{d}\, x_i$ to $\mathrm{d}\, x_i$, the same is true for λ''.

$$\begin{aligned}
&\lambda\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})\Big) \\
&= \lambda''\Big(\xi_X\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda''\Big(g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \\
&\qquad\quad + g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big) \\
&\qquad\quad + g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} + g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \\
&\qquad\quad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2\Big) \\
&= g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \lambda\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) \\
&\quad + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \lambda\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) \\
&\quad + g_{\mathrm{both}} \cdot \lambda\big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}\big) \\
&\quad + g_{\mathrm{both}} \cdot \lambda\big(\underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big) \\
&\quad + g_{\mathrm{same},x_1} \cdot \lambda\big(\underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}\big) \\
&\quad + g_{\mathrm{same},x_2} \cdot \lambda\big(\underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}\big) \\
&\quad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2 \\
&= g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big) \\
&\quad + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big) \\
&\quad + g_{\mathrm{both}} \cdot \frac{1}{2}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}\big)\big)\big) \\
&\quad + g_{\mathrm{both}} \cdot \frac{1}{2}\big(+\Phi_X\big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big)\big)\big) \\
&\quad - g_{\mathrm{both}} \cdot \frac{1}{2}\,\mathrm{d}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big)\big) \\
&\quad + g_{\mathrm{both}} \cdot \frac{1}{2}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}\big)\big)\big) \\
&\quad + g_{\mathrm{both}} \cdot \frac{1}{2}\big(\Phi_X\big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big)\big)\big) \\
&\quad + g_{\mathrm{both}} \cdot \frac{1}{2}\,\mathrm{d}\big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big)\big) \\
&\quad + g_{\mathrm{same},x_1} \cdot \Phi_X\big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}\big)\big) \\
&\quad + g_{\mathrm{same},x_2} \cdot \Phi_X\big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}\big)\big) \\
&\quad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2 \\
&= g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big) \\
&\quad + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \Phi_X\big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big) \\
&\quad + g_{\mathrm{both}} \cdot \big(\Phi_X\big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}\big)\big)\big) \\
&\quad + g_{\mathrm{both}} \cdot \big(+\Phi_X\big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big)\big)\big) \\
&\quad + g_{\mathrm{same},x_1} \cdot \Phi_X\big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}\big)\big) \\
&\quad + g_{\mathrm{same},x_2} \cdot \Phi_X\big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}\big)\big) \\
&\quad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2
\end{aligned}
$$

Now we use that $\Phi \circ \Xi_X = \Phi''_X \circ \xi_X$.

$$\begin{aligned}
&= g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \Phi''_X\big(\xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big) \\
&\quad + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \Phi''_X\big(\xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\big) \\
&\quad + g_{\mathrm{both}} \cdot \big(\Phi''_X\big(\xi_X\big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}\big)\big)\big) \\
&\quad + g_{\mathrm{both}} \cdot \big(+\Phi''_X\big(\xi_X\big(\underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big)\big)\big) \\
&\quad + g_{\mathrm{same},x_1} \cdot \Phi''_X\big(\xi_X\big(\underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}\big)\big) \\
&\quad + g_{\mathrm{same},x_2} \cdot \Phi''_X\big(\xi_X\big(\underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}\big)\big) \\
&\quad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2
\end{aligned}$$

We now use that Φ''_X is multiplicative and maps x_i to x_i and $\mathrm{d}\,x_i$ to $\mathrm{d}\,x_i$. The latter two properties follow from $\Phi_X \circ \Xi_X$ mapping x_i to x_i by construction of Ξ_X (see Construction 7.4.9.1), and then also mapping $\mathrm{d}\,x_i$ to $\mathrm{d}\,x_i$ by Proposition 7.4.8.1. Furthermore we can evaluate ξ_X.

$$\begin{aligned}
&= \Phi''_X\big(g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) + \Phi''_X\big(g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) \\
&\quad + \Phi''_X\big(g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big)\big) \\
&\quad + \Phi''_X\big(g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}\big) \\
&\quad + \Phi''_X\big(g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}\big) \\
&\quad + \Phi''_X\big(g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2\big) \\
&= \Phi''_X\Big(g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \\
&\qquad + g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big) \\
&\qquad + g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} + g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \\
&\qquad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2\Big) \\
&= \Phi''_X\Big(\xi_X\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})\Big)\Big) \\
&= \Phi_X\Big(\Xi_X\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})\Big)\Big)
\end{aligned}$$

It thus only remains to show that

$$\Phi_X\Big(\Xi_X\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})\Big)\Big) = 0$$

holds. Note that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\Omega'^{\bullet}_{k[t]/k}\Big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Omega'^{\bullet}_{F/k})"] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma)(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_{\{t\}}))"] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_X))"', "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}(\{t\})\Big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\widetilde{\mathrm{C}}(F))"] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma)(\Phi_{\{t\}})"] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}(X)\Big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)(\Phi_X)"', "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Omega^{\bullet}_{F/k})"] & \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where the top square is obtained from the transpose of the diagram from Proposition 7.5.2.5 by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ and using compatibility with γ_{Mixed} (see Construction 4.4.1.1), and the bottom square is from Proposition 7.4.7.1. The underlying differential graded k-algebra of $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant by Propositions 7.4.5.11 and 4.2.2.12, so we can conclude by [Hov99, 1.2.10 (ii)], Propositions A.1.0.1 and 4.2.2.17 that there exists a homotopy of differential graded k-algebras h from $\Phi_X \circ \Xi_X \circ \Omega'^{\bullet}_{F/k}$ to $\Omega^{\bullet}_{F/k} \circ \Phi_{\{t\}} \circ \Xi_{\{t\}}$ (we omit forgetful functors in the notation here) in the sense of Proposition 4.2.2.17. We can thus conclude the proof with the following calculation.

$$\begin{aligned}
&\Phi_X\Big(\Xi_X\Big(\Omega'^{\bullet}_{F/k}(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t})\Big)\Big)\\
&= \Omega^{\bullet}_{F/k}\big(\Phi_{\{t\}}\big(\Xi_{\{t\}}(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t})\big)\big)\\
&\quad + h(\partial(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t})) + \partial(h(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}))
\end{aligned}$$

$\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}$ is an element of degree 2, while $\Omega^2_{k[t]/k} = 0$. Thus purely for degree reasons we have $\Phi_{\{t\}}(\Xi_{\{t\}}(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t})) = 0$ so that the first summand is zero.

$$\begin{aligned}
&= 0 + h(t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t) + 0\\
&= h(t)\cdot\Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(\mathrm{d}\,t) + \Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(t)\cdot h(\mathrm{d}\,t)\\
&\quad - h(\mathrm{d}\,t)\cdot\Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(t) + \Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(\mathrm{d}\,t)\cdot h(t)\\
&= h(t)\cdot\Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(\mathrm{d}\,t) - h(t)\cdot\Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(\mathrm{d}\,t)\\
&\quad + h(\mathrm{d}\,t)\cdot\Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(t)\cdot -h(\mathrm{d}\,t)\cdot\Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(t)\\
&= h(t)\cdot\Big(\Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(\mathrm{d}\,t) - \Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(\mathrm{d}\,t)\Big)\\
&\quad + h(\mathrm{d}\,t)\cdot\Big(\Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(t) - \Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(t)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= h(t) \cdot (-\partial(h(\mathrm{d}\,t)) - h(\partial(\mathrm{d}\,t))) \\
&\quad + h(\mathrm{d}\,t) \cdot (\partial(h(t)) + h(\partial(t))) \\
&= h(t) \cdot (-0 - h(0)) + h(\mathrm{d}\,t) \cdot (0 + h(0)) \\
&= 0
\end{aligned}
$$

□

7.5.3. Conjecture D

In this short section we deduce Conjecture D in certain cases from Proposition 7.5.2.6.

Proposition 7.5.3.1. *Let X be a set and f an element of $k[X]$. Assume that one of the following holds.*

(1) $|X| \leq 1$.

(2) $|X| = 2$ *and* 2 *is invertible in* k.

Then Conjecture D holds for f. ♡

Proof. Apply Proposition 7.5.2.6 for f as well as for the element 0 of k (as the polynomial ring generated by an empty set of variables) and combine the commutative squares. Note that it is crucial here that Proposition 7.5.2.6 constructs the commutative square (7.25) with the top horizontal equivalence not depending on X or f, which is what allows us to glue the two squares together. □

7.5.4. Conjecture C for two variables in the domain

In this section we show Conjecture C for morphisms out of polynomial k-algebras in two variables into k using some of the same arguments that also went into Proposition 7.5.2.6.

Proposition 7.5.4.1. *Let X be a totally ordered set satisfying $|X| \leq 2$ and $F\colon k[X] \to k$ a morphism of commutative k-algebras.*

Then there exists a commutative square

$$
\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[X]/k}\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega^\bullet_{F/k})"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k/k}\right)
\end{tikzcd}
\tag{7.26}
$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ *such that the horizontal equivalences are the ones from Corollary 7.4.9.3. In particular, Conjecture C holds for* F. ♡

Proof. The cases $|X| = 0$ and $|X| = 1$ are already contained in Proposition 7.5.1.1 and Proposition 7.5.2.6, respectively. For the case $|X| = 1$ this requires a small argument for why the lower horizontal equivalence we obtain from Proposition 7.5.2.6 is homotopic to the one from Corollary 7.4.9.3, but as $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$ is an initial object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ (see the proof of Proposition 7.5.1.1) this is automatic.

So now assume that $|X| = 2$ and denote the elements of X by $x_1 < x_2$. As in Proposition 7.5.2.6, we begin by considering the following (non-commuting) diagram in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, that will be explained below.

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[rr, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"] \arrow[d, dash, "\simeq"] & & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(X)\Big) \arrow[rr, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(\widetilde{\mathrm{C}}(F)\big)"] & & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(\emptyset)\Big) \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big) \arrow[u, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)", "\simeq"'] \arrow[rr, dashed] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Theta_X)"', "\simeq"] & & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k/k}\Big) \arrow[u, "\simeq", "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_\emptyset)"'] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Theta_\emptyset)"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[rr, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(\Omega^{\bullet}_{F/k}\big)"] & & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k/k}\Big)
\end{tikzcd} \qquad (*)$$

The top square has a filler given by the (transpose of) commutative diagram (7.10) from the definition of $\widetilde{\mathrm{C}}(F)$ in Construction 7.4.2.5, and Ξ and Θ are as in Construction 7.4.9.1 and Definition 7.4.5.9. By Corollary 7.4.9.3 the vertical compositions are the horizontal equivalences in diagram (7.26) from the statement, so that it suffices to find a filler for the lower rectangle in the above diagram.

As $\Omega'^{\bullet}_{k[X]/k}$ is cofibrant as an object of $\mathrm{Alg}(\mathsf{Mixed})$ by Proposition 7.4.5.11, we can lift the composition

$$\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_\emptyset)^{-1} \circ \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(F)\Big) \circ \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)$$

to a morphism

$$\Omega'^{\bullet}_{F/k} \colon \Omega'^{\bullet}_{k[X]/k} \to \Omega'^{\bullet}_{k/k}$$

so that if we let the dashed morphism in the above diagram be the morphism $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega'^{\bullet}_{F/k})$ there will be a filler for the middle square of diagram $(*)$.

It thus suffices to show that

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[X]/k} \arrow[r, "\Omega'^{\bullet}_{F/k}"] \arrow[d, "\Theta_X"'] & \Omega'^{\bullet}_{k/k} \arrow[d, "\Theta_\emptyset"] \\
\Omega^{\bullet}_{k[X]/k} \arrow[r, "\Omega^{\bullet}_{F/k}"'] & \Omega^{\bullet}_{k/k}
\end{tikzcd}$$

commutes strictly. Note that as $\Omega^{\bullet}_{k/k}$ is concentrated in degree 0, it suffices to check that the two compositions agree on elements of degree 0, and as both compositions are multiplicative it even suffices to check the values on elements x in X. The composition over the bottom left maps x to $F(x)$, so this boils down to showing that $\Omega'^{\bullet}_{k[X]/k}(x) = F(x)$ for every element x in X.

For this we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Omega'^{\bullet}_{k[X]/k})"] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma)(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_X))"] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\Omega'^{\bullet}_{k/k}\Big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_\emptyset))"', "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}(X)\Big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\widetilde{\mathrm{C}}(F))"] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma)(\Phi_X)"] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}(\emptyset)\Big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)(\Phi_\emptyset)"', "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Omega^{\bullet}_{F/k})"] & \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k/k}\Big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where the top square is obtained from the middle square of diagram $(*)$ by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ and using compatibility with γ_{Mixed} (see Construction 4.4.1.1), and the bottom square is (the transpose of) the one from Proposition 7.4.7.1. The underlying differential graded k-algebra of $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant by Propositions 7.4.5.11 and 4.2.2.12, so we can conclude by [Hov99, 1.2.10 (ii)], Propositions A.1.0.1 and 4.2.2.17 that there exists a homotopy of differential graded k-algebras h from $\Phi_\emptyset \circ \Xi_\emptyset \circ \Omega'^{\bullet}_{k[X]/k}$ to $\Omega^{\bullet}_{F/k} \circ \Phi_X \circ \Xi_X$ (we omit forgetful functors in the notation here) in the sense of Proposition 4.2.2.17.

We can then carry out the following calculation for x an element of X, where we use that $\Phi_X \circ \Xi_X$ by definition in Construction 7.4.9.1 maps x to x.

$$\begin{aligned}
(\Phi_\emptyset \circ \Xi_\emptyset)\Big(\Omega'^{\bullet}_{k[X]/k}(x)\Big) &= \Omega^{\bullet}_{F/k}((\Phi_X \circ \Xi_X)(x)) + h(\partial(x)) + \partial(h(x)) \\
&= \Omega^{\bullet}_{F/k}(x) + h(0) + 0
\end{aligned}$$

$$= F(x)$$

Note that $\Omega'^{\bullet}_{k/k}$ is by Remark 7.4.5.2 given by $k \cdot \{1\}$ in degree 0, so that also using the analogous identification for degree 0 of $\Omega^{\bullet}_{k/k}$ we obtain that $\Phi_{\emptyset} \circ \Xi_{\emptyset}$ is given by the identity in degree 0. Hence we can conclude that $\Omega'^{\bullet}_{k[X]/k}(x) = F(x)$ holds for every element x in X. $\square$

Chapter 8.

Hochschild homology of certain quotients of commutative algebras

The goal of this chapter can be roughly summarized as giving a concrete formula for a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1, \ldots, x_n))$ as an object of $\mathcal{M}\mathrm{ixed}$, where R is a commutative k-algebra and $x_1, \ldots, x_n$ elements of R satisfying some conditions, given a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$.

More specifically, we require Conjecture C to hold for the morphism of k-algebras $k[t_1, \ldots, t_n] \to k$ mapping t_i to 0[1]. Furthermore we need as input an object M in $\mathrm{RMod}_{\Omega^{\bullet}_{k[t_1,\ldots,t_n]/k}}(\mathsf{Mixed}_{\mathrm{cof}})$ that represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ as an object in the ∞-category $\mathrm{RMod}_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])}(\mathcal{M}\mathrm{ixed})$, where the action is induced by the action of $k[t_1, \ldots, t_n]$ on R, where t_i acts by multiplication by x_i. Assuming Conjecture C as above and given such an object M, Proposition 8.3.0.1 can be roughly summarized as saying that (under some further conditions on R and $x_1, \ldots, x_n$), $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1, \ldots, x_n))$ is represented by a strict mixed complex that can be described as

$$M \otimes \Lambda(s_1, \ldots, s_n) \otimes \Gamma(\mathrm{d}\, s_1, \ldots, \mathrm{d}\, s_n)$$

with s_i of degree 1, $\mathrm{d}\, s_i$ of degree 2, and ∂ and d described by formulas given in Proposition 8.3.0.1. In particular, if m is a cycle in M_0 representing the unit 1 of R, then $\partial(m \otimes s_i) = (m \cdot t_i) \otimes 1$ and $\partial(m \otimes \mathrm{d}\, s_i^{[1]}) = -(m \cdot \mathrm{d}\, t_i) \otimes 1$, so we can think of s_i and $\mathrm{d}\, s_i^{[1]}$ as adding the relations that make x_i and $\mathrm{d}\, x_i$ zero.

To obtain such a formula, we proceed as follows. In Section 8.1 we start by showing that – under some assumptions – we can write the quotient $R/(x_1, \ldots, x_n)$ as a derived tensor product $R \otimes_{k[t_1,\ldots,t_n]} k$, with t_i acting by multiplication with x_i on the left and by multiplication with 0 on the right. Using that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is compatible with relative tensor products we then obtain an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1, \ldots, x_n)) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R) \otimes_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

[1]This is the case for $n \leq 2$ by Proposition 7.5.4.1.

so that the task becomes to find strict models for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n])$ (as an algebra in $\mathcal{M}$ixed) as well as for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$ (the latter two as modules over the strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n])$), and then calculating the derived relative tensor product. Assuming Conjecture B for $\{t_1, \ldots, t_n\}$ we can use the algebra in strict mixed complexes $\Omega^\bullet_{k[t_1,\ldots,t_n]/k}$ as a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n])$, and assuming that even Conjecture C holds for the morphism of commutative k-algebras $k[t_1, \ldots, t_n] \to k$ that maps t_i to 0 we can also use $\Omega^\bullet_{k/k}$ as a module over $\Omega^\bullet_{k[t_1,\ldots,t_n]/k}$ as a model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$. In order to be able to calculate the derived tensor product as an ordinary, underived tensor product, it will then be useful to replace $\Omega^\bullet_{k/k}$ with a weakly equivalent module over $\Omega^\bullet_{k[t_1,\ldots,t_n]/k}$ that is sufficiently cofibrant. Constructing such a module will be the goal of Section 8.2, and we will put everything together in Section 8.3.

8.1. Hochschild homology of certain quotients as relative tensor products

In Section 8.1.1 we will show that if R is a commutative k-algebra and $x_1, \ldots, x_n$ are elements of R satisfying some conditions[2], then the object $\gamma(R/(x_1, \ldots, x_n))$ in $\mathrm{CAlg}(\mathcal{D}(k))$ is equivalent to a relative tensor product $\gamma(R) \otimes_{\gamma(k[t_1,\ldots,t_n])} k$. Using compatibility of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ with relative tensor products, we can thus write $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\gamma(R/(x_1, \ldots, x_n)))$ as a relative tensor product as well, as we will make explicit in Section 8.1.2.

8.1.1. Certain quotients as relative tensor products

Proposition 8.1.1.1. *Let R be a commutative algebra in $\mathsf{Ch}(k)$ and let $x_1, \ldots, x_n$ be elements of R_0. We obtain a morphism of commutative algebras in $\mathsf{Ch}(k)$*

$$k[t_1, \ldots, t_n] \to R, \qquad t_i \mapsto x_i$$

that determines a $k[t_1, \ldots, t_n]$-module structure on the commutative differential graded algebra R (see Construction E.8.0.4). Assume that R is cofibrant as an object of $\mathrm{RMod}_{k[t_1,\ldots,t_n]}(\mathsf{Ch}(k))$ with respect to the model structure of Theorem 4.2.2.1.

Consider the commutative diagram

$$\begin{tikzcd} k[t_1, \ldots, t_n] \arrow[r, "t_i \mapsto x_i"] \arrow[d, "t_i \mapsto 0"'] & R \arrow[d] \\ k \arrow[r] & R/(x_1, \ldots, x_n) \end{tikzcd} \tag{8.1}$$

[2] Roughly, $x_1, \ldots, x_n$ need to act sufficiently nicely on R by multiplication.

in $\mathrm{CAlg}(\mathsf{Ch}(k))$*, where the right vertical morphism is the canonical quotient morphism. Then the following hold.*

(1) Diagram (8.1) *is a pushout diagram in* $\mathrm{CAlg}(\mathsf{Ch}(k))$*.*

(2) All four objects in diagram (8.1) *have cofibrant underlying chain complex.*

(3) The functor

$$\mathrm{CAlg}(\gamma)\colon \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \to \mathrm{CAlg}(\mathcal{D}(k))$$

maps diagram (8.1) *to a pushout diagram in* $\mathrm{CAlg}(\mathcal{D}(k))$*.*

(4) There is an equivalence

$$\mathrm{CAlg}(\gamma)(R/(x_1,\ldots,x_n)) \simeq \mathrm{CAlg}(\gamma)(R) \otimes_{\mathrm{CAlg}(\gamma)(k[t_1,\ldots,t_n])} k$$

in $\mathrm{CAlg}(\mathcal{D}(k))$*, where the module structures that are used for the relative tensor product arise from the two morphisms* $k[t_1,\ldots,t_n] \to R$ *and* $k[t_1,\ldots,t_n] \to k$ *in* (8.1) *by applying* $\mathrm{CAlg}(\gamma)$*, Construction E.8.0.4, and identifying* $\mathrm{CAlg}(\gamma)(k)$ *with* k*.* ♡

Proof. Proof of claim (1): This is well-known and can be shown by repeatedly applying the $n = 1$ case[3], which can be shown using Proposition E.8.0.5[4].

[3] For this one decomposes the (transposed) square (8.1) as

$$\begin{tikzcd}
k[t_1,\ldots,t_n] \arrow[r, "t_i \mapsto \begin{cases} 0 & i=1 \\ t_i & i>1 \end{cases}"] \arrow[d, "t_i \mapsto x_i"'] & k[t_2,\ldots,t_n] \arrow[r] \arrow[d, "t_i \mapsto x_i"'] & \cdots \arrow[r] \arrow[d] & k \arrow[d] \\
R \arrow[r] & R/(x_1) \arrow[r] & \cdots \arrow[r] & R/(x_1,\ldots,x_n)
\end{tikzcd}$$

so that it suffices to show that for each $1 \le j \le n$ the square

$$\begin{tikzcd}
k[t_j,\ldots,t_n] \arrow[r] \arrow[d] & k[t_{j+1},\ldots,t_n] \arrow[d] \\
R/(x_1,\ldots,x_{j-1}) \arrow[r] & R/(x_1,\ldots,x_j)
\end{tikzcd}$$

is a pushout square. The transpose of this square is the right square in the following commutative diagram.

$$\begin{tikzcd}
k[t_j] \arrow[r] \arrow[d] & k[t_j,\ldots,t_n] \arrow[r] \arrow[d] & R/(x_1,\ldots,x_{j-1}) \arrow[d] \\
k \arrow[r] & k[t_{j+1},\ldots,t_n] \arrow[r] & R/(x_1,\ldots,x_j)
\end{tikzcd}$$

It thus suffices to show that the outer rectangle and the left square are pushouts, but as $k[t_j,\ldots,t_n]/(t_j) \cong k[t_{j+1},\ldots,t_n]$ and $(R/(x_1,\ldots,x_{j-1}))/(x_j) \cong R/(x_1,\ldots,x_j)$, this follows from the $n = 1$ case.

[4] Using Proposition E.8.0.5, it suffices to show that the morphism

$$R \to R \otimes_{k[t_1]} k$$

Proof of claim (2): $k[t_1, \ldots, t_n]$ and k are free as k-modules and hence cofibrant as chain complexes [Hov99, 2.3.6]. We assumed that R is cofibrant as an object of $\mathrm{RMod}_{k[t_1,\ldots,t_n]}(\mathsf{Ch}(k))$, and as the underlying chain complex of $k[t_1, \ldots, t_n]$ is cofibrant as just mentioned, Theorem 4.2.2.1 (8) implies that the underlying chain complex of R is cofibrant as well. By (1) and Proposition E.8.0.5 the underlying chain complex of $R/(x_1, \ldots, x_n)$ is isomorphic to the relative tensor product $R \otimes_{k[t_1,\ldots,t_n]} k$, which is cofibrant as a chain complex by Proposition 6.3.3.3.

Proof of claim (3) and (4): Combining (1) and (2) with Proposition E.8.0.5 (applied to both $\mathsf{Ch}(k)^{\mathrm{cof}}$ as well as $\mathcal{D}(k)$) we only need to show that $\mathrm{CAlg}(\gamma)$ preserves the relative tensor product $R \otimes_{k[t_1,\ldots,t_n]} k$. As the forgetful functors $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}}) \to \mathsf{Ch}(k)^{\mathrm{cof}}$ and $\mathrm{CAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$ are conservative and preserve relative tensor products by Proposition E.8.0.1 and [HA, 3.2.3.1 (4)], it suffices to show that $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ preserves this relative tensor product, which follows from Proposition 6.3.3.3[5]. $\square$

8.1.2. Application to Hochschild homology

Combining Proposition 8.1.1.1 with $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserving relative tensor products by Proposition 6.2.3.1 we obtain the following result.

Proposition 8.1.2.1. *Let R and $x_1, \ldots, x_n$ be as in Proposition 8.1.1.1. Then we can consider R as an object in $\mathrm{RMod}_{k[t_1,\ldots,t_n]}(\mathsf{Ch}(k)^{\mathrm{cof}})$, with t_i acting by multiplication with x_i, and k as an object in $\mathrm{LMod}_{k[t_1,\ldots,t_n]}(\mathsf{Ch}(k)^{\mathrm{cof}})$, with t_i acting by multiplication with 0.*

As $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is a monoidal functor, $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ obtains the structure of an object in $\mathrm{RMod}_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])}(\mathcal{M}\mathrm{ixed})$ and similarly $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$ obtains the structure of an object in $\mathrm{LMod}_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])}(\mathcal{M}\mathrm{ixed})$.

Let P_n be an object of $\mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})$ coming with an equivalence

$$\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(P_n) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n]) \tag{8.2}$$

exhibits $R \otimes_{k[t_1]} k$ as the quotient $R/(x_1)$. As the forgetful functor $\mathrm{CAlg}(\mathsf{Ch}(k)) \to \mathsf{Ch}(k)$ is conservative and preserves relative tensor products (see Proposition E.8.0.1), we can take the relative tensor product in $\mathsf{Ch}(k)$.

There is a short exact sequence

$$0 \longrightarrow k[t_1] \xrightarrow{1 \mapsto t_1} k[t_1] \xrightarrow{1 \mapsto 1} k \longrightarrow 0$$

of left-$k[t_1]$-modules in $\mathsf{Ch}(k)$, so as $R \otimes_{k[t_1]} -$ is right exact [Wei94, 2.6.2], we obtain an exact sequence

$$R \otimes_{k[t_1]} k[t_1] \longrightarrow R \otimes_{k[t_1]} k[t_1] \longrightarrow R \otimes_{k[t_1]} k \longrightarrow 0$$

that can be identified with

$$R \xrightarrow{x_1 \cdot -} R \longrightarrow R \otimes_{k[t_1]} k \longrightarrow 0$$

which shows the claim.

[5]It is here were we really use the assumption that R is cofibrant as a $k[t_1, \ldots, t_n]$-module.

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$*. Let furthermore* M *be a right-*P_n*-module and* A_n *a left-*P_n*-module in* $\mathsf{Mixed}_{\mathrm{cof}}$ *such that there are equivalences*

$$\mathrm{RMod}(\gamma_{\mathsf{Mixed}})(M) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R) \tag{8.3}$$

and

$$\mathrm{LMod}(\gamma_{\mathsf{Mixed}})(A_n) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \tag{8.4}$$

in $\mathrm{RMod}(\mathcal{M}\mathrm{ixed})$ *and* $\mathrm{LMod}(\mathcal{M}\mathrm{ixed})$ *such that the underlying equivalences of algebras are given by equivalence* (8.2)*. Assume that* A_n *is cofibrant as an object in*[6]*.* $\mathrm{LMod}_{P_n}(\mathsf{Ch}(k))$*.*

Then the underlying chain complex of the relative tensor product $M \otimes_{P_n} A_n$ *(taken in* Mixed*) is cofibrant. Furthermore, there is an equivalence*

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1, \dots, x_n)) \simeq \gamma_{\mathsf{Mixed}}(M \otimes_{P_n} A_n)$$

in $\mathcal{M}\mathrm{ixed}$*.* ♡

Proof. By Proposition E.8.0.1 the forgetful functor $\mathrm{ev}_{\mathfrak{m}} \colon \mathsf{Mixed} \to \mathsf{Ch}(k)$ preserves relative tensor products, so cofibrancy of the underlying chain complex of $M \otimes_{P_n} A_n$ follows from Proposition 6.3.3.3.

By Proposition 8.1.1.1 (4) there is an equivalence

$$\mathrm{CAlg}(\gamma)(R/(x_1, \dots, x_n)) \simeq \mathrm{CAlg}(\gamma)(R) \otimes_{\mathrm{CAlg}(\gamma)(k[t_1,\dots,t_n])} k$$

in $\mathrm{CAlg}(\mathcal{D}(k))$. As $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserves relative tensor products by Proposition 6.2.3.1 we obtain an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1, \dots, x_n)) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R) \otimes_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\dots,t_n])} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

in $\mathcal{M}\mathrm{ixed}$, and the equivalences (8.2), (8.3), and (8.4) induce an equivalence in $\mathcal{M}\mathrm{ixed}$ as follows.

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R) \otimes_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\dots,t_n])} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \simeq \gamma_{\mathsf{Mixed}}(M) \otimes_{\gamma_{\mathsf{Mixed}}(P_n)} \gamma_{\mathsf{Mixed}}(A_n)$$

There is a comparison morphism

$$\gamma_{\mathsf{Mixed}}(M) \otimes_{\gamma_{\mathsf{Mixed}}(P_n)} \gamma_{\mathsf{Mixed}}(A_n) \to \gamma_{\mathsf{Mixed}}(M \otimes_{P_n} A_n)$$

in $\mathcal{M}\mathrm{ixed}$ just like in Remark 6.3.3.2, and it suffices to show that this is an equivalence. As the forgetful functors $\mathrm{ev}_{\mathfrak{m}} \colon \mathcal{M}\mathrm{ixed} \to \mathcal{D}(k)$ as well as $\mathrm{ev}_{\mathfrak{m}} \colon \mathsf{Mixed} \to \mathsf{Ch}(k)$ are conservative and preserve relative tensor products by Proposition E.8.0.1, it suffices to show that the comparison morphism

$$\gamma(M) \otimes_{\gamma(P_n)} \gamma(A_n) \to \gamma(M \otimes_{P_n} A_n)$$

in $\mathcal{D}(k)$ from Remark 6.3.3.2 is an equivalence. But this is precisely what we obtain from Proposition 6.3.3.3, as A_n was assumed to be cofibrant as a left-P_n-module. □

[6] We are using here that the forgetful functor $\mathrm{ev}_{\mathfrak{m}} \colon \mathsf{Mixed} \to \mathsf{Ch}(k)$ is monoidal.

8.2. A sufficiently cofibrant strict model of k

Proposition 7.5.4.1 implies that the morphism

$$\Omega^{\bullet}_{k[t_1,\ldots,t_n]/k} \to \Omega^{\bullet}_{k/k}$$

in $\mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})$ that is induced by the morphism of commutative algebras $k[t_1,\ldots,t_n] \to k$ that sends t_i to 0, represents the morphism

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n]) \to \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ induced by the same morphism, as long as $n \leq 2$. For $n > 2$ we have encapsulated this statement as Conjecture C for this morphism, and we will assume that Conjecture C holds for the results of this chapter.

Unfortunately, we can not directly use $\Omega^{\bullet}_{k/k}$, as the left-module P_n over $A_n = \Omega^{\bullet}_{k[t_1,\ldots,t_n]/k}$ as in Proposition 8.1.2.1, as this would require $\Omega^{\bullet}_{k/k}$ to be cofibrant as a module over $\Omega^{\bullet}_{k[t_1,\ldots,t_n]/k}$ in chain complexes, which is not necessarily the case.

The goal of this section is thus to construct a commutative diagram

$$\begin{tikzcd} & A_n \arrow[dd] \\ \Omega^{\bullet}_{k[t_1,\ldots,t_n]/k} \arrow[ur] \arrow[dr] & \\ & \Omega^{\bullet}_{k/k} \end{tikzcd}$$

in $\mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})$ such that the lower morphism is the one discussed above, the vertical morphism on the right is a quasiisomorphism, and such that A_n is cofibrant when considered as an object in $\mathrm{LMod}_{\Omega^{\bullet}_{k[t_1,\ldots,t_n]/k}}(\mathsf{Ch}(k))$. We will construct A_n and morphisms as in the diagram above in Section 8.2.1, show that A_k has the required cofibrancy property in Section 8.2.2, and show that the right vertical morphism is a quasiisomorphism in Section 8.2.3.

8.2.1. Construction of the strict model

Before we construct A_n we need a small result on the Leibniz rule and compositions.

Proposition 8.2.1.1. *Let R be a commutative differential graded algebra, and let f and g be two operators of odd degree on R that both satisfy the Leibniz rule. Then $f \circ f$ as well as $fg+gf$ satisfy the Leibniz rule as well*[7]. ♡

Proof. Let x and y be two elements in R. Then we can calculate as follows.

$$f(g(x \cdot y)) = f\Big(g(x)y + (-1)^{\deg_{\mathsf{Ch}}(x)} x g(y)\Big)$$

[7]Note that $f \circ f$ and $fg + gf$ will be of even degree, so there will be no sign.

$$
\begin{aligned}
&= f(g(x))y + (-1)^{\deg_{\mathsf{Ch}}(x)+1} g(x)f(y) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)g(y) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(x)+\deg_{\mathsf{Ch}}(x)} x f(g(y)) \\
&= f(g(x))y + x f(g(y)) \\
&\quad - (-1)^{\deg_{\mathsf{Ch}}(x)} g(x)f(y) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)g(y)
\end{aligned}
$$

Applying this to $g = f$ we immediately obtain the claim for $f \circ f$. For $fg+gf$ there is the following calculation.

$$
\begin{aligned}
(fg + gf)(x \cdot y) &= f(g(x))y + x f(g(y)) \\
&\quad - (-1)^{\deg_{\mathsf{Ch}}(x)} g(x)f(y) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)g(y) \\
&\quad + g(f(x))y + x g(f(y)) \\
&\quad - (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)g(y) + (-1)^{\deg_{\mathsf{Ch}}(x)} g(x)f(y) \\
&= (fg + gf)(x)y + x(fg + gf)(y)
\end{aligned}
$$
□

Construction 8.2.1.2. We define P_1 and A_1 to be the strict commutative graded k-modules given by[8]

$$
\begin{gathered}
P_1 \coloneqq k[t] \otimes \Lambda(\mathrm{d}\, t) \qquad \text{and} \qquad A_1 \coloneqq k[t] \otimes \Lambda(\mathrm{d}\, t) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s) \\
\deg_{\mathsf{Ch}}(t) = 0, \quad \deg_{\mathsf{Ch}}(\mathrm{d}\, t) = 1, \quad \deg_{\mathsf{Ch}}(s) = 1, \quad \deg_{\mathsf{Ch}}(\mathrm{d}\, s^{[m]}) = 2m
\end{gathered}
$$

and let $g_1 \colon P_1 \to A_1$ be the inclusion. Note that there is a commutative triangle of commutative graded k-modules

$$
\begin{tikzcd}
 & A_1 \arrow[d, "p_1"] \\
P_1 \arrow[ur, "g_1"] \arrow[r, "g_1'"'] & k
\end{tikzcd}
\tag{8.5}
$$

where g_1' and p_1 map t, $\mathrm{d}\, t$, s, and $\mathrm{d}\, s^{[m]}$ to 0.

We will now upgrade diagram (8.5) to a commutative diagram in the category $\mathrm{CAlg}(\mathsf{Mixed})$. For this we define ∂ and d on P_1 and A_1 by

$$
\begin{gathered}
\partial(t) = 0, \qquad \partial(\mathrm{d}\, t) = 0, \qquad \partial(s) = t, \qquad \partial\left(\mathrm{d}\, s^{[m]}\right) = -\,\mathrm{d}\, t\, \mathrm{d}\, s^{[m-1]} \\
\mathrm{d}(t) = \mathrm{d}\, t, \qquad \mathrm{d}(\mathrm{d}\, t) = 0, \qquad \mathrm{d}(s) = \mathrm{d}\, s^{[1]}, \qquad \mathrm{d}\left(\mathrm{d}\, s^{[m]}\right) = 0
\end{gathered}
$$

and extending by k-linearity and the Leibniz rule. It is clear that if this equips A_1 with the structure of a commutative algebra in strict mixed complexes, then this structure restricts to P_1 and makes g_1 into a morphism in $\mathrm{CAlg}(\mathsf{Mixed})$. What we need to show is that this definition of $\partial(\mathrm{d}\, s^{[m]})$ and

[8] For now $\mathrm{d}\, t$ and $\mathrm{d}\, s$ are just names, but we will in a moment define a strict mixed complex structure that will justify this notation.

$\mathrm{d}(\mathrm{d}\, s^{[m]})$ is well-defined[9] and that d and ∂ satisfy $\partial \circ \partial = 0$, $\mathrm{d} \circ \mathrm{d} = 0$, and $\mathrm{d}\partial + \partial\mathrm{d} = 0$ on A_1, see Remark 4.2.1.4 and Remark 4.2.1.12.

But first, let us state the formulas for d and ∂ for a k-linear basis of A_1 (obtained by applying k-linearity and the Leibniz rule), so we may refer to them later[10].

$$\begin{aligned}
&t^{n_1}\,\mathrm{d}\,t^{\epsilon_1} s^{\eta_1}\,\mathrm{d}\,s^{[m_1]} \cdot t^{n_2}\,\mathrm{d}\,t^{\epsilon_2} s^{\eta_2}\,\mathrm{d}\,s^{[m_2]} \\
&\qquad = (-1)^{\eta_1\cdot\epsilon_2}\binom{m_1+m_2}{m_1} t^{n_1+n_2}\,\mathrm{d}\,t^{\epsilon_1+\epsilon_2} s^{\eta_1+\eta_2}\,\mathrm{d}\,s^{[m_1+m_2]} \\
&\mathrm{d}\Big(t^n\,\mathrm{d}\,t^{\epsilon} s^{\eta}\,\mathrm{d}\,s^{[m]}\Big) \\
&\qquad = n\cdot t^{n-1}\,\mathrm{d}\,t^{\epsilon+1} s^{\eta}\,\mathrm{d}\,s^{[m]} + (-1)^{\epsilon}\cdot\eta\cdot(1+m)\cdot t^n\,\mathrm{d}\,t^{\epsilon}\,\mathrm{d}\,s^{[1+m]} \\
&\partial\Big(t^n\,\mathrm{d}\,t^{\epsilon} s^{\eta}\,\mathrm{d}\,s^{[m]}\Big) = (-1)^{\epsilon}\cdot\eta\cdot t^{n+\eta}\,\mathrm{d}\,t^{\epsilon}\,\mathrm{d}\,s^{[m]} - t^n\,\mathrm{d}\,t^{\epsilon+1} s^{\eta}\,\mathrm{d}\,s^{[m-1]}
\end{aligned} \tag{8.6}$$

For well-definedness, nothing needs to be done for d. For ∂, evaluating on

$$\mathrm{d}\,s^{[m_1]}\cdot\mathrm{d}\,s^{[m_2]} = \binom{m_1+m_2}{m_1}\mathrm{d}\,s^{[m_1+m_2]}$$

using the left hand side and the Leibniz rule we obtain

$$\begin{aligned}
&\Big(-\mathrm{d}\,t\,\mathrm{d}\,s^{[m_1-1]}\Big)\,\mathrm{d}\,s^{[m_2]} + \mathrm{d}\,s^{[m_1]}\Big(-\mathrm{d}\,t\,\mathrm{d}\,s^{[m_2-1]}\Big) \\
&= -\mathrm{d}\,t\left(\binom{m_1+m_2-1}{m_1-1}\mathrm{d}\,s^{[m_1+m_2-1]} + \binom{m_1+m_2-1}{m_1}\mathrm{d}\,s^{[m_1+m_2-1]}\right)
\end{aligned}$$

and using the right hand side we obtain

$$-\binom{m_1+m_2}{m_1}\mathrm{d}\,t\,\mathrm{d}\,s^{[m_1+m_2-1]}$$

which are equal by the binomial identity $\binom{m_1+m_2-1}{m_1-1}+\binom{m_1+m_2-1}{m_1}=\binom{m_1+m_2}{m_1}$.

We now check $\partial\circ\partial = 0$, $\mathrm{d}\circ\mathrm{d} = 0$, and $\mathrm{d}\partial + \partial\mathrm{d} = 0$. Note that Proposition 8.2.1.1 implies that we only need to check this on multiplicative generators. That $\mathrm{d}\circ\mathrm{d} = 0$ on multiplicative generators is clear from the definition, and for $\partial\circ\partial = 0$ the only case to consider is

$$\partial\Big(\partial\Big(\mathrm{d}\,s^{[m]}\Big)\Big) = \partial\Big(-\mathrm{d}\,t\,\mathrm{d}\,s^{[m-1]}\Big) = \mathrm{d}\,t\,\mathrm{d}\,t\,\mathrm{d}\,s^{[m-2]}$$

which is 0 as $(\mathrm{d}\,t)^2 = 0$. Finally, we verify that $\mathrm{d}\partial + \partial\mathrm{d} = 0$.

$$(\mathrm{d}\partial + \partial\mathrm{d})(t) = 0 + \partial(\mathrm{d}\,t) = 0$$

---

[9] I. e. compatible with the relation $\mathrm{d}\,s^{[m_1]}\cdot\mathrm{d}\,s^{[m_2]} = \binom{m_1+m_2}{m_1}\mathrm{d}\,s^{[m_1+m_2]}$.

[10] In the formulas, some summands may contain factors that are undefined, such as $\mathrm{d}\,s^{[-1]}$. Those summands are to be interpreted as 0.

$$\begin{aligned}
(\mathrm{d}\partial + \partial\mathrm{d})(\mathrm{d}\,t) &= 0 + 0\\
(\mathrm{d}\partial + \partial\mathrm{d})(s) &= \mathrm{d}(t) + \partial\Big(\mathrm{d}\,s^{[1]}\Big) = \mathrm{d}\,t - \mathrm{d}\,t = 0\\
(\mathrm{d}\partial + \partial\mathrm{d})\Big(\mathrm{d}\,s^{[m]}\Big) &= \mathrm{d}\Big(-\mathrm{d}\,t\,\mathrm{d}\,s^{[m-1]}\Big)\\
&= -\mathrm{d}(\mathrm{d}\,t)\,\mathrm{d}\,s^{[m-1]} + \mathrm{d}\,t\,\mathrm{d}\Big(\mathrm{d}\,s^{[m-1]}\Big) = 0 + 0 = 0
\end{aligned}$$

It is clear that the two morphisms to k in diagram (8.5) are compatible with d and ∂, so (8.5) is a commutative diagram in $\mathrm{CAlg}(\mathsf{Mixed})$.

For n a positive integer we denote by

$$A_n := A_1^{\otimes n} = k[t_1,\ldots,t_n]\otimes\Lambda(\mathrm{d}\,t_1,\ldots,\mathrm{d}\,t_n)\otimes\Lambda(s_1,\ldots,s_n)\otimes\Gamma(\mathrm{d}\,s_1,\ldots,\mathrm{d}\,s_n)$$

the n-fold tensor product of A_1 in $\mathrm{CAlg}(\mathsf{Mixed})$. We will also let

$$P_n := P_1^{\otimes n} = k[t_1,\ldots,t_n]\otimes\Lambda(\mathrm{d}\,t_1,\ldots,\mathrm{d}\,t_n)$$

be the n-fold tensor product of P_1. The n-fold tensor product of diagram (8.5) then yields a commutative diagram

$$\begin{tikzcd}
 & A_n \arrow[d, "p_n"] \\
P_n \arrow[ur, "g_n"] \arrow[r, "g'_n"'] & k
\end{tikzcd} \tag{8.7}$$

in $\mathrm{CAlg}(\mathsf{Mixed})$. ♢

8.2.2. Cofibrancy

Proposition 8.2.2.1. *Let n be a positive integer. Then A_n from Construction 8.2.1.2 is cofibrant (with respect to the model structure from Theorem 4.2.2.1) as an object in $\mathrm{LMod}_{P_n}(\mathsf{Ch}(k))$, where the module structure is the one arising from the morphism of differential graded algebras g_n from Construction 8.2.1.2.* ♡

Proof. Considered first as just a graded module over the graded algebra P_n, it is clear that A_n is a free P_n-module and that

$$\mathcal{B} := \left\{ s^{\vec{\epsilon}}\,\mathrm{d}\,s^{[\vec{i}\,]} \;\middle|\; \vec{i} \in \mathbb{Z}_{\geq 0}^n,\ \vec{\epsilon} \in \{0,1\}^n \right\}$$

forms a basis.

Let $\preceq$ be the lexicographic[11] well-order on $(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n \times \{0,1\}^n$. For an element $(\vec{j},\vec{\eta})$ in $(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n \times \{0,1\}^n$ we define

$$\mathcal{B}_{\vec{j},\vec{\eta}} := \left\{ s^{\vec{\epsilon}}\,\mathrm{d}\,s^{[\vec{i}\,]} \;\middle|\; \vec{i} \in \mathbb{Z}_{\geq 0}^n,\ \vec{\epsilon} \in \{0,1\}^n,\ (\vec{i},\vec{\epsilon}) \preceq (\vec{j},\vec{\eta}) \right\}$$

[11] In $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ we let ∞ be greater than any integer. The lexicographic order is then defined such that $(\vec{i},\vec{\epsilon}) \preceq (\vec{j},\vec{\eta})$ if and only if there is an index $1 \leq l \leq n$ with $i_1 = j_1,\ldots,i_{l-1} = j_{l-1}$ and $i_l < j_l$, or $\vec{i} = \vec{j}$ and there is an index $1 \leq l \leq n$ with $\epsilon_1 = \eta_1,\ldots,\epsilon_{l-1} = \eta_{l-1}$ and $\epsilon_l < \eta_l$.

and let $X_{\vec{j},\vec{\eta}}$ be the sub-P_n-module (still as just a graded module over a graded algebra) generated by $\mathcal{B}_{\vec{j},\vec{\eta}}$. It is clear from the definition of the differential on A_n that $X_{\vec{j},\vec{\eta}}$ is actually a subcomplex of A_n, and that $A_n = X_{(\infty,\ldots,\infty),(1,\ldots,1)}$.

Considering $(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n \times \{0,1\}^n$ as a category with a unique morphism $(\vec{i},\vec{\epsilon}) \to (\vec{j},\vec{\eta})$ if and only if $(\vec{i},\vec{\epsilon}) \preceq (\vec{j},\vec{\eta})$, we obtain a functor

$$(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n \times \{0,1\}^n \to \mathrm{LMod}_{P_n}(\mathsf{Ch}(k)) \tag{$*$}$$

that sends $(\vec{j},\vec{\eta})$ to $X_{\vec{j},\vec{\eta}}$ and the morphisms to the respective inclusions. One can see that this functor is colimit-preserving, which boils down to the fact that

$$\mathcal{B}_{\vec{j},\vec{0}} = \bigcup_{(\vec{i},\vec{\epsilon}) \prec (\vec{j},\vec{0})} \mathcal{B}_{\vec{i},\vec{\epsilon}}$$

for every $\vec{j}$ in $(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n$ such that there is an $1 \leq l \leq n$ with $j_l = \infty$ and $j_{l'} = 0$ for $l' > l$[12]. Thus the functor $(*)$ exhibits A_n as a transfinite composition, and so to show that A_n is cofibrant in $\mathrm{LMod}_{P_n}(\mathsf{Ch}(k))$ it suffices to show that $X_{\vec{0},\vec{0}}$ is cofibrant, and that for each $(\vec{i},\vec{\epsilon})$ and $(\vec{j},\vec{\eta})$ in $(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n \times \{0,1\}^n$, such that $(\vec{j},\vec{\eta})$ is the successor of $(\vec{i},\vec{\epsilon})$, the inclusion $X_{\vec{i},\vec{\epsilon}} \to X_{\vec{j},\vec{\eta}}$ is a cofibration.

As $X_{\vec{0},\vec{0}}$ is isomorphic to P_n, and hence free on the cofibrant chain complex $k[0]$ as a P_n-module in $\mathsf{Ch}(k)$, it is cofibrant. Furthermore, with $(\vec{i},\vec{\epsilon})$ and $(\vec{j},\vec{\eta})$ as above, the difference $\mathcal{B}_{\vec{j},\vec{\eta}} \setminus \mathcal{B}_{\vec{i},\vec{\epsilon}}$ consists of precisely the element $s^{\vec{\eta}}\,\mathrm{d}\,s^{[\vec{j}]}$. The diagram

$$\begin{tikzcd}
\mathrm{Free}^{\mathrm{LMod}_{P_n}}\left(S_{2\deg_{\mathsf{Ch}}(\vec{j})+\deg_{\mathsf{Ch}}(\vec{\eta})-1}\right) \arrow[r] \arrow[d] & X_{\vec{i},\vec{\epsilon}} \arrow[d] \\
\mathrm{Free}^{\mathrm{LMod}_{P_n}}\left(D_{2\deg_{\mathsf{Ch}}(\vec{j})+\deg_{\mathsf{Ch}}(\vec{\eta})}\right) \arrow[r] & X_{\vec{j},\vec{\eta}}
\end{tikzcd}$$

is a pushout, where we use use the notation from [Hov99, 2.3.3][13], with the morphism on the left being induced by the usual inclusion[14]. The morphism on the top sends the generator 1 in degree $2\deg_{\mathsf{Ch}}(\vec{j}) + \deg_{\mathsf{Ch}}(\vec{\eta}) - 1$ to $\partial\left(s^{\vec{\eta}}\,\mathrm{d}\,s^{[\vec{j}]}\right)$, and the morphism at the bottom sends the new generator in degree $2\deg_{\mathsf{Ch}}(\vec{j}) + \deg_{\mathsf{Ch}}(\vec{\eta})$ to $s^{\vec{\eta}}\,\mathrm{d}\,s^{[\vec{j}-\vec{\eta}]}$. It is crucial here that even though $s^{\vec{\eta}}\,\mathrm{d}\,s^{[\vec{j}]}$ is not an element of $X_{\vec{i},\vec{\epsilon}}$, its boundary is. □

[12] I. e. we consider those $(\vec{j},\vec{\eta})$ that are not successors or $(\vec{0},\vec{0})$.

[13] So S_l is the complex with k concentrated in degree l and D_l is the acyclic complex with k in degree l and $l-1$, with boundary operator the identity.

[14] Which is the identity in degree $2\deg_{\mathsf{Ch}}(\vec{j}) + \deg_{\mathsf{Ch}}(\vec{\eta})$.

8.2.3. Quasiisomorphism

Proposition 8.2.3.1. *Let n be a positive integer. Then the morphism*

$$p_n : A_n \to k$$

from Construction 8.2.1.2 is a quasiisomorphism. ♡

Proof. By Proposition 8.2.2.1 and Theorem 4.2.2.1 (8)[15], A_m is cofibrant as a chain complex for every positive integer m. By the pushout-product property for $\mathsf{Ch}(k)$ (see Fact 4.1.3.1) and Ken Brown's lemma [Hov99, 1.1.12], the tensor product of a cofibrant chain complex with a quasiisomorphism between cofibrant chain complexes is again a quasiisomorphism. Writing $p_n : A_n \to k$ as the composition

$$A_1 \otimes A_1^{n-1} \xrightarrow{p_1 \otimes \mathrm{id}_{A_1^{n-1}}} k \otimes A_1 \otimes A_1^{n-2} \xrightarrow{\mathrm{id}_k \otimes p_1 \otimes \mathrm{id}_{A_1^{n-2}}} k \otimes k \otimes A_1^{n-2}$$
$$\to \cdots \to k^n \cong k$$

it suffices to show that $p_1 : A_1 \to k$ is a quasiisomorphism.

As a morphism of chain complexes p_1 has a section ι that maps 1 to 1, so it suffices to give an homotopy ϑ between the id_{A_1} and $\iota \circ p_1$. As a graded abelian group, A_1 is free with basis $\left\{ t^n \,\mathrm{d}\, t^\epsilon s^\eta \,\mathrm{d}\, s^{[m]} \;\middle|\; n, m \in \mathbb{Z}_{\geq 0},\ \epsilon, \eta \in \{0,1\} \right\}$, and we will define ϑ on this basis. Define

$$\vartheta(t^n \,\mathrm{d}\, t^\epsilon s^\eta \,\mathrm{d}\, s^{[m]}) = \begin{cases} (-1)^\epsilon t^{n-1} \,\mathrm{d}\, t^\epsilon s^{\eta+1} \,\mathrm{d}\, s^{[m]} & \text{if } n > 0 \\ -\,\mathrm{d}\, s^{[m+1]} & \text{if } n = 0,\ \eta = 0, \text{ and } \epsilon = 1 \\ 0 & \text{otherwise} \end{cases}$$

We now check that $\vartheta\partial + \partial\vartheta = \iota p_1$ on basis elements $t^n \,\mathrm{d}\, t^\epsilon s^\eta \,\mathrm{d}\, s^{[m]}$ by distinguishing a couple of cases.

Case $n > 0, \eta = 0$:

$$\begin{aligned}
&(\vartheta\partial + \partial\vartheta)(t^n \,\mathrm{d}\, t^\epsilon \,\mathrm{d}\, s^{[m]}) \\
&= \vartheta\Big((-1) \cdot (-1)^\epsilon t^n \,\mathrm{d}\, t^{\epsilon+1} \,\mathrm{d}\, s^{[m-1]}\Big) + \partial\Big((-1)^\epsilon t^{n-1} \,\mathrm{d}\, t^\epsilon s \,\mathrm{d}\, s^{[m]}\Big) \\
&= (-1) \cdot (-1)^\epsilon \cdot (-1)^{\epsilon+1} t^{n-1} \,\mathrm{d}\, t^{\epsilon+1} s \,\mathrm{d}\, s^{[m-1]} \\
&\quad + (-1)^\epsilon \cdot (-1)^\epsilon t^n \,\mathrm{d}\, t^\epsilon \,\mathrm{d}\, s^{[m]} \\
&\quad + (-1)^\epsilon \cdot (-1) \cdot (-1)^\epsilon t^{n-1} \,\mathrm{d}\, t^{\epsilon+1} s \,\mathrm{d}\, s^{[m-1]} \\
&= t^n \,\mathrm{d}\, t^\epsilon \,\mathrm{d}\, s^{[m]}
\end{aligned}$$

Case $n > 0, \eta = 1$:

$$\begin{aligned}
&(\vartheta\partial + \partial\vartheta)(t^n \,\mathrm{d}\, t^\epsilon s \,\mathrm{d}\, s^{[m]}) \\
&= \vartheta\Big((-1)^\epsilon t^{n+1} \,\mathrm{d}\, t^\epsilon \,\mathrm{d}\, s^{[m]} - t^n \,\mathrm{d}\, t^{\epsilon+1} s \,\mathrm{d}\, s^{[m-1]}\Big) + \partial(0)
\end{aligned}$$

[15] This is applicable because P_m has cofibrant underlying chain complex by [Hov99, 2.3.6], as P_m is concentrated in nonnegative degrees and free as a graded k-module.

$$
\begin{aligned}
&= (-1)^{\epsilon} \cdot (-1)^{\epsilon} t^n \,\mathrm{d}\, t^{\epsilon} s \,\mathrm{d}\, s^{[m]} - 0 + 0 \\
&= t^n \,\mathrm{d}\, t^{\epsilon} s \,\mathrm{d}\, s^{[m]}
\end{aligned}
$$

Case $n = 0, \eta = 0, \epsilon = 1$:

$$
\begin{aligned}
(\vartheta\partial + \partial\vartheta)(\mathrm{d}\, t \,\mathrm{d}\, s^{[m]}) &= \vartheta(0) + \partial\Big(-\,\mathrm{d}\, s^{[m+1]}\Big) \\
&= \mathrm{d}\, t \,\mathrm{d}\, s^{[m]}
\end{aligned}
$$

Case $n = 0, \eta = 0, \epsilon = 0$:

$$
\begin{aligned}
(\vartheta\partial + \partial\vartheta)(\mathrm{d}\, s^{[m]}) &= \vartheta\Big(-\,\mathrm{d}\, t \,\mathrm{d}\, s^{[m-1]}\Big) + \partial(0) \\
&= \begin{cases} \mathrm{d}\, s^{[m]} & \text{if } m > 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Note that the case $n = m = \eta = \epsilon = 0$ is special, as 1 is the only basis element on which ιp is acts as the identity, rather than zero, so this is the expected result.

Case $n = 0, \eta = 1$:

$$
\begin{aligned}
(\vartheta\partial + \partial\vartheta)(\mathrm{d}\, t^{\epsilon} s \,\mathrm{d}\, s^{[m]}) &= \vartheta\Big((-1)^{\epsilon} t \,\mathrm{d}\, t^{\epsilon} \,\mathrm{d}\, s^{[m]} - \mathrm{d}\, t^{\epsilon+1} s \,\mathrm{d}\, s^{[m-1]}\Big) + \partial(0) \\
&= (-1)^{\epsilon} \cdot (-1)^{\epsilon} \,\mathrm{d}\, t^{\epsilon} s \,\mathrm{d}\, s^{[m]} + 0 \\
&= \mathrm{d}\, t^{\epsilon} s \,\mathrm{d}\, s^{[m]}
\end{aligned}
$$

□

8.3. A formula for Hochschild homology of certain quotients

In this section we combine Sections 8.1 and 8.2 to obtain a somewhat more concrete formula for a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of certain quotients than in Proposition 8.1.2.1.

Proposition 8.3.0.1. *Let $n \geq 1$ be an integer and assume*[16] *that Conjecture C holds for the morphism of commutative k-algebras $T\colon k[t_1, \ldots, t_n] \to k$ that maps t_i to 0, and fix a commutative square*

$$
\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n]) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(T)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[t_1,\ldots,t_n]/k}\Big) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega^{\bullet}_{T/k})"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k/k}\Big)
\end{tikzcd}
\tag{8.8}
$$

[16]If $n \leq 2$ this holds by Proposition 7.5.4.1, making this result unconditional.

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ *such that the horizontal morphisms are equivalences.*

Let R be a commutative algebra in $\mathsf{Ch}(k)$ and let $x_1, \ldots, x_n$ be elements of R_0. Assume that R is cofibrant as an object of $\mathrm{RMod}_{k[t_1,\ldots,t_n]}(\mathsf{Ch}(k))$ *with respect to the model structure of Theorem 4.2.2.1, where t_i acts by multiplication with x_i. Note that as* $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ *is monoidal,* $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ *obtains an induced structure of a right module over* $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n])$ *in* $\mathcal{M}\mathrm{ixed}$.

Let $P_n = k[t_1, \ldots, t_n] \otimes \Lambda(\mathrm{d}\, t_1, \ldots, \mathrm{d}\, t_n)$ be as in Construction 8.2.1.2 and M a right-P_n-module in $\mathsf{Mixed}_{\mathrm{cof}}$ such that there is an equivalence

$$\mathrm{RMod}(\gamma_{\mathsf{Mixed}})(M) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$$

in $\mathrm{RMod}(\mathcal{M}\mathrm{ixed})$ *such that the underlying equivalence of algebras is the composition*

$$\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(P_n) \simeq \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[t_1,\ldots,t_n]/k}\right) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n]) \tag{8.9}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$*, where the first equivalence is induced by the identification*

$$\Omega^{\bullet}_{k[t_1,\ldots,t_n]/k} \cong k[t_1, \ldots, t_n] \otimes \Lambda(\mathrm{d}\, t_1, \ldots, \mathrm{d}\, t_n)$$

from the start of Section 7.1 and the second equivalence is the one from (8.8).

Then there is an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1, \ldots, x_n)) \simeq \gamma_{\mathsf{Mixed}}(M')$$

in $\mathcal{M}\mathrm{ixed}$*, where M' is the strict mixed complex defined as follows. As a graded k-module, M' is given by*

$$M' := M \otimes \Lambda(s_1, \ldots, s_n) \otimes \Gamma(\mathrm{d}\, s_1, \ldots, \mathrm{d}\, s_n)$$

with $s_1, \ldots, s_n$ of degree 1 *and $\mathrm{d}\, s_1, \ldots, \mathrm{d}\, s_n$ of degree* 2*. The boundary operator ∂ and differential* d *are given by k-linearly extending the following formulas for $m \in M$, $\vec{\epsilon} \in \{0,1\}^n$, and $\vec{i} \in \mathbb{Z}^n_{\geq 0}$.*

$$\begin{aligned}
\partial\left(m \otimes s^{\vec{\epsilon}}\, \mathrm{d}\, s^{[\vec{i}]}\right) &= \left(\partial^M(m) \otimes s^{\vec{\epsilon}}\, \mathrm{d}\, s^{[\vec{i}]}\right) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(m)} \sum_{a=1}^{n} (-1)^{\sum_{b=1}^{a-1} \epsilon_b} \left(m \cdot t_a \otimes s^{\vec{\epsilon} - \vec{e_a}}\, \mathrm{d}\, s^{[\vec{i}]}\right) \\
&\quad - (-1)^{\deg_{\mathsf{Ch}}(m)} \sum_{a=1}^{n} \left(m \cdot \mathrm{d}\, t_a \otimes s^{\vec{\epsilon}}\, \mathrm{d}\, s^{[\vec{i} - \vec{e_a}]}\right) \\
\mathrm{d}\left(m \otimes s^{\vec{\epsilon}}\, \mathrm{d}\, s^{[\vec{i}]}\right) &= \left(\mathrm{d}^M(m) \otimes s^{\vec{\epsilon}}\, \mathrm{d}\, s^{[\vec{i}]}\right) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(m)} \sum_{a=1}^{n} (-1)^{\sum_{b=1}^{a-1} \epsilon_b} (i_a + 1) \left(m \otimes s^{\vec{\epsilon} - \vec{e_a}}\, \mathrm{d}\, s^{[\vec{i} + \vec{e_a}]}\right)
\end{aligned}$$

In the above formulas, summands in which a vector occurs with a component that is negative are to be interpreted as zero. ♡

Proof. We first apply Proposition 8.1.2.1, where we are using the specific model A_n that was constructed in Section 8.2 for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$ as a module over $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])$. To do so, we only need to check that A_n has the properties required of it in Proposition 8.1.2.1. Concretely, we need an equivalence

$$\mathrm{LMod}(\gamma_{\mathsf{Mixed}})(A_n) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

in LMod($\mathcal{M}$ixed) such that the underlying equivalence of algebras is (8.9), and we need that A_n is cofibrant as an object of $\mathrm{LMod}_{P_n}(\mathsf{Ch}(k))$. The latter is precisely Proposition 8.2.2.1, and for the former we use the following composite equivalence.

$$\begin{aligned}\mathrm{LMod}(\gamma_{\mathsf{Mixed}})(A_n) \xrightarrow{\simeq} \mathrm{LMod}(\gamma_{\mathsf{Mixed}})(k) &\simeq \mathrm{LMod}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k/k}\right)\\ &\simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)\end{aligned}$$

The first morphism is induced by the morphism of P_n-algebras $p_n\colon A_n \to k$ as defined in Construction 8.2.1.2, and lies over the identity morphism of $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(P_n)$ in Alg($\mathcal{M}$ixed). The second equivalences uses naturality of the isomorphism from Section 7.1, which ensures that the underlying equivalence of algebras is the first equivalence in (8.9). Finally, the third equivalence arises from the commutative square (8.8), and the underlying equivalence of algebras is the second one in (8.9).

By Proposition 8.1.2.1 we now obtain an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1,\ldots,x_n)) \simeq \gamma_{\mathsf{Mixed}}(M \otimes_{P_n} A_n)$$

in $\mathcal{M}$ixed. It thus remains to evaluate the relative tensor product $M \otimes_{P_n} A_n$ in Mixed.

As the forgetful functor from strict mixed complexes to graded k-modules is conservative, symmetric monoidal, and preserves colimits, we obtain an isomorphism of underlying graded k-modules[17]

$$\begin{aligned}M \otimes_{P_n} A_n = M \otimes_{k[t_1,\ldots,t_n]\otimes\Lambda(\mathrm{d}\,t_1,\ldots,\mathrm{d}\,t_n)} \Big(&k[t_1,\ldots,t_n] \otimes \Lambda(\mathrm{d}\,t_1,\ldots,\mathrm{d}\,t_n)\\ &\otimes \Lambda(s_1,\ldots,s_n) \otimes \Gamma(\mathrm{d}\,s_1,\ldots,\mathrm{d}\,s_n)\Big)\\ \cong M \otimes \Lambda(s_1,\ldots,s_n) \otimes \Gamma(\mathrm{d}\,s_1,\ldots,\mathrm{d}\,s_n)&\end{aligned}$$

where the isomorphism maps an element of the form $m \otimes t^{\vec{i}}\,\mathrm{d}\,t^{\vec{\epsilon}}\,s^{\vec{\eta}}\,\mathrm{d}^{[\vec{j}]}$ to $m \cdot (t^{\vec{i}}\,\mathrm{d}\,t^{\vec{\epsilon}}) \otimes s^{\vec{\eta}}\,\mathrm{d}^{[\vec{j}]}$. We can lift this isomorphism to an isomorphism of

[17] The point is that in graded k-modules,

$$k[t_1,\ldots,t_n] \otimes \Lambda(\mathrm{d}\,t_1,\ldots,\mathrm{d}\,t_n) \otimes \Lambda(s_1,\ldots,s_n) \otimes \Gamma(\mathrm{d}\,s_1,\ldots,\mathrm{d}\,s_n)$$

really is the tensor product of $k[t_1,\ldots,t_n] \otimes \Lambda(\mathrm{d}\,t_1,\ldots,\mathrm{d}\,t_n)$ and $\Lambda(s_1,\ldots,s_n) \otimes \Gamma(\mathrm{d}\,s_1,\ldots,\mathrm{d}\,s_n)$, whereas this is not the case as chain complexes.

strict mixed complex, and it then remains to determine d and ∂, for which we use the morphism of strict mixed complexes

$$M \otimes A_n \to M \otimes_{P_n} A_n \cong M \otimes \Lambda(s_1, \ldots, s_n) \otimes \Gamma(\mathrm{d}\, s_1, \ldots, \mathrm{d}\, s_n)$$

where the first morphism is the canonical one and the isomorphism the one just described. One can then read off the formulas claimed in the statement using Definition 4.1.2.1, Remark 4.2.1.10, and Construction 8.2.1.2 $\square$

Chapter 9.

Hochschild homology of certain quotients of polynomial algebras

In Chapter 8 we obtained a general result that helps to produce strict mixed complexes that represent $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of some quotients of commutative algebras. In this chapter we specialize to quotients of polynomial algebras by a single monic polynomial f of positive degree. The crucial input that we will need for this is that Conjecture D holds for f. After verifying the necessary requirements to apply the result, we will in Section 9.2 be able to specialize Proposition 8.3.0.1 to the case $k[x_1, \ldots, x_n]/f$ for n a positive integer and f a monic polynomial of positive degree satisfying Conjecture D, obtaining a strict mixed complex X_f that is a model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{M}$ixed. The underlying graded k-module of X_f is of the form

$$X_f \coloneqq k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$$

where x_i, $\mathrm{d}\, x_i$, s, and $\mathrm{d}\, s$ are of degree 0, 1, 1, and 2, respectively.

In our goal to obtain a strict mixed complex that represents the object $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ in $\mathcal{M}$ixed and that is as small as possible, this is already a significant improvement on the standard Hochschild complex $\mathrm{C}(k[x_1, \ldots, x_n]/f)$ that we discussed in Section 6.3.1. To underline this, note that X_f can be given the structure of a graded $k[x_1, \ldots, x_n]$-module, with $k[x_1, \ldots, x_n]$ acting through the leftmost tensor factor. X_f is then degree-wise free as a $k[x_1, \ldots, x_n]$-module, so we can consider the rank[1]. We find that $\mathrm{rank}_{k[x_1,\ldots,x_n]}((X_f)_i)$ (where i is an integer) is finite, and furthermore bounded, i. e. there is an integer r such that

$$\mathrm{rank}_{k[x_1,\ldots,x_n]}\big((X_f)_i\big) \leq r$$

holds for all integers i. This is very far from the situation for the standard Hochschild complex $\mathrm{C}(k[x_1, \ldots, x_n]/f)$. While $k[x_1, \ldots, x_n]$ doesn't act freely on the leftmost tensor factor, $k[x_1, \ldots, x_n]/f$ does, and

$$\mathrm{rank}_{k[x_1,\ldots,x_n]/f}(\mathrm{C}(k[x_1, \ldots, x_n]/f)_i)$$

[1] If we wanted to make the following discussion regarding ranks precise, we would define bases for the various modules and discuss their cardinalities (the modules we consider all have a relatively obvious basis to use for this). We omit such a detour, as this discussion is only for purpose of motivation.

$$\begin{aligned}
&= \operatorname{rank}_{k[x_1,\ldots,x_n]/f}\Big((k[x_1,\ldots,x_n]/f)^{\otimes(i+1)}\Big)\\
&= \operatorname{rank}_k\Big((k[x_1,\ldots,x_n]/f)^{\otimes(i)}\Big)\\
&= \operatorname{rank}_k((k[x_1,\ldots,x_n]/f))^i
\end{aligned}$$

for $i \geq 0$. For $n > 1$, $\operatorname{rank}_k((k[x_1,\ldots,x_n]/f))$ will already be infinite, and additionally it would also be reasonable to consider the rank to grow exponentially in the degree i.

So X_f is an improvement over the standard Hochschild complex. It is though certainly not optimal for specific polynomials. For example, for $f = x_1$ the quotient $k[x_1]/f$ is isomorphic to k, so we can by Corollary 7.4.9.3 use $\Omega^\bullet_{k/k} \cong k$ as a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1]/f)$, and k is significantly smaller than $X_f = k[x_1] \otimes \Lambda(\mathrm{d}\,x_1) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\,s)$.

The main goal of this chapter will thus be to improve on the size of X_f while relaxing what the result covers. This can be done in two directions: Firstly, we can reduce the amount of structure we consider, which we do by asking only for a sub-chain-complex of X_f that represents $\mathrm{HH}(k[x_1,\ldots,x_n]/f)$ as an object of $\mathcal{D}(k)$, rather than as a mixed complex, which we will do in Section 9.3. Secondly, we can insist on a sub-strict-mixed-complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$ as an object of $\mathcal{M}\mathrm{ixed}$, while reducing the set of polynomials f that we consider. This will be done in Section 9.5.

The results of this chapter should themselves also only be considered as stepping stones, just like Proposition 8.3.0.1 and X_f was a stepping stone for the results of this chapter. So for actual calculations that need a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$ for specific polynomials f, one would begin with the strict mixed complex obtained in Section 9.5 (if the relevant result is applicable) and then simplify it further, making use of the specific form of f. In Chapter 10 we will discuss the concrete example of $f = x_1^2 - x_2x_3$ in details along those lines[2].

Let us now say some more on the individual sections of this chapter.

As we stated at the beginning of this introduction, we will consider *monic* multivariable polynomials f to divide out of a polynomial algebra. For polynomials in a single variable there is precisely one standard definition of what it means to be monic, but this is not the case for multivariable polynomials, where there are multiple sensible definitions. What we will mean by *monic* is *monic with respect to a chosen monomial order*, and this notion will be introduced in Section 9.1. It will also be very important in this chapter to have a good handle of moving back and forth between $k[x_1,\ldots,x_n]$ and $k[x_1,\ldots,x_n]/f$, for example by producing canonical representatives in $k[x_1,\ldots,x_n]$ of elements in the quotient $k[x_1,\ldots,x_n]/f$. For this we will also discuss division with remainder for multivariable polynomials in Section 9.1.

In Section 9.2 we will then combine previous results to obtain X_f as a

[2] This polynomial has however so far not been proven to satisfy Conjecture D. The strict mixed complex X_f can nevertheless be constructed.

strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{M}$ixed, assuming that Conjecture D holds for f. Heavily using constructions discussed in Section 9.1 that are built on top of the division with remainder for multivariable polynomials, we will also describe a new basis for X_f as well as calculate some formulas for the boundary operator and differential in terms of that new basis.

In Section 9.3 we will discuss $\mathrm{HH}(k[x_1, \ldots, x_n]/f)$ as only an object of $\mathcal{D}(k)$. A chain complex representing it has already been obtained in the previous work of the Buenos Aires Cyclic Homology Group in [BACH]. For k a commutative ring and f an element of $k[x_1, \ldots, x_n]$ satisfying relatively mild conditions, they give a quite small differential graded algebra together with a multiplicative inclusion into the normalized standard Hochschild complex for $\overline{\mathrm{C}}(k[x_1, \ldots, x_n]/f)$, as well as a homotopy inverse to this inclusion, as a morphism of chain complexes. Using the basis for X_f and the formulas for the boundary operator in this basis obtained in Section 9.2, it will be relatively straightforward in Section 9.3 to define a subcomplex $X^e_{f,0}$ of X_f such that the inclusion into X_f is a quasiisomorphism, thereby obtaining a smaller chain complex than X_f representing $\mathrm{HH}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{D}(k)$. We will also show that $X^e_{f,0}$ is isomorphic to the chain complex described in [BACH]. Assuming that Conjecture D holds for f, the rest of the assumptions we need to make for f are the same as in [BACH], so this amounts to giving a new proof for one of the main results of [BACH], using a quite different approach, for the range in which Conjecture D has been proven, so $n \leq 2$ as long as 2 is invertible in k by Proposition 7.5.3.1.

Unfortunately the definition of the comparison morphisms used in [BACH] between the smaller chain complex and the normalized standard Hochschild complex are quite complicated, making them difficult to unwrap for transferring additional structure. Trying to transfer the strict mixed complex structure to the smaller chain complex from the normalized standard Hochschild complex additionally runs into the problem that one does not obtain a strict mixed complex structure; the necessary identities will only be satisfied up to homotopy for general f, and it is not possible to upgrade either of the two quasiisomorphisms between the small chain complex and the normalized standard Hochschild complex to a morphism of strict mixed complexes, as we show in Section 9.6.

However, for some polynomials f, the strict mixed structure on X_f restricts to $X^e_{f,0}$, so that $X^e_{f,0}$ even represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{M}$ixed. To properly formulate a condition for when the strict mixed structure restricts we introduce the notions of logarithm and the *log dimension* for multivariable polynomials in Section 9.4. In particular, we will prove a criterion that can be easily checked for multivariable polynomials f and that implies that $\mathrm{logdim}_f(\mathrm{d}\, f) \leq 1$.

In Section 9.5 we will then show that if f satisfies $\mathrm{logdim}_f(\mathrm{d}\, f) \leq 1$, then the strict mixed structure of X_f restricts to $X^e_{f,0}$, making the inclusion of $X^e_{f,0}$ into X_f into a morphism of strict mixed complexes that is a weak equivalence.

Under some stronger assumptions on f a strict mixed complex isomorphic to $X^e_{f,0}$ was already constructed by Larsen in [Lar95]. In the two-variable case Larsen furthermore constructs a strongly homotopy linear quasiisomorphism[3] from this strict mixed complex into the normalized standard Hochschild complex. The result in Section 9.5 can thus be seen as a generalization of one of the main results of [Lar95].[4] A number of constructions relating to polynomials that we use in order to simplify X_f are inspired by their use in [Lar95].

In Section 9.7 we discuss the relationship between our results and the main result of [Lar95], as well as how, assuming Conjecture D, our results provide an affirmative answer to a question posed by Larsen in [Lar95].

9.1. Prerequisites on polynomials and dividing with remainder

Given the non-zero polynomial f in n variables by which we want to divide the polynomial algebra $k[x_1, \ldots, x_n]$, it will be important for us to define uniquely determined remainders of dividing an arbitrary polynomial P by f, i. e. we would like to have a procedure obtain a unique decomposition of P as $P = Q \cdot f + R$ for other polynomials Q and R. In the one-variable case with f an element of $k[x]$ it is relatively straightforward to come up with an idea of how this decomposition should look like: We would like P to uniquely decompose as $P = Q \cdot f + R$ where R has smaller degree than f. It is not difficult to see that if the leading coefficient of f is not a zero-divisor, then this determines Q and R uniquely as long as such a decomposition exists. However, such a decomposition may not exist for all f and P – as a counterexample consider $f = 2$ and $P = 3$ for $k = \mathbb{Z}$. However it turns out that such a decomposition does exist if the polynomial f is *monic*, that is the leading coefficient is 1. In that case, one can perform the Euclidean algorithm, iteratively eliminating the highest power of x remaining with the leading term of f, i. e. if we have given $f = x^n + f'$ with f' of degree less than n, and $P = \sum_{i=0}^{m} a_i x^i$ with $m \geq n$, then the first step will be to write

$$P = (a_m x^{m-n}) \cdot f + \left(\left(\sum_{i=0}^{m} a_i x^i \right) - (a_m x^{m-n}) \cdot f' \right)$$

and in this decomposition the term in brackets is of degree less than m, so iterating this process will eventually come to a stop.

If we wish to generalize this procedure to the multi-variable case, we are confronted with an obvious question: Which term of P should we start eliminating? What is the leading term of f that we should use to do so? There is

[3]See Definition 4.2.3.1 for a definition. By Remark 4.4.4.2 a strongly homotopy linear quasiisomorphism induces an equivalence in $\mathcal{M}$ixed.

[4]However introducing the new assumption that 2 is invertible in k.

no obviously correct choice for a definition of leading terms of multivariable polynomials but multiple equally good competing ones. Thus we will have to codify what we require of such a definition to be nice enough to allow us to define the kind of decompositions described, and then require that f be monic with respect to that choice. The results will then also depend on that choice.

We will start in Section 9.1.1 by discussing *monomial orders*, which provide a consistent way of determining which of two monomials is to be considered the larger one. This will allow us to define a notion of degree of a multivariable polynomial in Section 9.1.2. Finally, we will discuss division with remainder for multivariable polynomials in Section 9.1.3.

9.1.1. Monomial orders

In this section we introduce the concept of monomial orders and discuss some easy consequences of the definition. We start in Section 9.1.1.1 by recalling the notions of partial, total, and well-orders. The important example of the pointwise partial order on $\mathbb{Z}^n_{\geq 0}$ will be discussed in Section 9.1.1.2, before we define monomial orders in Section 9.1.1.3. We end this section by proving some easy properties of monomial orders in Section 9.1.1.4.

9.1.1.1. Partial, total, and well-orders

We recall the following notions.

Definition 9.1.1.1. Let X be a set and $\preceq$ a binary relation on X. Recall the following properties that $\preceq$ may have.

Antisymmetry For any $a, b \in X$, if $a \preceq b$ and $b \preceq a$, then $a = b$.

Transitivity For any $a, b, c \in X$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

Reflexivity For any $a \in X$ it holds that $a \preceq a$.

Connectivity For any $a, b \in X$, it holds that $a \preceq b$ or $b \preceq a$.

Well-foundedness If X' is nonempty subset of X, then X' has a least element, that is an element $x \in X'$ such that for all $y \in X'$ it holds that $x \preceq y$.

Note that connectivity implies reflexivity.

The relation $\preceq$ is called a

partial order if it is antisymmetric, transitive, and reflexive.

total order if it is antisymmetric, transitive, and connected.

well-order if it is antisymmetric, transitive, connected, and well-founded.

A set equipped with a partial order (total order, well-order) on it will be called a *partially ordered set* (*totally ordered set*, *well-ordered set*). ♢

Notation 9.1.1.2. Let X be a set and $\preceq$ a binary relation on X. If x and y are elements of X such that $x \preceq y$ and $x \neq y$, then we will say that x is smaller than y and y is bigger than x.

We will use the notation $x \succeq y$ to mean $y \preceq x$. Furthermore, we will use $x \succ y$ and $y \prec x$ to mean $y \preceq x$ and $x \neq y$. ♢

Remark 9.1.1.3. The important consequence of well-foundedness is that we can prove statements about every element of X by transfinite induction: If we prove that any element of X has some property if every smaller element has that property, then it follows that *every* element of X has that property[5]. ♢

9.1.1.2. The standard partial order on $\mathbb{Z}^n_{\geq 0}$

We now define an important example of a partial order on $\mathbb{Z}^n_{\geq 0}$.

Definition 9.1.1.4. Let n be a positive integer. We define a relation $\leq$ on $\mathbb{Z}^n_{\geq 0}$ by letting $\vec{a} \leq \vec{b}$ if and only if $a_i \leq b_i$ for all $1 \leq i \leq n$. ♢

Remark 9.1.1.5. The relation $\leq$ as defined in Definition 9.1.1.4 is a partial order.

Note that a monomial $x^{\vec{i}}$ divides $x^{\vec{j}}$ for $\vec{i}, \vec{j} \in \mathbb{Z}^n_{\geq 0}$ if and only if $\vec{i} \leq \vec{j}$. This is the reason why the partial order $\leq$ is of relevance for us. ♢

Proposition 9.1.1.6. *Let n be a positive integer. For the partial order $\leq$ defined on $\mathbb{Z}^n_{\geq 0}$ as in Definition 9.1.1.4 and $\vec{a}, \vec{b}, \vec{c} \in \mathbb{Z}^n_{\geq 0}$, if $\vec{a} \leq \vec{b}$, then $\vec{a} + \vec{c} \leq \vec{b} + \vec{c}$.* ♡

Proof. Follows directly from the definition. □

9.1.1.3. Definition of monomial orders

The partial order $\leq$ encodes intuition on how some monomials definitely should compare: Certainly the monomial $x^{\vec{j}}$ should be "bigger" than $x^{\vec{i}}$ if $x^{\vec{i}}$ divides $x^{\vec{j}}$, or equivalently if $\vec{i} \leq \vec{j}$. But what if neither $\vec{i} \leq \vec{j}$ nor $\vec{j} \leq \vec{i}$? In order to be able to define notions such as degrees and leading terms for all elements of $k[x_1, \ldots, x_n]$, we are thus led to ask for a total order $\preceq$ on $\mathbb{Z}^n_{\geq 0}$ that extends $\leq$.

A finite subset of a totally ordered set has a maximum element. If we have a total order $\preceq$ on $\mathbb{Z}^n_{\geq 0}$ given, then we can now provisionally define what the

[5] Proof: Let $X' \subseteq X$ be the subset of X of elements that do *not* have the property in question. By well-foundedness, if X' were non-empty, it would need to have a least element x. But this would mean that every element smaller than x has the property, so x must have had it as well, so X' must have been empty.

leading term of a polynomial $f \in k[x_1, \ldots, x_n]$ should be: If f is given by

$$f = \sum_{\vec{i} \in \mathbb{Z}^n_{\geq 0}} f_{\vec{i}} x^{\vec{i}}$$

then we can declare $f_{\vec{j}} x^{\vec{j}}$ to be the leading term of f if $\vec{j}$ is the maximal element of $\left\{ \vec{i} \in \mathbb{Z}^n_{\geq 0} \;\middle|\; f_{\vec{i}} \neq 0 \right\}$.

However this is not quite enough to obtain the kind of decomposition we described in the introduction to Section 9.1. Firstly, in the one-variable case the procedure to iteratively eliminate the highest degree has to eventually terminate because there is no infinite strictly decreasing sequence of nonnegative integers. For the multivariable case we should thus require that $\preceq$ is a well-order. Secondly, in the one-variable case we need to argue that if f' has degree smaller than m, then $x^{l-m} \cdot f'$ has degree smaller than l, and we need an analogue of this in the multivariable case as well. This leads us to the following definition, which is also used in [BACH, 2.2].

Definition 9.1.1.7. Let n be a positive integer. A *monomial order (for n variables)* is a well-order $\preceq$ on $\mathbb{Z}^n_{\geq 0}$ satisfying the following property: For every $\vec{a}, \vec{b}, \vec{c} \in \mathbb{Z}^n_{\geq 0}$ such that $\vec{a} \preceq \vec{b}$ it also holds that $\vec{a} + \vec{c} \preceq \vec{b} + \vec{c}$. ♢

That a monomial order indeed extends $\leq$ will follow from this, and is shown below in Proposition 9.1.1.8.

9.1.1.4. Properties of monomial orders

Proposition 9.1.1.8. *Let n be a positive integer and $\preceq$ a monomial order for n variables. Then the following hold.*

(1) Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{Z}^n_{\geq 0}$ such that $\vec{a} + \vec{c} \preceq \vec{b} + \vec{c}$. Then it also holds that $\vec{a} \preceq \vec{b}$.

(2) $\vec{0}$ is minimal in $\mathbb{Z}^n_{\geq 0}$ with respect to $\preceq$, i. e. for every $\vec{a} \in \mathbb{Z}^n_{\geq 0}$ it holds that $\vec{0} \preceq \vec{a}$.

(3) $\preceq$ extends $\leq$, i. e. if $\vec{a}, \vec{b} \in \mathbb{Z}^n_{\geq 0}$ such that $\vec{a} \leq \vec{b}$, then $\vec{a} \preceq \vec{b}$. ♡

Proof. Proof of claim (1): If it is not true that $\vec{a} \preceq \vec{b}$, then we must have $\vec{a} \succeq \vec{b}$ by connectivity, and so $\vec{a} + \vec{c} \succeq \vec{b} + \vec{c}$ as $\preceq$ is a monomial order. But by antisymmetry this implies that $\vec{a} + \vec{c} = \vec{b} + \vec{c}$ and so $\vec{a} = \vec{b}$, from which $\vec{a} \preceq \vec{b}$ follows by reflexivity.

Proof of claim (2): Let $\vec{m}$ be an element of $\mathbb{Z}^n_{\geq 0}$. We need to show that $\vec{0} \preceq \vec{m}$, but by connectivity and reflexivity it suffices to show that if $\vec{0} \succeq \vec{m}$, then $\vec{m} = \vec{0}$. So assume that $\vec{0} \succeq \vec{m}$. By adding $l \cdot \vec{m}$ to this inequality we obtain $l \cdot \vec{m} \succeq (l+1) \cdot \vec{m}$, so that we obtain an infinite descending chain

$$\vec{0} \succeq \vec{m} \succeq 2 \cdot \vec{m} \succeq \cdots$$

in $\mathbb{Z}^n_{\geq 0}$. Well-foundedness of $\preceq$ implies that this chain must eventually stabilize, so there must be an $l \geq 0$ with $(l+1) \cdot \vec{m} = l \cdot \vec{m}$, which implies $\vec{m} = 0$.

Proof of (3): $\vec{a} \leq \vec{b}$ implies that $\vec{b} - \vec{a}$ still lies in $\mathbb{Z}^n_{\geq 0}$. Applying (2) we obtain $\vec{0} \preceq \vec{b} - \vec{a}$, and adding $\vec{a}$ to this inequality we conclude that $\vec{a} \preceq \vec{b}$. □

Remark 9.1.1.9. If $\preceq$ is a monomial order for 1 variable, then Proposition 9.1.1.8 (3) implies that $\preceq$ is equal to $\leq$. ♢

Remark 9.1.1.10. Let n be a positive integer. The assumptions made on the binary relation $\leq_T$ on $\mathbb{Z}^n_{\geq 0}$ considered in [BACH, 2.2] are that $\leq_T$ is a monomial order in the sense of Definition 9.1.1.7, and that $\leq_T$ extends $\leq$. Proposition 9.1.1.8 (3) shows that the latter assumption is unnecessary. ♢

Construction 9.1.1.11. Let n be a positive integer and $\preceq$ a monomial order for n variables. Let $m \leq n$ be another positive integer and

$$\varphi\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$$

an injection. Then we can define an additive injection $\mathbb{Z}^m_{\geq 0} \to \mathbb{Z}^n_{\geq 0}$ as follows.

$$\psi\colon \mathbb{Z}^m_{\geq 0} \to \mathbb{Z}^n_{\geq 0}\ , \qquad \psi(\vec{a})_i := \begin{cases} a_j & \text{if } \psi(j) = i \\ 0 & \text{if } i \text{ is not in the image of } \psi \end{cases}$$

For example if φ is the inclusion of $\{1\}$ into $\{1, 2\}$, then ψ maps (a) to $(a, 0)$.

We can then define a binary relation $\preceq$ on $\mathbb{Z}^m_{\geq 0}$ as follows. For $\vec{a}, \vec{b} \in \mathbb{Z}^m_{\geq 0}$ we let $\vec{a} \preceq \vec{b}$ if and only if $\psi(\vec{a}) \preceq \psi(\vec{b})$. It follows immediately from ψ being additive and injective that this defines a monomial order for m variables, which we will call the *restricted* monomial order.

Let $\{i_1, \ldots, i_{n-m}\}$ be the elements of $\{1, \ldots, n\}$ that are not in the image of φ. Define k' to be the commutative k-algebra $k' = k[x_{i_1}, \ldots, x_{i_{n-m}}]$. Then there is an isomorphism of k-algebras

$$k'[y_1, \ldots, y_m] \xrightarrow{\cong} k[x_1, \ldots, x_n]$$

that maps x_{i_j} to x_{i_j} and y_j to $x_{\varphi(j)}$. Note that this morphism then maps $y^{\vec{j}}$ to $x^{\psi(\vec{j})}$. We will make use of this isomorphism on some occasions when inducting on the number of variables. ♢

9.1.2. Degrees for multivariable polynomials

In this section we define a notation of degree of multivariable polynomials, dependent on a monomial order.

Definition 9.1.2.1. Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f \in k[x_1, \ldots, x_n]$ a polynomial. We define

$$\deg_{\preceq}(f) = \begin{cases} \max\left\{\vec{i} \in \mathbb{Z}^n_{\geq 0} \middle| f_{\vec{i}} \neq 0\right\} & \text{if } f \neq 0 \\ -\infty & \text{if } f = 0 \end{cases}$$

where the maximum is taken with respect to the order $\preceq$. We call $\deg_{\preceq}(f)$ the *degree of f (with respect to the monomial order $\preceq$)*. We call $f_{\deg_{\preceq}(f)} x^{\deg_{\preceq}(f)}$ the *leading term* and $f_{\deg_{\preceq}(f)}$ the *leading coefficient* of f (with respect to the monomial order $\preceq$).

If $f, g \in k[x_1, \ldots, x_n]$, then we write $f \preceq g$ if $\deg_{\preceq}(f) \preceq \deg_{\preceq}(g)$. ♢

Remark 9.1.2.2. It follows from Remark 9.1.1.9 and the definition that the degree as defined in Definition 9.1.2.1 recovers the usual notion in the case $n = 1$. ♢

The degree of multivariable polynomials as defined above satisfies the usual properties with respect to addition and multiplication of polynomials, as we record below.

Proposition 9.1.2.3. *Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f, g \in k[x_1, \ldots, x_n]$. Then the following hold.*

(1) $\deg_{\preceq}(f + g) \preceq \max\left\{\deg_{\preceq}(f), \deg_{\preceq}(g)\right\}$.

(2) If $\deg_{\preceq}(f) \succ \deg_{\preceq}(g)$*, then* $\deg_{\preceq}(f + g) = \deg_{\preceq}(f)$.

(3) $\deg_{\preceq}(f \cdot g) \preceq \deg_{\preceq}(f) + \deg_{\preceq}(g)$.

(4) If at least one of f or g is zero, or it holds that both are nonzero and $f_{\deg_{\preceq}(f)} \cdot g_{\deg_{\preceq}(g)} \neq 0$*, then* $\deg_{\preceq}(f \cdot g) = \deg_{\preceq}(f) + \deg_{\preceq}(g)$.

With respect to max *we interpret $-\infty$ as smaller than all elements of $\mathbb{Z}^n_{\geq 0}$, and we interpret the sum of $-\infty$ with $-\infty$ or an integer to be $-\infty$ again.* ♡

Proof. Proof of claim (1): By definition

$$f_{\deg_{\preceq}(f+g)} + g_{\deg_{\preceq}(f+g)} = (f + g)_{\deg_{\preceq}(f+g)} \neq 0$$

holds, so one of $f_{\deg_{\preceq}(f+g)}$ and $g_{\deg_{\preceq}(f+g)}$ must be non-zero, which directly implies that $\deg_{\preceq}(f) \succeq \deg_{\preceq}(f + g)$ or $\deg_{\preceq}(g) \succeq \deg_{\preceq}(f + g)$.

Proof of claim (2): In this case $\max\{\deg_{\preceq}(f), \deg_{\preceq}(g)\} = \deg_{\preceq}(f)$, so using (1) it suffices to show that $\deg_{\preceq}(f + g) \succeq \deg_{\preceq}(f)$. The assumption $\deg_{\preceq}(f) \succ \deg_{\preceq}(g)$ also implies $g_{\deg_{\preceq}(f)} = 0$ and thus

$$(f + g)_{\deg_{\preceq}(f)} = f_{\deg_{\preceq}(f)} + g_{\deg_{\preceq}(f)} = f_{\deg_{\preceq}(f)} \neq 0$$

from which $\deg_{\preceq}(f + g) \succeq \deg_{\preceq}(f)$ follows.

Proof of claim (3) and (4): We can write

$$f = \sum_{\vec{i} \preceq \deg_{\preceq}(f)} f_{\vec{i}} x^{\vec{i}} \qquad \text{and} \qquad g = \sum_{\vec{j} \preceq \deg_{\preceq}(g)} g_{\vec{j}} x^{\vec{j}}$$

and thus obtain the following description of the product fg.

$$f \cdot g = \sum_{\substack{\vec{i} \preceq \deg_{\preceq}(f) \\ \vec{j} \preceq \deg_{\preceq}(g)}} f_{\vec{i}} f_{\vec{j}} x^{\vec{i}+\vec{j}}$$

As $\preceq$ is not just a well-order, but even a monomial order, it follows from $\vec{i} \preceq \deg_{\preceq}(f)$ and $\vec{j} \preceq \deg_{\preceq}(g)$ that $\vec{i} + \vec{j} \preceq \deg_{\preceq}(f) + \deg_{\preceq}(g)$, and if one (or both) of the former two inequalities is strict, then so is the latter inequality. This implies both claims. □

Proposition 9.1.2.4. *Assume that we are in the situation of Construction 9.1.1.11. Let f be an element of $k[x_1, \ldots, x_n]$, and assume that $\deg_{\preceq}(f)$ is in the image of ψ. Let f' be the element of $k'[y_1, \ldots, y_m]$ corresponding to f under the isomorphism from Construction 9.1.1.11. Then*

$$\deg_{\preceq}(f) = \psi\big(\deg_{\preceq}(f')\big)$$

where on the right hand side $\preceq$ refers to the restricted monomial order as defined in Construction 9.1.1.11. Furthermore, $f'_{\deg_{\preceq}(f')}$ is an element of k and the leading coefficients of f and f' agree, i. e. $f'_{\deg_{\preceq}(f')} = f_{\deg_{\preceq}(f)}$. ♡

Proof. Let $\vec{j} \in \mathbb{Z}^m_{\geq 0}$ be such that $\psi(\vec{j}) = \deg_{\preceq}(f)$. Then $f_{\psi(\vec{j})} \neq 0$ implies that $f'_{\vec{j}} \neq 0$ and hence $\deg_{\preceq}(f') \succeq \vec{j}$, from which we can conclude that $\psi(\deg_{\preceq}(f')) \succeq \deg_{\preceq}(f)$. On the other hand, $f'_{\deg_{\preceq}(f')} \neq 0$, so there must be some $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ with $i_l = 0$ for l in the image of φ such that

$$f_{\psi\left(\deg_{\preceq}(f')\right)+\vec{i}} = (f'_{\deg_{\preceq}(f')})_{\vec{i}} \neq 0$$

from which

$$\deg_{\preceq}(f) \succeq \psi\big(\deg_{\preceq}(f')\big) + \vec{i} \succeq \psi\big(\deg_{\preceq}(f')\big) \tag{$*$}$$

follows. Antisymmetry now implies that $\deg_{\preceq}(f) = \psi\big(\deg_{\preceq}(f')\big)$.

Furthermore, this implies that if $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ with $i_l = 0$ for l in the image of φ such that $(f'_{\deg_{\preceq}(f')})_{\vec{i}} \neq 0$, then $\vec{i}$ must actually be $\vec{0}$, as otherwise the inequality $(*)$ would by *strict* by Proposition 9.1.1.8 ((2)). It follows that $f'_{\deg_{\preceq}(f')}$ is in k and that $f'_{\deg_{\preceq}(f')} = f_{\deg_{\preceq}(f)}$ as elements of k. □

9.1.3. Dividing multivariable polynomials with remainder

In this section we discuss a generalization of division with remainder of polynomials from the one-variable case as discussed in the introduction to Section 9.1 to the multivariable case. If we want to have a chance of dividing polynomials P with remainder by some polynomial f, then we should require that f is *monic*, and we discuss the multivariable notion of monic polynomials that we will use in Section 9.1.3.1. If f is a monic polynomial, then division with remainder will yield a decomposition of P as $P = Qf + R$, where R is in some sense "small" with respect to f. In the one-variable case, R will have smaller degree than f. In the multivariable case, R will be *f-reduced*, and we discuss what this means in Section 9.1.3.2. We will then be able to tackle division with remainder for multivariable polynomials in Section 9.1.3.3, and discuss decomposing P as $P = \sum_{i\geq 0} r_f^i(P) f^i$ with $r_f^i(P)$ being f-reduced polynomials in Section 9.1.3.4.

9.1.3.1. Monic polynomials

After the discussions in Sections 9.1.1 and 9.1.2, we can now give a definition of monic polynomials that generalizes the usual definition for the univariable case.

Definition 9.1.3.1. Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f \in k[x_1, \ldots, x_n]$ a polynomial. Then f is *monic with respect to* $\preceq$ if $f_{\deg_{\preceq}(f)} = 1$. In particular a monic polynomial is nonzero. ♢

Convention 9.1.3.2. From here on we will introduce a monomial order $\preceq$ in statements which depend on one, but will drop reference to $\preceq$ when this will not cause confusion. For example we will write "Let f be a monic polynomial." rather than "Let f be a monic polynomial with respect to $\preceq$." when there is only one polynomial degree order in context. ♢

Remark 9.1.3.3. If $n = 1$, then f is monic as defined in Definition 9.1.3.1 if and only if it is monic in the usual sense. See Remarks 9.1.1.9 and 9.1.2.2. ♢

Proposition 9.1.3.4. *Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f, g \in k[x_1, \ldots, x_n]$ monic polynomials. Then $f \cdot g$ is also monic.* ♡

Proof. Follows immediately from Proposition 9.1.2.3 (4). □

Proposition 9.1.3.5. *Assume that we are in the situation of Construction 9.1.1.11, and that f and f' are as in Proposition 9.1.2.4. Then f is monic with respect to the monomial order on $\mathbb{Z}^n_{\geq 0}$ if and only if f' is monic with respect to the restricted monomial order on $\mathbb{Z}^m_{\geq 0}$.* ♡

Proof. Follows immediately from Proposition 9.1.2.4. □

We end this section with a useful statement we will use later.

Proposition 9.1.3.6. *Let n be a positive integer, $\preceq$ a monomial order for n variables, $f \in k[x_1, \ldots, x_n]$ a monic polynomial, and $g \in k[x_1, \ldots, x_n]$ any polynomial. Then $g = 0$ if and only if $fg = 0$.* ♡

Proof. It is clear that $g = 0$ implies $fg = 0$, so it remains to show that $g \neq 0$ implies $fg \neq 0$. But if $g \neq 0$, then we can apply Proposition 9.1.2.3 (4)[6], to conclude that

$$\deg_{\preceq}(fg) = \deg_{\preceq}(f) + \deg_{\preceq}(g)$$

where the right hand side, and thus also the left hand side, is a nonnegative integer. Thus fg must be nonzero. □

9.1.3.2. Reduced polynomials

Let f be a monic polynomial in a single variable, i. e. an element of $k[x]$. Then we can write any polynomial $P \in k[x]$ as $P = Q \cdot f + R$ for $Q, R \in k[x]$ such that the degree of R is smaller than the degree of f. If we want to generalize this to the multivariable case we should find an analogous condition for R. A first guess might be to use the condition that $\deg_{\preceq}(R) \prec \deg_{\preceq}(f)$, but this turns out not to work. Consider for example the case of two variables and the lexicographic order, so where $(a_1, a_2) \preceq (b_1, b_2)$ if $a_1 < b_1$ or if $a_1 = b_1$ and $a_2 \leq b_2$. If we then consider $f = x_1 x_2$ and $P = x_1^2$, then it is impossible to find a decomposition $P = Q \cdot f + R$ such that $\deg_{\preceq}(R) \prec \deg_{\preceq}(f)$. So this condition is too strong. The reason is that we can only eliminate the lead term of P if $\deg_{\preceq}(f) \leq \deg_{\preceq}(P)$. We should thus ask R to be f-reduced in the following sense.

Definition 9.1.3.7. Let n be a positive integer, $\preceq$ a monomial order for n variables, $\overrightarrow{j} \in \mathbb{Z}^n_{\geq 0}$, and $P \in k[x_1, \ldots, x_n]$ a polynomial. P is called $\overrightarrow{j}$*-reduced* if $P_{\overrightarrow{i}} = 0$ for all $\overrightarrow{i} \geq \overrightarrow{j}$.

If $f \in k[x_1, \ldots, x_n]$ is a nonzero polynomial, then P is called f-reduced if and only if P is $\deg_{\preceq}(f)$-reduced. ♢

Remark 9.1.3.8. If $f \neq 0$ and P are elements of $k[x]$, then P is f-reduced in the sense of Definition 9.1.3.7 if and only if the degree of P is smaller than the degree of f. ♢

Remark 9.1.3.9. Assume we are in the situation of Construction 9.1.1.11. Let f and P be elements of $k[x_1, \ldots, x_n]$ and assume that $\deg_{\preceq}(f)$ is in the image of ψ. Let f' and P' be the elements of $k'[y_1, \ldots, y_m]$ corresponding to f and P under the isomorphism from Construction 9.1.1.11.

Then P is f-reduced if and only if P' is f'-reduced. This can be seen by combining Proposition 9.1.2.4 with arguments very similar to those used in the proof of Proposition 9.1.2.4. ♢

[6] Both f and g are nonzero, and as f is monic we also have $f_{\deg_{\preceq}(f)} \cdot g_{\deg_{\preceq}(g)} = g_{\deg_{\preceq}(g)} \neq 0$.

9.1.3.3. Division with remainder

We are now ready to discuss division with remainders for multivariable polynomials.

Proposition 9.1.3.10. *Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f \in k[x_1, \ldots, x_n]$ a monic polynomial. Let $P \in k[x_1, \ldots, x_n]$ be another polynomial. Then there exist unique polynomials $Q, R \in k[x_1, \ldots, x_n]$ such that $P = Q \cdot f + R$ and R is f-reduced.* ♡

Proof. We first prove uniqueness. Assume that

$$P = Q_1 \cdot f + R_1 \qquad \text{and} \qquad P = Q_2 \cdot f + R_2$$

are two such decompositions. Then the equation

$$(Q_1 - Q_2) \cdot f = R_2 - R_1 \tag{$*$}$$

holds. We have to show that $Q_1 = Q_2$ and $R_1 = R_2$, but applying Proposition 9.1.3.6 to $(*)$ it suffices to show that $R_1 = R_2$.

We show $R_1 = R_2$ by contradiction and assume that $R_1 \neq R_2$. Without loss of generality we can additionally assume that $R_1 \prec R_2$. By Proposition 9.1.3.6 $Q_1 - Q_2 \neq 0$, so we can apply Proposition 9.1.2.3 (4) to $(*)$ and obtain the following formula relating the degrees.

$$\deg_{\preceq}(R_2 - R_1) = \deg_{\preceq}(Q_1 - Q_2) + \deg_{\preceq}(f)$$

As we assumed $R_1 \prec R_2$, we can also apply Proposition 9.1.2.3 (2) to obtain

$$\deg_{\preceq}(R_2 - R_1) = \deg_{\preceq}(R_2)$$

which implies that

$$\deg_{\preceq}(R_2) = \deg_{\preceq}(Q_1 - Q_2) + \deg_{\preceq}(f)$$

and thus in particular $\deg_{\preceq}(R_2) \geq \deg_{\preceq}(f)$, contradicting the assumption that R_2 is f-reduced.

It remains to show existence of the claimed decomposition. So for every polynomial $P \in k[x_1, \ldots, x_n]$ we have to prove the following claim.

Claim There exist $Q, R \in k[x_1, \ldots, x_n]$ such that R is f-reduced and $P = Qf + R$.

To do so, we first define the map

$$\begin{aligned} \Theta\colon k[x_1, \ldots, x_n] &\to \mathbb{Z}^n_{\geq 0} \cup \{-\infty\} \\ P &\mapsto \max\left\{ \vec{i} \in \mathbb{Z}^n_{\geq 0} \;\middle|\; P_{\vec{i}} \neq 0 \text{ and } \vec{i} \geq \deg_{\preceq}(f) \right\} \end{aligned}$$

where the maximum is to be interpreted as $-\infty$ if the set is empty, and the set the maximum is taken over is always finite[7], so the maximum exists if the set is nonempty. Note that $R \in k[x_1, \ldots, x_n]$ is f-reduced if and only if $\Theta(R) = -\infty$. We can extend the well-order $\preceq$ on $\mathbb{Z}^n_{\geq 0}$ to $\mathbb{Z}^n_{\geq 0} \cup \{-\infty\}$ by letting $-\infty$ be the minimal element, and will prove the claim stated above for every element P of $k[x_1, \ldots, x_n]$ by transfinite induction on $\Theta(P)$.

So we let P be an element of $k[x_1, \ldots, x_n]$ and assume that the claim holds for any $P' \in k[x_1, \ldots, x_n]$ with $\Theta(P') \preceq \Theta(P)$. We have to show that then P also satisfies the claim.

If $\Theta(P) = -\infty$, then P itself is reduced and so we can take $Q = 0$, $R = P$ and are done.

So assume now that $\Theta(P) \neq -\infty$. Note that the definition of $\Theta(P)$ and the assumption that $\Theta(P) \neq -\infty$ together imply that $\Theta(P) \geq \deg_{\preceq}(f)$, so that in particular $\Theta(P) - \deg_{\preceq}(f)$ is an element of $\mathbb{Z}^n_{\geq 0}$. We can thus define a new polynomial P' as follows.

$$P' = P - P_{\Theta(P)} \cdot x^{\Theta(P) - \deg_{\preceq}(f)} \cdot f \tag{$**$}$$

We claim that $\Theta(P') \prec \Theta(P)$. Let us for the moment assume this and explain how the claim for P follows. As $\Theta(P') \prec \Theta(P)$ we can use the induction hypothesis and obtain $Q', R' \in k[x_1, \ldots, x_n]$ such that R' is f-reduced and $P' = Q'f + R'$. Combining this with $(**)$ we obtain

$$P = \left(Q' + P_{\Theta(P)} x^{\Theta(P) - \deg_{\preceq}(f)}\right) \cdot f + R'$$

so that setting $Q = Q' + P_{\Theta(P)} x^{\Theta(P) - \deg_{\preceq}(f)}$ and $R = R'$ shows the claim for P.

We are left to show that $\Theta(P') \prec \Theta(P)$. Note that $(**)$ implies that for $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ we have

$$P'_{\vec{i}} = P_{\vec{i}} - P_{\Theta(P)} \cdot f_{\vec{i} + \det_{\preceq}(f) - \Theta(P)} \tag{$***$}$$

where $f_{\vec{i} + \det_{\preceq}(f) - \Theta(P)}$ is to be interpreted as 0 if $\vec{i} + \det_{\preceq}(f) - \Theta(P)$ is not in $\mathbb{Z}^n_{\geq 0}$. Plugging in $\vec{i} = \Theta(P)$ we obtain

$$P'_{\Theta(P)} = P_{\Theta(P)} - P_{\Theta(P)} \cdot f_{\det_{\preceq}(f)} = P_{\Theta(P)} - P_{\Theta(P)} = 0$$

so if $\Theta(P') \succeq \Theta(P)$ then we actually must have $\Theta(P') \succ \Theta(P)$. So now assume that $\vec{i}$ is an element of $\mathbb{Z}^n_{\geq 0}$ such that the following holds.

(1) $\vec{i} \geq \deg_{\preceq}(f)$

(2) $\vec{i} \succ \Theta(P)$

[7] As polynomials only have finitely many nonzero components.

What we have to show is that then $P'_{\overrightarrow{i}} = 0$. The two assumptions imply that $P_{\overrightarrow{i}} = 0$, so if $\overrightarrow{i} + \det_{\preceq}(f) - \Theta(P)$ is not in $\mathbb{Z}^n_{\geq 0}$, then equation $(***)$ implies $P'_{\overrightarrow{i}} = 0$. So assume that $\overrightarrow{i} + \det_{\preceq}(f) - \Theta(P)$ is in $\mathbb{Z}^n_{\geq 0}$. (2) implies that

$$\overrightarrow{i} + \deg_{\preceq}(f) \succ \Theta(P) + \deg_{\preceq}(f)$$

which – using that $\overrightarrow{i} + \det_{\preceq}(f) - \Theta(P)$ is in $\mathbb{Z}^n_{\geq 0}$ – implies that

$$\overrightarrow{i} + \deg_{\preceq}(f) - \Theta(P) \succ \deg_{\preceq}(f)$$

from which we can deduce that $f_{\overrightarrow{i} + \deg_{\preceq}(f) - \Theta(P)} = 0$. It again follows from equation $(***)$ that $P'_{\overrightarrow{i}} = 0$. $\square$

Remark 9.1.3.11. Assume we are in the situation of Construction 9.1.1.11. Let f and P be elements of $k[x_1, \ldots, x_n]$ and assume that $\deg_{\preceq}(f)$ is in the image of ψ. Let f' and P' be the elements of $k'[y_1, \ldots, y_m]$ corresponding to f and P under the isomorphism from Construction 9.1.1.11. Then the decompositions of P and P' with respect to f and f' correspond to each other under the isomorphism from Construction 9.1.1.11. Concretely, if Q, R are elements of $k[x_1, \ldots, x_n]$ such that $P = Qf + R$ and R is f-reduced, and Q' and R' are the elements of $k'[y_1, \ldots, y_m]$ corresponding to Q and R under the isomorphism from Construction 9.1.1.11, then $P' = Q'f + R$ as the isomorphism is an isomorphism of R-algebras, and R' is f'-reduced by Remark 9.1.3.9. $\diamondsuit$

9.1.3.4. Full sum decomposition

If f is a monic polynomial and P any polynomial, we saw in Proposition 9.1.3.10 that we can divide P by f with remainder to obtain a decomposition $P = Qf + R_0$ for polynomials Q and R_0 such that R_0 is f-reduced. We can then also divide Q by f with remainder and obtain a decomposition of Q as $Q = Q'f + R_1$, so that we can write P as $P = Q'f^2 + R_1 f + R_0$. We would like this process to eventually stop (i. e. eventually arrive at an R_i that is already f-reduced), to obtain a decomposition of P as $P = \sum_{i \geq 0} R_i \cdot f^i$, such that each R_i is f-reduced and all but finitely many are zero. For this we however need one extra assumption: If $f = 1$, then the decomposition from Proposition 9.1.3.10 will be $P = P \cdot 1 + 0$, so iterating this process will never yield an f-reduced R_i unless $P = 0$. We thus arrive at the following proposition.

Proposition 9.1.3.12. *Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f \in k[x_1, \ldots, x_n]$ a monic polynomial with $\deg_{\preceq}(f) > \overrightarrow{0}$ (equivalently, $f \neq 1$). Let $P \in k[x_1, \ldots, x_n]$ be another polynomial. Then there exist unique $R_i \in k[x_1, \ldots, x_n]$ for $i \in \mathbb{Z}_{\geq 0}$ of which all but finitely many are zero such that*

$$P = \sum_{i \geq 0} R_i \cdot f^i$$

and all R_i are f-reduced. ♡

Proof. We first show uniqueness. So assume we are given two such decompositions as follows.

$$P = \sum_{i \geq 0} R_i \cdot f^i \qquad \text{and} \qquad P = \sum_{i \geq 0} R'_i \cdot f^i$$

We can rewrite this as

$$\left(\sum_{i \geq 1} R_i \cdot f^{i-1} \right) \cdot f + R_0 = \left(\sum_{i \geq 1} R'_i \cdot f^{i-1} \right) \cdot f + R'_0$$

and hence by Proposition 9.1.3.10 we can conclude that $R_0 = R'_0$ and

$$\sum_{i \geq 1} R_i \cdot f^{i-1} = \sum_{i \geq 1} R'_i \cdot f^{i-1}$$

as well. Iterating this argument now yields $R_1 = R'_1$, $R_2 = R'_2$, and so on.

We prove existence by transfinite induction on $\deg_{\preceq}(P)$ and assume that the statement has already been proven for all polynomials P' that satisfy $\deg_{\preceq}(P') \prec \deg_{\preceq}(P)$. By Proposition 9.1.3.10 there are polynomials Q and R_0 such that $P = Qf + R_0$ and R_0 is f-reduced. If $Q = 0$ we are already done, so assume that $Q \neq 0$. As R_0 is f-reduced we must have $(R_0)_{\deg_{\preceq}(Q)+\deg_{\preceq}(f)} = 0$ and hence, using Proposition 9.1.2.3 (4),

$$P_{\deg_{\preceq}(Q)+\deg_{\preceq}(f)} = (Qf)_{\deg_{\preceq}(Q)+\deg_{\preceq}(f)} \neq 0$$

so that we can conclude that $\deg_{\preceq}(P) \succeq \deg_{\preceq}(Q) + \deg_{\preceq}(f)$. As we assumed $\vec{0} \prec \deg_{\preceq}(f)$ this implies the following inequality.

$$\deg_{\preceq}(Q) \prec \deg_{\preceq}(Q) + \deg_{\preceq}(f) \preceq \deg_{\preceq}(P)$$

By the induction hypothesis we can thus find f-reduced polynomials R_i for $i \geq 1$, all but finitely many zero, such that

$$Q = \sum_{i \geq 1} R_i f^{i-1}$$

which implies that

$$P = Q \cdot f + R_0 = \left(\sum_{i \geq 1} R_i f^{i-1} \right) \cdot f + R_0 = \sum_{i \geq 0} R_i f^i$$

and thus proves the claim. □

The assumptions made in Proposition 9.1.3.12 will be used a lot in the rest of this chapter. To improve readability and reduce unnecessary repetitions, we thus package them together.

Assumption MonOrdMonicPoly. *Wherever we invoke this assumption, we let n be a positive integer, $\preceq$ a monomial order for n variables, and $f \in k[x_1, \ldots, x_n]$ a monic polynomial with* $\deg_{\preceq}(f) > 0$. ♢

We next introduce some notation to help us refer to the polynomials R_i occurring in the decomposition from Proposition 9.1.3.12.

Definition 9.1.3.14. Assume MonOrdMonicPoly. We define maps

$$r_f^j, r_f^{\leq j}, r_f^{<j}, q_f^j \colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$$

for each integer j in the following way.

For $P \in k[x_1, \ldots, x_n]$, let

$$P = \sum_{i \geq 0} R_i f^i$$

be the decomposition from Proposition 9.1.3.12, i. e. R_i is an f-reduced element of $k[x_1, \ldots, x_n]$ for each $i \geq 0$. We then define r_f^j, $r_f^{\leq j}$, $r_f^{<j}$, and q_f^j for $j \geq 0$ as follows.

$$r_f^j(P) \coloneqq R_j \qquad r_f^{\leq j}(P) \coloneqq \sum_{i=0}^{j} r_f^i(P) f^i \qquad r_f^{<j}(P) \coloneqq \sum_{i=0}^{j-1} r_f^i(P) f^i$$

$$q_f^j(P) \coloneqq \sum_{i \geq j} r_f^i(P) \cdot f^{i-j}$$ ♢

If $j < 0$, then we define r_f^j, $r_f^{\leq j}$, and $r_f^{<j}$ to map P to 0, and define $q_f^j(P) \coloneqq P \cdot f^{-j}$.

9.1.3.5. Properties of remainders

In the following proposition we collect a number of useful properties of the maps from Definition 9.1.3.14.

Proposition 9.1.3.15. *Assume MonOrdMonicPoly. Then the following hold for each $i, j \geq 0$ and $P, Q \in k[x_1, \ldots, x_n]$.*

(1) $r_f^j(P)$ is f-reduced.

(2) $P = q_f^j(P) \cdot f^j + r_f^{<j}(P)$.

(3) r_f^j, $r_f^{\leq j}$, $r_f^{<j}$, and q_f^j are k-linear.

(4) $r_f^j(P \cdot f^i) = r_f^{j-i}(P)$ and $q_f^j(P \cdot f^i) = q_f^{j-i}(P)$.

(5) $r_f^j(P \cdot Q) = \sum_{a+b+c=j} r_f^a\Big(r_f^b(P) \cdot r_f^c(Q)\Big)$.

(6) $q_f^i\Big(q_f^j(P)\Big) = q_f^{i+j}(P)$. ♡

Proof. Proof of claims (1), (2), and (4): Clear by definition.

Proof of claim (3): Follows immediately from uniqueness of the decomposition in Proposition 9.1.3.12, as k-linear combinations of f-reduced polynomials are again f-reduced.

Proof of claim (5): First note that both sides are k-linear in both P and Q. It hence suffices to consider the case $P = R \cdot f^e$, $Q = R' \cdot f^{e'}$ with f-reduced polynomials R and R' and nonnegative integers e and e'. In this case we can read off

$$r_f^b(P) = \begin{cases} R & \text{if } b = e \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad r_f^c(Q) = \begin{cases} R' & \text{if } c = e' \\ 0 & \text{otherwise} \end{cases}$$

so that we obtain

$$\sum_{a+b+c=j} r_f^a\big(r_f^b(P) \cdot r_f^c(Q)\big) = r_f^{j-e-e'}(RR')$$

which is equal to $r_f^j(P \cdot Q) = r_f^j(RR'f^{e+e'})$ by (4).

Proof of claim (6): This follows from the previous claims, as in the following calculation.

$$\begin{aligned} q_f^{i+j}(P) &= q_f^{i+j}\Big(q_f^j(P)f^j + r_f^{<j}(P)\Big) \\ &= q_f^{i+j}\Big(q_f^j(P)f^j\Big) + q_f^{i+j}\Big(r_f^{<j}(P)\Big) \\ &= q_f^i\Big(q_f^j(P)\Big) + 0 \end{aligned}$$

□

As r_f^j, $r_f^{\leq j}$, $r_f^{<j}$, and q_f^j are k-linear by Proposition 9.1.3.15 (3), we can extend their definitions as follows.

Convention 9.1.3.16. Assume MonOrdMonicPoly. Let M be a (graded) k-module. Then for any integer j we obtain a morphism of (graded) k-modules

$$r_f^j \otimes_k \mathrm{id}_M \colon k[x_1, \ldots, x_n] \otimes_k M \to k[x_1, \ldots, x_n] \otimes_k M \tag{9.1}$$

which we will also call r_f^j. Similarly for $r_f^{\leq j}, r_f^{<j}$, and q_f^j. ♢

9.2. A strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of medium size

In this section we will give a description of a strict mixed complex that represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{M}$ixed under assumptions MonOrdMonicPoly and Conjecture D for f.

We will start in Section 9.2.1 by showing that $k[x_1, \ldots, x_n]$ satisfies the necessary conditions as a module over $k[t]$ in order to apply the more general result Proposition 8.3.0.1 on a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of quotients. In Section 9.2.2 we will then spell out Proposition 8.3.0.1 specialized to $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$. While there is an obvious basis of the resulting strict mixed complex, that basis is not well adapted to further simplification steps that we will want to do in later sections. We thus describe a new, more useful, basis in Section 9.2.3.

9.2.1. $k[x_1, \ldots, x_n]$ as a module over $k[t]$

In this short section we show that multiplication with f acts on the commutative k-algebra $k[x_1, \ldots, x_n]$ in a way that satisfies the requirements to apply Proposition 8.3.0.1.

Proposition 9.2.1.1. *Assume MonOrdMonicPoly. Then the subset*

$$\left\{ x^{\vec{i}} \;\middle|\; \vec{i} \in \mathbb{Z}^n_{\geq 0},\ \ \vec{i} \ngeq \deg_{\preceq}(f) \right\} \tag{9.2}$$

of $k[x_1, \ldots, x_n]$ is a basis of $k[x_1, \ldots, x_n]$ as a right-$k[t]$-module, where t acts by multiplication with f. In particular, $k[x_1, \ldots, x_n]$ is free as a right-$k[t]$-module. ♡

Proof. The sub-k-module of $k[x_1, \ldots, x_n]$ spanned by $x^{\vec{i}}$ for $\vec{i} \ngeq \deg_{\preceq}(f)$ is a basis of the sub-k-module of f-reduced polynomials, so it follows from Proposition 9.1.3.12 that (9.2) generates $k[x_1, \ldots, x_n]$ as a right-$k[t]$-module.

For linear independence, assume that $p_{\vec{i}}$ are elements of $k[t]$ for each $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ such that $\vec{i} \ngeq \deg_{\preceq}(f)$, with all but finitely many $p_{\vec{i}}$ zero. We can write $p_{\vec{i}}$ as $p_{\vec{i}} = \sum_{j \geq 0} a_{\vec{i},j} t^j$, with $a_{\vec{i},j}$ elements of k, all but finitely many (for fixed $\vec{i}$) zero. Furthermore, assume that the following holds.

$$\sum_{\substack{\vec{i} \in \mathbb{Z}^n_{\geq 0},\\ \vec{i} \ngeq \deg_{\preceq}(f)}} x^{\vec{i}} \cdot \left(\sum_{j \geq 0} a_{\vec{i},j} f^j \right) = 0$$

Then the uniqueness part of Proposition 9.1.3.12 implies

$$\sum_{\substack{\vec{i} \in \mathbb{Z}^n_{\geq 0},\\ \vec{i} \ngeq \deg_{\preceq}(f)}} a_{\vec{i},j} x^{\vec{i}} = 0$$

for every $j \geq 0$, but as the $x^{\vec{i}}$ are k-linearly independent, this implies that all $a_{\vec{i},j}$ are zero. □

Proposition 9.2.1.2. *Assume MonOrdMonicPoly. Then $k[x_1, \ldots, x_n]$ is cofibrant as an object in $\mathrm{RMod}_{k[t]}(\mathsf{Ch}(k))$, where t acts by multiplication with f.* ♡

Proof. As $k[x_1, \ldots, x_n]$ is free as a right-$k[t]$-module by Proposition 9.2.1.1, this follows from Theorem 4.2.2.1 (5) and [Hov99, 2.3.6]. $\square$

9.2.2. A strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$

We can now specialize Proposition 8.3.0.1 to obtain a first strict mixed complex X_f that represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$. While the result is conditional on Conjecture D holding for f, we can construct X_f in greater generality.

Construction 9.2.2.1. Assume MonOrdMonicPoly. We will construct a strict mixed complex X_f. As a $\mathbb{Z}$-graded k-module[8], X_f is given by

$$X_f \coloneqq k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$$

with $x_1, \ldots, x_n$ of degree 0, $\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n$ and s of degree 1 and $\mathrm{d}\, s$ of degree 2. The boundary operator ∂ and differential d are given by k-linearly extending the following formulas for $\vec{i} \in \mathbb{Z}^n_{\geq 0}$, $\vec{\epsilon} \in \{0,1\}^n$, and $m \geq 0$.

$$\begin{aligned}
\partial\Big(x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}} s\, \mathrm{d}\, s^{[m]}\Big) &= (-1)^{|\vec{\epsilon}|} x^{\vec{i}} f\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m]} \\
&\quad - (-1)^{|\vec{\epsilon}|} x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, f \cdot s\, \mathrm{d}\, s^{[m-1]} \\
\partial\Big(x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m]}\Big) &= -(-1)^{|\vec{\epsilon}|} x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, f\, \mathrm{d}\, s^{[m-1]} \\
\mathrm{d}\Big(x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}} s\, \mathrm{d}\, s^{[m]}\Big) &= \mathrm{d}\Big(x^{\vec{i}}\Big)\, \mathrm{d}\, x^{\vec{\epsilon}} s\, \mathrm{d}\, s^{[m]} \\
&\quad + (-1)^{|\vec{\epsilon}|}(m+1) x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m+1]} \\
\mathrm{d}\Big(x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m]}\Big) &= \mathrm{d}\Big(x^{\vec{i}}\Big)\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m]}
\end{aligned}$$

In the formulas above, d applied to elements of

$$k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)$$

is defined as in $\Omega^{\bullet}_{k[x_1,\ldots,x_n]/k}$[9], and $\mathrm{d}\, s^{[-1]}$ is to be interpreted as zero.

To see that ∂ and d as defined really upgrade X_f to a strict mixed complex we need to check that ∂ and d square to 0, and that $\partial \circ \mathrm{d} + \mathrm{d} \circ \partial = 0$ holds. We check all of these on basis elements. Using that $\mathrm{d}\, f \cdot \mathrm{d}\, f = 0$ in the $\mathbb{Z}$-graded k-algebra underlying X_f we obtain the following calculations for $\vec{i} \in \mathbb{Z}^n_{\geq 0}$, $\vec{\epsilon} \in \{0,1\}^n$, and $m \geq 0$.

$$\begin{aligned}
&\partial\Big(\partial\Big(x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}} s\, \mathrm{d}\, s^{[m]}\Big)\Big) \\
&= \partial\Big((-1)^{|\vec{\epsilon}|} x^{\vec{i}} f\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m]} - (-1)^{|\vec{\epsilon}|} x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, f \cdot s\, \mathrm{d}\, s^{[m-1]}\Big)
\end{aligned}$$

[8] We will use the structure of a commutative $\mathbb{Z}$-graded k-algebra on X_f to write down elements, but X_f itself is only considered as a strict mixed complex.

[9] So extending from $\mathrm{d}(x_i) \coloneqq \mathrm{d}\, x_i$ using k-linearity and the Leibniz rule.

$$
\begin{aligned}
&= -(-1)^{|\vec{\epsilon}|}(-1)^{|\vec{\epsilon}|} x^{\vec{i}} f \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \,\mathrm{d}\, s^{[m-1]} \\
&\quad - (-1)^{|\vec{\epsilon}|}\Big((-1)^{|\vec{\epsilon}|+1} x^{\vec{i}} f \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \,\mathrm{d}\, s^{[m-1]}\Big) \\
&\quad - (-1)^{|\vec{\epsilon}|}\Big(-(-1)^{|\vec{\epsilon}|+1} x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \cdot \mathrm{d}\, f \cdot s \,\mathrm{d}\, s^{[m-1]}\Big) \\
&= -x^{\vec{i}} f \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \,\mathrm{d}\, s^{[m-1]} + x^{\vec{i}} f \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \,\mathrm{d}\, s^{[m-1]} - 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&\partial\Big(\partial\Big(x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]}\Big)\Big) \\
&= \partial\Big(-(-1)^{|\vec{\epsilon}|} x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \,\mathrm{d}\, s^{[m-1]}\Big) \\
&= \Big(-(-1)^{|\vec{\epsilon}|}\Big) \cdot \Big(-(-1)^{|\vec{\epsilon}|+1}\Big) \cdot x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \,\mathrm{d}\, f \,\mathrm{d}\, s^{[m-2]} \\
&= 0
\end{aligned}
$$

Using that d squares to 0 in $\Omega^{\bullet}_{k[x_1,\ldots,x_n]/k}$ we obtain the following calculations.

$$
\begin{aligned}
&\mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} s \,\mathrm{d}\, s^{[m]}\Big)\Big) \\
&= \mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big) \,\mathrm{d}\, x^{\vec{\epsilon}} s \,\mathrm{d}\, s^{[m]} + (-1)^{|\vec{\epsilon}|}(m+1) x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m+1]}\Big) \\
&= \mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\Big) \,\mathrm{d}\, x^{\vec{\epsilon}} s \,\mathrm{d}\, s^{[m]} + (-1)^{|\vec{\epsilon}|+1}(m+1) \,\mathrm{d}\Big(x^{\vec{i}}\Big) \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m+1]} \\
&\quad + (-1)^{|\vec{\epsilon}|}(m+1) \,\mathrm{d}\Big(x^{\vec{i}}\Big) \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m+1]} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]}\Big)\Big) \\
&= \mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big) \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]}\Big) \\
&= \mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\Big) \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]} \\
&= 0
\end{aligned}
$$

Finally, using that d satisfies the Leibniz rule on $\Omega^{\bullet}_{k[x_1,\ldots,x_n]/k}$ we can carry out the following calculations showing that $\partial \circ \mathrm{d} + \mathrm{d} \circ \partial = 0$.

$$
\begin{aligned}
&\partial\Big(\mathrm{d}\Big(x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} s \,\mathrm{d}\, s^{[m]}\Big)\Big) + \mathrm{d}\Big(\partial\Big(x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} s \,\mathrm{d}\, s^{[m]}\Big)\Big) \\
&= \partial\Big(\mathrm{d}\Big(x^{\vec{i}}\Big) \,\mathrm{d}\, x^{\vec{\epsilon}} s \,\mathrm{d}\, s^{[m]} + (-1)^{|\vec{\epsilon}|}(m+1) x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m+1]}\Big) \\
&\quad + \mathrm{d}\Big((-1)^{|\vec{\epsilon}|} x^{\vec{i}} f \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]} - (-1)^{|\vec{\epsilon}|} x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \cdot s \,\mathrm{d}\, s^{[m-1]}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{|\vec{\epsilon}|+1}\,\mathrm{d}\Big(x^{\vec{i}}\Big) f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]} - (-1)^{|\vec{\epsilon}|+1}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f \cdot s\,\mathrm{d}\,s^{[m-1]}\\
&\quad - (-1)^{|\vec{\epsilon}|}(-1)^{|\vec{\epsilon}|}(m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\\
&\quad + (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\cdot f\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\\
&\quad - (-1)^{|\vec{\epsilon}|}\Big(\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\Big)\cdot s\,\mathrm{d}\,s^{[m-1]}\Big)\\
&\quad - (-1)^{|\vec{\epsilon}|}\Big((-1)^{|\vec{\epsilon}|+1}\cdot m\cdot x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\Big)\\
&= -(-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\Big) f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]} + (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\cdot s\,\mathrm{d}\,s^{[m-1]}\\
&\quad - (m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\\
&\quad + (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\cdot f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]} + (-1)^{|\vec{\epsilon}|}x^{\vec{i}}\cdot \mathrm{d}(f)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\\
&\quad - (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\cdot s\,\mathrm{d}\,s^{[m-1]} + mx^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\\
&= -(m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]} + (-1)^{|\vec{\epsilon}|}x^{\vec{i}}\cdot\mathrm{d}(f)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\\
&\quad + mx^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\\
&= -(m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]} + x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]} + mx^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&\partial\Big(\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big)\Big) + \mathrm{d}\Big(\partial\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big)\Big)\\
&= \partial\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big) - (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]}\Big)\\
&= -(-1)^{|\vec{\epsilon}|+1}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]} - (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]}\\
&= 0
\end{aligned}
$$

Note that as X_f is free as a $\mathbb{Z}$-graded k-module, it follows from [Hov99, 2.3.6] that the underlying chain complex of X_f is cofibrant. ♢

Proposition 9.2.2.2. *Assume MonOrdMonicPoly and that Conjecture D*[10] *holds for f. Then there is an equivalence*

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/(f)) \simeq \gamma_{\mathsf{Mixed}}(X_f)$$

in $\mathcal{M}$ixed, *where X_f is as in Construction 9.2.2.1.* ♡

Proof. This is a specialization of Proposition 8.3.0.1 for $R = k[x_1,\ldots,x_n]$, the x_1 from Proposition 8.3.0.1 being f and the n from Proposition 8.3.0.1 being 1. The requirement on R was verified with Proposition 9.2.1.2. That

[10] Note that Conjecture D holds if $n = 1$ or $n = 2$ with 2 invertible in k by Proposition 7.5.3.1.

Conjecture D holds for f yields a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, dash, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k/k}\right) \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t]) \arrow[r, dash, "\simeq"] \arrow[u, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(G)"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[t]/k}\right) \arrow[u, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{G/k}\right)"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{F/k}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[X]/k}\right)
\end{tikzcd} \qquad (*)$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ such that the horizontal morphisms are equivalences. We can use the top square as the one witnessing Conjecture C for Proposition 8.3.0.1.

Naturality of the identification at the start of Section 7.1 yields a commutative diagram

$$\begin{tikzcd}
k[t] \otimes \Lambda(\mathrm{d}\, t) \arrow[r] \arrow[d, dash, "\cong"'] & k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots \mathrm{d}\, x_n) \arrow[d, dash, "\cong"] \\
\Omega^\bullet_{k[t]/k} \arrow[r] & \Omega^\bullet_{k[x_1,\ldots,x_n]/k}
\end{tikzcd}$$

in $\mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})$ with the vertical morphisms the isomorphisms from Section 7.1 and the horizontal morphisms induced by $t \mapsto f$. Combining this with the bottom square in diagram $(*)$, we obtain a commutative diagram as follows in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$

$$\begin{tikzcd}
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k[t] \otimes \Lambda(\mathrm{d}\, t)) \arrow[r] \arrow[d, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[t]/k}\right) \arrow[r] \arrow[d, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[x_1,\ldots,x_n]/k}\right) \arrow[d, dash, "\simeq"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t]) \arrow[r] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n])
\end{tikzcd}$$

where the left column is precisely (8.9), and the horizontal morphisms are all induced by $t \mapsto f$. Letting M be $k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots \mathrm{d}\, x_n)$, as a right-$k[t] \otimes \Lambda(\mathrm{d}\, t)$-module in $\mathsf{Mixed}_{\mathrm{cof}}$, with the module action arising from the above morphism of algebras, M thus satisfies the requirements for applying Proposition 8.3.0.1. □

9.2.3. A basis for the strict model

In this section we describe a new basis for

$$k[x_1,\ldots,x_n]\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n)\otimes\Lambda(s)\otimes\Gamma(\mathrm{d}\,s)$$

in which the formulas for ∂ and d will take a form that will make it easier to construct smaller strict models in later sections.

9.2.3.1. Interaction of q_f^1 with d and multiplication

We will need two small results on the interaction of q_f^1 and q_f^2 with products and the differentiation.

Proposition 9.2.3.1. *Assume MonOrdMonicPoly. Then the following hold for P and Q elements of the strict mixed complex*

$$k[x_1,\ldots,x_n]\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n)$$

(see Section 7.1).

(1) If P is f-reduced, then $\mathrm{d}\,P$ is f-reduced as well.

(2) $-q_f^1(\mathrm{d}\,f\cdot\mathrm{d}\,P)=q_f^1\Big(\mathrm{d}\,f\cdot q_f^1(\mathrm{d}\,f\cdot P)\Big)+\mathrm{d}\Big(q_f^1(\mathrm{d}\,f\cdot P)\Big)$

(3) $q_f^2(PQ)=q_f^1\Big(P\cdot q_f^1(Q)\Big)+q_f^2\Big(P\cdot r_f^0(Q)\Big)$ ♡

Proof. Proof of claim (1): It suffices to consider the case $P=x^{\vec{i}}$ for $\vec{i}\in\mathbb{Z}^n_{\geq 0}$. In this case, $\mathrm{d}\,P=\sum_{j=1}^n i_j x^{\vec{i}-\vec{e_j}}$, and the claim follows from $\vec{i}-\vec{e_j}\leq\vec{i}$.

Proof of claim (2): By definition we have

$$\mathrm{d}\,f\cdot P=f\cdot q_f^1(\mathrm{d}\,f\cdot P)+r_f^0(\mathrm{d}\,f\cdot P)$$

so that applying d yields the following.

$$-\mathrm{d}\,f\cdot\mathrm{d}\,P=\mathrm{d}\,f\cdot q_f^1(\mathrm{d}\,f\cdot P)+f\,\mathrm{d}\big(q_f^1(\mathrm{d}\,f\cdot P)\big)+\mathrm{d}\big(r_f^0(\mathrm{d}\,f\cdot P)\big)$$

We can now apply q_f^1, to obtain the following.

$$\begin{aligned}-q_f^1(\mathrm{d}\,f\cdot\mathrm{d}\,P)&=q_f^1\big(\mathrm{d}\,f\cdot q_f^1(\mathrm{d}\,f\cdot P)\big)+q_f^1\big(f\,\mathrm{d}\big(q_f^1(\mathrm{d}\,f\cdot P)\big)\big)\\&\quad+q_f^1\big(\mathrm{d}\big(r_f^0(\mathrm{d}\,f\cdot P)\big)\big)\end{aligned}$$

$r_f^0(\mathrm{d}\,f\cdot P)$ is f-reduced, so the third summand is zero by (1). We use Proposition 9.1.3.15 (4) for the second summand.

$$=q_f^1\big(\mathrm{d}\,f\cdot q_f^1(\mathrm{d}\,f\cdot P)\big)+\mathrm{d}\big(q_f^1(\mathrm{d}\,f\cdot P)\big)$$

Proof of claim (3): By definition of q_f^1 and r_f^0, the following holds.

$$Q=q_f^1(Q)\cdot f+r_f^0(Q)$$

We can now multiply with P on the left.

$$PQ = P \cdot q_f^1(Q) \cdot f + P \cdot r_f^0(Q)$$

Applying q_f^2 and using Proposition 9.1.3.15 (4) on the first summand on the right hand side we obtain the following.

$$q_f^2(PQ) = q_f^1\big(P \cdot q_f^1(Q)\big) + q_f^2\big(P \cdot r_f^0(Q)\big) \qquad \square$$

9.2.3.2. The basis

Definition 9.2.3.2. Assume MonOrdMonicPoly and let m be an integer. We define two k-linear maps

$$\begin{aligned} C^{[m]}\colon\ & k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \\ & \to k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s) \end{aligned}$$

and

$$\begin{aligned} E^{[m]}\colon\ & k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \\ & \to k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s) \end{aligned}$$

as follows. If $m < 0$, then we let $C^{[m]}$ and $E^{[m]}$ be constant with value 0. If $m \geq 0$, then we define them as follows.

$$\begin{aligned} C^{[m]}(g) &:= sg\, \mathrm{d}\, s^{[m]} \\ E^{[m]}(g) &:= g\, \mathrm{d}\, s^{[m]} + s q_f^1(\mathrm{d}\, f \cdot g)\, \mathrm{d}\, s^{[m-1]} = g\, \mathrm{d}\, s^{[m]} + C^{[m-1]}\big(q_f^1(\mathrm{d}\, f \cdot g)\big) \end{aligned}$$

In the formulas above, we interpret $\mathrm{d}\, s^{[-1]}$ as zero.

Let $\mathcal{J}$ be the defined as

$$\mathcal{J} := \left\{ \left(\vec{i}, l, \vec{\epsilon}, m\right) \in \mathbb{Z}_{\geq 0}^n \times \mathbb{Z}_{\geq 0} \times \{0,1\}^n \times \mathbb{Z}_{\geq 0} \;\middle|\; \vec{i} \not\geq \deg_{\preceq}(f) \right\}$$

and for $(\vec{i}, l, \vec{\epsilon}, m) \in \mathcal{J}$, define $c_{\vec{i},l,\vec{\epsilon},m}$ and $e_{\vec{i},l,\vec{\epsilon},m}$ as follows.

$$\begin{aligned} c_{\vec{i},l,\vec{\epsilon},m} &:= C^{[m]}\left(x^{\vec{i}} f^l\, \mathrm{d}\, x^{\vec{\epsilon}}\right) = s x^{\vec{i}} f^l\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m]} \\ e_{\vec{i},l,\vec{\epsilon},m} &:= E^{[m]}\left(x^{\vec{i}} f^l\, \mathrm{d}\, x^{\vec{\epsilon}}\right) \\ &= x^{\vec{i}} f^l\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m]} + C^{[m-1]}\left(q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\, \mathrm{d}\, x^{\vec{\epsilon}}\right)\right) \end{aligned} \qquad \diamondsuit$$

Proposition 9.2.3.3. *Assume MonOrdMonicPoly. Then*

$$\left\{ c_{\vec{i},l,\vec{\epsilon},m} \;\middle|\; \left(\vec{i}, l, \vec{\epsilon}, m\right) \in \mathcal{J} \right\} \cup \left\{ e_{\vec{i},l,\vec{\epsilon},m} \;\middle|\; \left(\vec{i}, l, \vec{\epsilon}, m\right) \in \mathcal{J} \right\}$$

is a k-basis for the $\mathbb{Z}$-graded k-module

$$k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$$

$\heartsuit$

Proof. The set

$$\left\{ \mathrm{d}\, s^{[m]} \;\middle|\; m \in \mathbb{Z}_{\geq 0} \right\} \cup \left\{ s \,\mathrm{d}\, s^{[m]} \;\middle|\; m \in \mathbb{Z}_{\geq 0} \right\}$$

is a k-basis for $\Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$, so there is a sum decomposition as follows.

$$\begin{aligned} & k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s) \\ \cong & \bigoplus_{m \geq 0} \mathrm{Im}\Big(C^{[m]}\Big) \oplus \bigoplus_{m \geq 0} \mathrm{Im}\Big(E^{[m]}\Big) \end{aligned}$$

As $C^{[m]}$ and $E^{[m]}$ are clearly injective for $m \geq 0$, it thus suffices to show that

$$\left\{ x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}} \;\middle|\; \vec{i} \in \mathbb{Z}^n_{\geq 0},\, \vec{i} \not\geq \deg_{\preceq}(f), l \in \mathbb{Z}_{\geq 0},\, \vec{\epsilon} \in \{0, 1\}^n \right\}$$

is a k-basis of $k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)$, which follows from Proposition 9.2.1.1. □

9.2.3.3. Description of boundary and differential

Proposition 9.2.3.4. *Assume MonOrdMonicPoly, recall the notation from Definition 9.2.3.2, and let $(\vec{i}, l, \vec{\epsilon}, m) \in \mathfrak{J}$. Then the following formulas hold in the strict mixed complex X_f from Construction 9.2.2.1.*

$$\begin{aligned} \partial\Big(c_{\vec{i}, l, \vec{\epsilon}, m}\Big) &= e_{\vec{i}, l+1, \vec{\epsilon}, m} \\ \partial\Big(e_{\vec{i}, l, \vec{\epsilon}, m}\Big) &= \begin{cases} -E^{[m-1]}\Big(r^0_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big) & \text{if } l = 0 \\ 0 & \text{if } l > 0 \end{cases} \\ \mathrm{d}\Big(e_{\vec{i}, l, \vec{\epsilon}, m}\Big) &= E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big) \,\mathrm{d}\, x^{\vec{\epsilon}} + m q^1_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\Big) \,\mathrm{d}\, x^{\vec{\epsilon}}\Big) \\ &\quad + (m-1) C^{[m-1]}\Big(q^2_f\Big(\mathrm{d}\, f \cdot r^0_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big) \qquad \heartsuit \end{aligned}$$

Proof. We start with $\partial\Big(c_{\vec{i}, l, \vec{\epsilon}, m}\Big)$ and obtain the following by reordering the factors and applying the formula from Construction 9.2.2.1.

$$\begin{aligned} \partial\Big(c_{\vec{i}, l, \vec{\epsilon}, m}\Big) &= \partial\Big(s x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]}\Big) \\ &= \partial\Big((-1)^{|\vec{\epsilon}|} x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}} s \,\mathrm{d}\, s^{[m]}\Big) \\ &= (-1)^{|\vec{\epsilon}|}\Big((-1)^{|\vec{\epsilon}|} x^{\vec{i}} f^{l+1} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]} \\ &\qquad\qquad - (-1)^{|\vec{\epsilon}|} x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \cdot s \,\mathrm{d}\, s^{[m-1]}\Big) \\ &= x^{\vec{i}} f^{l+1} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]} - x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \cdot s \,\mathrm{d}\, s^{[m-1]} \\ &= x^{\vec{i}} f^{l+1} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]} + s \,\mathrm{d}\, f x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m-1]} \end{aligned}$$

If follows from Proposition 9.1.3.15 (4) that

$$q^1_f(\mathrm{d}\, f x^{\vec{i}} f^{l+1}\,\mathrm{d}\, x^{\vec{\epsilon}}) = q^0_f(\mathrm{d}\, f x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}) = \mathrm{d}\, f x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}$$

so that we obtain the following (continuing for $\partial(c_{\vec{i},l,\vec{\epsilon},m})$).

$$\begin{aligned}
&= x^{\vec{i}} f^{l+1}\,\mathrm{d}\, x^{\vec{\epsilon}}\,\mathrm{d}\, s^{[m]} + s q^1_f\Big(\mathrm{d}\, f x^{\vec{i}} f^{l+1}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\,\mathrm{d}\, s^{[m-1]}\\
&= e_{\vec{i},l+1,\vec{\epsilon},m}
\end{aligned}$$

We next consider $\partial(e_{\vec{i},l,\vec{\epsilon},m})$.

$$\begin{aligned}
&\partial\Big(e_{\vec{i},l,\vec{\epsilon},m}\Big)\\
&= \partial\Big(x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\,\mathrm{d}\, s^{[m]} + s q^1_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\,\mathrm{d}\, s^{[m-1]}\Big)\\
&= -(-1)^{|\vec{\epsilon}|} x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\,\mathrm{d}\, f\,\mathrm{d}\, s^{[m-1]}\\
&\quad + (-1)^{1+|\vec{\epsilon}|}\partial\Big(q^1_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big) s\,\mathrm{d}\, s^{[m-1]}\Big)\\
&= -(-1)^{|\vec{\epsilon}|} x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\,\mathrm{d}\, f\,\mathrm{d}\, s^{[m-1]}\\
&\quad + (-1)^{2(1+|\vec{\epsilon}|)}\Bigg(q^1_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big) f\,\mathrm{d}\, s^{[m-1]}\\
&\qquad\qquad - q^1_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\,\mathrm{d}\, f\cdot s\cdot\mathrm{d}\, s^{[m-2]}\Bigg)\\
&= \Big(-\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}} + q^1_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big) f\Big)\cdot\mathrm{d}\, s^{[m-1]}\\
&\quad + s\,\mathrm{d}\, f q^1_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\,\mathrm{d}\, s^{[m-2]}
\end{aligned}$$

Before we continue with $\partial(e_{\vec{i},l,\vec{\epsilon},m})$, we carry out the following small calculation.

$$q^1_f\Big(\mathrm{d}\, f\cdot\Big(-\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}} + q^1_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big) f\Big)\Big)$$

Using that $\mathrm{d}\, f$ squares to 0.

$$= q^1_f\Big(\mathrm{d}\, f\cdot q^1_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big) f\Big)$$

Applying Proposition 9.1.3.15 (4) to the outer q^1_f.

$$= \mathrm{d}\, f\cdot q^1_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)$$

Note that by definition we also have the following equality.

$$-\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}} + q^1_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big) f = -r^0_f\Big(\mathrm{d}\, f\cdot x^{\vec{i}} f^{l}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)$$

Continuing with $\partial(e_{\vec{i},l,\vec{\epsilon},m})$, we can plug in the above two calculations to obtain the following.

$$\begin{aligned}
&\partial(e_{\vec{i},l,\vec{\epsilon},m})\\
&= -r_f^0\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\,\mathrm{d}\,s^{[m-1]} - sq_f^1\Big(\mathrm{d}\,f\cdot r_f^0\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big)\,\mathrm{d}\,s^{[m-2]}\\
&= -E^{[m-1]}\Big(r_f^0\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big)
\end{aligned}$$

It follows from Proposition 9.1.3.15 (4) that this is zero for $l > 0$.

We now turn towards the mixed structure.

$$\begin{aligned}
&\mathrm{d}\Big(e_{\vec{i},l,\vec{\epsilon},m}\Big)\\
&= \mathrm{d}\Big(x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]} + sq_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\,\mathrm{d}\,s^{[m-1]}\Big)\\
&= \mathrm{d}\Big(x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big)\\
&\quad + (-1)^{1+|\vec{\epsilon}|}\,\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)s\,\mathrm{d}\,s^{[m-1]}\Big)
\end{aligned}$$

Applying the definition in Construction 9.2.2.1.

$$\begin{aligned}
&= \mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\\
&\quad + (-1)^{1+|\vec{\epsilon}|}\,\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big)s\,\mathrm{d}\,s^{[m-1]}\\
&\quad + mq_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\,\mathrm{d}\,s^{[m]}\\
&= \Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} + mq_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\,\mathrm{d}\,s^{[m]}\\
&\quad - s\,\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big)\,\mathrm{d}\,s^{[m-1]}
\end{aligned}$$

Replacing the first summand by $E^{[m]} - C^{[m-1]}(q_f^1(\mathrm{d}\,f\cdot -))$ and the second summand by $C^{[m-1]}$.

$$\begin{aligned}
&= E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} + mq_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\\
&\quad - C^{[m-1]}\Big(q_f^1\Big(\mathrm{d}\,f\cdot \mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} + m\,\mathrm{d}\,f\cdot q_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big)\\
&\quad - C^{[m-1]}\Big(\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big)\Big)\\
&= E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} + mq_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\\
&\quad - C^{[m-1]}\Big(q_f^1\Big(\mathrm{d}\,f\cdot \mathrm{d}\Big(x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big)\Big)\\
&\quad - C^{[m-1]}\Big(q_f^1\Big(m\,\mathrm{d}\,f\cdot q_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big)\\
&\quad - C^{[m-1]}\Big(\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\,f\cdot x^{\vec{i}} f^l\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big)\Big)
\end{aligned}$$

We now apply Proposition 9.2.3.1 (2) for $P = x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}$ to the second summand.

$$\begin{aligned}
&= E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}} + mq_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\\
&\quad + C^{[m-1]}\Big(q_f^1\Big(\mathrm{d}\, f \cdot q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big)\\
&\quad + C^{[m-1]}\Big(\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big)\\
&\quad - C^{[m-1]}\Big(q_f^1\Big(m\,\mathrm{d}\, f \cdot q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\\
&\quad - C^{[m-1]}\Big(\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big)\\
&= E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}} + mq_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\\
&\quad - (m-1)C^{[m-1]}\Big(q_f^1\Big(\mathrm{d}\, f \cdot q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big)
\end{aligned}$$

We apply Proposition 9.2.3.1 (3) to the second summand for $P = \mathrm{d}\, f$ and $Q = x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}$

$$\begin{aligned}
&= E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}} + mq_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\\
&\quad - (m-1)C^{[m-1]}\Big(q_f^2\Big(\mathrm{d}\, f \cdot \mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\\
&\quad + (m-1)C^{[m-1]}\Big(q_f^2\Big(\mathrm{d}\, f \cdot r_f^0\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big)
\end{aligned}$$

Finally, we use that $\mathrm{d}\, f$ squares to 0.

$$\begin{aligned}
&= E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}} + mq_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\\
&\quad + (m-1)C^{[m-1]}\Big(q_f^2\Big(\mathrm{d}\, f \cdot r_f^0\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big) \qquad \square
\end{aligned}$$

9.3. A smaller strict model for the underlying complex

Assume MonOrdMonicPoly and that Conjecture D holds for the polynomial f. Then Proposition 9.2.2.2 shows that the strict mixed complex X_f constructed in Construction 9.2.2.1 represents the Hochschild homology $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$. This strict mixed model is significantly "smaller" than the standard Hochschild complex that we discussed in Section 6.3.1, but we would nevertheless like to obtain an even smaller model.

There are two ways in which we can relax the problem in the hope of being able to make progress on this. We could impose stronger conditions on f (so make the result less general), or we could consider less structure. It is the latter that we do in this section. Instead of asking for a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ as an object in $\mathcal{M}\mathrm{ixed}$, we merely ask for a chain complex representing $\mathrm{HH}(k[x_1, \ldots, x_n]/f)$ as an object in $\mathcal{D}(k)$.

Such a chain complex was already given in [BACH], obtained by identifying a decomposition of the normalized standard Hochschild complex[11] as a sum of a small chain complex with a very large acyclic chain complex.

We will instead start from the chain complex X_f from Construction 9.2.2.1 and Propositions 9.2.2.2 and 9.2.3.4, and similarly show that a chain complex isomorphic to the one obtained in [BACH] is a subcomplex and that the inclusion is a quasiisomorphism. This gives a new, different proof of the result in [BACH] (albeit requiring the additional assumption of Conjecture D, which we only showed for $n = 1$ and $n = 2$, additionally assuming that 2 is invertible in k, in Proposition 7.5.3.1).

We will describe the smaller model as a subcomplex of the complex X_f from Construction 9.2.2.1 in Section 9.3.1, and then show that this subcomplex is isomorphic to the one described in [BACH] in Section 9.3.2.

9.3.1. The smaller strict model as a subcomplex

In this section we define a subcomplex of X_f from Construction 9.2.2.1 and show that the inclusion of this subcomplex is a quasiisomorphism.

Definition 9.3.1.1. Assume MonOrdMonicPoly. Let

$$X_f \coloneqq k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$$

be the strict mixed complex from Construction 9.2.2.1.

We then define the following sub-graded-k-modules of X_f for every integer $l \geq 0$, where $c_{\vec{i},l',\vec{\epsilon},m}$ and $e_{\vec{i},l',\vec{\epsilon},m}$ are the basis elements defined in Definition 9.2.3.2.

$$X^c_{f,l} \coloneqq \bigoplus_{\substack{(\vec{i},l',\vec{\epsilon},m)\in\mathcal{J} \\ l'=l}} k \cdot c_{\vec{i},l',\vec{\epsilon},m} \qquad X^c_{f,\geq l} \coloneqq \bigoplus_{l'\geq l} X^c_{f,l'} \qquad X^c_{f,\leq l} \coloneqq \bigoplus_{l'\leq l} X^c_{f,l'}$$

$$X^e_{f,l} \coloneqq \bigoplus_{\substack{(\vec{i},l',\vec{\epsilon},m)\in\mathcal{J} \\ l'=l}} k \cdot e_{\vec{i},l',\vec{\epsilon},m} \qquad X^e_{f,\geq l} \coloneqq \bigoplus_{l'\geq l} X^e_{f,l'} \qquad X^e_{f,\leq l} \coloneqq \bigoplus_{l'\leq l} X^e_{f,l'}$$

$\diamondsuit$

Proposition 9.3.1.2. *Assume MonOrdMonicPoly and let $l \geq 0$. Then the following hold for the sub-graded-k-modules of the strict mixed complex X_f from Construction 9.2.2.1 that were defined in Definition 9.3.1.1.*

$$\begin{aligned}
\partial\big(X^c_{f,l}\big) &\subseteq X^e_{f,l+1} \\
\partial\big(X^e_{f,0}\big) &\subseteq X^e_{f,0} \\
\partial\big(X^e_{f,l}\big) &\subseteq 0 \quad \textit{if } l > 0
\end{aligned}$$

[11] See Section 6.3.1.5.

In particular, $X^e_{f,0}$ as well as the sum

$$X^c_{f,\geq l} \oplus X^e_{f,\geq l+1}$$

are closed under the boundary operator and hence subcomplexes of X_f. Both of these chain complexes are cofibrant, and $X^c_{f,\geq l} \oplus X^e_{f,\geq l+1}$ is acyclic. ♡

Proof. The statement about the images of the differential follow immediately from the description of ∂ in Proposition 9.2.3.4.

That $X^e_{f,0}$ and $X^c_{f,\geq l} \oplus X^e_{f,\geq l+1}$ are cofibrant as chain complexes follows from [Hov99, 2.3.6], as they are concentrated in nonnegative degree and by definition free as graded k-modules.

Finally, that $X^c_{f,\geq l} \oplus X^e_{f,\geq l+1}$ is acyclic also immediately follows from the description of ∂ in Proposition 9.2.3.4;

$$\begin{aligned} e_{\vec{i},l',\vec{\epsilon},m} &\mapsto c_{\vec{i},l'-1,\vec{\epsilon},m} && \text{for } \left(\vec{i}, l', \vec{\epsilon}, m\right) \in \mathfrak{J},\ l' \geq l+1 \\ c_{\vec{i},l',\vec{\epsilon},m} &\mapsto 0 && \text{for } \left(\vec{i}, l', \vec{\epsilon}, m\right) \in \mathfrak{J},\ l' \geq l \end{aligned}$$

defines a contracting homotopy, see Definition 9.2.3.2 and Propositions 9.2.3.3 and 9.2.3.4. □

Proposition 9.3.1.3. *Assume MonOrdMonicPoly and that Conjecture D*[12] *holds for f. Then there is an equivalence*

$$\mathrm{HH}(k[x_1, \ldots, x_n]/f) \simeq \gamma\big(X^e_{f,0}\big)$$

in $\mathcal{D}(k)$, where $X^e_{f,0}$ is the cofibrant chain complex defined in Definition 9.3.1.1 and Proposition 9.3.1.2. ♡

Proof. It follows from Proposition 9.2.3.3 that, as a graded k-module, X_f decomposes as the direct sum of $X^e_{f,0}$ and $X^c_{f,\geq 0} \oplus X^e_{f,\geq 1}$. As both summands are subcomplexes of X_f by Proposition 9.3.1.2, with the latter chain complex acyclic, it follows that the inclusion

$$X^e_{f,0} \to X_f$$

is a quasiisomorphism. We hence obtain equivalences

$$\gamma\big(X^e_{f,0}\big) \simeq \gamma(X_f) \simeq \mathrm{HH}(k[x_1, \ldots, x_n]/f)$$

in $\mathcal{D}(k)$, where the first equivalence is induced by the just mentioned quasiisomorphism, and the second equivalence is the one from Proposition 9.2.2.2. □

[12] Note that Conjecture D holds if $n = 1$ or $n = 2$ with 2 invertible in k by Proposition 7.5.3.1.

9.3.2. A different description of the smaller model

In Proposition 9.3.1.3 we showed that the chain complex $X^e_{f,0}$ defined in Definition 9.3.1.1 is a model for $\mathrm{HH}(k[x_1,\ldots,x_n]/f)$ as an object in $\mathcal{D}(k)$, assuming some conditions on f. As $X^e_{f,0}$ was defined as a subcomplex of X_f generated by some basis elements, it is slightly unexplicit, and in this section we give a somewhat more direct description of this complex. In particular, our description will be nearly the same as the one in [BACH, 2.3 and 3.2][13].

Construction 9.3.2.1. Assume MonOrdMonicPoly.

We let

$$p\colon k[x_1,\ldots,x_n] \to k[x_1,\ldots,x_n]/f$$

be the canonical quotient map. Note that p is a morphism of k-algebras. If M is a graded k-module, then we will also denote the morphism of graded k-modules

$$p \otimes \mathrm{id}_M : k[x_1,\ldots,x_n] \otimes M \to k[x_1,\ldots,x_n]/f \otimes M$$

by p again.

Consider the commutative graded k-algebra

$$k[x_1,\ldots,x_n]/f \otimes \Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n) \otimes \Gamma(t)$$

with x_i of degree 0, $\mathrm{d}\,x_i$ of degree 1 and t of degree 2. We define an operator ∂ decreasing degree by 1 by extending the following formulas by k-linearity and the Leibniz rule, where $P \in k[x_1,\ldots,x_n]/f$, $1 \leq i \leq n$, and $m \geq 0$.

$$\partial(P) = 0, \qquad \partial(\mathrm{d}\,x_i) = 0, \qquad \partial\left(t^{[m]}\right) = -p(\mathrm{d}\,f)t^{[m-1]}$$

To show that ∂ is well-defined we need to verify that the formula for $\partial(t^{[m]})$ is compatible with the Leibniz rule, so as for $m, m' \geq 0$ we have

$$t^{[m]} \cdot t^{[m']} = \binom{m+m'}{m} t^{[m+m']}$$

we have to show that the following equality holds.

$$-p(\mathrm{d}\,f)t^{[m-1]} \cdot t^{[m']} - t^{[m]} \cdot p(\mathrm{d}\,f)t^{[m'-1]} = -\binom{m+m'}{m} p(\mathrm{d}\,f)t^{[m+m'-1]} \quad (*)$$

The left hand side is given by

$$-p(\mathrm{d}\,f)t^{[m-1]} \cdot t^{[m']} - t^{[m]} \cdot p(\mathrm{d}\,f)t^{[m'-1]}$$

[13] The complex constructed here differs from the one in [BACH] in the very minor detail that our external generators are the the additive inverses of the external generators they consider. We do this because we will in Section 9.5 also define a mixed structure on this complex, and prefer the exterior generators to be given by $\mathrm{d}\,x_i$ rather than $-\mathrm{d}\,x_i$.

$$= -p(\mathrm{d}\, f)\Big(t^{[m-1]} \cdot t^{[m']} + t^{[m]} \cdot t^{[m'-1]}\Big)$$
$$= -p(\mathrm{d}\, f)\left(\binom{m+m'-1}{m-1} t^{[m+m'-1]} + \binom{m+m'-1}{m} t^{[m+m'-1]}\right)$$

so $(*)$ follows from $\binom{m+m'-1}{m-1} + \binom{m+m'-1}{m} = \binom{m+m'}{m}$.

As $\mathrm{d}\, f \cdot \mathrm{d}\, f = 0$, the operator ∂ squares to zero, and thus makes

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

into a commutative differential graded k-algebra. It is isomorphic to the one considered in [BACH, 2.3 and 3.2][14], where it is shown that this complex is quasiisomorphic to the normalized standard Hochschild complex for $k[x_1, \ldots, x_n]/f$.

Now let

$$\varphi\colon X^e_{f,0} \to k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

be the morphism of graded k-modules defined on basis elements as follows.

$$\varphi\Big(e_{\vec{i},0,\vec{\epsilon},m}\Big) := p\Big(x^{\vec{i}}\Big)\, \mathrm{d}\, x^{\vec{\epsilon}}\, t^{[m]} \qquad \text{for } \Big(\vec{i}, 0, \vec{\epsilon}, m\Big) \in \mathcal{J} \qquad \diamondsuit$$

Proposition 9.3.2.2. *Assume MonOrdMonicPoly. Then the morphism of graded k-modules φ from Construction 9.3.2.1 is an isomorphism of chain complexes.* ♡

Proof. We first check that φ is compatible with the boundary operator. So let the tuple $(\vec{i}, 0, \vec{\epsilon}, m)$ be an element of $\mathcal{J}$.

$$\varphi\Big(\partial\Big(e_{\vec{i},0,\vec{\epsilon},m}\Big)\Big)$$

We first use Proposition 9.2.3.4.

$$= \varphi\Big(-E^{[m-1]}\Big(r^0_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big) = -p\Big(r^0_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\Big)\Big)\, \mathrm{d}\, x^{\vec{\epsilon}}\, t^{[m-1]}$$

We can now use that p sends the ideal generated by f to 0 and hence satisfies $p \circ r^0_f = p$, and furthermore that p is multiplicative.

$$= -p\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\Big)\, \mathrm{d}\, x^{\vec{\epsilon}}\, t^{[m-1]} = -p(\mathrm{d}\, f) p\Big(x^{\vec{i}}\Big)\, \mathrm{d}\, x^{\vec{\epsilon}}\, t^{[m-1]}$$
$$= \partial\Big(p\Big(x^{\vec{i}}\Big)\, \mathrm{d}\, x^{\vec{\epsilon}}\, t^{[m]}\Big) = \partial\Big(\varphi\Big(e_{\vec{i},0,\vec{\epsilon},m}\Big)\Big)$$

It now remains to show that φ is an isomorphism of graded k-modules. For this it is enough to show that the restriction of the quotient map

$$p\colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/f$$

[14] As noted before, our description deviates in the signs of the external generators, but this does not change the fact that the differential graded k-algebras themselves are isomorphic, via an isomorphism from our complex to the one in [BACH, 2.3 and 3.2] mapping x_i to X_i, $\mathrm{d}\, x_i$ to $-e_i$, and $t^{[m]}$ to $t^{(m)}$. .

to the sub-graded-k-module of f-reduced polynomials is an isomorphism. But this follows immediately from Proposition 9.1.3.10, which shows that every element of $k[x_1, \ldots, x_n]/f$ has a unique f-reduced representative in $k[x_1, \ldots, x_n]$. □

The following corollary alternatively follows easily from the main result of [BACH], without requiring the assumption that Conjecture D holds for f. Our approach gives a different, independent, proof for those cases in which Conjecture D holds for f.

Corollary 9.3.2.3. *Assume MonOrdMonicPoly and that Conjecture D*[15] *holds for f. Then there is an equivalence*

$$\mathrm{HH}(k[x_1, \ldots, x_n]/f) \simeq \gamma(k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t))$$

in $\mathcal{D}(k)$, where

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

is the cofibrant chain complex defined in Construction 9.3.2.1. ♡

Proof. Combine Proposition 9.3.1.3 with Proposition 9.3.2.2. □

9.4. Logarithmic dimension of polynomials

Assume MonOrdMonicPoly and that Conjecture D holds for f. In Section 9.3.1 we constructed a subcomplex $X^e_{f,0}$ of the strict mixed complex X_f from Construction 9.2.2.1 such that the inclusion is a quasiisomorphism, which implied that $X^e_{f,0}$ represents the Hochschild homology

$$\mathrm{HH}(k[x_1, \ldots, x_n]/f)$$

as an object of $\mathcal{D}(k)$.

We would like to show that the strict mixed structure on X_f restricts to $X^e_{f,0}$, which would allow us to conclude that $X^e_{f,0}$ even represents

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$$

as an object of $\mathcal{M}$ixed.

Unfortunately the formula for d we obtained in Proposition 9.2.3.4 is somewhat more complicated than those we obtained for ∂ and it is not obvious that $X^e_{f,0}$ is closed under d. In particular, there is a term of the form

$$C^{[m-1]}\left(q^2_f\left(\mathrm{d}\, f \cdot r^0_f\left(\mathrm{d}\, f \cdot x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\right) \tag{9.3}$$

that we would need to vanish, and there is no reason to assume this is always the case. Indeed, the following example shows that this term can be nonzero.

[15] Note that Conjecture D holds if $n = 1$ or $n = 2$ and 2 is invertible in k by Proposition 7.5.3.1.

Example 9.4.0.1. Let $k = \mathbb{Z}$, $n = 2$, and consider the following polynomial.

$$f = x_1 x_2 - x_2^2$$

If we let $\preceq$ be the lexicographic monomial order[16], then f is monic and of degree $(1, 1)$.

We claim that

$$q_f^2\big(\mathrm{d}\, f \cdot r_f^0\big(\mathrm{d}\, f \cdot x_1^2\big)\big)$$

is nonzero, even though x_1^2 is f-reduced. Let us calculate this step by step.

$$r_f^0\big(\mathrm{d}\, f \cdot x_1^2\big) = r_f^0\big(x_1^2 x_2 \,\mathrm{d}\, x_1 + x_1^3 \,\mathrm{d}\, x_2 - 2x_1^2 x_2 \,\mathrm{d}\, x_2\big)$$

To calculate for example $r_f^0(x_1^2 x_2)$ we start by writing $x_1^2 x_2 = x_1 f + x_1 x_2^2$ and then continue with $x_1 x_2^2 = x_2 f + x_2^3$.

$$= x_2^3 \,\mathrm{d}\, x_1 + x_1^3 \,\mathrm{d}\, x_2 - 2x_2^3 \,\mathrm{d}\, x_2$$

We next need to multiply by $\mathrm{d}\, f$, and obtain the following.

$$\mathrm{d}\, f \cdot r_f^0\big(\mathrm{d}\, f \cdot x_1^2\big) = \big(x_1^3 x_2 - 2x_2^4 - x_1 x_2^3 + 2x_2^4\big) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2$$

Applying q_f^2 amounts to applying q_f^1 twice by Proposition 9.1.3.15 (6), so we obtain the following.

$$\begin{aligned}
& q_f^2\big(\mathrm{d}\, f \cdot r_f^0\big(\mathrm{d}\, f \cdot x_1^2\big)\big) \\
&= q_f^1\big(q_f^1\big(x_1^3 x_2 - 2x_2^4 - x_1 x_2^3 + 2x_2^4\big)\big) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \\
&= q_f^1\big(\big(x_1^2 + x_1 x_2 + x_2^2\big) - 2 \cdot (0) - \big(x_2^2\big) + 2 \cdot (0)\big) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \\
&= q_f^1\big(x_1^2 + x_1 x_2\big) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \\
&= (0 + 1) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 = \mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \neq 0 \qquad \diamondsuit
\end{aligned}$$

The goal of this section is to describe a criterion for f that is easy to check and that implies that terms of the form (9.3) that need to be zero for $X_{f,0}^{e}$ to be closed under d are indeed zero. For this we will generalize $r_f^0(\mathrm{d}\, f \cdot x^{\vec{i}} \,\mathrm{d}\, x^{\vec{\epsilon}})$ to an arbitrary f-reduced polynomial R and ask what the largest integer i is such that $q_f^i(\mathrm{d}\, f \cdot R)$ can be nonzero for an f-reduced polynomial R (with f fixed). We will call this number the *log dimension of* $\mathrm{d}\, f$ *to basis* f and will give an easy to check criterion that implies that this number is at most 1 in Proposition 9.4.2.5 and Corollary 9.4.2.6.

We will start this section with Section 9.4.1, where we discuss the logarithm for polynomials, before we turn towards the log dimension in Section 9.4.2.

[16] So $(i_1, i_2) \preceq (j_1, j_2)$ if $i_1 < j_1$ or $i_1 = j_1$ and $i_2 < j_2$.

9.4.1. Logarithm for polynomials

In this section we introduce a notion of logarithm for multivariable polynomials and point out some basic properties and consistency results.

Definition 9.4.1.1. Assume MonOrdMonicPoly. We define a map

$$\log_f : k[x_1, \ldots, x_n] \to \mathbb{Z}_{\geq 0}$$

as follows. For P an element of $k[x_1, \ldots, x_n]$, we let

$$\log_f(P) := \max\bigl(\{\, i \in \mathbb{Z}_{\geq 0} \mid r_f^i(P) \neq 0 \,\}\bigr)$$

and call $\log_f(P)$ the *logarithm to base f of P (with respect to the monomial order $\preceq$)*. Note that the set over which we take the maximum is finite, as all but finitely many summands in the decomposition from Proposition 9.1.3.12 are zero, so attains a maximum in $\mathbb{Z}_{\geq 0}$. ♢

Remark 9.4.1.2. Assume MonOrdMonicPoly and let P be an element of $k[x_1, \ldots, x_n]$. Then P is f-reduced if and only if $\log_f(P) = 0$. ♢

Remark 9.4.1.3. Assume MonOrdMonicPoly and that we are in the situation of Construction 9.1.1.11 and that $\deg_{\preceq}(f)$ is in the image of ψ. Let P be an element of $k[x_1, \ldots, x_n]$ and let f' and P' be the elements of $k'[y_1, \ldots, y_m]$ corresponding to f and P under the isomorphism of Construction 9.1.1.11. It then follows from Remark 9.1.3.11 that $\log_{f'}(P') = \log_f(P)$. ♢

Proposition 9.4.1.4. *Assume MonOrdMonicPoly and let P and Q be elements of $k[x_1, \ldots, x_n]$. Then the following holds.*

$$\log_f(P + Q) \leq \max\bigl(\{\, \log_f(P), \log_f(Q) \,\}\bigr) \qquad \heartsuit$$

Proof. By Proposition 9.1.3.15 (3), r_f^i is additive for every $i \geq 0$, so if $r_f^i(P + Q) \neq 0$ for some $i \geq 0$, then at least one of $r_f^i(P)$ and $r_f^i(Q)$ must be nonzero as well. □

9.4.2. Logarithmic dimension for polynomials

Let f be an element of $\mathbb{R}_{>1}$, i. e. a real number bigger than 1, and let us for a moment consider the logarithm function

$$\log_f : \mathbb{R}_{>0} \to \mathbb{R}$$

for the real numbers. This function satisfies a compatibility relation with multiplication; if P and Q are positive real numbers, then

$$\log_f(P \cdot Q) = \log_f(P) + \log_f(Q)$$

holds. In Section 9.4.1 we defined a logarithm for (multivariable) polynomials, and we would like to better understand how the logarithm of products relates to the individual logarithms as well. The logarithm for polynomials does not take real values, so to improve the analogy we should first replace $\log_f$ with the function

$$\log'_f \colon \mathbb{R}_{>0} \to \mathbb{Z}_{\geq 0}, \qquad x \mapsto \begin{cases} \lfloor \log_f(x) \rfloor & \text{if } \log_f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

so we round down the logarithm, but set it to 0 should it be negative. The rounding destroys the precise property of the logarithm of a product being the sum of the logarithms, but as for any real number x it holds that $x - 1 < \lfloor x \rfloor \leq x$, we still obtain an inequality

$$\log'_f(P) + \log'_f(Q) \leq \log'_f(P \cdot Q) \leq \log'_f(P) + \log'_f(Q) + 1 \tag{9.4}$$

for every P and Q in $\mathbb{R}_{>0}$.

If we now let f be an element of $k[x_1]$ that is a monic polynomial of positive degree, and P and Q any elements of $k[x_1]$, then the analogue of (9.4) holds, at least as long k is an integral domain. Indeed, for one-variable polynomials, it is actually not difficult to see that

$$\log_f(P) = \left\lfloor \frac{\deg(P)}{\deg(f)} \right\rfloor$$

from which the inequality

$$\log_f(P) + \log_f(Q) \leq \log_f(P \cdot Q) \leq \log_f(P) + \log_f(Q) + 1$$

follows as long as k is an integral domain. The inequality

$$\log_f(P \cdot Q) \leq \log_f(P) + \log_f(Q) + 1$$

holds for any commutative ring k. We can restate this as saying that the expression

$$\log_f(P \cdot Q) - \log_f(P) - \log_f(Q) \tag{9.5}$$

is bounded above by 1 as we let f, P, and Q vary.

Let us now consider multivariable polynomials and assume MonOrdMonicPoly. The first question we can then ask is whether (9.5) is still bounded above while letting f, P, and Q range over $k[x_1, \ldots, x_n]$ with f satisfying the assumptions in MonOrdMonicPoly.

Unfortunately, this is not the case as soon as $n \geq 2$. Consider the example $f = x_1 x_2$, with $P = x_1^m$ and $Q = x_2^m$, where $m \geq 1$. In this case, $\log_f(P) = \log_f(Q) = 0$, but $\log_f(P \cdot Q) = m$, so the value of (9.5) is unbounded if we let f, P, and Q vary.

However, if we fix f, then it is not difficult to find examples where the value of (9.5) *is* bounded while letting P and Q range over $k[x_1, \ldots, x_n]$. For example consider $f = x_1$. In this case the value of $\log_f(P)$ is given by the highest exponent of x_1 appearing in the monomials of P, and the value of (9.5) is bounded above by 0.

So we can instead ask, given fixed f, whether the value of (9.5), as P and Q range over the elements of $k[x_1, \ldots, x_n]$, is bounded above, and if so, what the maximum value is. In this section we go one step further, and fix both f as well as P, and consider the supremum of (9.5) when varying Q, calling it the *log dimension to base f of P*. In particular, we will establish a condition that ensures that the log dimension of a polynomial is at most 1.

Definition 9.4.2.1. Assume MonOrdMonicPoly. For P an element of the polynomial k-algebra $k[x_1, \ldots, x_n]$ we let $\mathrm{logdim}_f(P)$ be the element of the set $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ that is defined as

$$\mathrm{logdim}_f(P) \coloneqq \sup\bigl(\{\, \log_f(P \cdot Q) - \log_f(P) - \log_f(Q) \bigm| Q \in k[x_1, \ldots, x_n] \,\}\bigr)$$

and call $\mathrm{logdim}_f(P)$ the *log dimension to base f of P*. ♢

Remark 9.4.2.2. Assume MonOrdMonicPoly, that we are in the situation of Construction 9.1.1.11, and that $\deg_{\preceq}(f)$ is in the image of ψ. Let P be an element of $k[x_1, \ldots, x_n]$ and let f' and P' be the elements of $k'[y_1, \ldots, y_m]$ corresponding to f and P under the isomorphism of Construction 9.1.1.11. It then follows from Remark 9.4.1.3 and Remark 9.1.3.9 that $\mathrm{logdim}_{f'}(P') = \mathrm{logdim}_f(P)$. ♢

Proposition 9.4.2.3. *Assume MonOrdMonicPoly and let $P \in k[x_1, \ldots, x_n]$ be a polynomial. Then it suffices to consider f-reduced polynomials Q in the definition of* $\mathrm{logdim}_f(P)$*, i. e. there is an equality as follows.*

$$\begin{aligned}&\mathrm{logdim}_f(P)\\ &= \sup\bigl(\{\, \log_f(P \cdot R) - \log_f(P) \bigm| R \in k[x_1, \ldots, x_n],\ R \text{ is } f\text{-reduced} \,\}\bigr)\end{aligned}$$ ♡

Proof. For the moment let us denote the right hand side of the equality in the statement by $\mathrm{logdim}^{\mathrm{red}}_f(P)$. The inequality $\mathrm{logdim}^{\mathrm{red}}_f(P) \leq \mathrm{logdim}_f(P)$ is clear, so it suffices to show that $\mathrm{logdim}_f(P) \leq \mathrm{logdim}^{\mathrm{red}}_f(P)$ also holds.

So let Q be any element of $k[x_1, \ldots, x_n]$. It suffices to find an f-reduced polynomial R such that

$$\log_f(P \cdot Q) - \log_f(P) - \log_f(Q) \leq \log_f(P \cdot R) - \log_f(P)$$

holds, which is equivalent to the following inequality.

$$\log_f(P \cdot Q) - \log_f(Q) \leq \log_f(P \cdot R)$$

For this, let us write Q as

$$Q = \sum_{i=0}^{\log_f(Q)} r^i_f(Q) f^i$$

so that we obtain the following chain of inequalities.

$$\begin{aligned}
&\log_f(P \cdot Q) - \log_f(Q)\\
&= \log_f\left(\sum_{i=0}^{\log_f(Q)} P \cdot r_f^i(Q) \cdot f^i\right) - \log_f(Q)
\end{aligned}$$

Using Proposition 9.4.1.4.

$$\leq \max\bigl(\bigl\{\, \log_f\bigl(P \cdot r_f^i(Q) \cdot f^i\bigr) \bigm| 0 \leq i \leq \log_f(Q) \,\bigr\}\bigr) - \log_f(Q)$$

Using Proposition 9.1.3.15 (4).

$$\begin{aligned}
&\leq \max\bigl(\bigl\{\, \log_f\bigl(P \cdot r_f^i(Q)\bigr) + i \bigm| 0 \leq i \leq \log_f(Q) \,\bigr\}\bigr) - \log_f(Q)\\
&\leq \max\bigl(\bigl\{\, \log_f\bigl(P \cdot r_f^i(Q)\bigr) \bigm| 0 \leq i \leq \log_f(Q) \,\bigr\}\bigr) + \log_f(Q) - \log_f(Q)\\
&= \max\bigl(\bigl\{\, \log_f\bigl(P \cdot r_f^i(Q)\bigr) \bigm| 0 \leq i \leq \log_f(Q) \,\bigr\}\bigr)
\end{aligned}$$

We can thus take R to be the f-reduced polynomial $r_f^i(Q)$, where the integer $0 \leq i \leq \log_f(Q)$ is chosen to maximize $\log_f\Bigl(P \cdot r_f^i(Q)\Bigr)$. □

Proposition 9.4.2.4. *Assume MonOrdMonicPoly, and assume furthermore that the degree of f satisfies $\deg_{\preceq}(f) \geq (1, \ldots, 1)$ and that $f_{\vec{i}} = 0$ for any $\vec{i} \in \mathbb{Z}_{\geq 0}^n$ such that $\vec{i} \not\leq \deg_{\preceq}(f)$, i. e. every variable divides the leading monomial of f and every monomial appearing in f divides the leading monomial.*

Let $P \in k[x_1, \ldots, x_n]$ be an f-reduced polynomial such that $P_{\vec{i}} = 0$ for every $\vec{i} \in \mathbb{Z}_{\geq 0}^n$ such that $\vec{i} \not\leq \deg_{\preceq}(f)$, i. e. every monomial in P divides the lead monomial of f.

Then $\operatorname{logdim}_f(P) \leq 1$. ♡

Proof. By Proposition 9.4.2.3, it suffices to show that for any f-reduced polynomial Q the inequality $\log_f(P \cdot Q) \leq 1$ holds. Using Proposition 9.4.1.4 we can furthermore reduce to the case $P = x^{\vec{j}}$ with $\vec{j} < \deg_{\preceq}(f)$ and $Q = x^{\vec{i}}$ with $\vec{i} \ngeq \deg_{\preceq}(f)$.

By Proposition 9.1.3.12 we can write the product $P \cdot Q = x^{\vec{j}+\vec{i}}$ as

$$x^{\vec{j}+\vec{i}} = R_2 f^2 + R_1 f + R_0 \tag{$*$}$$

such that R_1 and R_0 are f-reduced polynomials, and R_2 is any polynomial. What we have to show is then that $R_2 = 0$. We prove this by contradiction and assume that $R_2 \neq 0$. It then follows from Proposition 9.1.2.3 (4) that

$$\bigl(R_2 f^2\bigr)_{\deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)} \neq 0$$

so that it suffices to show that

$$\begin{aligned}
\Bigl(x^{\vec{j}+\vec{i}}\Bigr)_{\deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)} &= (R_1 f)_{\deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)}\\
&= (R_0)_{\deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)} = 0
\end{aligned}$$

in contradiction to $(*)$.

We start with $(x^{\vec{j}+\vec{i}})_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)}$, which could only be nonzero if the following equation would hold.

$$\vec{j} + \vec{i} = \deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)$$

However, as $\vec{j} < \deg_{\preceq}(f)$ we would then obtain

$$\begin{aligned} \vec{i} &= \left(\vec{j} + \vec{i}\right) - \vec{j} \\ &> \left(\deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)\right) - \deg_{\preceq}(f) \\ &\geq \deg_{\preceq}(f) \end{aligned}$$

which would contradict $\vec{i} \ngeq \deg_{\preceq}(f)$. Thus $(x^{\vec{j}+\vec{i}})_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)} = 0$ must hold.

Next, if $(R_1 f)_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)}$ were nonzero, then there would exist two tuples $\vec{a}, \vec{b} \in \mathbb{Z}^n_{\geq 0}$ such that $(R_1)_{\vec{a}} \neq 0$ and $f_{\vec{b}} \neq 0$ and such that the equation

$$\vec{a} + \vec{b} = \deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)$$

holds. Using that, by assumption on f, the inequality $\vec{b} \leq \deg_{\preceq}(f)$ must hold, we obtain completely like in the previous case, with $\vec{b}$ taking the place of $\vec{j}$, that

$$\vec{a} \geq \deg_{\preceq}(f)$$

which contradicts the assumption that R_2 is f-reduced.

Finally, that

$$(R_0)_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)} = 0$$

follows directly from R_0 being f-reduced. □

Proposition 9.4.2.5. *Assume MonOrdMonicPoly, and let $P \in k[x_1, \ldots, x_n]$ be an f-reduced polynomial. Assume that for every $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ such that $f_{\vec{i}} \neq 0$ or $P_{\vec{i}} \neq 0$ the following property holds: If $1 \leq j \leq n$ and $\deg_{\preceq}(f)_j \neq 0$, then $\vec{i}_j \leq \deg_{\preceq}(f)_j$. In other words, we require that every monomial appearing in f or P divides the leading monomial of f after setting those variables that do not appear in the leading monomial of f to 1.*

Then $\operatorname{logdim}_f(P) \leq 1$. ♡

Proof. Let $\{i'_1, \ldots, i'_r\}$ be the subset of $\{1, \ldots, n\}$ of elements for which $\deg_{\preceq}(f)_{i'_j} = 0$, let $\{i_1, \ldots, i_l\}$ be the complement, and let

$$\varphi\colon \{i_1, \ldots, i_l\} \to \{1, \ldots, n\}$$

be the inclusion. Note that $\deg_{\preceq}(f)$ is then in the image of ψ from Construction 9.1.1.11. Denote by f' and P' the elements of $(k[x_{i'_1}, \ldots, x_{i'_r}])[x_{i_1}, \ldots, x_{i_l}]$

corresponding to f and P under the isomorphism of Construction 9.1.1.11. Note that by Proposition 9.1.3.5, f' is monic and $\deg_{\preceq}(f) = \psi\big(\deg_{\preceq}(f')\big)$ by Proposition 9.1.2.4. Then the assumptions on f and P then translate to f' and P' satisfying the assumptions required in Proposition 9.4.2.4. We can thus conclude that $\operatorname{logdim}_{f'}(P') \leq 1$. As by Remark 9.4.2.2 we also have $\operatorname{logdim}_f(P) = \operatorname{logdim}_{f'}(P')$, we are done. □

Corollary 9.4.2.6. *Assume MonOrdMonicPoly, and assume that f satisfies the property required in Proposition 9.4.2.5.*

Then for every $1 \leq i \leq n$ the partial derivative $\frac{\partial f}{\partial x_j}$ satisfies the property required of P in Proposition 9.4.2.5, and so $\operatorname{logdim}_f\left(\frac{\partial f}{\partial x_j}\right) \leq 1$. In particular, $q_f^2(\mathrm{d}\, f \cdot P) = 0$ for every f-reduced polynomial P. ♡

Proof. Every monomial in $\frac{\partial f}{\partial x_j}$ divides a monomial in f. □

Notation 9.4.2.7. Assume MonOrdMonicPoly. Then we define $\operatorname{logdim}_f(\mathrm{d}\, f)$ as follows.

$$\operatorname{logdim}_f(\mathrm{d}\, f) \coloneqq \max\left(\left\{ \operatorname{logdim}_f\left(\frac{\partial f}{\partial x_i}\right) \,\middle|\, 1 \leq i \leq n \right\}\right)$$

In particular, using this convention the conclusion of Corollary 9.4.2.6 can be phrased as $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$, and $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$ implies $q_f^2(\mathrm{d}\, f \cdot P) = 0$ for every f-reduced polynomial P. ♢

9.5. A smaller strict model for the mixed complex

Assume MonOrdMonicPoly and that Conjecture D holds for f. As was already discussed in the introduction of Section 9.4, we would like to show that the strict mixed structure on X_f from Construction 9.2.2.1 restricts to the subcomplex $X_{f,0}^e$ that we constructed in Section 9.3.1, which would allow us to conclude that $X_{f,0}^e$ even represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{M}$ixed.

The work of Section 9.4 now allows us to concisely state a condition on f that implies that the strict mixed structure restricts like that, namely the condition $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$. We show that this indeed implies that the strict mixed structure of X_f restricts to $X_{f,0}^e$ in the short section Section 9.5.1.

In continuation to Section 9.3.2, in which we gave a different (independent from X_f) description of the chain complex $X_{f,0}^e$ by constructing an isomorphism between $X_{f,0}^e$ and a chain complex with underlying graded k-module

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t) \tag{9.6}$$

we will upgrade that isomorphism to an isomorphism of strict mixed complexes in Section 9.5.2.

9.5.1. Restricting the strict mixed structure

Proposition 9.5.1.1. *Assume MonOrdMonicPoly and* $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$.

Then the strict mixed structure of X_f *from Construction 9.2.2.1 restricts to the subcomplex*[17] $X^e_{f,0}$. *Thus the inclusion* $X^e_{f,0} \to X_f$ *is a quasiisomorphism of strict mixed complexes.* ♡

Proof. That the inclusion $X^e_{f,0} \to X_f$ is a quasiisomorphism was already shown in Proposition 9.3.1.3, so it suffices to show that $X^e_{f,0}$ is closed under d. Unpacking the definition of $X^e_{f,0}$ and using the formula for d obtained in Proposition 9.2.3.4 this means that we need to show that for $\vec{i} \not\geq \deg_{\preceq}(f)$, $\vec{\epsilon} \in \{0,1\}^n$ and $m \geq 0$ the element

$$\begin{aligned}\mathrm{d}\Big(e_{\vec{i},0,\vec{\epsilon},m}\Big) = E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} + m q^1_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\\ + (m-1)C^{[m-1]}\Big(q^2_f\Big(\mathrm{d}\, f \cdot r^0_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big)\Big)\end{aligned}$$

is again in $X^e_{f,0}$. For this it suffices to show the following.

(1) $\mathrm{d}\Big(x^{\vec{i}}\Big)$ is f-reduced.

(2) $q^1_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\Big)$ is f-reduced.

(3) $q^2_f(\mathrm{d}\, f \cdot R) = 0$ if R is f-reduced.

Claim (1) follows immediately from Proposition 9.2.3.1 (1), claim (2) follows from $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$ with Proposition 9.1.3.15 (6), and claim (3) follows from $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$. □

9.5.2. An alternative description of the smaller strict mixed model

We can now transfer the strict mixed structure on $X^e_{f,0}$ via the isomorphism of chain complexes φ from Construction 9.3.2.1 and Proposition 9.3.2.2. We first describe the resulting d, and then show that φ is compatible with it.

Construction 9.5.2.1. Assume MonOrdMonicPoly and $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$.

Recall the commutative differential graded k-algebra

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t) \tag{9.7}$$

as well as the morphisms p defined in Construction 9.3.2.1.

[17] See Definition 9.3.1.1 for the definition and Proposition 9.3.1.2 for being a subcomplex.

We will define a k-linear operator[18] d that increases degree by 1 on (9.7) by

$$\mathrm{d}\Big(p(P)\,\mathrm{d}\,x^{\vec{\epsilon}}t^{[m]}\Big) := \big(p\big(\mathrm{d}\big(r_f^0(P)\big)\big) + mp\big(q_f^1\big(\mathrm{d}\,f \cdot r_f^0(P)\big)\big)\big)\,\mathrm{d}\,x^{\vec{\epsilon}}t^{[m]} \quad (9.8)$$

for $P \in k[x_1, \ldots, x_n]$, $\vec{\epsilon} \in \{0,1\}^n$, and $m \geq 0$. Note that r_f^0 is zero on the ideal generated by f, so d as defined above is well-defined. $\diamondsuit$

Proposition 9.5.2.2. *Assume MonOrdMonicPoly and* $\operatorname{logdim}_f(\mathrm{d}\,f) \leq 1$. *Then the isomorphism*

$$\varphi\colon X_{f,0}^e \to k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\,x_1, \ldots, \mathrm{d}\,x_n) \otimes \Gamma(t)$$

of chain complexes from Construction 9.3.2.1 and Proposition 9.3.2.2 is compatible with the operators d *defined on either side. In particular,* d *as defined in Construction 9.5.2.1 on the codomain defines a strict mixed complex structure on*

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\,x_1, \ldots, \mathrm{d}\,x_n) \otimes \Gamma(t)$$

and this strict mixed complex is isomorphic as a mixed complex to $X_{f,0}^e$. $\heartsuit$

Proof. Using the description for d on $X_{f,0}^e$ obtained in the proof of Proposition 9.5.1.1, we obtain for $\vec{i} \not\geq \deg_{\preceq}(f)$, $\vec{\epsilon} \in \{0,1\}^n$ and $m \geq 0$ the following calculation.

$$\begin{aligned}
\varphi\Big(\mathrm{d}\Big(e_{\vec{i},0,\vec{\epsilon},m}\Big)\Big) &= \varphi\Big(E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} + mq_f^1\Big(\mathrm{d}\,f \cdot x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\Big)\Big) \\
&= \Big(p\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\Big) + mp\Big(q_f^1\Big(\mathrm{d}\,f \cdot x^{\vec{i}}\Big)\Big)\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}t^{[m]} \\
&= \Big(p\Big(\mathrm{d}\Big(r_f^0\Big(x^{\vec{i}}\Big)\Big)\Big) + mp\Big(q_f^1\Big(\mathrm{d}\,f \cdot r_f^0\Big(x^{\vec{i}}\Big)\Big)\Big)\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}t^{[m]} \\
&= \mathrm{d}\Big(p\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}t^{[m]}\Big) \\
&= \mathrm{d}\Big(\varphi\Big(e_{\vec{i},0,\vec{\epsilon},m}\Big)\Big)
\end{aligned}$$

$\square$

We can now put everything together to obtain the main result.

Proposition 9.5.2.3. *Assume MonOrdMonicPoly and* $\operatorname{logdim}_f(\mathrm{d}\,f) \leq 1$[19]. *Furthermore assume that Conjecture D*[20] *holds for* f.

---

[18] We will later show that under the isomorphism φ this operator agrees with the d that is part of the strict mixed complex structure on $X_{f,0}^e$, so that the operator d defined here defines a strict mixed complex structure will then be automatic.

[19] Recall from Corollary 9.4.2.6 and Proposition 9.4.2.5 that this holds in particular if for every $\vec{i} \in \mathbb{Z}_{\geq 0}^n$ such that $f_{\vec{i}} \neq 0$ the following property holds: If $1 \leq j \leq n$ and $\deg_{\preceq}(f)_j \neq 0$, then $\vec{i}_j \leq \deg_{\preceq}(f)_j$.

[20] Note that Conjecture D holds if $n = 1$ or $n = 2$ and 2 is invertible in k by Proposition 7.5.3.1.

Then there is an equivalence

$$\begin{aligned}&\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)\\ &\simeq \gamma_{\mathsf{Mixed}}(k[x_1,\ldots,x_n]/f\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n)\otimes\Gamma(t))\end{aligned}$$

in $\mathcal{M}$ixed*, where*

$$k[x_1,\ldots,x_n]/f\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n)\otimes\Gamma(t)$$

is the mixed complex described in Construction 9.3.2.1, Construction 9.5.2.1, and Proposition 9.5.2.2. ♡

Proof. Combine Proposition 9.2.2.2 with Proposition 9.5.1.1 and Proposition 9.5.2.2. □

Proposition 9.5.2.3 is the last missing piece to prove Theorem A that was stated in the introduction.

Proof of Theorem A. Combine Proposition 9.5.2.3 with Proposition 7.5.3.1 and Corollary 9.4.2.6. □

9.6. On the quasiisomorphisms constructed by the Buenos Aires Cyclic Homology Group

Assume MonOrdMonicPoly and let $A \coloneqq k[x_1,\ldots,x_n]/f$. In [BACH], an $A\otimes A$-free resolution $R_s(A)$ of A is constructed, together with morphisms of $A\otimes A$-chain complexes

$$h\colon R_s(A)\to \overline{\mathrm{C}}^{\mathrm{Bar}}(A)\qquad\text{and}\qquad g\colon \overline{\mathrm{C}}^{\mathrm{Bar}}(A)\to R_s(A)$$

where $\overline{\mathrm{C}}^{\mathrm{Bar}}(A)$ refers to the normalized bar construction that relates to the bar construction defined in Construction 6.3.2.1 as the normalized standard Hochschild complex relates to the standard Hochschild complex; in chain degree $n\geq 0$ the complex $\overline{\mathrm{C}}^{\mathrm{Bar}}(A)$ is given by $A\otimes(A/k\cdot\{1\})^{\otimes n}\otimes A$. It it shown in [BACH, 2.5.11] that g and h are mutual homotopy inverses. Tensoring over $A\otimes A$ from the left with A one then obtains quasiisomorphisms[21]

$$\overline{h}\colon \overline{R_s}(A)\to \overline{\mathrm{C}}(A)\qquad\text{and}\qquad \overline{g}\colon \overline{\mathrm{C}}(A)\to \overline{R_s}(A)$$

so that $\gamma(\overline{R_s}(A))\simeq\gamma(\overline{\mathrm{C}}(A))$ in $\mathcal{D}(k)$. By Propositions 6.3.1.10 and 6.3.4.1 the chain complex $\overline{R_s}(A)$ is thus a strict model for $\mathrm{HH}(A)$ as an object of $\mathcal{D}(k)$. As was remarked in Section 9.3.2, the chain complex $\overline{R_s}(A)$ is isomorphic to the chain complex $k[x_1,\ldots,x_n]/f\otimes\Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n)\otimes\Gamma(t)$ described in

[21] Compare with Proposition 6.3.2.4 for the identification $\overline{\mathrm{C}}(A)\cong A\otimes_{A\otimes A}\overline{\mathrm{C}}^{\mathrm{Bar}}(A)$.

Construction 9.3.2.1. Corollary 9.3.2.3 could thus also be deduced directly from the results of [BACH].

The question now arises whether one could similarly give an alternative proof of Proposition 9.5.2.3 and Theorem A, perhaps even without requiring the assumption that Conjecture D holds for f and that $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$, by showing that $\overline{g}$ or $\overline{h}$ can be lifted to a morphism of strict mixed complexes, and using that the normalized standard Hochschild complex $\overline{\mathrm{C}}(A)$ represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(A)$ even as an object in $\mathcal{M}$ixed by Propositions 6.3.1.10 and 6.3.4.1.

The following two propositions show that this is in general not possible; there is in general no strict mixed complex structure on $\overline{R_s}(A)$ that makes $\overline{g}$ or $\overline{h}$ into a morphism of strict mixed complexes. The counterexamples we use are $f = x_1x_2x_3$ for g and $f = x_1x_2$ for h. Note that both of these polynomials satisfy $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$ by Corollary 9.4.2.6.

This leaves open the question of whether it is possible to prove that $\overline{g}$ or $\overline{h}$ can be upgraded to a strongly homotopy linear morphism of strict mixed complexes (see Section 4.2.3). This is what the author tried originally for $f = x_1x_2x_3$, but without succeeding. The amount of data required for the higher homotopies combined with the complicated definitions of $\overline{g}$ and $\overline{h}$ may make this infeasible as n gets large.

In the rest of this section we will assume that the reader is familiar with the definitions and notation from [BACH]. We will however deviate from the notation from [BACH] when we have already established notation for the same thing. In particular, if P is an element of $k[x_1, \ldots, x_n]$, then we will write $q_f^1(P)$ rather than $\overline{P}$ used in [BACH, 2.2.1], and we denote by $\overline{P}$ the residue class of P in $A/k \cdot \{1\}$, as in Proposition 6.3.1.10. We will denote by $\overline{\varphi}$ the morphism $A \otimes_{A \otimes A} \varphi$, with φ as in [BACH, 2.5.1].

Proposition 9.6.0.1. *Let $f = x_1x_2x_3$ and $A \coloneqq k[x_1, x_2, x_3]/f$. Then there is no strict mixed structure on $\overline{R_s}(A)$ such that $\overline{g}$ is a morphism of strict mixed complexes.* ♡

Proof. If $\overline{g}$ were a morphism of strict mixed complexes, then the following equation would need to hold.

$$\mathrm{d}(\overline{g}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2})) = \overline{g}(\mathrm{d}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2}))$$

However, we will show that this is not possible no matter what the strict mixed complex structure on $\overline{R_s}(A)$ is, as $\overline{g}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2})$ is already zero, making the left hand side zero, while the right hand side is nonzero.

We begin by showing that $\overline{g}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2}) = 0$. We begin with the definition of g from [BACH, 2.5.4].

$$\begin{aligned}
&\overline{g}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2}) \\
&= x_2x_3\overline{g}_2(1 \otimes \overline{x_1} \otimes \overline{x_2}) \\
&= x_2x_3 \cdot \left(-q_f^1(x_1x_2)\overline{g}_0(1) \blacksquare\, t + (-1)^1 \sum_{i_1 < i_2} \overline{\varphi}_{i_2 i_1}(1 \otimes \overline{x_1} \otimes \overline{x_2}) e_{i_1 i_2} \right)
\end{aligned}$$

$q_f^1(x_1x_2) = 0$, so the first summand vanishes. We plug in the definition of $\overline{\varphi}$ from [BACH, 2.5.1].

$$= x_2x_3 \cdot \left(- \sum_{i_1<i_2} \left(\overline{\varphi}^0_{i_2i_1}(1 \otimes \overline{x_1} \otimes \overline{x_2}) + \overline{\varphi}^1_{i_2i_1}(1 \otimes \overline{x_1} \otimes \overline{x_2})\right) e_{i_1i_2} \right)$$
$$= x_2x_3 \cdot \left(- \sum_{i_1<i_2} \left(\frac{\partial x_1}{\partial x_{i_2}} \cdot \frac{\partial x_2}{\partial x_{i_1}} + \overline{\varphi}^1_{i_2i_1}(1 \otimes \overline{x_1} \otimes \overline{x_2}) \right) e_{i_1i_2} \right)$$

The first summand can only be nonzero if both $i_2 = 1$ and $i_1 = 2$, but this does not actually occur as $i_1 < i_2$.

$$= x_2x_3 \cdot \left(- \sum_{i_1<i_2} \left(\overline{\varphi}^1_{i_2i_1}(1 \otimes \overline{x_1} \otimes \overline{x_2})\right) e_{i_1i_2} \right)$$
$$= x_2x_3 \cdot \left(- \sum_{i_1<i_2} \left(-1 \cdot \frac{\partial q_f^1(x_1x_2)}{\partial x_{i_2}} \cdot \overline{\varphi}^0_{i_1}(1 \otimes f) \right) e_{i_1i_2} \right)$$

This is zero as $q_f^1(x_1x_2) = 0$.

It remains to show that $\overline{g}(\mathrm{d}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2}))$ is not zero. We begin by evaluating $\mathrm{d}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2})$ using Proposition 6.3.1.10.

$$\begin{aligned}
&\overline{g}_3(\mathrm{d}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2})) \\
&= \overline{g}_3(1 \otimes \overline{x_2x_3} \otimes \overline{x_1} \otimes \overline{x_2} + 1 \otimes \overline{x_1} \otimes \overline{x_2} \otimes \overline{x_2x_3} + 1 \otimes \overline{x_2} \otimes \overline{x_2x_3} \otimes \overline{x_1}) \\
&= -\Big(q_f^1(x_1x_2x_3)\overline{g}_1(1 \otimes \overline{x_2}) + q_f^1(x_1x_2)\overline{g}_1(1 \otimes \overline{x_2x_3}) \\
&\qquad + q_f^1\big(x_2^2x_3\big)\overline{g}_1(1 \otimes \overline{x_1}) \Big) \blacksquare\, t \\
&\quad + (\overline{\varphi}_{321}(1 \otimes \overline{x_2x_3} \otimes \overline{x_1} \otimes \overline{x_2}) + \overline{\varphi}_{321}(1 \otimes \overline{x_1} \otimes \overline{x_2} \otimes \overline{x_2x_3}) \\
&\qquad + \overline{\varphi}_{321}(1 \otimes \overline{x_2} \otimes \overline{x_2x_3} \otimes \overline{x_1})) \cdot e_{123}
\end{aligned}$$

We have three elements to which $\overline{\varphi}_{321} = \overline{\varphi}^0_{321} + \overline{\varphi}^1_{321}$ is applied. The $\overline{\varphi}^0_{321}$ component is zero for all three terms; for the first one because $\frac{\partial x_2}{\partial x_1} = 0$, for the second one because $\frac{\partial x_1}{\partial x_3} = 0$, and for the last one because $\frac{\partial x_2}{\partial x_3} = 0$.

$$\begin{aligned}
&= -\overline{g}_1(1 \otimes \overline{x_2}) \blacksquare\, t \\
&\quad + \overline{\varphi}^1_{321}(1 \otimes \overline{x_2x_3} \otimes \overline{x_1} \otimes \overline{x_2}) \cdot e_{123} \\
&\quad + \overline{\varphi}^1_{321}(1 \otimes \overline{x_1} \otimes \overline{x_2} \otimes \overline{x_2x_3}) \cdot e_{123} \\
&\quad + \overline{\varphi}^1_{321}(1 \otimes \overline{x_2} \otimes \overline{x_2x_3} \otimes \overline{x_1}) \cdot e_{123}
\end{aligned}$$

The definition of $\overline{\varphi}^1_{321}$ has a factor that is a partial derivative of q_f^1 of the product of two neighboring tensor factors. q_f^1 of such a product can only possibly be nonzero if we multiply the first two tensor factors in $1 \otimes \overline{x_2x_3} \otimes \overline{x_1} \otimes \overline{x_2}$ or the last two in $1 \otimes \overline{x_2} \otimes \overline{x_2x_3} \otimes \overline{x_1}$. In both cases the product is $x_1x_2x_3$, so that the value of q_f^1 will be 1. Forming any partial derivative then yields zero.

$$= -\overline{g}_1(1 \otimes \overline{x_2}) \blacksquare\, t$$

$$= (\overline{\varphi}_1(1 \otimes \overline{x_2}) \cdot e_1 + \overline{\varphi}_2(1 \otimes \overline{x_2}) \cdot e_2 + \overline{\varphi}_3(1 \otimes \overline{x_2}) \cdot e_3) \blacksquare\, t$$
$$= \left(\frac{\partial x_2}{\partial x_1} \cdot e_1 + \frac{\partial x_2}{\partial x_2} \cdot e_2 + \frac{\partial x_2}{\partial x_3} \cdot e_3\right) \blacksquare\, t$$
$$= e_2 t \qquad \square$$

Proposition 9.6.0.2. *Let $f = x_1 x_2$ and $A \coloneqq k[x_1, x_2]/f$. Then there is no strict mixed structure on $\overline{R_s}(A)$ such that $\overline{h}$ is a morphism of strict mixed complexes.* ♡

Proof. If $\overline{h}$ were a morphism of strict mixed complexes, then the following equation would need to hold.

$$\overline{h}(\mathrm{d}(x_1 t)) = \mathrm{d}\big(\overline{h}(x_1 t)\big)$$

However, we will show that this is not possible no matter what the strict mixed complex structure on $\overline{R_s}(A)$ is, as $\mathrm{d}\big(\overline{h}(x_1 t)\big)$ does not lie in the image of $\overline{h}$.

We begin by calculating $h(t)$, for which we have the following by [BACH, After 2.4.5, 2.2.4 (g), and 1.1].

$$\begin{aligned}
&h(t) \\
&= \epsilon_0\left(-\frac{T_1(x_1x_2)}{T(x_1)}(1 \otimes \overline{x_1} \otimes 1) - \frac{T_2(x_1x_2)}{T(x_2)}(1 \otimes \overline{x_2} \otimes 1)\right) \\
&= \epsilon_0(-(1 \otimes x_2)(1 \otimes \overline{x_1} \otimes 1) - (x_1 \otimes 1)(1 \otimes \overline{x_2} \otimes 1)) \\
&= \epsilon_0(-(1 \otimes \overline{x_1} \otimes x_2) - (x_1 \otimes \overline{x_2} \otimes 1)) \\
&= -\big(1 \otimes \overline{1} \otimes \overline{x_1} \otimes x_2\big) - (1 \otimes \overline{x_1} \otimes \overline{x_2} \otimes 1) \\
&= -(1 \otimes \overline{x_1} \otimes \overline{x_2} \otimes 1)
\end{aligned}$$

We can thus conclude the following for $\overline{h}(t)$.

$$\overline{h}(t) = -1 \otimes \overline{x_1} \otimes \overline{x_2}$$

We can now evaluate $\mathrm{d}\big(\overline{h}(x_1 t)\big)$ as follows, using Proposition 6.3.1.10.

$$\begin{aligned}
&\mathrm{d}\big(\overline{h}(x_1 t)\big) \\
&= -\,\mathrm{d}(x_1 \otimes \overline{x_1} \otimes \overline{x_2}) \\
&= -1 \otimes \overline{x_1} \otimes \overline{x_1} \otimes \overline{x_2} - 1 \otimes \overline{x_1} \otimes \overline{x_2} \otimes \overline{x_1} - 1 \otimes \overline{x_2} \otimes \overline{x_1} \otimes \overline{x_1}
\end{aligned}$$

Note that $\overline{\mathrm{C}}_3(A)$ is a free k-module that has a basis that is given by elements of the following form.

$$x^{\vec{i}} \otimes \overline{x^{\vec{j_1}}} \otimes \overline{x^{\vec{j_2}}} \otimes \overline{x^{\vec{j_3}}} \text{ for } \vec{i}, \vec{j_1}, \vec{j_2}, \vec{j_3} \in \mathbb{Z}^2_{\geq 0}$$
$$\text{such that } \vec{i}, \vec{j_1}, \vec{j_2}, \vec{j_3} \ngeq (1,1) \text{ and } \vec{j_1}, \vec{j_2}, \vec{j_3} \neq \vec{0}$$

We can define a submodule J spanned by the basis elements of the above form such that there exist $1 \leq a < b \leq 3$ such that $\overrightarrow{j_a} = (1,0)$ and $\overrightarrow{j_b} = (0,1)$. In other words, J is spanned elements in which two of the last three tensor factors are x_1 and x_2, and appearing in that order. Note that $\mathrm{d}(\overline{h}(x_1 t))$ is a linear combination of three basis elements of $\overline{\mathrm{C}}_3(A)$, and while the first two lie in J, this is not the case for $1 \otimes \overline{x_2} \otimes \overline{x_1} \otimes \overline{x_1}$. This implies that $\mathrm{d}(\overline{h}(x_1 t))$ does not lie in J, so it suffices to show that the image of $\overline{h}_3$ is a submodule of J.

$\overline{R_s}(A)_3$ is generated by elements of the form $x^{\vec{i}} e_j t$ with $\vec{i} \in \mathbb{Z}^2_{\geq 0}$ and $j \in \{1,2\}$. The image of $\overline{h}_3$ is thus generated by elements of the following form, using Propositions 6.3.2.10 and 6.3.2.11.

$$\begin{aligned}
&\overline{h}_3\left(x^{\vec{i}} e_j t\right) \\
&= x^{\vec{i}} \cdot (-1 \otimes \overline{x_j}) \cdot (-1 \otimes \overline{x_1} \otimes \overline{x_2}) \\
&= x^{\vec{i}} \otimes \overline{x_j} \otimes \overline{x_1} \otimes \overline{x_2} - x^{\vec{i}} \otimes \overline{x_1} \otimes \overline{x_j} \otimes \overline{x_2} + x^{\vec{i}} \otimes \overline{x_1} \otimes \overline{x_2} \otimes \overline{x_j}
\end{aligned}$$

This shows that the image of $\overline{h}_3$ is contained in J. □

9.7. On a question of Larsen

Let n be a positive integer and f an element of $k[x_1, \ldots, x_n]$ that is monic and of positive degree when considered as a polynomial in the single variable x_1 with coefficients in $k[x_2, \ldots, x_n]$. Then Larsen constructs in [Lar95, 2.11] a strict mixed complex and asks the question whether it gives the cyclic homology of $k[x_1, \ldots, x_n]/f$, having answered this question in the affirmative for $n = 2$ in [Lar95, 2.10].

In the $n = 2$ case, what Larsen actually shows is that there is a strongly homotopy linear[22] quasiisomorphism from the strict mixed complex Larsen constructs to the normalized standard Hochschild complex. As the normalized standard Hochschild complex as well as the strict mixed complex Larsen constructs are bounded below, it follows from [Kas87, 2.3] using the argument of the proof of [Kas87, 2.6] that this strongly homotopy linear quasiisomorphism induces an isomorphism of cyclic homology groups.

By Remark 4.4.4.2, the strongly homotopy linear quasiisomorphism constructed by Larsen induces an equivalence in $\mathcal{M}$ixed, and as the normalized standard Hochschild complex represents Hochschild homology as a mixed complex by Propositions 6.3.4.1 and 6.3.1.10, this implies that Larsen's strict mixed complex represents the Hochschild homology $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, x_2]/f)$ as an object of $\mathcal{M}$ixed. Applying [Hoy18, 2.1, 2.2, and 2.3] this in turn also implies the statement regarding cyclic homology groups, without invoking [Kas87, 2.3].

[22] See Definition 4.2.3.1 for a definition. The definition stated in [Lar95, 1.4.1] differs slightly, likely due to a mistake, see a discussion in Remark 9.7.0.1.

Using Corollary 9.4.2.6 it is easy to see that the conditions stated at the start of this section for f imply that $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$. If we assume that Conjecture D holds for f, then Proposition 9.5.2.3 will thus provide a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{M}\mathrm{ixed}$.

We claim that the strict mixed complex

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

used in Proposition 9.5.2.3 and described in Construction 9.3.2.1, Construction 9.5.2.1, and Proposition 9.5.2.2 is in fact isomorphic as a strict mixed complex to the strict mixed complex constructed by Larsen in [Lar95, 2.11], so that proving Conjecture D will result in an affirmative answer to Larsen's question. This is essentially clear if one understands both definitions, but due to the very different notations used, we say some words about this.

That the underlying commutative graded k-algebras are isomorphic via an isomorphism that maps our x_i, $\mathrm{d}\, x_i$, and $t^{[m]}$ to Larsen's x_i, dx_i, and $(-1)^m z^{[m]}$ is clear by looking at [Lar95, 2.11]. Comparing the formulas for the boundary operator (denoted by b in [Lar95]) given in Construction 9.3.2.1 and [Lar95, 2.11], it is also clear that this isomorphism is compatible with the boundary operators.

The differential d is denoted by B in [Lar95], and defined in [Lar95, 2.11] by the following formula.

$$B(\alpha) \coloneqq \mathrm{d}\, \alpha + \left[df, z \frac{\partial \alpha}{\partial z} \right] \tag{9.9}$$

Let $\alpha = p(P) dx^{\vec{\epsilon}} z^{[m]}$ for $P \in k[x_1, \ldots, x_n]$, $\vec{\epsilon} \in \{0,1\}^n$, and $m \geq 0$. The summand $\mathrm{d}\, \alpha$ is then notation for $p(\mathrm{d}(r_f^0(P))) dx^{\vec{\epsilon}} z^{[m]}$, so corresponds to the first summand in the formula (9.8) in Construction 9.5.2.1.

The term $z \frac{\partial \alpha}{\partial z}$ is given by[23]

$$z \cdot \frac{\partial p(P) dx^{\vec{\epsilon}} z^{[m]}}{\partial z} = z \cdot p(P) dx^{\vec{\epsilon}} z^{[m-1]} = m \cdot p(P) dx^{\vec{\epsilon}} z^{[m]}$$

so that we are left to consider the term $\left[df, m \cdot p(P) dx^{\vec{\epsilon}} z^{[m]} \right]$.

The notation $[-,-]$ is defined in [Lar95, 2.1.1], and in our notation

$$\left[df, m \cdot p(P) dx^{\vec{\epsilon}} z^{[m]} \right]$$

corresponds to[24]

$$q_f^1 \Big(\mathrm{d}\, f \cdot r_f^0(m \cdot P) dx^{\vec{\epsilon}} z^{[m]} \Big)$$

so that the second summand in (9.9) corresponds to the second summand in (9.8) in Construction 9.5.2.1.

[23] Recall that $z^{[m]}$ is $\frac{1}{m!} z^m$.

[24] We use that $\mathrm{d}\, f$ is f-reduced.

Remark 9.7.0.1. A definition of what we call strongly homotopy linear morphisms of strict mixed complexes is given around [Lar95, 1.4.1], which however differs in signs from the one we gave in Definition 4.2.3.1, with a plus sign on the left hand side. It is noted just after [Lar95, 1.4.1] that the sign conventions differ from those of [Kas87]. However, this changed sign does not seem to be a matter of convention but rather a mistake, with the definition of [Lar95] leading to a different notion, making the results of [Kas87] inapplicable. Luckily the inductive method to construct $i^{(2k+2)}$ in [Lar95, Display between (1.4.1) and (1.4.2)] works with the correct definition (4.15), while the first step of the induction actually fails when using [Lar95, 1.4.1]. Thus the results of [Lar95] should hold with the corrected definition.

In the following we construct a morphism of chain complexes $f\colon X \to Y$ between strict mixed complexes that can be extended to a strongly homotopy linear morphism using the definition we gave in Definition 4.2.3.1 and that is also used in [Kas87, 2.2] and [Lod98, 2.5.14], but that can not be extended using the definition of [Lar95, 1.4.1], thereby showing that the sign difference is not just a matter of conventions.

Let X be the strict mixed complex whose underlying $\mathbb{Z}$-graded k-module is free with generator x in degree 0 and y in degree 1, with $\mathrm{d} = 0$ and $\partial(y) = x$. As the underlying chain complex is cofibrant and acyclic, we should expect that every chain morphism out of it can be extended to a strongly homotopy linear morphism. Indeed, this is the case with the definition we give here. Let $f\colon X \to Y$ be a morphism of chain complexes to any other strict mixed complex Y. Then setting

$$\begin{aligned} f^{(1)}(x) &:= \mathrm{d}(f(y)) \\ f^{(1)}(y) &:= 0 \\ f^{(n)} &:= 0 \qquad \text{for } n > 1 \end{aligned}$$

and extending k-linearly defines the necessary data to extend f to a strongly homotopy linear morphism, as it is easy to check that (4.15) is satisfied.

Let us now consider the strict mixed complex Y whose underlying $\mathbb{Z}$-graded k-module is free on a in degree 0, on c and e in degree 1, and on g in degree 2, with d and ∂ defined by extending k-linearly from the following definitions.

$$\begin{array}{cccc} \partial(a) := 0 & \partial(c) := a & \partial(e) := 0 & \partial(g) := -e \\ \mathrm{d}(a) := e & \mathrm{d}(c) := g & \mathrm{d}(e) := 0 & \mathrm{d}(g) := 0 \end{array}$$

The following diagram depicts the strict mixed complex Y using the conventions from Convention 4.2.1.7.

$$\begin{tikzcd} & g \arrow[dr, "-1"] & \\ c \arrow[ur, squiggly, red, "1"] \arrow[dr, "1"'] & & e \\ & a \arrow[ur, squiggly, red, "1"'] & \end{tikzcd}$$

Now define a morphism of chain complexes $f\colon X \to Y$ by k-linearly extending $f(x) \coloneqq a$ and $f(y) \coloneqq c$. Assume that $f^{(1)}$ were a morphism of $\mathbb{Z}$-graded k-modules from X to Y increasing degree by 2 and satisfying the following equation.

$$f^{(1)} \circ \partial + \partial \circ f^{(1)} = f \circ \mathrm{d} - \mathrm{d} \circ f$$

Then we obtain

$$\partial\Big(f^{(1)}(x)\Big) = f(\mathrm{d}(x)) - \mathrm{d}(f(x)) - f^{(1)}(\partial(x)) = f(0) - \mathrm{d}(a) - f^{(1)}(0) = -e$$

which implies that $f^{(1)}(x) = g$. We then need

$$\begin{aligned}\partial\Big(f^{(1)}(y)\Big) &= f(\mathrm{d}(y)) - \mathrm{d}(f(y)) - f^{(1)}(\partial(y))\\ &= f(0) - \mathrm{d}(c) - f^{(1)}(x)\\ &= -g - g = -2g\end{aligned}$$

to hold. However, if $2 \neq 0$ in k, then this is impossible, as $2g$ is then not a boundary in Y. This shows that the notion defined by [Lar95, 1.4.1] is genuinely different to the notion of strongly homotopy linear morphisms as defined in (4.15) as well as [Kas87, 2.2] and [Lod98, 2.5.14]. $\diamondsuit$

Remark 9.7.0.2. In [HN20, Theorem 1], a description is given of an object of $\mathcal{D}(\mathbb{Z})^{\mathrm{B}\mathbb{T}}$ related to $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, x_2]/f)$ for $f = x_1^a - x_2^b$ for $a, b \geq 2$ relatively prime integers. It is stated that this description follows from the results of [Lar95], but as so far there was no proof in the literature that strongly homotopy linear quasiisomorphisms induce equivalences in $\mathcal{M}\mathrm{ixed}$, this constituted a gap in [HN20], which is filled by Sections 4.2.3 and 4.4.4 and in particular Remark 4.4.4.2.[25]

If 2 is in invertible in k then one can now also use Proposition 9.5.2.3 in combination with Proposition 7.5.3.1, which gives a new proof of the statement that the strict mixed complex constructed by Larsen represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, x_2]/f)$ in $\mathcal{M}\mathrm{ixed}$. However to use this for [HN20, Theorem 1] slightly more work would be needed to also identify the decomposition – see Section 1.6 (3). $\diamondsuit$

[25] However the construction of the higher homotopies of the strongly homotopy linear map constructed in [Lar95] ultimately depends on the choice of a contracting homotopy K^t in [Lar95, Lemma 1.3]. It is unclear to the author which choice should be used as the canonical one to obtain a canonical equivalence in [HN20, Theorem 1] as claimed.

Chapter 10.

Example: $x_1^2 - x_2x_3$

Just like Proposition 8.3.0.1 was a stepping stone for Theorem A, we also view Theorem A as a stepping stone; for any particular polynomial f of interest one will most likely want to further simplify the strict mixed complex provided by Theorem A before using it as input for further calculations.

In this chapter we thus go through one relatively simple but nontrivial example in detail: Conditional on Conjecture D[1] holding for for f we describe $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}[x_1, x_2, x_3]/f)$, where f is the polynomial $f = x_1^2 - x_2x_3$ that geometrically defines a cone. We will describe the process step by step in the order one might proceed in when first working out the example.

10.1. Applying Theorem A

In order to be able to apply Theorem A, f needs to be in particular monic with respect to a chosen monomial order. While f is monic with respect to any monomial order, which one we choose matters with regards to what the degree of f will be – either x_2x_3 or x_1^2 could be chosen as the leading term.

We choose $\preceq$ to be the lexicographic monomial ordering on three variables so that x_1^2 is the leading term. We then have $\deg_{\preceq}(f) = (2, 0, 0)$, and for $\overrightarrow{i} \in \mathbb{Z}^3_{\geq 0}$ the monomial $x^{\overrightarrow{i}}$ is f-reduced if and only if $i_1 \leq 1$. We can now apply Theorem A to obtain a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1, x_2, x_3]/(x_1^2 - x_2x_3)\big)$, conditional on Conjecture D holding for for f.

Proposition 10.1.0.1. *Let $f = x_1^2 - x_2x_3$ as an element of $\mathbb{Z}[x_1, x_2, x_3]$, and assume that Conjecture D holds for f. Then there is an equivalence*

$$\begin{aligned}&\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}[x_1, x_2, x_3]/f)\\ \simeq\ &\gamma_{\mathsf{Mixed}}(\mathbb{Z}[x_1, x_2, x_3]/f \otimes \Lambda(\mathrm{d}\,x_1, \mathrm{d}\,x_2, \mathrm{d}\,x_3) \otimes \Gamma(t))\end{aligned}$$

in $\mathcal{M}\mathrm{ixed}$*, where*

$$Y \coloneqq \mathbb{Z}[x_1, x_2, x_3]/f \otimes \Lambda(\mathrm{d}\,x_1, \mathrm{d}\,x_2, \mathrm{d}\,x_3) \otimes \Gamma(t)$$

[1] That the discussion of the example in this chapter is conditional on a conjecture is of course slightly unsatisfactory, but allows us to discuss an illustrative example with nontrivial features.

is the strict mixed complex with underlying graded abelian group as indicated, with x_i of degree 0, $\mathrm{d}\,x_i$ of degree 1 and t of degree 2, and with boundary operator and differential given by the following formulas[2]*, for $a, b \geq 0$, $\vec{\epsilon} \in \{0,1\}^3$, and $m \geq 0$.*

$$\begin{aligned}
\partial\Big(p\big(x_2^a x_3^b\big)\,\mathrm{d}\,x^{\vec{\epsilon}}\, t^{[m]}\Big) &= \Big(-2\cdot p\big(x_1x_2^ax_3^b\big)\,\mathrm{d}\,x_1 + p\big(x_2^ax_3^{b+1}\big)\,\mathrm{d}\,x_2 \\
&\qquad + p\big(x_2^{a+1}x_3^b\big)\,\mathrm{d}\,x_3\Big)\cdot \mathrm{d}\,x^{\vec{\epsilon}}\, t^{[m-1]} \\
\partial\Big(p\big(x_1x_2^a x_3^b\big)\,\mathrm{d}\,x^{\vec{\epsilon}}\, t^{[m]}\Big) &= \Big(-2\cdot p\big(x_2^{a+1}x_3^{b+1}\big)\,\mathrm{d}\,x_1 + p\big(x_1x_2^ax_3^{b+1}\big)\,\mathrm{d}\,x_2 \\
&\qquad + p\big(x_1x_2^{a+1}x_3^b\big)\,\mathrm{d}\,x_3\Big)\cdot \mathrm{d}\,x^{\vec{\epsilon}}\, t^{[m-1]} \\
\mathrm{d}\Big(p\big(x_2^a x_3^b\big)\,\mathrm{d}\,x^{\vec{\epsilon}}\, t^{[m]}\Big) &= \Big(a\cdot p\big(x_2^{a-1}x_3^b\big)\,\mathrm{d}\,x_2 \\
&\qquad + b\cdot p\big(x_2^ax_3^{b-1}\big)\,\mathrm{d}\,x_3\Big)\cdot \mathrm{d}\,x^{\vec{\epsilon}}\, t^{[m]} \\
\mathrm{d}\Big(p\big(x_1x_2^a x_3^b\big)\,\mathrm{d}\,x^{\vec{\epsilon}}\, t^{[m]}\Big) &= \Big((1+2m)\cdot p\big(x_2^ax_3^b\big)\,\mathrm{d}\,x_1 + a\cdot p\big(x_1x_2^{a-1}x_3^b\big)\,\mathrm{d}\,x_2 \\
&\qquad + b\cdot p\big(x_1x_2^ax_3^{b-1}\big)\,\mathrm{d}\,x_3\Big)\,\mathrm{d}\,x^{\epsilon}t^{[m]}
\end{aligned}$$

In the formulas above, terms involving negative exponents of a variable are to be interpreted as 0. ♡

Proof. As x_1 is the only variable occurring in the leading term of f and the exponent of x_1 in the other term x_2x_3 is 0, the assumptions of Theorem A are satisfied, so that it suffices to check that the formulas for ∂ and d from Theorem A specialize to the ones given in the statement above. We have

$$\mathrm{d}\,f = 2x_1\,\mathrm{d}\,x_1 - x_3\,\mathrm{d}\,x_2 - x_2\,\mathrm{d}\,x_3$$

so the two formulas for ∂ follow directly from their description in Theorem A, where in the second formula we need only note that $p(x_1^2x_2^ax_3^b) = p(x_2^{a+1}x_3^{b+1})$.

The formula for d from Theorem A is as follows, for $\eta \in \{0,1\}$.

$$\mathrm{d}\Big(p\big(x_1^\eta x_2^a x_3^b\big)\,\mathrm{d}\,x^{\epsilon} t^{[m]}\Big) = \Big(p\big(\mathrm{d}\big(x_1^\eta x_2^a x_3^b\big)\big) + m\cdot p\big(q_f^1\big(\mathrm{d}\,f\cdot x_1^\eta x_2^a x_3^b\big)\big)\Big)\,\mathrm{d}\,x^{\epsilon} t^{[m]}$$

As the maximum exponent of x_1 occurring in $x_2^ax_3^b$ and $\mathrm{d}\,f$ is 0 and 1, respectively, $\mathrm{d}\,f\cdot x_2^ax_3^b$ is f-reduced and thus $q_f^1(\mathrm{d}\,f\cdot x_2^ax_3^b) = 0$, so that the first formula for d follows.

[2] We use p as notation for the quotient morphism $\mathbb{Z}[x_1,x_2,x_3] \to \mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)$, like in Construction 9.3.2.1

For the second formula for d, we first note that

$$\begin{aligned}
\mathrm{d}\, f \cdot x_1 x_2^a x_3^b &= 2x_1^2 x_2^a x_3^b \,\mathrm{d}\, x_1 - x_1 x_2^a x_3^{b+1} \,\mathrm{d}\, x_2 - x_1 x_2^{a+1} x_3^b \,\mathrm{d}\, x_3 \\
&= \big(2x_2^a x_3^b \,\mathrm{d}\, x_1\big) \cdot f + 2x_2^{a+1} x_3^{b+1} \,\mathrm{d}\, x_1 \\
&\quad - x_1 x_2^a x_3^{b+1} \,\mathrm{d}\, x_2 - x_1 x_2^{a+1} x_3^b \,\mathrm{d}\, x_3
\end{aligned}$$

which implies that

$$q_f^1\big(\mathrm{d}\, f \cdot x_1 x_2^a x_3^b\big) = 2x_2^a x_3^b \,\mathrm{d}\, x_1$$

The following calculation then shows the second formula for d from the statement.

$$\begin{aligned}
&\mathrm{d}\Big(p\big(x_1 x_2^a x_3^b\big) \,\mathrm{d}\, x^\epsilon t^{[m]}\Big) \\
&= \Big(p\big(\mathrm{d}\big(x_1 x_2^a x_3^b\big)\big) + m \cdot p\big(2x_2^a x_3^b \,\mathrm{d}\, x_1\big)\Big) \,\mathrm{d}\, x^\epsilon t^{[m]} \\
&= \Big(p\big(x_2^a x_3^b\big) \,\mathrm{d}\, x_1 + a \cdot p\big(x_1 x_2^{a-1} x_3^b\big) \,\mathrm{d}\, x_2 + b \cdot p\big(x_1 x_2^a x_3^{b-1}\big) \,\mathrm{d}\, x_3 \\
&\quad + 2m \cdot p\big(x_2^a x_3^b\big) \,\mathrm{d}\, x_1\Big) \,\mathrm{d}\, x^\epsilon t^{[m]} \\
&= \Bigg((1+2m) \cdot p\big(x_2^a x_3^b\big) \,\mathrm{d}\, x_1 + a \cdot p\big(x_1 x_2^{a-1} x_3^b\big) \,\mathrm{d}\, x_2 \\
&\quad + b \cdot p\big(x_1 x_2^a x_3^{b-1}\big) \,\mathrm{d}\, x_3\Bigg) \,\mathrm{d}\, x^\epsilon t^{[m]}
\end{aligned}$$

□

10.2. Comparison with the mixed complex of de Rham forms

To describe Y it will be useful to compare it to the mixed complex of de Rham forms. We first note the following about $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$.

Remark 10.2.0.1. It follows from [Wei94, 9.2.7] that the identification

$$\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]\,/\,\mathbb{Z}} \cong \mathbb{Z}[x_1,x_2,x_3] \otimes \Lambda(\mathrm{d}\, x_1, \mathrm{d}\, x_2, \mathrm{d}\, x_3)$$

from Section 7.1 induces an isomorphism

$$\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}} \cong (\mathbb{Z}[x_1,x_2,x_3]/f \otimes \Lambda(\mathrm{d}\, x_1, \mathrm{d}\, x_2, \mathrm{d}\, x_3))/\,\mathrm{d}\, f$$

of strict mixed complexes[3]. ♢

We next define a morphism

$$Y \to \Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$$

of strict mixed complexes.

[3] The boundary operators are zero, and the differential d maps x_i to $\mathrm{d}\, x_i$ and satisfies the Leibniz rule.

Definition 10.2.0.2. Consider the following morphism of graded abelian groups.

$$\varphi\colon Y \to \Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$$
$$p\left(x^{\vec{i}}\right) \mathrm{d}\, x^{\vec{\epsilon}} t^{[m]} \mapsto \begin{cases} 0 & m > 0 \\ p(x^{\vec{i}}) \, \mathrm{d}\, x^{\vec{\epsilon}} & m = 0 \end{cases} \quad \text{for } \vec{i} \in \mathbb{Z}^3_{\geq 0}, \vec{\epsilon} \in \{0,1\}^3, m \geq 0$$

It is clear from Proposition 10.1.0.1 that φ is compatible with the chain complex and mixed structure so that φ is a morphism of strict mixed complexes.

We furthermore define the morphism of strict mixed complexes

$$\psi\colon K \to Y$$

to be the kernel of φ. $\diamondsuit$

10.3. Grading

To make it easier to discuss K and Y, we equip them with a $\mathbb{Z}^2_{\geq 0}$-grading.

Construction 10.3.0.1. We upgrade $\mathbb{Z}[x_1, x_2, x_3]$ to a $\mathbb{Z}^2_{\geq 0}$-graded ring by declaring $\deg_{\mathrm{gr}}(x_1) = (1,1)$, $\deg_{\mathrm{gr}}(x_2) = (2,0)$, and $\deg_{\mathrm{gr}}(x_3) = (0,2)$. This makes f into a homogeneous polynomial of grading $\deg_{\mathrm{gr}}(f) = (2,2)$, so $\mathbb{Z}[x_1, x_2, x_3]/(f)$ inherits a grading where $\deg_{\mathrm{gr}}(p(x^{\vec{i}})) = \deg_{\mathrm{gr}}(r^0_f(x^{\vec{i}}))$ (note that f being homogeneous ensures that $r^0_f(x^{\vec{i}})$ is homogeneous). Declaring $\deg_{\mathrm{gr}}(\mathrm{d}\, x_i) = \deg_{\mathrm{gr}}(x_i)$ and $\deg_{\mathrm{gr}}(t^{[m]}) = m \cdot (2,2)$ makes both Y and $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$ into $\mathbb{Z}^2_{\geq 0}$-graded strict mixed complexes, as one can easily see by inspecting the formulas for ∂ and d in Proposition 10.1.0.1. Furthermore, $\varphi\colon Y \to \Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$ clearly respects the grading, so the kernel K obtains an induced grading, making $\psi\colon K \to Y$ into a morphism of $\mathbb{Z}^2_{\geq 0}$-graded strict mixed complexes as well.

Let us denote the sub-mixed-complex of Y (of $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$, of K) of homogeneous elements of grading $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ by $Y(\vec{j})$ (by $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}(\vec{j})$, by $K(\vec{j})$), so that we obtain a sum decomposition as a strict mixed complex

$$Y \cong \bigoplus_{\vec{j} \in \mathbb{Z}^2_{\geq 0}} Y\left(\vec{j}\right)$$

and similarly for $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$ and K. $\diamondsuit$

Remark 10.3.0.2. Note that the additive submonoid of $\mathbb{Z}^2_{\geq 0}$ generated by $(1,1)$, $(2,0)$, and $(0,2)$ is not equal to all of $\mathbb{Z}^2_{\geq 0}$; it contains precisely those elements (a, b) for which the sum $a + b$ is even[4]. It follows that

$$Y(\overrightarrow{j}) \cong \Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}(\overrightarrow{j}) \cong K(\overrightarrow{j}) \cong 0$$

if $\overrightarrow{j} \in \mathbb{Z}^2_{\geq 0}$ such that $j_1 + j_2$ is odd.

Note that the mixed complexes $Y(\overrightarrow{j})$ for $\overrightarrow{j} \in \mathbb{Z}^2_{\geq 0}$ such that $j_1 + j_2$ is even might look different depending on the parity of j_1; In the even case, x_1 and $\mathrm{d}\, x_1$ must always "occur together", while in the odd case they never do. Indeed, one consequence is that the summand $(1 + 2m) \cdot p\big(x_2^a x_3^b\big)\, \mathrm{d}\, x_1$ in the second formula for d in Proposition 10.1.0.1 vanishes in the even case, as $\mathrm{d}\, x_1 \cdot \mathrm{d}\, x_1 = 0$. $\diamondsuit$

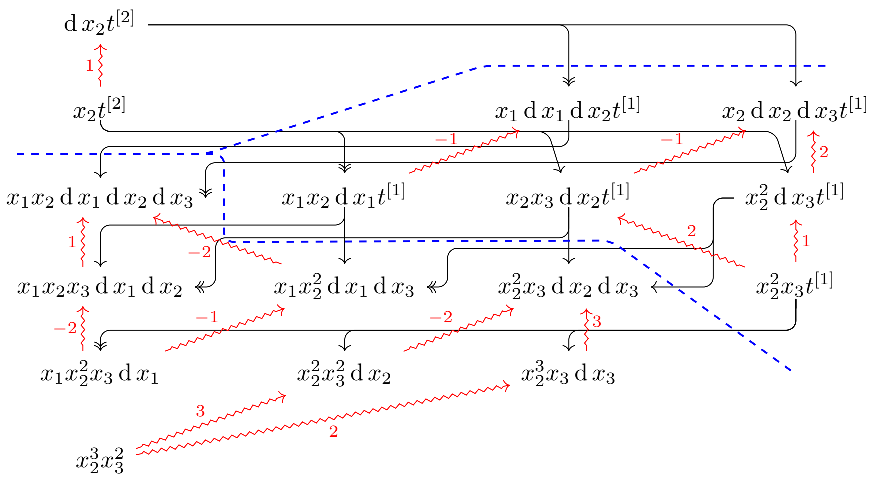
10.4. Non-diagonal pieces

10.4.1. A first look at $Y((6, 4))$ and $Y((7, 5))$

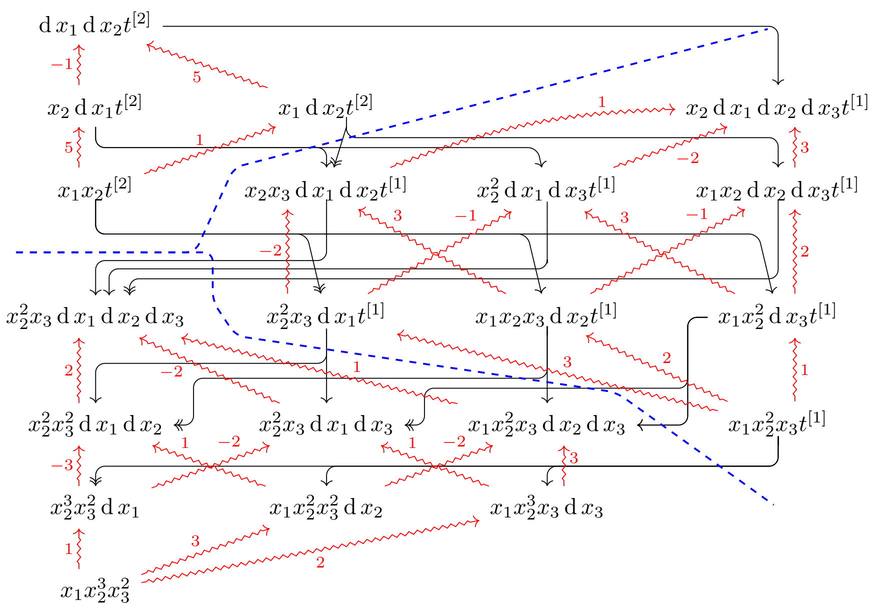
We will next look at two illustrative examples to understand the mixed complexes $Y(\overrightarrow{j})$ better, one where j_1 is even and one where it is odd. We will depict the strict mixed complexes diagrammatically in the manner introduced in Convention 4.2.1.7, with respect to the basis given by elements of the form $p(x^{\overrightarrow{i}})\, \mathrm{d}\, x^{\overrightarrow{\epsilon}} t^{[m]}$. In this basis, the components of ∂ all have absolute value 0, 1, or 2. To make the diagram more readable, we omit the labels to the respective arrows and instead use a normal arrowhead to indicate an absolute value of 1, and a double arrowhead to indicate an absolute value of 2, while not indicating the sign to avoid overloading the diagram. We also omit p from the notation and write e. g. $x_2^3 x_3^2$ instead of $p(x_2^3 x_3^2)$.

We first consider $Y((6, 4))$.

[4] This is obviously an additive condition, so as it holds for the three generators it holds for the full submonoid. On the other hand, if $(a, b) \in \mathbb{Z}^2_{\geq 0}$ with $a + b = 2c$ even, and without loss of generality say $b > a$, then $(a, b) = a \cdot (1, 1) + (c - a) \cdot (0, 2)$.



Next, the following diagram depicts $Y((7,5))$ as representative of the odd case.



Looking at these diagrams we can see that in both cases we can split of a large acyclic subcomplex (ignoring the mixed structure for now). Let us discuss the first case $Y((6,4))$. Starting from the top, we can first replace the basis element $p(x_2)\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3t^{[1]}$ with the following element

$$\partial(\mathrm{d}\,x_2t^{[2]}) = -p(x_2)\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3t^{[1]} - 2\cdot p(x_1)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2t^{[1]}$$

Then $\mathrm{d}\,x_2 t^{[2]}$ and the new basis element generate a subcomplex that splits off as an acyclic summand. Continuing downward, we can replace $p(x_2^2)\,\mathrm{d}\,x_3 t^{[1]}$ with $\partial(p(x_2)t^{[2]})$, and so on. In the end, the only basis elements that "survive" are $p(x_2^3 x_3^2)$, $p(x_1 x_2^2 x_3)\,\mathrm{d}\,x_1$, $p(x_2^2 x_3^2)\,\mathrm{d}\,x_2$, and $p(x_1 x_2 x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2$.

10.4.2. A new basis

In general, we would like to do the following. For $a, \epsilon_1, \epsilon_2 \in \{0,1\}$ and $b, c, m \geq 0$, we would like to replace the basis element

$$p(x_1^a x_2^b x_3^c)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2}\,\mathrm{d}\,x_3 t^{[m]}$$

of $Y(\overrightarrow{j})$ by the element $\partial(p(x_1^a x_2^{b-1} x_3^c)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} t^{[m+1]})$. Roughly, we divide by $x_2\,\mathrm{d}\,x_3$, increase the divided power of t by one, and then take the boundary. This is of course not possible if $b = 0$. So when could $b = 0$ happen? If $p(x_1^a x_3^c)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2}\,\mathrm{d}\,x_3 t^{[m]}$ is in $Y(\overrightarrow{j})$, then we have $j_1 = 2m + a + \epsilon_1 + 2\epsilon_2$ and $j_2 = 2m + a + 2c + \epsilon_1 + 2$. As $\epsilon_2 \leq 1$ this implies that such an element can only occur in $Y(\overrightarrow{j})$ if $j_1 \leq j_2$.

So we are lead to distinguish three cases: For $Y(\overrightarrow{j})$ with $j_1 > j_2$, we can "eliminate" basis elements divisible by $\mathrm{d}\,x_3$, and for $Y(\overrightarrow{j})$ with $j_1 < j_2$, we can analogously "eliminate" basis elements divisible by $\mathrm{d}\,x_2$, leaving the case of $Y(\overrightarrow{j})$ with $j_1 = j_2$ to still be analyzed (and which will indeed turn out to be more interesting).

We will now carry out the idea we just sketched and first construct the indicated new basis for $Y(\overrightarrow{j})$ for $\overrightarrow{j} \in \mathbb{Z}_{\geq 0}^2$ with $j_1 \neq j_2$. We will then be able to use this to show that $K(\overrightarrow{j})$ is acyclic.

Definition 10.4.2.1. To ease notation in the following, we make the following definitions for $\overrightarrow{j} \in \mathbb{Z}_{\geq 0}^2$.

$$\begin{aligned}
V &= \{0,1\} \times \mathbb{Z}_{\geq 0}^2 \times \{0,1\}^3 \times \mathbb{Z}_{\geq 0} \\
V' &= \{0,1\} \times \mathbb{Z}_{\geq 0}^2 \times \{0,1\}^2 \times \mathbb{Z}_{\geq 0} \\
V\left(\overrightarrow{j}\right) &= \Big\{ (a,b,c,\epsilon_1,\epsilon_2,\epsilon_3,m) \in V \;\Big|\; \\
&\qquad \deg_{\mathrm{gr}}\Big(p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2}\,\mathrm{d}\,x_3^{\epsilon_3} t^{[m]}\Big) = \overrightarrow{j} \Big\} \\
V_2\left(\overrightarrow{j}\right) &= \Big\{ (a,b,c,\epsilon_1,\epsilon_2,m) \in V' \;\Big|\; \\
&\qquad \deg_{\mathrm{gr}}\Big(p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} t^{[m]}\Big) = \overrightarrow{j} \Big\} \\
V_3\left(\overrightarrow{j}\right) &= \Big\{ (a,b,c,\epsilon_1,\epsilon_3,m) \in V' \;\Big|\; \\
&\qquad \deg_{\mathrm{gr}}\Big(p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_3^{\epsilon_3} t^{[m]}\Big) = \overrightarrow{j} \Big\}
\end{aligned}$$

$\diamondsuit$

Proposition 10.4.2.2. *Let $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ with $j_1 > j_2$. Then the set*

$$\begin{aligned}\mathcal{B}_2(\vec{j}) = &\left\{\, p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} t^{[m]} \,\middle|\, (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}\,) \right\}\\ &\cup \left\{\, \partial\Big(p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} t^{[m]}\Big) \,\middle|\, (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}\,), m > 0 \right\}\end{aligned}$$

forms a basis of $Y(\vec{j}\,)$. Analogously, let $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ with $j_1 < j_2$. Then the set

$$\begin{aligned}\mathcal{B}_3(\vec{j}) = &\left\{\, p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_3^{\epsilon_3} t^{[m]} \,\middle|\, (a,b,c,\epsilon_1,\epsilon_3,m) \in V_3(\vec{j}\,) \right\}\\ &\cup \left\{\, \partial\Big(p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_3^{\epsilon_3} t^{[m]}\Big) \,\middle|\, (a,b,c,\epsilon_1,\epsilon_3,m) \in V_3(\vec{j}\,), m > 0 \right\}\end{aligned}$$

forms a basis of $Y(\vec{j}\,)$. ♡

Proof. We only discuss the statement for $j_1 > j_2$, the other is completely analogous. We will refer to the basis given by elements of the form $p(x^{\vec{i}}\,)\,\mathrm{d}\,x^{\vec{\epsilon}} t^{[m]}$ used up to now as the *monomial basis*. We wrote $\mathcal{B}_2(\vec{j}\,)$ as a union, and will call elements of the first set elements of the first type and elements of the second set elements of the second type.

Note that the monomial basis can be written as follows, following the discussion before Definition 10.4.2.1 showing that any element of the monomial basis divisible by $\mathrm{d}\,x_3$ must have x_2 as a factor as well.

$$\begin{aligned}&\left\{\, p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} t^{[m]} \,\middle|\, (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}\,) \right\}\\ \cup &\left\{\, p\big(x_1^a x_2^{b+1} x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2}\,\mathrm{d}\,x_3 t^{[m-1]} \,\middle|\, (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}\,), m > 0 \right\}\end{aligned}$$

In this subdivision of the basis elements of the monomial basis, the first subset is exactly equal to the elements of $\mathcal{B}_2(\vec{j}\,)$ of the first type.

For the elements of the second type we note that for $(a,b,c,\epsilon_1,\epsilon_2,m)$ an element of $V_2(\vec{j}\,)$ with $m > 0$, they have the following form.

$$\begin{aligned}&\partial\Big(p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} t^{[m]}\Big)\\ = &\,(-1)^{\epsilon_1+\epsilon_2} p\big(x_1^a x_2^{b+1} x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2}\,\mathrm{d}\,x_3 t^{[m-1]}\\ &(-1)^{\epsilon_1} p\big(x_1^a x_2^b x_3^{c+1}\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2+1} t^{[m-1]} - 2p\big(x_1^{a+1} x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1+1}\,\mathrm{d}\,x_2^{\epsilon_2} t^{[m-1]}\end{aligned}$$

Note that the first summand is always the negative of the corresponding (also indexed by $(a,b,c,\epsilon_1,\epsilon_2,m)$) basis element of second type in the monomial basis, while the other two summands are multiples of elements of the first type. This shows the claim. □

10.4.3. Non-diagonal pieces of K are acyclic

Proposition 10.4.3.1. *Let $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ with $j_1 \neq j_2$. Then $K(\vec{j}\,)$ is acyclic.* ♡

Proof. We again only discuss the case $j_1 > j_2$, as the other case is completely analogous.

Using Remark 10.2.0.1 and the same kind of argument as in the proof of Proposition 10.4.2.2 shows that

$$\left\{ p\big(x_1^a x_2^b x_3^c\big) \,\mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} \;\middle|\; (a, b, c, \epsilon_1, \epsilon_2, 0) \in V_2(\vec{j}\,) \right\}$$

is a basis of $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}(\vec{j}\,)$. It thus follows immediately from Proposition 10.4.2.2 that

$$\begin{aligned} &\left\{ p\big(x_1^a x_2^b x_3^c\big) \,\mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a, b, c, \epsilon_1, \epsilon_2, m) \in V_2(\vec{j}\,), m > 0 \right\} \\ \cup &\left\{ \partial\Big(p\big(x_1^a x_2^b x_3^c\big) \,\mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\Big) \;\middle|\; (a, b, c, \epsilon_1, \epsilon_2, m) \in V_2(\vec{j}\,), m > 0 \right\} \end{aligned}$$

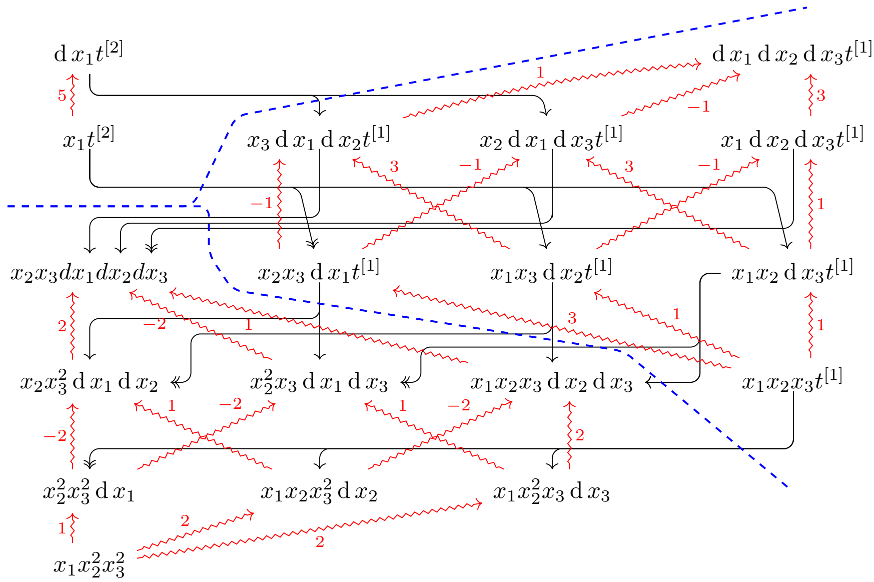
is a basis for $K(\vec{j}\,)$. We can thus easily define a contracting homotopy h of $K(\vec{j}\,)$ as follows, where $(a, b, c, \epsilon_1, \epsilon_2, m) \in V_2(\vec{j}\,)$ with $m > 0$.

$$\begin{aligned} h\Big(p\big(x_1^a x_2^b x_3^c\big) \,\mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\Big) &\coloneqq 0 \\ h\Big(\partial\Big(p\big(x_1^a x_2^b x_3^c\big) \,\mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\Big)\Big) &\coloneqq p\big(x_1^a x_2^b x_3^c\big) \,\mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \end{aligned}$$ □

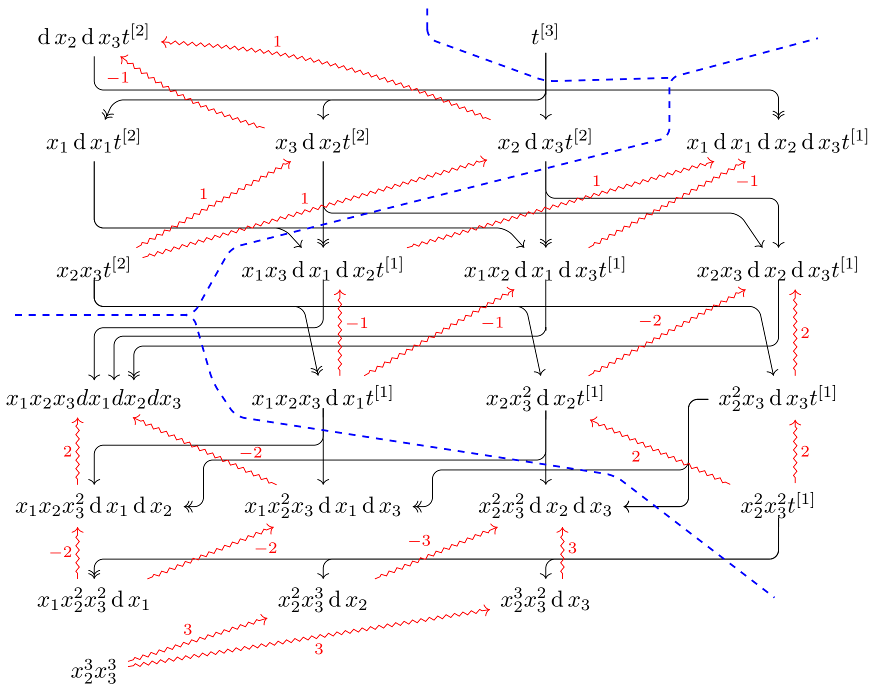
10.5. Diagonal pieces

10.5.1. A first look at $Y((5,5))$ and $Y((6,6))$

Let us now look at what happens when when $j_1 = j_2$. The following is the diagram for $Y((5,5))$. We use the same conventions as we did for $Y((6,4))$ and $Y((7,5))$ above.



As mentioned in Remark 10.3.0.2, $Y((j,j))$ may differ in character depending on the parity of j, so let us also look at $Y((6,6))$.



We can already see the difference between these two cases as well as $Y(\vec{j})$ with $j_1 \neq j_2$ in these two examples. Indeed, note how in the diagrams for both $Y((5,5))$ and $Y((6,6))$ the upper element in the rightmost column represents a nonzero element in the homology of K, showing that $K(\vec{j})$ is in general not acyclic for $j_1 = j_2$, in contrast to the case $j_1 \neq j_2$ (see Proposition 10.4.3.1). In $Y((6,6))$ this element in homology is of order 2, in contrast to $Y((5,5))$, where it is of infinite order.

10.5.2. A new basis

To simplify $Y(\vec{j})$ for $j_1 = j_2$ we make a similar base change as we did for $j_1 \neq j_2$. We again try to eliminate replace basis elements from the monomial basis that are divisible by $\mathrm{d}\, x_3$, as in Proposition 10.4.2.2. This time, we will not be able to write all of the relevant elements as boundaries, however the formulas themselves still make sense.

Notation 10.5.2.1. Let $j \geq 0$ be an integer and $(a, b, c, \epsilon_1, \epsilon_2, 1, m)$ an element of $V((j,j))$. We will define an element $b_{(a,b,c,\epsilon_1,\epsilon_2,m)}$ of $Y((j,j))$, by distinguishing three cases. If $b > 0$, then we define $b_{(a,b,c,\epsilon_1,\epsilon_2,m)}$ as follows.

$$\begin{aligned}
b_{(a,b,c,\epsilon_1,\epsilon_2,m)} &\coloneqq \partial\Big(p\big(x_1^a x_2^{b-1} x_3^c\big)\,\mathrm{d}\, x_1^{\epsilon_1}\,\mathrm{d}\, x_2^{\epsilon_2} t^{[m+1]}\Big) \\
&= (-1)^{\epsilon_1+\epsilon_2} p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\, x_1^{\epsilon_1}\,\mathrm{d}\, x_2^{\epsilon_2}\,\mathrm{d}\, x_3 t^{[m]} \\
&\quad + (-1)^{\epsilon_1} p\big(x_1^a x_2^{b-1} x_3^{c+1}\big)\,\mathrm{d}\, x_1^{\epsilon_1}\,\mathrm{d}\, x_2^{1+\epsilon_2} t^{[m]} \\
&\quad - 2p\big(x_1^{a+1} x_2^{b-1} x_3^c\big)\,\mathrm{d}\, x_1^{1+\epsilon_1}\,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}
\end{aligned}$$

If instead $b = 0$, then note that this implies $\epsilon_2 = 1$ and $c = 0$. We then make the following definitions.

$$\begin{aligned}
b_{(0,0,0,\epsilon_1,1,m)} &\coloneqq (-1)^{\epsilon_1+1}\,\mathrm{d}\, x_1^{\epsilon_1}\,\mathrm{d}\, x_2\,\mathrm{d}\, x_3 t^{[m]} \\
b_{(1,0,0,\epsilon_1,1,m)} &\coloneqq (-1)^{\epsilon_1+1} p(x_1)\,\mathrm{d}\, x_1^{\epsilon_1}\,\mathrm{d}\, x_2\,\mathrm{d}\, x_3 t^{[m]} - 2p(x_3)\,\mathrm{d}\, x_1^{1+\epsilon_1}\,\mathrm{d}\, x_2 t^{[m]} \quad \diamondsuit
\end{aligned}$$

Proposition 10.5.2.2. *Let $j \geq 0$. Then the following form a basis for $Y((j,j))$.*

$$\begin{aligned}
\mathcal{B}((j,j)) = \Big\{\, & p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\, x_1^{\epsilon_1}\,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\Big|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2((j,j)) \,\Big\} \\
& \cup \Big\{\, b_{(a,b,c,\epsilon_1,\epsilon_2,m)} \;\Big|\; (a,b,c,\epsilon_1,\epsilon_2,1,m) \in V((j,j)) \,\Big\} \qquad \heartsuit
\end{aligned}$$

Proof. The proof is very similar to Proposition 10.4.2.2. The monomial basis can be written as follows.

$$\begin{aligned}
& \Big\{\, p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\, x_1^{\epsilon_1}\,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\Big|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}) \,\Big\} \\
\cup & \Big\{\, p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\, x_1^{\epsilon_1}\,\mathrm{d}\, x_2^{\epsilon_2}\,\mathrm{d}\, x_3 t^{[m]} \;\Big|\; (a,b,c,\epsilon_1,\epsilon_2,1,m) \in V((j,j)) \,\Big\}
\end{aligned}$$

Again the elements of $\mathcal{B}((j,j))$ of the first type correspond to elements of the monomial basis of the first type, and the element of the second type indexed by $(a, b, c, \epsilon_1, \epsilon_2, 1, m)$ is – up to sign – the sum of the corresponding element of the second type indexed by $(a, b, c, \epsilon_1, \epsilon_2, 1, m)$ of the monomial basis and a linear combination of elements of the first type. $\square$

We can record the following behavior of the new basis with respect to the boundary operator.

Proposition 10.5.2.3. *Let $j \geq 0$. Then the following holds in $Y((j,j))$ for elements $(a, b, c, \epsilon_1, \epsilon_2, m)$ of $V_2((j,j))$.*

$$\partial\Big(p\big(x_1^a x_2^b x_3^c\big) \,\mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\Big) = \begin{cases} 0 & \text{if } m = 0 \\ b_{(a,b+1,c,\epsilon_1,\epsilon_2,m-1)} & \text{if } m > 0 \end{cases}$$

Furthermore, the following holds for $(a, b, c, \epsilon_1, \epsilon_2, 1, m) \in V((j,j))$.

$$\partial\big(b_{(a,b,c,\epsilon_1,\epsilon_2,m)}\big) = \begin{cases} 2 \cdot b_{(1,0,0,1,1,m-1)} & \text{if } (a, b, \epsilon_1) = (0,0,0) \text{ and } m > 0 \\ 0 & \text{otherwise} \end{cases}$$

$\heartsuit$

Proof. The first formula follows immediately from the definitions in Notation 10.5.2.1. The second formula follows from $\partial^2 = 0$ if $b > 0$ and from Proposition 10.1.0.1 if $m = 0$. So we can assume that $b = 0$ and $m > 0$. We distinguish three cases: first $\epsilon_1 = 1$, then $(a, \epsilon_1) = (0, 0)$, and finally $(a, \epsilon_1) = (1, 0)$. In each case the formula follows by writing out the elements and using Proposition 10.1.0.1[5].

$$\begin{aligned}
\partial\big(b_{(a,0,0,1,1,m)}\big) &= \partial\Big(p(x_1^a) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \,\mathrm{d}\, x_3 t^{[m]}\Big) = 0 \\
\partial\big(b_{(0,0,0,0,1,m)}\big) &= \partial\Big(-\,\mathrm{d}\, x_2 \,\mathrm{d}\, x_3 t^{[m]}\Big) \\
&= 2p(x_1) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \,\mathrm{d}\, x_3 t^{[m-1]} = 2 \cdot b_{(1,0,0,1,1,m-1)} \\
\partial\big(b_{(1,0,0,0,1,m)}\big) &= \partial\Big(-p(x_1) \,\mathrm{d}\, x_2 \,\mathrm{d}\, x_3 t^{[m]} - 2p(x_3) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 t^{[m]}\Big) \\
&= 2p(x_2x_3) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \,\mathrm{d}\, x_3 t^{[m-1]} \\
&\quad - 2p(x_2x_3) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \,\mathrm{d}\, x_3 t^{[m-1]} \\
&= 0
\end{aligned}$$

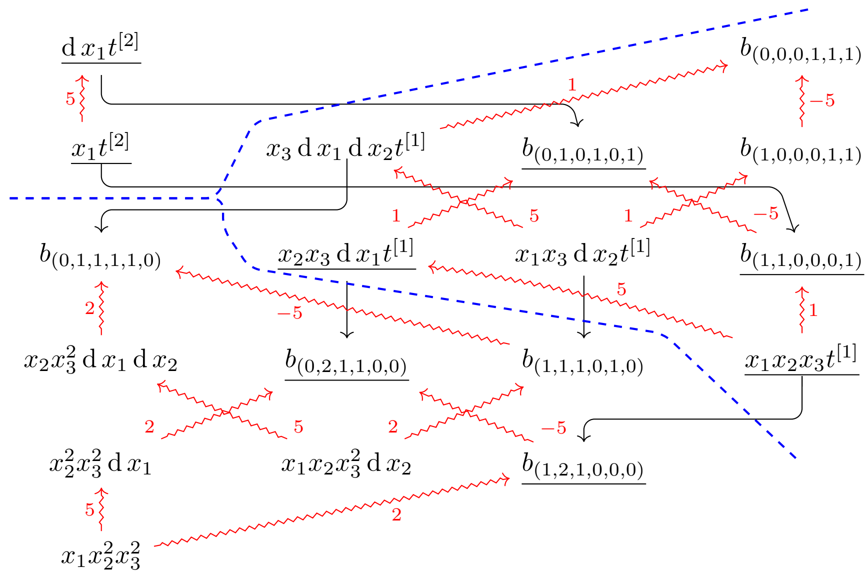
$\square$

Note that we can see a distinction between the cases of $Y((j,j))$ with j odd and even in Proposition 10.5.2.3; the first (non-zero) case in the formula for $\partial\big(b_{(a,b,c,\epsilon_1,\epsilon_2,m)}\big)$ only occurs if j is even.

[5] Note that $b = 0$ implies $\epsilon_2 = 1$ and $c = 0$.

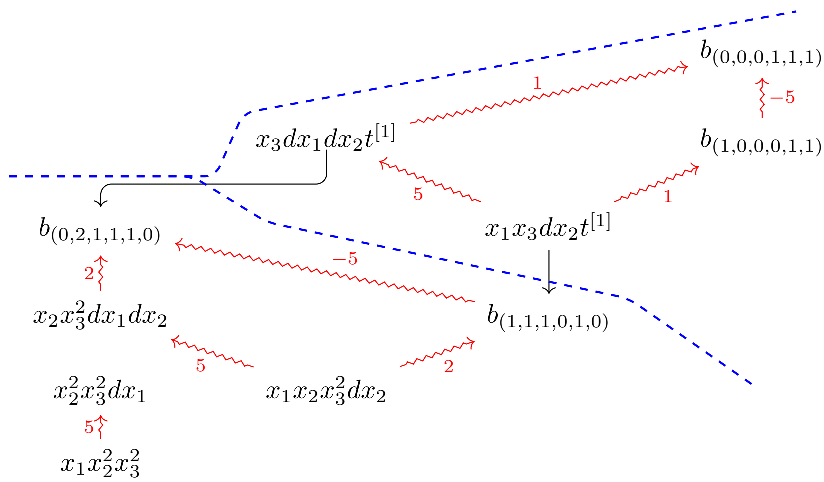
10.5.3. Another look at $Y((5,5))$

We can now look at $Y((5,5))$ again, but in this new basis.



(10.1)

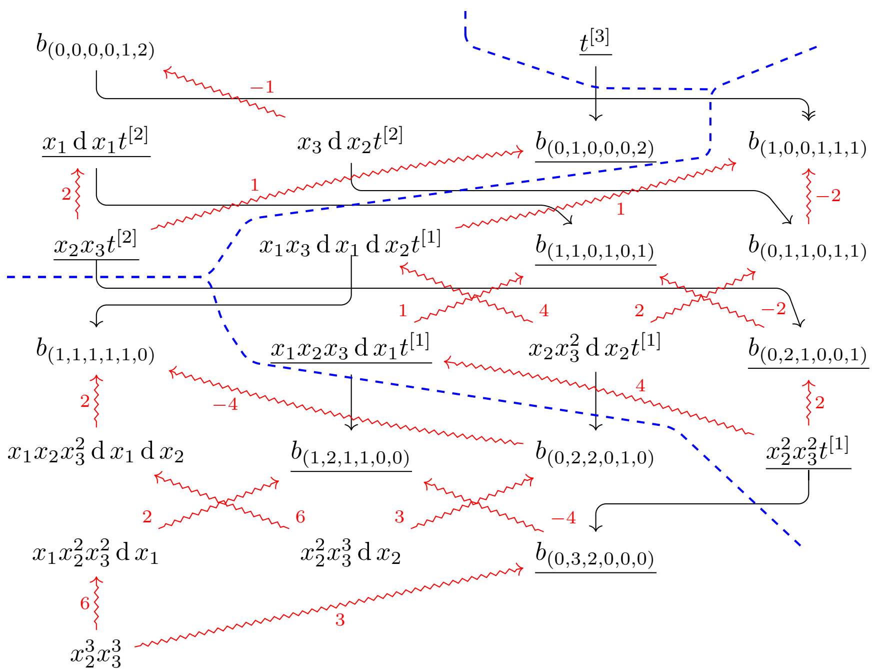
Note that the $\mathbb{Z}$-graded-abelian group generated by the underlined basis elements is closed under both boundary operator and differential. It is also acyclic, so the the quotient map from $Y((5,5))$ obtained by dividing out this sub-mixed-complex is a quasiisomorphism. The following diagram depicts the resulting strict mixed complex.



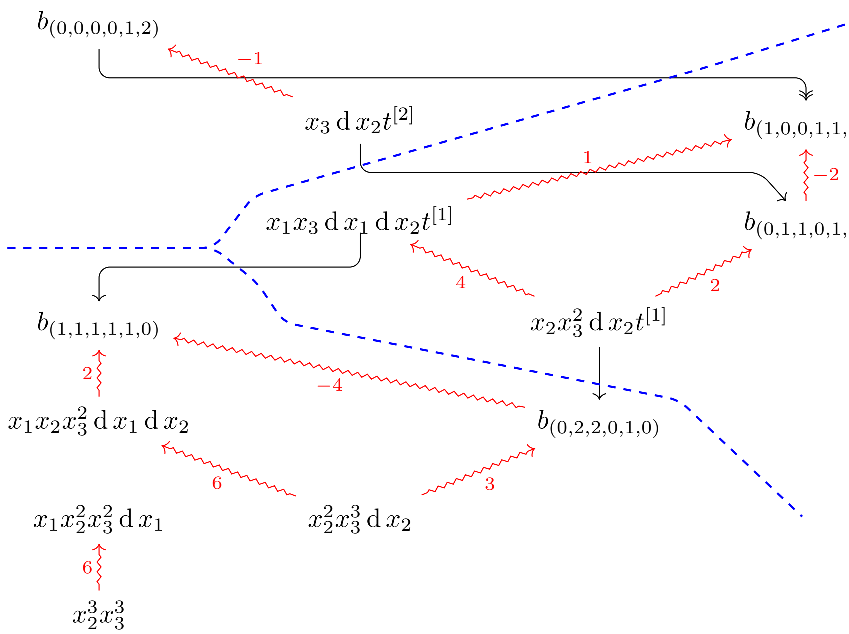
From this we can read off that $K((5,5))$ will not be acyclic, but rather equivalent to the strict mixed subcomplex generated by $b_{(1,0,0,0,1,1)}$ and $b_{(0,0,0,1,1,1)}$, which is isomorphic to $\mathrm{D}_{-5}[4]$, where we use notation from Definition 4.2.1.5.

10.5.4. Another look at $Y((6,6))$

Let us now consider the even case. The following diagram depicts $Y((6,6))$ in the basis from Proposition 10.5.2.2.



We again underlined basis elements that generate an acyclic sub-mixed-complex that we can divide out, obtaining the strict mixed complex depicted in the diagram below.



This time we see that $K((6,6))$ will be equivalent to $\mathbb{Z}/2[5]$, generated by $b_{(1,0,0,1,1,1)}$.

10.5.5. A basis for $K((j,j))$

We will now show how the description above of $K((j,j))$ generalizes to $j \geq 5$ other than 5 and 6, whereas $K((j,j))$ for $j < 5$ is acyclic. We start by describing a basis of $K((j,j))$.

Proposition 10.5.5.1. *Let $j \geq 5$. Then a basis of $K((j,j))$ is given by the following set.*

$$\left\{ p\big(x_1^a x_2^b x_3^c\big) \,\mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2((j,j)), m > 0 \right\}$$
$$\cup \left\{ b_{(a,b,c,\epsilon_1,\epsilon_2,m)} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,1,m) \in V((j,j)) \right\}$$

Furthermore, $K((0,0)) \cong 0$, $K((1,1)) \cong 0$, a basis of $K((2,2))$ is given by

$$\left\{ b_{(0,1,0,0,0,0)}, t^{[1]} \right\}$$

a basis of $K((3,3))$ is given by

$$\left\{ b_{(1,1,0,0,0,0)}, b_{(0,1,0,1,0,0)}, x_1 t^{[1]}, \mathrm{d}\, x_1 t^{[1]} \right\}$$

and a basis of $K((4,4))$ is given by the following set.

$$\begin{aligned} &\left\{ b_{(0,2,1,0,0,0)}, b_{(1,1,0,1,0,0)}, b_{(0,1,1,0,1,0)}, 2 \cdot b_{(1,0,0,1,1,0)} \right\} \\ \cup &\left\{ p(x_2x_3)t^{[1]}, p(x_1)\,\mathrm{d}\,x_1t^{[1]}, p(x_3)\,\mathrm{d}\,x_2t^{[1]}, b_{(0,1,0,0,0,1)}, b_{(0,0,0,0,1,1)}, t^{[2]} \right\} \end{aligned}$$

♡

Proof. We first consider the case $j \geq 5$. This assumption implies that if $(a, b, c, \epsilon_1, \epsilon_2, \epsilon_3, 0)$ is an element of $V((j,j))$, then $b > 0$. In other words, every element of grading (j,j) of the monomial basis of

$$\mathbb{Z}[x_1, x_2, x_3]/f \otimes \Lambda(\mathrm{d}\,x_1, \mathrm{d}\,x_2, \mathrm{d}\,x_3)$$

is divisible by x_2. Like in Proposition 10.4.3.1 we can thus conclude that

$$\left\{ p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,0) \in V_2((j,j)) \right\}$$

is a basis of $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}((j,j))$.

By Proposition 10.5.2.2 the set

$$\begin{aligned} &\left\{ p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,0) \in V_2((j,j)) \right\} \\ \cup &\left\{ p\big(x_1^a x_2^b x_3^c\big)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2}t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2((j,j)), m > 0 \right\} \\ \cup &\left\{ b_{(a,b,c,\epsilon_1,\epsilon_2,m)} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,1,m) \in V((j,j)) \right\} \end{aligned}$$

is a basis of $Y((j,j))$, and elements of the first type (of this decomposition into three subsets) are mapped by φ to the corresponding element of $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$. It thus suffices to show that elements of the second and third type are mapped to 0 by φ. If $m > 0$ in either of the two types of elements, then this is clear. So it remains to consider elements of the form $b_{(a,b,c,\epsilon_1,\epsilon_2,0)}$ for $(a,b,c,\epsilon_1,\epsilon_2,1,0) \in V((j,j))$. As mentioned at the start, this implies $b > 0$. It thus follows from Proposition 10.5.2.3 that

$$b_{(a,b,c,\epsilon_1,\epsilon_2,0)} = \partial\Big(p(x_1^a x_2^{b-1} x_3^c\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2}t^{[m+1]})\Big)$$

from which $\varphi(b_{(a,b,c,\epsilon_1,\epsilon_2,0)}) = 0$ follows from the $m > 0$ case as φ is a morphism of chain complexes.

The cases of $0 \leq j \leq 4$ can be done by inspecting each case individually. The difference to the case $j \geq 5$ is that terms that are divisible by $\mathrm{d}\,x_3$ but not by t need not automatically be divisible by x_2 as well. This means that for example $b_{(1,0,0,1,1,0)}$ is not in the kernel of φ (but $2 \cdot b_{(1,0,0,1,1,0)}$ is), whereas the analogous element of $Y((6,6))$, namely $b_{(1,1,1,1,1,0)}$, *does* lie in the kernel. □

10.5.6. $K((j,j))$ for $j < 5$

We can now already finish the case of $j < 5$.

Proposition 10.5.6.1. *Let $0 \leq j < 5$. Then $K((j,j))$ is acyclic.* ♡

Proof. This follows immediately from Proposition 10.5.5.1 in combination with Proposition 10.5.2.3. □

10.5.7. Splitting an acyclic summand off of $K((j,j))$ for $j > 5$

We now turn back to $K((j,j))$ for $j \geq 5$. We start by splitting off an acyclic summand.

Proposition 10.5.7.1. *Let $j \geq 5$. Then define $K_{\mathrm{acyc}}((j,j))$ to be the sub-$\mathbb{Z}$-graded-abelian-group of $K((j,j))$ with basis the following set (compare Proposition 10.5.5.1).*

$$\left\{ p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2((j,j)), m > 0 \right\}$$
$$\cup \left\{ b_{(a,b,c,\epsilon_1,\epsilon_2,m)} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,1,m) \in V((j,j)), b > 0 \right\}$$

Furthermore, we define $K'((j,j))$ to be the sub-$\mathbb{Z}$-graded-abelian-group of $K((j,j))$ with basis the following set.

$$\left\{ b_{(a,0,0,\epsilon_1,1,m)} \;\middle|\; (a,0,0,\epsilon_1,1,1,m) \in V((j,j)) \right\}$$

Then the following hold.

(1) $K_{\mathrm{acyc}}((j,j))$ is a subcomplex of $K((j,j))$.

(2) $K_{\mathrm{acyc}}((j,j))$ is acyclic.

(3) $K'((j,j))$ a subcomplex of $K((j,j))$.

(4) $K((j,j))$ is the sum of $K_{\mathrm{acyc}}((j,j))$ and $K'((j,j))$ as chain complexes.

(5) The inclusion of $K'((j,j))$ into $K((j,j))$ is a quasiisomorphism. ♡

Proof. Proof of claims (1), (2) and (3): Follows immediately from Proposition 10.5.2.3.

Proof of claim (4): If $(a,0,c,\epsilon_1,\epsilon_2,1,m)$ is an element of $V((j,j))$, then this implies that $c = 0$ and $\epsilon_2 = 1$. The claim now follows from Proposition 10.5.5.1.

Proof of claim (5): Immediate consequence of the preceding claims. □

10.5.8. Description of the strict mixed structure

We next need to understand how d acts on $K'((j,j))$.

Proposition 10.5.8.1. *Let $j \geq 5$. Then a basis of $K'((j,j))$ is given by the following set.*

$$\begin{aligned} &\left\{ b_{(0,0,0,0,1,\frac{j-2}{2})}, b_{(1,0,0,1,1,\frac{j-4}{2})} \right\} && \text{if } 2 \mid j \\ &\left\{ b_{(0,0,0,1,1,\frac{j-3}{2})}, b_{(1,0,0,0,1,\frac{j-3}{2})} \right\} && \text{if } 2 \nmid j \end{aligned}$$

Furthermore, the following holds for $m \geq 0$.

$$\begin{aligned} \mathrm{d}\big(b_{(0,0,0,0,1,m)}\big) &= 0 \\ \mathrm{d}\big(b_{(1,0,0,1,1,m)}\big) &= 0 \\ \mathrm{d}\big(b_{(0,0,0,1,1,m)}\big) &= 0 \\ \mathrm{d}\big(b_{(1,0,0,0,1,m)}\big) &= -(2m+3) \cdot b_{(0,0,0,1,1,m)} \end{aligned}$$

♡

Proof. The claim about the basis follows directly from Proposition 10.5.7.1, it merely involves spelling out what a, ϵ_1, and m can be such that the tuple $(a, 0, 0, \epsilon_1, 1, 1, m)$ is an element of $V((j,j))$.

For the formulas for d, we use the definition from Notation 10.5.2.1 and then apply Proposition 10.1.0.1.

$$\begin{aligned} \mathrm{d}\big(b_{(0,0,0,0,1,m)}\big) &= \mathrm{d}\Big(-\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]}\Big) = 0 \\ \mathrm{d}\big(b_{(1,0,0,1,1,m)}\big) &= \mathrm{d}\Big(p(x_1)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]}\Big) = 0 \\ \mathrm{d}\big(b_{(0,0,0,1,1,m)}\big) &= \mathrm{d}\Big(\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]}\Big) = 0 \\ \mathrm{d}\big(b_{(1,0,0,0,1,m)}\big) &= \mathrm{d}\Big(-p(x_1)\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]} - 2 \cdot p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[m]}\Big) \\ &= -(1+2m)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]} - 2 \cdot \mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]} \\ &= -(2m+3) b_{(0,0,0,1,1,m)} \end{aligned}$$

□

10.5.9. A smaller model for $K((j,j))$ for $j > 5$

Proposition 10.5.8.1 implies that $K'((j,j))$ is equivalent as a strict mixed complex to $K((j,j))$ for $j \geq 5$, as we record next.

Proposition 10.5.9.1. *Let $j \geq 5$. Then the strict mixed structure of $K((j,j))$ restricts to $K'((j,j))$ and the inclusion $K'((j,j)) \to K((j,j))$ is a weak equivalence of strict mixed complexes.*

Furthermore, if j is even, then $K'((j,j))$ is isomorphic to the mapping cone of $\mathbb{Z}[j-1] \xrightarrow{2\cdot -} \mathbb{Z}[j-1]$. If j is odd, then $K'((j,j))$ is isomorphic to $\mathrm{D}_j[j-1]$ (see Definition 4.2.1.5 for the notation). ♡

Proof. That the strict mixed structure of $K((j,j))$ restricts to $K'((j,j))$ follows directly from Proposition 10.5.8.1, and that the inclusion is a weak equivalence of strict mixed complexes then follows from Proposition 10.5.7.1 (5).

The identification of $K'((j,j))$ up to isomorphism follows from Proposition 10.5.2.3 and Proposition 10.5.8.1. For the isomorphism to $\mathrm{D}_j[j-1]$, note that $\mathrm{D}_j \cong \mathrm{D}_{-j}$. □

10.6. $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2-x_2x_3)\big)$ as a non-split extension

We can now sum up all the results by coming back to Hochschild homology.

Proposition 10.6.0.1. *Assume that Conjecture D holds for the polynomial $f = x_1^2 - x_2x_3$ in $\mathbb{Z}[x_1,x_2,x_3]$. Then there is a cofiber sequence*

$$\left(\bigoplus_{j\geq 5, 2\mid j} \mathbb{Z}/2[j-1]\right) \oplus \left(\bigoplus_{j\geq 5, 2\nmid j} \gamma_{\mathsf{Mixed}}(\mathrm{D}_j[j-1])\right)$$
$$\to \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2-x_2x_3)\big) \to \gamma_{\mathsf{Mixed}}\left(\left(\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right)$$

in $\mathcal{M}\mathrm{ixed}$ *that does not split.* ♡

Proof. By definition of K we have a pullback square

$$\begin{tikzcd} K \arrow[r, "\psi"] \arrow[d] & Y \arrow[d, "\varphi"] \\ 0 \arrow[r] & \Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}} \end{tikzcd}$$

in Mixed. It is clear from Definition 10.2.0.2 and Remark 10.2.0.1 that φ is levelwise surjective and hence a fibration in Mixed. As every object in Mixed is fibrant, it follows from [HTT, A.2.4.4] that the above square is also a homotopy pullback square.

We can apply $\gamma_{\mathsf{Mixed}}(-^{\mathrm{cof}})$ (where $-^{\mathrm{cof}}$ is the cofibrant replacement functor for Mixed) to this diagram to obtain a commutative square in $\mathcal{M}\mathrm{ixed}$ that is a pullback square by [HA, 1.3.4.23][6] By Proposition 4.4.3.1 $\mathcal{M}\mathrm{ixed}$ is stable, so said square is also a pushout square, so we have shown existence of a cofiber sequence as follows.

$$\gamma_{\mathsf{Mixed}}\big(K^{\mathrm{cof}}\big) \to \gamma_{\mathsf{Mixed}}\big(Y^{\mathrm{cof}}\big) \to \gamma_{\mathsf{Mixed}}\left(\left(\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right)$$

[6]See Propositions 4.4.1.7 and 4.4.2.2 for $\mathcal{M}\mathrm{ixed}$ being the underlying ∞-category of the model category Mixed.

We can identify $\gamma_{\mathsf{Mixed}}(Y^{\mathrm{cof}})$ with $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}[x_1,x_2,x_3]/f)$ along the equivalence from Proposition 10.1.0.1, and for $\gamma_{\mathsf{Mixed}}(K^{\mathrm{cof}})$ we obtain a sequence of equivalences

$$
\begin{aligned}
&\gamma_{\mathsf{Mixed}}\left(K^{\mathrm{cof}}\right) \\
&\simeq \gamma_{\mathsf{Mixed}}\left(\bigoplus_{\vec{j}\in\mathbb{Z}^2_{\geq 0}} K(\vec{j}\,)^{\mathrm{cof}}\right) \\
&\simeq \bigoplus_{\vec{j}\in\mathbb{Z}^2_{\geq 0}} \gamma_{\mathsf{Mixed}}\left(K(\vec{j}\,)^{\mathrm{cof}}\right) \\
&\simeq \left(\bigoplus_{j\geq 5, 2\mid j} \mathrm{cofib}\left(\mathbb{Z}[j-1] \xrightarrow{2\cdot -} \mathbb{Z}[j-1]\right)\right) \oplus \left(\bigoplus_{j\geq 5, 2\nmid j} \gamma_{\mathsf{Mixed}}(\mathrm{D}_j[j-1])\right)
\end{aligned}
$$

where in the first equivalence we used the decomposition from Construction 10.3.0.1 and that coproducts of quasiisomorphisms are again quasiisomorphisms, in the second we used that coproducts of cofibrant objects are homotopy coproducts and [HA, 1.3.4.24], and in the third we used Propositions 10.4.3.1, 10.5.6.1 and 10.5.9.1[7]. This shows existence of a cofiber sequence as claimed.

It remains to show that this cofiber sequence does not split. So suppose that there is a morphism

$$
\gamma_{\mathsf{Mixed}}\left(\left(\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right) \to \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)\right)
$$

in $\mathcal{M}$ixed such that postcomposition with the morphism induced by φ is homotopic to the identity on $\gamma_{\mathsf{Mixed}}((\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}})^{\mathrm{cof}})$. By Propositions 4.4.1.7 and 4.4.2.2 and [Hov99, 1.2.10 (ii)] we can then lift this section to a triangle

$$
\begin{tikzcd}
 & \left(\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}} \arrow[dl, "s"'] \arrow[d, "i"] \\
Y \arrow[r, "\varphi"'] & \Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}
\end{tikzcd}
$$

in Mixed that commutes up to homotopy, with i a quasiisomorphism. We will denote $(\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}})^{\mathrm{cof}}$ by C below. The following argument will use $Y((5,5))$, and it will likely be helpful to follow along with diagram (10.1). We will in particular read off ∂ and d from that diagram; to verify those formulas one uses the formulas in Proposition 10.1.0.1 and the definition

[7] Note that D_n has cofibrant underlying chain complex.

of the basis elements in Notation 10.5.2.1. The diagram below provides an overview over the argument; The left column depicts elements of C and the right column of $Y((5,5))$ In both columns we depict where the elements are mapped by ∂ and d using the conventions of Convention 4.2.1.7, and the horizontal arrows correspond to application of s followed by the projection $Y \to Y((5,5))$ associated with the decomposition from Construction 10.3.0.1.

$$\begin{tikzcd}
\delta \arrow[r, mapsto] \arrow[d] & ? \arrow[d, dashed] \\
\mathrm{d}\,\beta \arrow[r, mapsto] & (2-5d)\cdot b_{(0,0,0,1,1,1)} \\
\beta \arrow[u, squiggly, red] \arrow[r, mapsto] \arrow[d] & 2\cdot p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[1]} + c\cdot b_{(0,1,0,1,0,1)} + d\cdot b_{(1,0,0,0,1,1)} \arrow[u, squiggly, red] \arrow[d] \\
\mathrm{d}\,\alpha \arrow[r, mapsto] & 2\cdot b_{(0,1,1,1,1,0)} \\
\alpha \arrow[u, squiggly, red] \arrow[r, mapsto] & p(x_2x_3^2)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 \arrow[u, squiggly, red]
\end{tikzcd}$$

As the homology of the fiber of φ is concentrated in degrees above 3 by the already obtained cofiber sequence, $\mathrm{H}_2(\varphi)$ is an isomorphism. From diagram (10.1) we can read off that $p(x_2x_3^2)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2$ is a cycle in Y_2 that represents a nontrivial homology class. There must thus be a cycle $\alpha \in C_2$ such that $s(\alpha) = p(x_2x_3^2)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 + \partial y$, with $y \in Y_3$.

As α is a cycle, we have

$$\partial(\mathrm{d}\,\alpha) = -\,\mathrm{d}(\partial\alpha) = 0$$

so $\mathrm{d}\,\alpha$ is a cycle. We furthermore obtain

$$\begin{aligned} s(\mathrm{d}\,\alpha) &= \mathrm{d}(s(\alpha)) = \mathrm{d}\big(p(x_2x_3^2)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 + \partial y\big) \\ &= 2\cdot b_{(0,1,1,1,1,0)} + \mathrm{d}\,\partial y = \partial\Big(p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[1]} - \mathrm{d}\,y\Big) \end{aligned}$$

so that $s(\mathrm{d}\,\alpha)$ is a boundary. As $\mathrm{H}_3(s)$ has to be injective, this implies that $\mathrm{d}\,\alpha$ must be a boundary. So let $\beta \in C_4$ be such that $\partial\beta = \mathrm{d}\,\alpha$.

Using the description of a basis for $Y_4((5,5))$ from Proposition 10.5.2.2 we can write $s(\beta)$ as

$$s(\beta) = a\cdot p(x_1)t^{[2]} + b\cdot p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[1]} + c\cdot b_{(0,1,0,1,0,1)} + d\cdot b_{(1,0,0,0,1,1)} + z$$

with $a,b,c,d \in \mathbb{Z}$, and $z \in \bigoplus_{\vec{j}\in\mathbb{Z}^2_{\geq 0},\, \vec{j}\neq(5,5)} Y_4(\vec{j}\,)$. It follows that

$$2\cdot b_{(0,1,1,1,1,0)} + \mathrm{d}\,\partial y = s(\mathrm{d}\,\alpha) = s(\partial\beta)$$

$$
\begin{aligned}
&= \partial(s(\beta)) \\
&= a \cdot b_{(1,1,0,0,0,1)} + b \cdot b_{(0,1,1,1,1,0)} + c \cdot 0 + d \cdot 0 + \partial z
\end{aligned}
$$

where both $\mathrm{d}\,\partial y$ and ∂z are elements of $\bigoplus_{\vec{j} \in \mathbb{Z}^2_{\geq 0}, \vec{j} \neq (5,5)} Y_3(\vec{j})$, so we can conclude that $a = 0$ and $b = 2$.

We have

$$\partial(\mathrm{d}\,\beta) = -\,\mathrm{d}(\partial\beta) = -\,\mathrm{d}(\mathrm{d}\,\alpha) = 0$$

so $\mathrm{d}\,\beta$ is a cycle. As $\mathrm{H}_5(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}) \cong 0$, it thus follows that $\mathrm{d}\,\beta$ must be of the form $\partial\delta$ for some element $\delta \in C_6$, and hence $\mathrm{d}\,s(\beta) = s(\mathrm{d}\,\beta)$ must be a cycle in Y_5 that is also a boundary. But we can calculate

$$
\begin{aligned}
\mathrm{d}\,s(\beta) &= \mathrm{d}\Big(2 \cdot p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[1]} + c \cdot b_{(0,1,0,1,0,1)} + d \cdot b_{(1,0,0,0,1,1)} + z\Big) \\
&= 2 \cdot b_{(0,0,0,1,1,1)} + 0 - 5d \cdot b_{(0,0,0,1,1,1)} + \mathrm{d}\,z \\
&= (2 - 5d) \cdot b_{(0,0,0,1,1,1)} + \mathrm{d}\,z
\end{aligned}
$$

which, as z lies in $\bigoplus_{\vec{j} \in \mathbb{Z}^2_{\geq 0}, \vec{j} \neq (5,5)} Y_4(\vec{j})$ and $(2-5d) \cdot b_{(0,0,0,1,1,1)}$ is a cycle representing a nontrivial homology class, is in contradiction to $\mathrm{d}\,s(\beta)$ being a boundary. □

10.7. Non-formality of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)\big)$

Let M be a strict mixed complex. Then the relation $\mathrm{d} \circ \partial + \partial \circ \mathrm{d} = 0$ ensures that $\mathrm{d}\colon M_* \to M_{*+1}$ maps cycles to cycles, and thus induces an operator increasing degree by 1 on homology. Equipping $\mathrm{H}_\bullet(X)$ with the zero boundary operator we can then consider $\mathrm{H}_\bullet(M)$ again as an object of Mixed.

Now let M be a mixed complex, i. e. an object in the ∞-category $\mathcal{M}\mathrm{ixed}$. Then we can make a similar construction using the functors

$$\mathrm{H}_*\colon\ \mathcal{D}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$$

defined in Definition 4.3.3.1. From the element d in $\mathrm{H}_1(\mathrm{D})$ we obtain with Proposition 4.3.2.1 (5) a morphism $k[1] \to \mathrm{D}$ in $\mathcal{D}(k)$ which induces a morphism

$$M[1] \simeq k[1] \otimes M \to \mathrm{D} \otimes M \to M$$

in $\mathcal{D}(k)$, where the second morphism is the action of D on M. This morphism induces an operator increasing degree by 1 in H_*, and $\mathrm{d}^2 = 0$ in $\mathrm{H}_*(\mathrm{D})$ implies that this operator squares to 0. Equipping $\mathrm{H}_\bullet(M)$ with this operator as d and the zero boundary operator we again obtain a strict mixed complex. Proposition 4.3.3.2 ensures that the just discussed two constructions agree, i. e. if M is a strict mixed complex with cofibrant underlying chain complex, then the strict mixed complexes $\mathrm{H}_\bullet(\gamma_{\mathsf{Mixed}}(M))$ and $\mathrm{H}_\bullet(M)$ are naturally isomorphic.

Given an object M of $\mathcal{M}$ixed, we can now ask whether M is *formal*, i. e. whether there is an equivalence

$$M \simeq \gamma_{\mathsf{Mixed}}\big(\mathrm{H}_\bullet(M)^{\mathrm{cof}}\big)$$

in $\mathcal{M}$ixed. In the next proposition we show that, still assuming that Conjecture D holds for the polynomial $x_1^2 - x_2x_3$ in $\mathbb{Z}[x_1,x_2,x_3]$, that

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2-x_2x_3)\big)$$

is *not* formal. Note that

$$\mathrm{H}_\bullet\Big(\gamma_{\mathsf{Mixed}}\Big(\mathrm{H}_\bullet(M)^{\mathrm{cof}}\Big)\Big) \cong \mathrm{H}_\bullet(M)$$

for every mixed complex M. This implies (under the assumption of Conjecture D) that there are at least two mixed complexes whose homology, as a strict mixed complex, is isomorphic to

$$\mathrm{H}_\bullet\big(\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2-x_2x_3)\big)\big)$$

so the mixed complex $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2-x_2x_3)\big)$ can not be recovered from its homology (even including the action of d) alone.

Proposition 10.7.0.1. *Assume that Conjecture D holds for the polynomial* $f = x_1^2 - x_2x_3$ *in* $\mathbb{Z}[x_1,x_2,x_3]$. *Then there is no equivalence between*

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2-x_2x_3)\big)$$

and

$$\gamma_{\mathsf{Mixed}}\Big(\mathrm{H}_\bullet\big(\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2-x_2x_3)\big)\big)^{\mathrm{cof}}\Big)$$

in $\mathcal{M}$ixed. ♡

Proof. We will make use of the cofiber sequence constructed in Proposition 10.6.0.1, for which we will use the following notation.

$$F \to Z \xrightarrow{\Phi} R$$

We show the claim by contradiction and assume that there is an equivalence

$$\Theta\colon \gamma_{\mathsf{Mixed}}\Big(\mathrm{H}_\bullet(Z)^{\mathrm{cof}}\Big) \xrightarrow{\simeq} Z$$

in $\mathcal{M}$ixed.

Note that F has homology concentrated in degrees ≥ 4, so Φ induces an isomorphism in homology on degrees ≤ 3. As R has homology concentrated in degrees ≤ 3, it follows that there is a unique morphism of underlying chain complexes $s\colon \mathrm{H}_\bullet(R) \to \mathrm{H}_\bullet(Z)$ such that $\mathrm{H}_\bullet(\Phi) \circ s$ is the identity.

We claim that s is in fact also compatible with d and thus a morphism in Mixed. As $\mathrm{H}_\bullet(\Phi)$ is an isomorphism in degrees ≤ 3, it automatically follows that $\mathrm{d} \circ s = s \circ \mathrm{d}$ on elements of degree ≤ 2. What remains to show is that d applied to every element of $\mathrm{H}_3(Z)$ is zero. From Proposition 10.6.0.1 and the previous discussion in this chapter we know that the elements of

$$\mathrm{H}_4(Z) \cong \mathrm{H}_4(Y)$$

are precisely represented by the integer multiples of the element $b_{(1,0,0,0,1,1)}$ of $Y((5,5))$ (see in particular Propositions 10.5.8.1 and 10.5.9.1). From the sum decomposition of Y it follows that it suffices to show that there is no cycle in $Y((5,5))$ that is mapped by d to a linear combination of basis elements of $Y((5,5))$ with respect to the basis from Proposition 10.5.2.2, in which $b_{(1,0,0,0,1,1)}$ has nonzero coefficient. But this follows from Proposition 10.5.2.3 and can be read off of the first diagram in Section 10.5.3.

Note that

$$R = \gamma_{\mathsf{Mixed}}\left(\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right)$$

and that are isomorphisms as follows in Mixed;

$$\mathrm{H}_\bullet\left(\gamma_{\mathsf{Mixed}}\left(\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right)\right) \cong \mathrm{H}_\bullet\left(\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right)$$

as discussed before this proposition,

$$\mathrm{H}_\bullet\left(\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right) \cong \mathrm{H}_\bullet\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)$$

induced by the cofibrant replacement, and

$$\mathrm{H}_\bullet\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right) \cong \Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$$

as $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$ has zero boundary operator.

Combining these isomorphisms and applying $\gamma_{\mathsf{Mixed}}(-^{\mathrm{cof}})$ we obtain an equivalence

$$\alpha\colon R \xrightarrow{\simeq} \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(R)^{\mathrm{cof}}\right)$$

in $\mathcal{M}\mathrm{ixed}$.

We can now consider the composition

$$\lambda\colon R \xrightarrow{\alpha} \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(R)^{\mathrm{cof}}\right) \xrightarrow{\gamma_{\mathsf{Mixed}}(s^{\mathrm{cof}})} \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(Z)^{\mathrm{cof}}\right) \xrightarrow{\Theta} Z \xrightarrow{\Phi} R$$

in $\mathcal{M}\mathrm{ixed}$. As α and Θ are equivalences, they induce isomorphisms on homology. The morphism s induces an isomorphism in homology in degrees ≤ 3, so

$\gamma_{\mathsf{Mixed}}(s^{\mathrm{cof}})$ does so too, and we already used above that Φ induces an isomorphism in homology in degrees ≤ 3. It follows that λ induces an isomorphism in degrees ≤ 3. As R has homology concentrated in those degrees, it follows that λ actually induces an isomorphism on homology in all degrees and is thus an equivalence.

Now define ϱ to be the composition

$$R \xrightarrow{\lambda^{-1}} R \xrightarrow{\alpha} \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(R)^{\mathrm{cof}}\right) \xrightarrow{\gamma_{\mathsf{Mixed}}(s^{\mathrm{cof}})} \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(Z)^{\mathrm{cof}}\right) \xrightarrow{\Theta} Z$$

in $\mathcal{M}\mathrm{ixed}$. Then it follows that

$$\Phi \circ \varrho \simeq \lambda \circ \lambda^{-1} \simeq \mathrm{id}_Z$$

so ϱ is a section of Φ. This contradicts the fact that the cofiber sequence from Proposition 10.6.0.1 does not split. $\square$

Appendix A.

∞-category theory

This is the first of two appendices in which we collect a number of small results on various basic staples of ∞-category theory, the second one being Appendix D[1].

In Section A.1 we will see that the homotopy category of the underlying ∞-category of a model category is canonically equivalent to the homotopy category of the model category. We will then discuss mapping spaces in ∞-categories in Section A.2, and collect some results relating to the $(\infty, 2)$-category of ∞-categories $\mathcal{C}\mathrm{at}_\infty$ in Section A.3.

A.1. Homotopy categories of model categories

Given a model category C with a class of weak equivalences W, we can form its homotopy category $\mathrm{Ho}_W(\mathsf{C})$ of C, as discussed for example in [Hov99, Section 1.2]. There is also another way to produce a 1-category called "homotopy category" from C: We can first pass to the underlying ∞-category $\mathsf{C}[W^{-1}]$ of C (see [HA, 1.3.4.1]), and then take the homotopy category $\mathrm{Ho}(\mathsf{C}[W^{-1}])$ of this ∞-category as explained in [HTT, 1.2.3]. The following proposition shows that these two notions of "homotopy category" are consistent with each other, i. e. they are canonically equivalent.

Proposition A.1.0.1. *Let C be a model category with class of weak equivalences W. Then there exists an equivalence $\mathrm{Ho}_W \mathsf{C} \simeq \mathrm{Ho}\big(\mathsf{C}[W^{-1}]\big)$ fitting into a commutative diagram as follows*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \mathsf{C}[W^{-1}] \arrow[d, "\beta"] \\ \mathrm{Ho}_W \mathsf{C} \arrow[r, dash, "\simeq"'] & \mathrm{Ho}\big(\mathsf{C}[W^{-1}]\big) \end{tikzcd}$$

where $\mathrm{Ho}_W \mathsf{C}$ is the homotopy category of the model category C (see [Hov99, 1.2]), $\mathrm{Ho}\big(\mathsf{C}[W^{-1}]\big)$ is the homotopy category of the ∞-category $\mathsf{C}[W^{-1}]$ (see [HTT, 1.2.3]), and the functors α, β, and γ are the canonical ones. ♡

[1]Some parts of Appendix D depend on Appendices B and C.

Proof. The functor α sends morphisms in W to equivalences[2], and β sends all equivalences to isomorphisms as $\mathrm{Ho}\big(\mathsf{C}[W^{-1}]\big)$ is a 1-category. The universal property of $\mathrm{Ho}_W \mathsf{C}$ (see [Hov99, 1.2.2]) furnishes us with a functor Φ making the following diagram commute.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \mathsf{C}[W^{-1}] \arrow[d, "\beta"] \\ \mathrm{Ho}_W \mathsf{C} \arrow[r, "\Phi"] & \mathrm{Ho}\big(\mathsf{C}[W^{-1}]\big) \end{tikzcd}$$

As isomorphisms are in particular equivalences, the universal property of $\mathsf{C}[W^{-1}]$ (see [HA, 1.3.4.1]) provides us with a functor $\psi\colon \mathsf{C}[W^{-1}] \to \mathrm{Ho}_W \mathsf{C}$ satisfying $\psi \circ \alpha \simeq \gamma$. Applying Ho we obtain a commuting diagram as follows.

$$\begin{tikzcd} & \mathsf{C} \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\ \mathsf{C}[W^{-1}] \arrow[rr, "\psi"] \arrow[d, "\beta"'] & & \mathrm{Ho}_W \mathsf{C} \arrow[d, "\cong"] \\ \mathrm{Ho}\,\mathsf{C}[W^{-1}] \arrow[rr] \arrow[urr, dashed, "\Psi"'] & & \mathrm{Ho}(\mathrm{Ho}_W \mathsf{C}) \end{tikzcd}$$

As $\mathrm{Ho}_W \mathsf{C}$ already is a 1-category, we can identify $\mathrm{Ho}(\mathrm{Ho}_W \mathsf{C})$ with $\mathrm{Ho}_W \mathsf{C}$. Call the resulting functor $\Psi\colon\ \mathrm{Ho}\,\mathsf{C}[W^{-1}] \to \mathrm{Ho}_W \mathsf{C}$.

Using the uniqueness part of the universal properties of α, β, and γ one concludes that the compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are naturally isomorphic to the respective identity functors. □

A.2. Mapping spaces

In this section we discuss two results relating to mapping spaces of ∞-categories. In Proposition A.2.0.1 we show that mapping spaces can be calculated as certain pullbacks in $\mathcal{C}\mathrm{at}_\infty$. We will then apply this result in Proposition A.2.0.2 to show that a pullback diagram in $\mathcal{C}\mathrm{at}_\infty$ induces pullback diagrams of the respective mapping spaces.

Proposition A.2.0.1. *Let $\mathcal{C}$ be an ∞-category and X and Y two objects of $\mathcal{C}$. Then there is a natural (in $\mathcal{C}$) pullback square in* $\mathcal{C}\mathrm{at}_\infty$

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d] & \mathcal{C}^{[1]} \arrow[d] \\ \{(X,Y)\} \arrow[r] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

where the right vertical functor sends a morphism $f\colon A \to B$ to (A,B). ♡

[2] See [HA, 1.3.4.1] for a definition of $\mathsf{C}[W^{-1}]$.

Proof. We give a proof for this claim in the setting of quasicategories. The discussion in [HTT, Discussion after 1.2.2.5 and 4.2.1.8] exhibits the mapping space as a pullback of quasicategories, so we need to argue why this is a homotopy pullback in the Joyal model structure, and then identify the resulting (iterated) homotopy pullback with the pullback square we claimed. So let $\mathtt{C}$ be a quasicategory modeling the ∞-category $\mathcal{C}$. In [HTT, 4.2.1.8], a model for $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is identified with the pullback in simplicial sets of the following diagram.

$$\mathtt{C}^{\{X\}/} \to \mathtt{C} \leftarrow \{Y\}$$

Applying [HTT, 4.2.1.6][3] to $X = \mathtt{C}$, $S = \{Y\}$, $K = \{X\}$, and $K_0 = \emptyset$, we obtain that

$$\mathtt{C}^{\{X\}/} \to \mathtt{C}^{\emptyset/} \times_{\{Y\}^{\emptyset/}} \{Y\}^{\{X\}/} \cong \mathtt{C} \times_{\{Y\}} \{Y\} \cong \mathtt{C}$$

is a left fibration. By [HTT, 2.4.2.4 and 3.3.1.4] this implies that the pullback of

$$\mathtt{C}^{\{X\}/} \to \mathtt{C} \leftarrow \{Y\}$$

is already a homotopy pullback in the Joyal model structure.

Unpacking the definition of $\mathtt{C}^{\{X\}/}$ (see [HTT, after 4.2.1.4]) one can write $\mathtt{C}^{\{X\}/}$ as the pullback in simplicial sets of the following diagram.

$$\{X\} \to \mathtt{C}^{\{0\}} \leftarrow \mathtt{C}^{\Delta^1}$$

It follows from [HTT, 2.4.7.12] (applied to $\mathrm{id}_{\mathtt{C}}$) that $\mathtt{C}^{\Delta^1} \to \mathtt{C}^{\{0\}}$ is a cartesian fibration, so again by [HTT, 3.3.1.4] the pullback in simplicial sets is already a homotopy pullback in the Joyal model structure. Together this implies that the ∞-groupoid $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is naturally equivalent to the iterated pullback

$$\left(\{X\} \times_{\mathcal{C}^{\{0\}}} \mathcal{C}^{[1]}\right) \times_{\mathcal{C}^{\{1\}}} \{Y\}$$

in $\mathcal{C}\mathrm{at}_\infty$. Using [HTT, 4.4.2.2] one can rewrite this iterated pullback into the form that was stated in the claim. $\square$

Proposition A.2.0.2. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[d, "H"] \\ \mathcal{E} \arrow[r] & \mathcal{F} \end{tikzcd} \tag{$*$}$$

be a pullback square in $\mathcal{C}\mathrm{at}_\infty$, and X, Y two objects in $\mathcal{C}$. Then the commutative diagram in $\mathcal{S}$ induced by $(*)$ on mapping spaces

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(F(X), F(Y)) \arrow[d] \\ \mathrm{Map}_{\mathcal{E}}(G(X), G(Y)) \arrow[r] & \mathrm{Map}_{\mathcal{F}}(H(F(X)), H(F(Y))) \end{tikzcd}$$

[3] $\mathtt{C} \to \{Y\}$ is a categorical fibration by [HTT, 2.4.6.1].

is a pullback diagram. ♡

Proof. As $\mathcal{C}$ is given as a pullback in $\mathcal{C}\mathrm{at}_\infty$ and products as well as $\mathrm{Fun}([1], -)$ preserve limits, we can write $\mathrm{Map}_{\mathcal{C}}(X, Y)$ as a pullback of pullbacks by applying Proposition A.2.0.1: The ∞-groupoid $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is the pullback of the following diagram.

$$\begin{tikzcd}
\{(G(X), G(Y))\} \times_{\{(H(F(X)),H(F(Y)))\}} \{(F(X), F(Y))\} \arrow[d] \\
(\mathcal{E} \times \mathcal{E}) \times_{(\mathcal{F}\times\mathcal{F})} (\mathcal{D} \times \mathcal{D}) \\
\mathcal{E}^{[1]} \times_{\mathcal{F}^{[1]}} \mathcal{D}^{[1]} \arrow[u]
\end{tikzcd}$$

Commuting the two limits [HTT, 5.5.2.3] and applying Proposition A.2.0.1 again we can conclude that the commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(F(X), F(Y)) \arrow[d] \\
\mathrm{Map}_{\mathcal{E}}(G(X), G(Y)) \arrow[r] & \mathrm{Map}_{\mathcal{F}}(H(F(X)), H(F(Y)))
\end{tikzcd}$$

induced by $(*)$ is a pullback diagram in $\mathcal{C}\mathrm{at}_\infty$, and hence a pullback diagram in $\mathcal{S}$ by [HTT, 1.2.13.7]. □

A.3. The $(\infty, 2)$-category of ∞-categories

In this section we discuss some results concerning the $(\infty, 2)$-category of ∞-categories. We will characterize pullbacks in the underlying ∞-category $\mathcal{C}\mathrm{at}_\infty$ in Section A.3.1, and show that checking that a natural transformation is an equivalence can be done pointwise in Section A.3.2.

A.3.1. Pullbacks in $\mathcal{C}\mathrm{at}_\infty$

Proposition A.3.1.1. *Let*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[d, "H"] \\
\mathcal{E} \arrow[r] & \mathcal{F}
\end{tikzcd} \tag{A.1}$$

be a commutative diagram in $\mathcal{C}\mathrm{at}_\infty$*. Then diagram* (A.1) *is a pullback diagram if and only if the induced diagram on* ∞*-groupoid cores*

$$\begin{tikzcd} \mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{D}^{\simeq} \arrow[d] \\ \mathcal{E}^{\simeq} \arrow[r] & \mathcal{F}^{\simeq} \end{tikzcd} \tag{A.2}$$

as well as the induced diagram on mapping spaces

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(F(X), F(Y)) \arrow[d] \\ \mathrm{Map}_{\mathcal{E}}(G(X), G(Y)) \arrow[r] & \mathrm{Map}_{\mathcal{F}}(H(F(X)), H(F(Y))) \end{tikzcd} \tag{A.3}$$

for every pair of objects X *and* Y *in* $\mathcal{C}$ *are pullback diagrams in* $\mathcal{S}$. ♡

Proof. The functor $(-)^{\simeq}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ is right adjoint to the inclusion (see [HTT, 1.2.5]) and thus preserves pullbacks, which together with Proposition A.2.0.2 shows the “only if”-direction.

For the “if”-direction, consider the following commutative diagram in $\mathcal{C}\mathrm{at}_\infty$ induced by (A.1), where the small square is to be a pullback diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[dr, dashed, "\Phi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ & \mathcal{E} \arrow[r] & \mathcal{F} \end{tikzcd}$$

It suffices to show that Φ is an equivalence. The already proven “only if”-direction and the assumption that (A.2) is a pullback diagram imply that $\Phi^{\simeq}$ is an equivalence of spaces, which implies that Φ is essentially surjective (see [HTT, 1.2.10.1]). Analogously we deduce from (A.3) that Φ is fully faithful (see [HTT, 1.2.10.1] and Definition B.2.0.1 below). Thus Φ is an equivalence. □

Remark A.3.1.2. In Proposition A.3.1.1, if diagrams (A.3) are pullback diagrams, then it follows immediately from the proof that we can replace the condition that diagram (A.2) is a pullback diagram with the a-priori weaker claim that the map $\Phi^{\simeq}$ constructed in the proof is induces a surjection on π_0. As $(-)^{\simeq}$ preserves pullbacks we can identify $\Phi^{\simeq}$ with the induced functor $\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq} \times_{\mathcal{E}^{\simeq}} \mathcal{F}^{\simeq}$. ♢

A.3.2. Natural transformations

Proposition A.3.2.1 ([Lur21, Theorem 01DK])**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be* ∞*-categories,* F *and* G *two functors* $\mathcal{C} \to \mathcal{D}$*, and* $\Phi\colon F \to G$ *a natural transformation.*

Then Φ *is an equivalence in* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *if and only if* $\Phi_X \colon F(X) \to F(X)$ *is an equivalence in* $\mathcal{D}$ *for every object* X *of* $\mathcal{C}$. ♡

Proof. Equivalences can be described via colimits; A morphism f in some ∞-category $\mathcal{E}$ is an equivalence if and only if the corresponding functor $[0]^{\triangleright} \simeq [1] \to \mathcal{E}$ is a colimit diagram, see [HTT, 4.4.1 and 1.2.4.1]. The claim now follows from the fact that colimits in functor categories can be detected pointwise by [HTT, 5.1.2.3 (2)]. □

Appendix B.

(Fully) faithful functors and monomorphisms in $\mathcal{C}\mathrm{at}_\infty$

In this appendix we discuss three important classes of functors of ∞-categories that are all in some sense analogues to the notion of injections of sets. These are the *faithful* functors, *fully faithful* functors, as well as *monomorphisms* in $\mathcal{C}\mathrm{at}_\infty$.

The notion of monomorphism can be defined in any ∞-category, not just $\mathcal{C}\mathrm{at}_\infty$, so we begin by discussing monomorphisms in this greater generality in Section B.1. We then define (fully) faithful functors Section B.2 and discuss some immediate consequences of the definitions. Before discussing these three classes of functors of ∞-categories further, we will need to show an intermediate result in Section B.3, stability of (fully) faithful functors under $\mathrm{Fun}(\mathcal{I}, -)$ for an ∞-category $\mathcal{I}$. We will then be ready to discuss monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ in detail in Section B.4. In Section B.5 we will cover a number of stability results, including descriptions of replete images, for (fully) faithful functors and monomorphisms in $\mathcal{C}\mathrm{at}_\infty$, under $\mathrm{Fun}(\mathcal{I}, -)$, pullbacks along another functor, and pullbacks. We will end this section with Section B.6, in which we will discuss the correspondence between monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ with codomain a fixed ∞-category $\mathcal{C}$ and replete subcategories of $\mathrm{Ho}\,\mathcal{C}$.

B.1. Monomorphisms

Let $\mathcal{C}$ be an ∞-category and $f\colon X \to Y$ a morphism in $\mathcal{C}$. Then f is called a *monomorphism*[1] if the morphism

$$f_*\colon \mathrm{Map}_{\mathcal{C}}(Z, X) \to \mathrm{Map}_{\mathcal{C}}(Z, Y)$$

is a monomorphism in $\mathcal{S}$ for every object Z of $\mathcal{C}$.

In Section B.1.1 we will give a number of equivalent characterizations for monomorphisms in $\mathcal{S}$, before discussing the interaction of monomorphisms with compositions in Section B.1.2 and with limits in Section B.1.3.

[1] See the definition given in [HTT, Between 5.5.6.13 and 5.5.6.14] as well as [HTT, 5.5.6.8].

B.1.1. Monomorphisms in the ∞-category $\mathcal{S}$

The following proposition recalls the notion of *monomorphisms* in the ∞-category $\mathcal{S}$.

Proposition B.1.1.1. *Let $f\colon X \to Y$ be a morphism in $\mathcal{S}$. Then the following are equivalent.*

(1) f is a monomorphism in the sense of [HTT, Directly after 5.5.6.13], i. e. if f is (-1)-truncated in the sense of [HTT, 5.5.6.8].

(2) For every point y in Y the fiber of f over y is (-1)-truncated, i. e. empty or contractible.

(3) For every point x in X the fiber of f over $f(x)$ is (-2)-truncated, i. e. contractible.

(4) For every point x in X the morphism induced by f

$$\pi_n(X, x) \to \pi_n(Y, f(x)) \tag{B.1}$$

is a bijection for $n > 0$ and an injection for $n = 0$.

(5) The induced morphism on path components $\pi_0(f)$ is injective and the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y) \end{tikzcd}$$

is a pullback diagram in $\mathcal{S}$.

(6) Considering f as a functor of ∞-categories (via the inclusion of ∞-groupoids into $\mathcal{C}\mathrm{at}_\infty$) the induced map on mapping spaces[2]

$$\mathrm{Map}_X(x, x') \to \mathrm{Map}_Y(f(x), f(x')) \tag{B.2}$$

is an equivalence for every pair of points x and x' in X.

♡

Proof. Proof that (1) is equivalent to (2): This is [HTT, 5.5.6.9].

Proof that (2) is equivalent to (3): Follows from the fact that points in Y are equivalent to $f(x)$ for a point x in X if and only if the fiber of f over y is not empty.

[2] These are the path spaces if we think of X and Y as spaces.

Proof that (5) implies (1): As any injective morphism of discrete spaces satisfies (3) and hence (1), and monomorphisms are stable under taking pullbacks by [HTT, 5.5.6.12], (5) implies (1).

Proof that (3) is equivalent to (4): Follows immediately from the long exact sequence of homotopy groups.

Proof that (3) implies (5): That $\pi_0(f)$ is injective is part of (4). Now consider the following diagram, where the small square is a pullback square.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\
& P \arrow[r, "\psi"] \arrow[d] & Y \arrow[d] \\
& \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y)
\end{tikzcd}$$

It suffices to show that φ is an equivalence. By the long exact sequence of homotopy groups, it suffices for this to show that $\pi_0(\varphi)$ is surjective and the fiber of φ over every point in P is contractible. As $Y \to \pi_0(Y)$ is 1-connective (see [HTT, 6.5.1.10] for a definition) we obtain that $P \to \pi_0(X)$ is 1-connective by [HTT, 6.5.1.16 (6)], and as $X \to \pi_0(X)$ is 1-connective as well it follows that $\pi_0(\varphi)$ must be an isomorphism.

Now let p be a point in P. Consider the following diagram of pullback squares.

$$\begin{tikzcd}
\mathrm{fib}_p(\varphi) \arrow[r] \arrow[d] & \{p\} \arrow[d] & \\
\mathrm{fib}_{\psi(p)}(f) \arrow[r] \arrow[d] & \mathrm{fib}_{\psi(p)}(\psi) \arrow[r] \arrow[d] & \{\psi(p)\} \arrow[d] \\
X \arrow[r, "\varphi"] & P \arrow[r, "\psi"] \arrow[d] & Y \arrow[d] \\
& \pi_0(X) \arrow[r] & \pi_0(Y)
\end{tikzcd}$$

As $\pi_0(\varphi)$ is surjective, $\psi(p)$ is equivalent to $f(x)$ for some point x in X, so it follows from the assumption that $\mathrm{fib}_{\psi(p)}(f)$ is contractible. Furthermore, as $\mathrm{fib}_{\psi(p)}(\psi)$ can be identified as a fiber of a map of discrete spaces, it is discrete as well. It follows, using the long exact sequence of homotopy groups associated to the fiber sequence

$$\mathrm{fib}_p(\varphi) \to \mathrm{fib}_{\psi(p)}(f) \to \mathrm{fib}_{\psi(p)}(\psi)$$

that $\mathrm{fib}_p(\varphi)$ is contractible.

Proof that (6) is equivalent to (4): Let x and x' be points of X. We distinguish two cases. If x and x' are not in the same path component, then $\mathrm{Map}_X(x, x')$ is empty, and so (B.2) is an equivalence if and only if $\mathrm{Map}_Y(f(x), f(x'))$ is empty. That this is the case for all points x and x' in different path components of X is equivalent to $\pi_0(f)$ being injective.

If x and x' are two points of X that lie in the same path component, then the map (B.2) can be identified with the induced map on loop spaces.

$$\Omega_x(X) \xrightarrow{\Omega_x(f)} \Omega_{f(x)}(Y)$$

As $\pi_n(\Omega_x(f)) \cong \pi_{n+1}(f)$ (where on the left we use the constant loop at x as the basepoint, and at the right the point x) we can conclude that (B.2) being an equivalence for all x and x' in the same path component of X is equivalent to (B.1) being an isomorphism for $n > 0$. □

B.1.2. Monomorphisms and composition

Proposition B.1.2.1. *Let $\mathcal{C}$ be an ∞-category and $f\colon X \to Y$ and $g\colon Y \to Z$ two morphisms in $\mathcal{C}$ such that g is a monomorphism. Then $g \circ f$ is a monomorphism if and only if f is a monomorphism.* ♡

Proof. The statement for $\mathcal{C} = \mathcal{S}$ follows immediately from criterion Proposition B.1.1.1 (4). The claim for general $\mathcal{C}$ now follows immediately from the definition. □

B.1.3. Monomorphisms and limits

Proposition B.1.3.1. *Let $\mathcal{I}$ and $\mathcal{C}$ be ∞-categories, $A, B\colon \mathcal{I} \to \mathcal{C}$ two functors, and F a natural transformation from A to B. Assume that for every object X of $\mathcal{I}$ the morphism $F(X)\colon A(X) \to B(X)$ in $\mathcal{C}$ is a monomorphism. Then the morphism $\lim_{\mathcal{I}} A \xrightarrow{\lim_{\mathcal{I}} F} \lim_{\mathcal{I}} B$ in $\mathcal{C}$ is a monomorphism as well.* ♡

Proof. We first prove the claim for $\mathcal{C} = \mathcal{S}$. Let y be a point in $\lim_{\mathcal{I}} B$. We have to show that $\mathrm{fib}_y(\lim_{\mathcal{I}} F)$ is (-1)-truncated. But as limits commute with limits, we have an equivalence

$$\mathrm{fib}_y\left(\lim_{\bullet \in \mathcal{I}} F(\bullet)\right) \simeq \lim_{\bullet \in \mathcal{I}}\bigl(\mathrm{fib}_{\mathrm{pr}_\bullet(y)} F(\bullet)\bigr)$$

so that the claim follows from the closure of (-1)-truncated objects under limits, see [HTT, 5.5.6.5].

The case of general $\mathcal{C}$ now follows from this special case using that for every object X of $\mathcal{C}$ the functor

$$\mathrm{Map}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathcal{S}$$

preserves limits. □

B.2. (Fully) faithful functors

In this section we define the notions of *(fully) faithful* functors of ∞-categories[3] and record some direct consequences of the definition.

Definition B.2.0.1. Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a functor of ∞-categories. We say that ι is *(fully) faithful* if for every pair of objects X and Y of $\mathcal{C}'$ the morphism in $\mathcal{S}$ induced by ι

$$\mathrm{Map}_{\mathcal{C}'}(X,Y) \to \mathrm{Map}_{\mathcal{C}}(F(X),F(Y))$$

is a monomorphism (is an equivalence). ♢

Remark B.2.0.2. It is clear from the definition, that the notions of (fully) faithfulness agree with the classical definitions on 1-categories. Furthermore, as $\pi_0\colon \mathcal{S} \to \mathsf{Set}$ sends equivalences to isomorphisms and monomorphisms to monomorphisms (see Proposition B.1.1.1), if a functor ι of ∞-categories is (fully) faithful, then the same is true for the functor $\mathrm{Ho}\,\iota$ of ordinary categories. ♢

Proposition B.2.0.3. *Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a faithful functor of ∞-categories. Then the commutative diagram*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\iota"] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{Ho}(\mathcal{C}') \arrow[r, "\mathrm{Ho}\,\iota"'] & \mathrm{Ho}(\mathcal{C}) \end{tikzcd} \tag{B.3}$$

is a pullback diagram in $\mathcal{C}\mathrm{at}_\infty$. ♡

Proof. We use Proposition A.3.1.1 and Remark A.3.1.2. Let X and Y be two object of $\mathcal{C}'$. Diagram (B.3) induces the following diagram of mapping spaces.

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}'}(X,Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(\iota X,\iota Y) \arrow[d] \\ \pi_0(\mathrm{Map}_{\mathcal{C}'}(X,Y)) \arrow[r] & \pi_0(\mathrm{Map}_{\mathcal{C}}(\iota X,\iota Y)) \end{tikzcd}$$

It follow from Proposition B.1.1.1 that this square is a pullback square in $\mathcal{S}$.
It now remains to show that

$$\mathcal{C}'^{\simeq} \to \mathcal{C}^{\simeq} \times_{\mathrm{Ho}(\mathcal{C})^{\simeq}} \mathrm{Ho}(\mathcal{C}')^{\simeq}$$

induces a surjection on π_0. The map[4]

$$\mathcal{C}'^{\simeq} \to \mathrm{Ho}(\mathcal{C}')^{\simeq} \simeq \mathrm{Ho}(\mathcal{C}'^{\simeq}) \simeq \tau_{\leq 1}(\mathcal{C}'^{\simeq})$$

[3]Fully faithful functors are defined in [HTT, 1.2.10.1].
[4]That $\mathrm{Ho}(\mathcal{C})^{\simeq} \simeq \mathrm{Ho}(\mathcal{C}^{\simeq})$ can be seen directly using the definitions, it boils down to the subspace of $\mathrm{Map}_{\mathcal{C}'}(X,Y)$ spanned by the equivalences consisting exactly of the

is 2-connective. Similarly, $\mathcal{C}^\simeq \to \mathrm{Ho}(\mathcal{C})^\simeq$ is 2-connective, so by [HTT, 6.5.1.16 (6)] the projection $\mathrm{pr}_2 \colon \mathcal{C}^\simeq \times_{\mathrm{Ho}(\mathcal{C})^\simeq} \mathrm{Ho}(\mathcal{C}')^\simeq \to \mathrm{Ho}(\mathcal{C}')^\simeq$ is 2-connective as well. We thus have a commuting triangle

$$\begin{tikzcd}
\mathcal{C}'^\simeq \arrow[rr] \arrow[dr] & & \mathcal{C}^\simeq \times_{\mathrm{Ho}(\mathcal{C})^\simeq} \mathrm{Ho}(\mathcal{C}')^\simeq \arrow[dl, "\mathrm{pr}_2"] \\
& \mathrm{Ho}(\mathcal{C}')^\simeq &
\end{tikzcd}$$

where the two non-horizontal maps are 2-connective, so the horizontal map must in particular induce a surjection on π_0. □

Proposition B.2.0.4. *Let $\iota \colon \mathcal{C}' \to \mathcal{C}$ be a faithful functor. Then for any objects X and Y of $\mathcal{C}'$, the induced map*

$$\mathrm{Map}_{\mathcal{C}'^\simeq}(X, Y) \to \mathrm{Map}_{\mathcal{C}^\simeq}(\iota X, \iota Y) \tag{B.4}$$

is a monomorphism in $\mathcal{S}$. ♡

Proof. The map in question is by definition the induced vertical map by taking limits of the horizontal diagrams in the following commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}'}(X, Y) \arrow[r] \arrow[d] & \pi_0(\mathrm{Map}_{\mathcal{C}'}(X, Y)) \arrow[d] & \pi_0(\mathrm{Map}_{\mathcal{C}'^\simeq}(X, Y)) \arrow[l] \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(X, Y) \arrow[r] & \pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y)) & \pi_0(\mathrm{Map}_{\mathcal{C}^\simeq}(X, Y)) \arrow[l]
\end{tikzcd}$$

where the vertical maps are induced by ι, the horizontal maps from the left to the middle are the canonical ones, and the horizontal maps from the right to the middle are the inclusions of the path components spanned by invertible morphisms.

As all vertical maps are monomorphisms, we can apply Proposition B.1.3.1 to conclude that (B.4) is a monomorphism as well. □

B.3. (Fully) Faithful functors and Fun

When we discuss monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ in Section B.4, we will need to use a first stability result for (fully) faithful functors that we prove in this

path components that as elements of $\pi_0(\mathrm{Map}_{\mathcal{C}'}(X, Y)) = \mathrm{Mor}_{\mathrm{Ho}\,\mathcal{C}'}(X, Y)$ correspond to isomorphisms in $\mathrm{Ho}\,\mathcal{C}'$. That $\mathrm{Ho}(\mathcal{C}'^\simeq) \simeq \tau_{\leq 1}(\mathcal{C}'^\simeq)$ amounts to the fact that the diagram of inclusions

$$\begin{tikzcd}
\mathcal{S}_{\leq 1} \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\
\mathsf{Cat} \arrow[r] & \mathcal{C}\mathrm{at}_\infty
\end{tikzcd}$$

is left adjointable in the sense of [HTT, 7.3.1.1]. However, in this situation this follows from the horizontal functors being fully faithful.

section; for an ∞-category $\mathcal{I}$, the functor $\mathrm{Fun}(\mathcal{I}, -)\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ preserves (fully) faithful functors.

Proposition B.3.0.1. *Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a (fully) faithful functor of ∞-categories and let $\mathcal{I}$ be some ∞-category. Then the induced functor*

$$\iota_*\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}') \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$$

is (fully) faithful as well. ♡

Proof. Let F and G be two objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{D}')$. Mapping spaces in functor categories can be written as ends, see [Gla16, 2.3]. Concretely, the map induced by ι_* on mapping spaces

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D}')}(F, G) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(\iota \circ F, \iota \circ G)$$

can be identified with the following map of ends, induced by the maps induced by ι on mapping spaces $\mathrm{Map}_{\mathcal{D}'}(\bullet, \bullet) \to \mathrm{Map}_{\mathcal{D}}(\iota(\bullet), \iota(\bullet))$.

$$\int_{\bullet\in\mathcal{C}} \mathrm{Map}_{\mathcal{D}'}(F(\bullet), G(\bullet)) \longrightarrow \int_{\bullet\in\mathcal{C}} \mathrm{Map}_{\mathcal{D}}(\iota(F(\bullet), \iota(G(\bullet)))$$

If ι is fully faithful, then this is an equivalence as ends, like other limits, are invariant under equivalences, so ι_* is fully faithful as well.

If ι is faithful, then we can use that limits commute with limits, so for $\varphi\colon F \to G$ a morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D}')$ we obtain

$$\begin{aligned}
&\mathrm{fib}_\varphi\Big(\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D}')}(F, G) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(\iota \circ F, \iota \circ G)\Big)\\
\simeq &\int_{\bullet\in\mathcal{C}} \mathrm{fib}_{\varphi_\bullet}(\mathrm{Map}_{\mathcal{D}'}(F(\bullet), G(\bullet)) \to \mathrm{Map}_{\mathcal{D}}(\iota(F(\bullet), \iota(G(\bullet))))\\
\simeq &\int_{\bullet\in\mathcal{C}} * \simeq *
\end{aligned}$$

where in the second-to-last step we use that ι is faithful in combination with criterion (3) of Proposition B.1.1.1, and in the last step we use that a limit of a diagram that is pointwise a terminal object (which is a limit over the empty diagram) is the terminal object (as limits commute with limits). Thus ι_* is again faithful. □

B.4. Monomorphisms in Cat_∞

In this section we discuss monomorphisms in Cat_∞. We will start in Section B.4.1 by giving several equivalent characterizations of monomorphisms in Cat_∞, that will be crucial for later results. In Section B.4.2 we will then discuss the analogue of monomorphism in Cat_∞ for 1-categories, the notion of *pseudomonic* functors, as well as the relation between monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ and pseudomonic functors in Cat. Section B.4.3 will provide the important criterion for lifting along monomorphisms in $\mathcal{C}\mathrm{at}_\infty$. Finally, we end this section with Section B.4.4, where we show that faithful functors are monomorphisms.

B.4.1. Equivalent characterizations of monomorphisms in $\mathcal{C}\mathrm{at}_\infty$

In this section we provide a number of equivalent characterizations of monomorphisms in $\mathcal{C}\mathrm{at}_\infty$. We also show that monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ are conservative functors, i. e. reflect equivalences.

Proposition B.4.1.1. *Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a functor of ∞-categories. Then the following are equivalent.*

(1) ι is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$ in the sense of [HTT, After 5.5.6.13].

(2) For every ∞-category $\mathcal{I}$, the induced map

$$(\iota_*)^{\simeq}\colon \operatorname{Fun}(\mathcal{I}, \mathcal{C}')^{\simeq} \to \operatorname{Fun}(\mathcal{I}, \mathcal{C})^{\simeq}$$

is a monomorphism in $\mathcal{S}$.

(3) ι is faithful and the induced functor on ∞-groupoid cores $\iota^{\simeq}\colon \mathcal{C}'^{\simeq} \to \mathcal{C}^{\simeq}$ is a monomorphism in $\mathcal{S}$.

(4) ι is faithful and for every two objects X and Y in $\mathcal{C}'$ and equivalence $f\colon \iota X \to \iota Y$ there is an equivalence $f'\colon X \to Y$ such that $\iota f'$ is homotopic to f.

♡

Proof. Proof that (1) is equivalent to (2): This follows immediately by unpacking the definition of monomorphisms and using that

$$\operatorname{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{I}, -) \simeq \operatorname{Fun}(\mathcal{I}, -)^{\simeq}$$

by definition [HTT, 3.0.0.1].

Proof that (2) implies (3): Applying the assumption to $\mathcal{I} = [0]$, we deduce immediately that $\iota^{\simeq}$ is a monomorphism in $\mathcal{S}$. Let X and Y be objects of $\mathcal{C}'$. Using that $(-)^{\simeq}$ preserves pullbacks as a right adjoint [HTT, 1.2.5] we obtain from Proposition A.2.0.1 that the map induced by ι

$$\operatorname{Map}_{\mathcal{C}'}(X, Y) \to \operatorname{Map}_{\mathcal{C}}(\iota X, \iota Y) \tag{$*$}$$

is the map induced on limits of the horizontal diagrams in the following commutative diagram.

$$\begin{tikzcd}
\operatorname{Fun}([1], \mathcal{C}')^{\simeq} \arrow[r] \arrow[d] & \operatorname{Fun}(\{0,1\}, \mathcal{C}')^{\simeq} \arrow[d] & \{(X, Y)\} \arrow[l] \arrow[d] \\
\operatorname{Fun}([1], \mathcal{C})^{\simeq} \arrow[r] & \operatorname{Fun}(\{0,1\}, \mathcal{C})^{\simeq} & \{(\iota X, \iota Y)\} \arrow[l]
\end{tikzcd}$$

where the vertical maps are induced by ι, the horizontal maps from the left to the middle are induced by precomposition with the inclusion of $\{0,1\}$ into $[1]$ and the horizontal maps from the right to the middle are the inclusions of the functors sending 0 to the first component of the tuple and 1 to the second component. The vertical map on the right is an equivalence and thus a monomorphism, and the other two vertical maps are monomorphisms by assumption. It follows from Proposition B.1.3.1 that $(*)$ is a monomorphism as well.

Proof that (3) implies (4): Follows immediately from description Proposition B.1.1.1 (6) of monomorphisms in $\mathcal{S}$ applied to $\iota^{\simeq}$.

Proof that (4) implies (2): Let $\mathcal{I}$ be an ∞-category. As

$$\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{I},-) \simeq \mathrm{Fun}(\mathcal{I},-)^{\simeq}$$

preserves limits, we obtain from Proposition B.2.0.3 a pullback diagram of spaces as follows

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I},\mathcal{C}')^{\simeq} \arrow[r, "(\iota_*)^{\simeq}"] \arrow[d] & \mathrm{Fun}(\mathcal{I},\mathcal{C})^{\simeq} \arrow[d] \\
\mathrm{Fun}(\mathcal{I},\mathrm{Ho}\,\mathcal{C}')^{\simeq} \arrow[r, "((\mathrm{Ho}\,\iota)_*)^{\simeq}"'] & \mathrm{Fun}(\mathcal{I},\mathrm{Ho}\,\mathcal{C})^{\simeq}
\end{tikzcd}$$

where the vertical maps are induced by postcomposition with the canonical functors. We have to show that the top map is a monomorphism, so as monomorphisms are stable under pullback by [HTT, 5.5.6.12], it suffices to show that $((\mathrm{Ho}\,\iota)_*)^{\simeq}$ is a monomorphism in $\mathcal{S}$. Note that as $\mathrm{Ho}\,\iota$ is a functor of 1-categories, we can identify $((\mathrm{Ho}\,\iota)_*)^{\simeq}$ with the following functor.

$$((\mathrm{Ho}\,\iota)_*)^{\simeq}\colon \mathrm{Fun}(\mathrm{Ho}\,\mathcal{I},\mathrm{Ho}\,\mathcal{C}')^{\simeq} \to \mathrm{Fun}(\mathrm{Ho}\,\mathcal{I},\mathrm{Ho}\,\mathcal{C})^{\simeq}$$

Let F and G be two functors from $\mathrm{Ho}\,\mathcal{I}$ to $\mathrm{Ho}\,\mathcal{C}'$, considered as objects of $\mathrm{Fun}(\mathrm{Ho}\,\mathcal{I},\mathrm{Ho}\,\mathcal{C}')^{\simeq}$. By criterion Proposition B.1.1.1 (6) it suffices to show that postcomposition with $\mathrm{Ho}\,\iota$ induces an equivalence on mapping spaces as follows.

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Ho}\,\mathcal{I},\mathrm{Ho}\,\mathcal{C}')^{\simeq}}(X,Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{Ho}\,\mathcal{I},\mathrm{Ho}\,\mathcal{C})^{\simeq}}(\iota\circ X,\iota\circ Y) \qquad (**)$$

By Remark B.2.0.2 together with Proposition B.3.0.1 the functor $(\mathrm{Ho}\,\iota)_*$ is faithful, so by Proposition B.2.0.4, the map $(**)$ is already a monomorphism, so that it suffices to show that it induces a surjection on π_0. So let $\Phi\colon \iota\circ F \to \iota\circ G$ be a natural isomorphism of functors from $\mathrm{Ho}\,\mathcal{I}$ to $\mathrm{Ho}\,\mathcal{C}$. We have to show that we can lift Φ to a natural transformation from F to G. Let X be an object of $\mathrm{Ho}\,\mathcal{I}$. Then we can apply the assumption on ι and lift the isomorphism $\Phi_X\colon \iota(F(X)) \to \iota(G(X))$ in $\mathrm{Ho}\,\mathcal{C}'$ to an isomorphism $\Phi'_X\colon F(X) \to G(X)$ in $\mathrm{Ho}\,\mathcal{C}$ such that $\iota(\Phi'_X) = \Phi_X$. It remains to check that

Φ' defines a natural transformation from F to G. As F and G are functors of 1-categories, this is a property, not data, and it suffices to check that for every morphism $f\colon X \to Y$ in $\mathcal{I}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\Phi'_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\Phi'_Y"'] & G(Y) \end{tikzcd}$$

commutes. But as $\mathrm{Ho}\,\iota$ is faithful, it suffices to check that ι applied to this square yields a commutative square, which is the case as Φ is a natural transformation. □

Proposition B.4.1.2. *Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a monomorphism in $\mathcal{C}\mathrm{at}_\infty$. Then ι is conservative, i. e. reflect equivalences.* ♡

Proof. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}'$ such that $\iota(f)$ is an equivalence. By Proposition B.4.1.1 (4) we can lift $\iota(f)$ to an equivalence $f'\colon X \to Y$ in $\mathcal{C}'$. But faithfulness of ι implies that $\pi_0(\mathrm{Map}_{\mathcal{C}'}(X, Y)) \to \pi_0(\mathrm{Map}_{\mathcal{C}}(\iota X, \iota Y))$ is injective, hence f and f' must be homotopic, so f is an equivalence as well. □

B.4.2. Pseudomonic functors and replete images

The notion of monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ corresponds to the notion of *pseudomonic* functors of 1-categories, as we discuss in this section. Like injective maps of sets, pseudomonic functors of 1-categories are, up to equivalence, determined by their image. In the case of pseudomonic functors we will usually consider a more invariant notion of image, the *replete image*, which we also introduce below.

Remark B.4.2.1. Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a monomorphism in $\mathcal{C}\mathrm{at}_\infty$. Then if follows immediately from Proposition B.4.1.1 (4) and Remark B.2.0.2 that $\mathrm{Ho}\,\iota\colon \mathrm{Ho}\,\mathcal{C}' \to \mathrm{Ho}\,\mathcal{C}$ is a pseudomonic functor, i. e. $\mathrm{Ho}\,\iota$ satisfies the following two conditions.

(1) $\mathrm{Ho}\,\iota$ is faithful.

(2) If X and Y are two objects of $\mathrm{Ho}\,\mathcal{C}'$ and $f\colon (\mathrm{Ho}\,\iota)(X) \to (\mathrm{Ho}\,\iota)(Y)$ is an isomorphism in $\mathrm{Ho}\,\mathcal{C}$, then f lifts to an isomorphism $f'\colon X \to Y$ in $\mathrm{Ho}\,\mathcal{C}'$ such that $(\mathrm{Ho}\,\iota)(f') = f$.

If $\iota\colon \mathsf{C}' \to \mathsf{C}$ is a pseudomonic functor of 1-categories, then it follows similarly that ι is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$. ♢

Definition B.4.2.2. Let C' be a subcategory of the 1-category C. We say that C' is a *replete* subcategory of C if the collection of morphisms in C' is closed under isomorphisms in the arrow category $\mathrm{Fun}([1], \mathsf{C})$.

If $F\colon \mathsf{C} \to \mathsf{D}$ is a functor of 1-categories, then the *replete image* $\operatorname{Im} F$ of F is the replete subcategory of D generated by the image of F, i. e. it consists of those objects isomorphic to an object of the form $F(X)$ for X in C, and those morphisms isomorphic in the arrow category of D to a morphism of the form $F(f)$ for f a morphism in C. ♢

Remark B.4.2.3. Let $\iota\colon \mathsf{C}' \to \mathsf{C}$ be a pseudomonic functor of 1-categories. Then it follows directly from the definitions that the induced functor

$$\iota'\colon \mathsf{C}' \to \operatorname{Im}\iota$$

is essentially surjective as well as fully faithful and thus an equivalence. ♢

B.4.3. Lifting along monomorphisms

We now show that monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ have the expected property: We can check whether two functors into the domain of a monomorphism ι are homotopic by checking their compositions with ι, and any functor into the target of ι can be lifted as long as its image is contained in the image of ι.

Proposition B.4.3.1. *Let $\iota\colon \mathcal{D}' \to \mathcal{D}$ be a monomorphism in $\mathcal{C}\mathrm{at}_\infty$ and $F\colon \mathcal{C} \to \mathcal{D}$ a functor of ∞-categories.*

Then F can be lifted along ι, i. e. there exists a commutative diagram as follows

$$\begin{tikzcd} & \mathcal{D}' \arrow[d, "\iota"] \\ \mathcal{C} \arrow[ur, dashed, "F'"] \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

if and only if $\operatorname{Im}(\operatorname{Ho} F)$ *is contained in* $\operatorname{Im}(\operatorname{Ho}\iota)$. *If this is the case, then the lift is essentially unique in the sense that the fiber over F of the map*

$$(\iota_*)^{\simeq}\colon\ \operatorname{Fun}(\mathcal{C}, \mathcal{D}')^{\simeq} \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$$

is contractible. ♡

Proof. Proof of the "only if"-direction: Clear.

Proof of the "if"-direction: By Proposition B.4.1.1 ι is faithful and so the right square in the following commutative diagram is a pullback square by Proposition B.2.0.3.

$$\begin{tikzcd} & \mathcal{D}' \arrow[r] \arrow[d, "\iota"] & \operatorname{Ho}\mathcal{D}' \arrow[d, "\operatorname{Ho}\iota"] \\ \mathcal{C} \arrow[ur, dashed, "F'"] \arrow[r, "F"'] & \mathcal{D} \arrow[r] & \operatorname{Ho}\mathcal{D} \end{tikzcd}$$

It thus suffices to show that the composition $\widetilde{F}$ of F with the canonical functor $\mathcal{D} \to \operatorname{Ho}\mathcal{D}$ can be lifted along $\operatorname{Ho}\iota$. But $\operatorname{Ho}\iota$ factors by Remark B.4.2.3

as an equivalence composed with the inclusion $\mathrm{Im}(\mathrm{Ho}\,\iota) \to \mathrm{Ho}\,\mathcal{C}$, and by assumption $\widetilde{F}$ factors over this inclusion.

Proof that the lift is essentially unique if it exists: As we assume a lift exists, the fiber can not be empty. That it is then contractible follows from Proposition B.4.1.1 (2). □

B.4.4. (Fully) faithful functors are monomorphisms

In this short section we show that (fully) faithful functors are monomorphisms.

Proposition B.4.4.1. *Fully faithful functors of* ∞*-categories are monomorphisms in* $\mathcal{C}\mathrm{at}_\infty$*.* ♡

Proof. Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a fully faithful functor. We will use criterion Proposition B.4.1.1 (4). That ι is faithful is clear. Let X and Y be objects of $\mathcal{C}'$ and $f\colon \iota X \to \iota Y$ an equivalence in $\mathcal{C}$. Let f^{-1} be an inverse of f. As ι is fully faithful, we can lift f to a morphism $f'\colon X \to Y$ and f^{-1} to a morphism $f''\colon Y \to X$. But as ι also induces an equivalence

$$\mathrm{Map}_{\mathcal{C}'}(X, X) \to \mathrm{Map}_{\mathcal{C}}(\iota X, \iota X)$$

we can also lift the homotopy $f^{-1} \circ f \simeq \mathrm{id}_{\iota X}$ to a homotopy $f'' \circ f' \simeq \mathrm{id}_X$, and similarly $f' \circ f'' \simeq \mathrm{id}_Y$, so $f'\colon X \to Y$ is an equivalence with $\iota f' \simeq f$. □

B.5. Stability properties of (fully) faithful functors and monomorphisms in $\mathcal{C}\mathrm{at}_\infty$

In this section we show that monomorphism in $\mathcal{C}\mathrm{at}_\infty$ as well as (fully) faithful functors are stable under various constructions. In Section B.5.1 we handle the case of induced functors on functor ∞-categories, and in Sections B.5.2 and B.5.3 we discuss two types of stability under taking pullbacks.

Section B.5.2 concerns taking the pullback along an arbitrary other functor, i. e. we show that if $\iota_{\mathcal{D}}$ is faithful (fully faithful, a monomorphism), then the functor $\iota_{\mathcal{C}}$, defined via a pullback diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\iota_{\mathcal{C}}"] \arrow[d, "F'"'] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r, "\iota_{\mathcal{D}}"'] & \mathcal{D} \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$, with F any functor, is so as well.

In Section B.5.3 we instead consider stability under taking pullbacks in the arrow ∞-category; in Proposition B.1.3.1 we already showed that a natural transformation between two diagrams that is pointwise a monomorphism induces a monomorphism between the two limits. Section B.5.3 specializes this to the case of pullbacks in $\mathcal{C}\mathrm{at}_\infty$, and adds additional information regarding the replete image of the induced functor.

B.5.1. Functor ∞-categories

Proposition B.5.1.1. *Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a monomorphism in $\mathcal{C}\mathrm{at}_\infty$ and $\mathcal{I}$ an ∞-category. Then the induced functor*

$$\iota_*\colon\ \mathrm{Fun}(\mathcal{I}, \mathcal{C}') \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$$

is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$ as well.

Let J be defined to be the replete subcategory of $\mathrm{Ho}\,\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ where

- *a functor $F\colon \mathcal{I} \to \mathcal{C}$ considered as an object of $\mathrm{Ho}\,\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ is in J if and only if $\mathrm{Im}(\mathrm{Ho}\,F)$ is contained in $\mathrm{Im}(\mathrm{Ho}\,\iota)$.*
- *a natural transformation $\Phi\colon F \to G$ of functors $\mathcal{I} \to \mathcal{C}$, considered as a morphism from F to G in $\mathrm{Ho}\,\mathrm{Fun}(\mathcal{I}, \mathcal{C})$, is in J if and only if F and G are objects of J and Φ_X is in $\mathrm{Im}(\mathrm{Ho}\,\iota)$ for every object X of $\mathcal{I}$.*

Then the replete image $\mathrm{Im}(\mathrm{Ho}\,\iota_*)$ of the functor

$$\mathrm{Ho}(\iota_*)\colon\ \mathrm{Ho}\,\mathrm{Fun}(\mathcal{I}, \mathcal{C}') \to \mathrm{Ho}\,\mathrm{Fun}(\mathcal{I}, \mathcal{C})$$

is equal to J. ♡

Proof. Proof that ι_* is a monomorphism: Follows from description Proposition B.4.1.1 (2) using that for any ∞-category $\mathcal{J}$ there is a natural equivalence as follows.

$$\mathrm{Fun}(\mathcal{J}, \mathrm{Fun}(\mathcal{I}, -)) \simeq \mathrm{Fun}(\mathcal{J} \times \mathcal{I}, -)$$

Proof that $\mathrm{Im}(\mathrm{Ho}\,\iota_*)$ is contained in J: Clear

Proof that J is contained in $\mathrm{Im}(\mathrm{Ho}\,\iota_*)$: It suffices to show an inclusion of the respective collection of morphisms, as the case of objects is covered by the identity morphisms. So let $\Phi\colon F \to G$ be a natural transformation of functors $\mathcal{I} \to \mathcal{C}$, considered as a morphism from F to G in $\mathrm{Ho}\,\mathrm{Fun}(\mathcal{I}, \mathcal{C})$, and assume that Φ lies in J. What we have to show is that Φ can be lifted along ι, i. e. that there is a natural transformation Φ' of functors $\mathcal{I} \to \mathcal{C}'$ such that $\iota \circ \Phi' \simeq \Phi$. But we can encode Φ as a functor $\widetilde{\Phi}\colon [1] \times \mathcal{I} \to \mathcal{C}$, and the assumptions mean precisely that $\mathrm{Im}(\mathrm{Ho}\,\widetilde{\Phi})$ is contained $\mathrm{Im}(\mathrm{Ho}\,\iota)$. That we can lift Φ along ι now follows from Proposition B.4.3.1. □

Remark B.5.1.2. Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a fully faithful functor. By B.4.4.1 ι is also a monomorphism in $\mathcal{C}\mathrm{at}_\infty$, so we can apply Proposition B.5.1.1. In this case, the replete subcategory J of $\mathrm{Ho}\,\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ appearing in the statement of Proposition B.5.1.1 has a simpler description, using that $\mathrm{Ho}\,\iota$ is full: J is the full subcategory of $\mathrm{Ho}\,\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ spanned by those functors $F\colon \mathcal{I} \to \mathcal{C}$ for which $F(X)$ is in the essential image of $\mathrm{Ho}\,\iota$ for every object X of $\mathcal{I}$. ♢

B.5.2. Pullbacks along another functor

Proposition B.5.2.1. *Let*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\iota_{\mathcal{C}}"] \arrow[d, "F'"'] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r, "\iota_{\mathcal{D}}"'] & \mathcal{D} \end{tikzcd} \tag{B.5}$$

be a pullback square in $\mathcal{C}\mathrm{at}_\infty$ and assume that $\iota_{\mathcal{D}}$ is faithful (fully faithful, a monomorphism). Then $\iota_{\mathcal{C}}$ is faithful (fully faithful, a monomorphism) as well.

Furthermore, if $\iota_{\mathcal{D}}$ is a monomorphism[5]*, then the induced diagram on homotopy categories*

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{C}') \arrow[r, "\mathrm{Ho}\,\iota_{\mathcal{C}}"] \arrow[d, "\mathrm{Ho}\,F'"'] & \mathrm{Ho}(\mathcal{C}) \arrow[d, "\mathrm{Ho}\,F"] \\ \mathrm{Ho}(\mathcal{D}') \arrow[r, "\mathrm{Ho}\,\iota_{\mathcal{D}}"'] & \mathrm{Ho}(\mathcal{D}) \end{tikzcd} \tag{B.6}$$

is a pullback[6]*.* ♡

Proof. That $\iota_{\mathcal{C}}$ is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$ if $\iota_{\mathcal{D}}$ is follows immediately from pullbacks of monomorphisms being pullbacks again, see [HTT, 5.5.6.12].

We next show the first statement for (fully) faithful functors. Let X and Y be objects of $\mathcal{C}'$. We have to show that the map in $\mathcal{S}$

$$\mathrm{Map}_{\mathcal{C}'}(X, Y) \to \mathrm{Map}_{\mathcal{C}}(\iota_{\mathcal{C}}(X), \iota_{\mathcal{C}}(Y))$$

induced by $\iota_{\mathcal{C}}$ is a monomorphism (is an equivalence). By Proposition A.2.0.2 the commutative square induced by (B.5)

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}'}(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(\iota_{\mathcal{C}}(X), \iota_{\mathcal{C}}(Y)) \arrow[d] \\ \mathrm{Map}_{\mathcal{D}'}(F'(X), F'(Y)) \arrow[r] & \mathrm{Map}_{\mathcal{D}}(\iota_{\mathcal{D}}(F'(X)), \iota_{\mathcal{D}}(F'(Y))) \end{tikzcd}$$

is a pullback diagram in $\mathcal{S}$. As $\iota_{\mathcal{D}}$ is (fully) faithful the lower horizontal map is a monomorphism (equivalence), and hence so is the upper horizontal map (see [HTT, 5.5.6.12] for monomorphisms being preserved by pullbacks) This shows that $\iota_{\mathcal{C}}$ is (fully) faithful.

Finally it remains to show that diagram (B.6) is a pullback diagram if $\iota_{\mathcal{D}}$ is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$. By Remark B.4.2.1, the functors $\mathrm{Ho}\,\iota_{\mathcal{D}}$ and $\mathrm{Ho}\,\iota_{\mathcal{C}}$ are pseudomonic, so this boils down to showing that the replete image of

[5] Recall that by Proposition B.4.4.1 fully faithful functors are automatically monomorphisms in $\mathcal{C}\mathrm{at}_\infty$.

[6] We take the pullback in the ∞-category of 1-categories.

$\mathrm{Ho}\,\iota_{\mathcal{C}}$ is equal to the $\mathrm{Ho}\,F$-preimage of the replete image of $\mathrm{Ho}\,\iota_{\mathcal{D}}$. It is clear that $\mathrm{Ho}\,F$ maps the replete image of $\mathrm{Ho}\,\iota_{\mathcal{C}}$ to the replete image of $\mathrm{Ho}\,\iota_{\mathcal{D}}$. On the other hand, if f is a morphism in $\mathcal{C}$ such that $\mathrm{Ho}\,F(f)$ is in the replete image of $\mathrm{Ho}\,\iota_{\mathcal{D}}$, then there must exist a morphism g' in $\mathcal{D}$ and an equivalence $\iota_{\mathcal{D}}(g') \simeq F(f)$ in $\mathrm{Fun}([1], \mathcal{D})$. We can interpret the situation as a commuting square as depicted as the outer square in the following diagram.

$$\begin{tikzcd}
{[1]} \arrow[dr, dashed] \arrow[drr, bend left, "\mathrm{const}_f"] \arrow[ddr, bend right, "\mathrm{const}_{g'}"'] & & \\
& \mathcal{C}' \arrow[r, "\iota_{\mathcal{C}}"] \arrow[d, "F'"'] & \mathcal{C} \arrow[d, "F"] \\
& \mathcal{D}' \arrow[r, "\iota_{\mathcal{D}}"'] & \mathcal{D}
\end{tikzcd}$$

As the small square is a pullback square we obtain the dashed functor, which we can interpret as a morphism in $\mathcal{C}'$ that is mapped by $\mathrm{Ho}\,\iota_{\mathcal{C}}$ to a morphism isomorphic to C. That the objects of the two replete subcategories we are to compare agree can be proven analogously, or deduced from this by considering identity morphisms. □

B.5.3. Pullbacks

Proposition B.5.3.1. *Let*

$$\begin{tikzcd}
& \mathcal{C}' \arrow[rr] \arrow[dl, "F_{\mathcal{C}}"'] \arrow[dd] & & \mathcal{D}' \arrow[dl, "F_{\mathcal{D}}"'] \arrow[dd, "R'"] \\
\mathcal{C} \arrow[rr, "P"] \arrow[dd, "Q"'] & & \mathcal{D} \arrow[dd, "R"] & \\
& \mathcal{E}' \arrow[rr, "S'"] \arrow[dl, "F_{\mathcal{E}}"'] & & \mathcal{F}' \arrow[dl, "F_{\mathcal{F}}"'] \\
\mathcal{E} \arrow[rr, "S"'] & & \mathcal{F} &
\end{tikzcd} \tag{B.7}$$

be a commuting cube of ∞-categories such that $F_{\mathcal{D}}$, $F_{\mathcal{E}}$, and $F_{\mathcal{F}}$ are faithful (fully faithful, monomorphisms) and the front and back squares are pullback squares in Cat_∞. Then the functor $F_{\mathcal{C}}$ is faithful (fully faithful, a monomorphism) as well.

Furthermore, if $F_{\mathcal{F}}$ is a monomorphism[7] *in Cat_∞, then an object (morphism) in $\mathrm{Ho}\,\mathcal{C}$ is in $\mathrm{Im}(\mathrm{Ho}\,F_{\mathcal{C}})$ if and only if it is mapped by $\mathrm{Ho}\,P$ and $\mathrm{Ho}\,Q$ to an object (morphism) in $\mathrm{Im}(\mathrm{Ho}\,F_{\mathcal{D}})$ and $\mathrm{Im}(\mathrm{Ho}\,F_{\mathcal{E}})$, respectively.* ♡

Proof. To show that $F_{\mathcal{C}}$ is again faithful or fully faithful we apply Proposition A.2.0.2 and use Proposition B.1.3.1 and that the formation of pullbacks

[7]By Proposition B.4.4.1, fully faithful functors are monomorphisms as well.

is invariant under equivalences. The case of monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ is even simpler, as it follows directly from Proposition B.1.3.1.

It remains to show the statement concerning replete images. The "only if"-direction is clear. We show that a morphism in $\operatorname{Ho}\mathcal{C}$ satisfying the assumption lies in $\operatorname{Im}(\operatorname{Ho}F_{\mathcal{C}})$, the statement for objects follows from this by considering identity morphisms. As the front of (B.7) is a pullback diagram, a morphism in $\mathcal{C}$ satisfying the assumptions corresponds to a commutative square

$$\begin{tikzcd}
{[1]} \arrow[r, "\Phi_{\mathcal{D}}"] \arrow[d, "\Phi_{\mathcal{E}}"'] & \mathcal{D} \arrow[d, "R"] \\
\mathcal{E} \arrow[r, "S"'] & \mathcal{F}
\end{tikzcd}$$

such that $\operatorname{Im}(\operatorname{Ho}\Phi_{\mathcal{D}})$ is contained in $\operatorname{Im}(F_{\mathcal{D}})$ and $\operatorname{Im}(\operatorname{Ho}\Phi_{\mathcal{E}})$ is contained in $\operatorname{Im}(F_{\mathcal{E}})$. What we have to show is that we can extend this square to a commutative cube as follows.

$$\begin{tikzcd}
& {[1]} \arrow[rr, dashed, "\Phi_{\mathcal{D}'}"] \arrow[dl, "\mathrm{id}_{[1]}"'] \arrow[dd, dashed, "\Phi_{\mathcal{E}'}"] & & \mathcal{D}' \arrow[dl, "F_{\mathcal{D}}"'] \arrow[dd, "R'"] \\
{[1]} \arrow[rr, "\Phi_{\mathcal{D}}"] \arrow[dd, "\Phi_{\mathcal{E}}"'] & & \mathcal{D} \arrow[dd, "R"] & \\
& \mathcal{E}' \arrow[rr, "S'"] \arrow[dl, "F_{\mathcal{E}}"'] & & \mathcal{F}' \arrow[dl, "F_{\mathcal{F}}"'] \\
\mathcal{E} \arrow[rr, "S"'] & & \mathcal{F} &
\end{tikzcd} \tag{$*$}$$

The assumptions on $\operatorname{Im}(\operatorname{Ho}\Phi_{\mathcal{D}})$ and $\operatorname{Im}(\operatorname{Ho}\Phi_{\mathcal{E}})$ imply that we can fill the dashed arrows together with the top and left squares by Proposition B.4.3.1, as $F_{\mathcal{D}}$ and $F_{\mathcal{D}}$ are monomorphisms. We are left to find a filler for the back square and the cube. But this amounts to lifting the homotopy between $F_{\mathcal{F}} \circ R' \circ \Phi_{\mathcal{D}'}$ and $F_{\mathcal{F}} \circ S' \circ \Phi_{\mathcal{E}'}$ encoded by the other sides to a homotopy between $R' \circ \Phi_{\mathcal{D}'}$ and $S' \circ \Phi_{\mathcal{E}'}$. This is possible as the following map induced by $F_{\mathcal{F}}$

$$\begin{gathered}
\operatorname{Map}_{(\operatorname{Fun}([1],\mathcal{F}')^{\simeq})}(R' \circ \Phi_{\mathcal{D}'}, S' \circ \Phi_{\mathcal{E}'}) \\
\to \operatorname{Map}_{(\operatorname{Fun}([1],\mathcal{F})^{\simeq})}(F_{\mathcal{F}} \circ R' \circ \Phi_{\mathcal{D}'}, F_{\mathcal{F}} \circ S' \circ \Phi_{\mathcal{E}'})
\end{gathered}$$

is an equivalence by Proposition B.4.1.1 (2) and Proposition B.1.1.1 (6). □

B.6. Subcategories

In this short section we briefly discuss how monomorphisms into a fixed ∞-category $\mathcal{C}$ correspond to replete subcategories of $\operatorname{Ho}\mathcal{C}$.

Remark B.6.0.1. Let $\mathcal{C}$ be an ∞-category and $(\mathrm{Ho}\,\mathcal{C})'$ a replete subcategory of $\mathrm{Ho}\,\mathcal{C}$. Then define $\iota\colon \mathcal{C}' \to \mathcal{C}$ as in the following pullback diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\iota"] \arrow[d] & \mathcal{C} \arrow[d] \\ (\mathrm{Ho}\,\mathcal{C})' \arrow[r, "\iota'"'] & \mathrm{Ho}\,\mathcal{C} \end{tikzcd}$$

where the right vertical functor is the canonical one. As the inclusion of a replete subcategory of a 1-category is a pseudomonic functor of 1-categories, it follows from Remark B.4.2.1 that ι' is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$. By Proposition B.5.2.1 ι is also a monomorphism, and furthermore the induced functor $\mathrm{Ho}(\mathcal{C}') \to \mathrm{Ho}(\mathcal{C})'$ is an equivalence[8], so $\mathrm{Im}(\mathrm{Ho}\,\iota) = (\mathrm{Ho}\,\mathcal{C})'$.

By Proposition B.4.3.1, two monomorphisms $\iota'\colon \mathcal{C}' \to \mathcal{C}$ and $\iota''\colon \mathcal{C}'' \to \mathcal{C}$ are equivalent as functors to $\mathcal{C}$ in the sense that there is a commutative triangle

$$\begin{tikzcd} \mathcal{C}' \arrow[dr, "\iota'"] \arrow[dd, dash, "\simeq"'] & \\ & \mathcal{C} \\ \mathcal{C}'' \arrow[ur, "\iota''"'] & \end{tikzcd}$$

if and only if $\mathrm{Im}(\mathrm{Ho}\,\iota') = \mathrm{Im}(\mathrm{Ho}\,\iota'')$. This implies that all monomorphisms arise up to equivalence from the above construction, and that there is a bijection between equivalence classes of monomorphisms with target $\mathcal{C}$ and replete subcategories of $\mathrm{Ho}\,\mathcal{C}$. $\diamondsuit$

[8] As $\mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}(\mathrm{Ho}\,\mathcal{C})$ is.

Appendix C.

(Co)Cartesian Fibrations

For many technical parts of this thesis, (co)cartesian fibrations play a crucial role. For a very readable model-independent introduction [Maz19] can be recommended. For a full introduction to (co)cartesian fibrations and their properties in the setting of quasicategories see [HTT, 2.4]. We will follow [Maz19] in deviating somewhat from Lurie's terminology by using a more model-independent definition: For us, a cartesian fibration is a morphism in $\mathcal{C}\mathrm{at}_\infty$ that can be represented by a morphism of quasicategories that is a cartesian fibration in Lurie's sense (see [HTT, 2.4.2.1]). Equivalently, those are the functors of ∞-categories which satisfy condition [HTT, 2.4.1.1 (2)], with the pullback in the definition of cartesian morphisms in [HTT, 2.4.1.1] replaced by the homotopy pullback in $\mathcal{C}\mathrm{at}_\infty$. For a definition along these lines, see [Maz19, 3]. It is shown in [Maz19, 4.3 and 4.4] that these two descriptions coincide, and we can thus use the latter model-independent definition while still making use of all the properties of (co)cartesian fibrations proved in [HTT].

In this appendix we collect some statements relating to (co)cartesian fibrations that we need; in Section C.1 we will show a number of stability statements, and in Section C.2 we will discuss compatibility of cocartesian fibrations with products.

C.1. Stability properties of (co)cartesian fibrations

In this section we discuss stability of (co)cartesian fibrations under some constructions. Concretely, in Section C.1.1 we consider pullbacks of cartesian fibrations along any other functor, in Section C.1.2 we discuss a condition under which restrictions of cartesian fibrations along fully faithful functors are again cartesian fibrations, and in Section C.1.3 we show that if $p\colon \mathcal{C} \to \mathcal{D}$ and $q\colon \mathcal{D} \to \mathcal{E}$ are cartesian fibrations, then p is also a morphism of cartesian fibrations from qp to q, i. e. maps qp-cartesian morphisms to q-cartesian morphisms.

Remark C.1.0.1. The definitions of cocartesian and cartesian fibrations are dual to each other: $p\colon \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration if and only if

$p^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a cartesian fibration [HTT, 2.4.2.1]. Because of this it suffices to prove many statements for only one of the two (usually cartesian fibrations), the other case following by passing to opposite ∞-categories. To avoid overly long statements we will not state the dual versions in the propositions below, but use them without further comment. ♢

C.1.1. Pullbacks

We record the following fact, that is clear from [HTT, 2.4.1 and 2.4.2], but not stated like this.

Proposition C.1.1.1. *Let*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "F"] \arrow[d, "p'"'] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D}' \arrow[r] & \mathcal{D} \end{tikzcd}$$

be a pullback diagram of ∞-categories where p is a cartesian fibration.

Then p' is also a cartesian fibration and a morphism $\varphi\colon X \to Y$ in $\mathcal{C}'$ is p'-cartesian if and only if $F(\varphi)$ is p-cartesian. ♡

Proof. That p' is also a cartesian fibration is [HTT, 2.4.2.3 (2)], which follows from [HTT, 2.4.1.3 (2)], which also covers the "if"-direction. For the "only if"-direction, let $\varphi\colon X \to Y$ be a p'-cartesian morphism in $\mathcal{C}'$. Then φ is in particular locally p'-cartesian[1], so we can apply [HTT, 2.4.1.12] to conclude that $F(\varphi)$ is locally p-cartesian. As p is a cartesian fibration we can then apply [HTT, 2.4.2.13] to show that $F(\varphi)$ is in fact p-cartesian. □

C.1.2. Restriction along fully faithful functors

Proposition C.1.2.1. *Let $p'\colon \mathcal{C}' \to \mathcal{D}$ be a cartesian fibration of ∞-categories and $\iota\colon \mathcal{C} \to \mathcal{C}'$ a fully faithful functor. Assume that for every object Y in $\mathcal{C}$ and every p'-cartesian morphism $f'\colon X' \to \iota(Y)$ in $\mathcal{C}'$ there is an object X in $\mathcal{C}$ with $\iota(X) \simeq X'$.*

Let $p = p'\iota$. Then p is also a cartesian fibration, and a morphism f in $\mathcal{C}$ is p-cartesian if and only if $\iota(f)$ is p'-cartesian. ♡

Proof. We start by noting that the "if"-direction, i. e. the criterion for checking when a morphism of $\mathcal{C}$ is p-cartesian, follows immediately from [HTT, 2.4.4.3].

We can now use this criterion to show that p has a sufficient supply of cartesian lifts to be a cartesian fibration. So let Y be an object in $\mathcal{C}$ and $g\colon X \to p(Y)$ a morphism in $\mathcal{D}$. Then there exists a p'-cartesian lift

[1] This follows from the already proved "if"-direction. See [HTT, 2.4.1.11] for a definition of locally p'-cartesian morphisms.

$\overline{g}' \colon \overline{X}' \to \iota(Y)$ in $\mathcal{C}'$, as p' is a cartesian fibration. By the assumption on ι, there exists an object $\overline{X}$ of $\mathcal{C}$ such that $\iota(\overline{X}) \simeq \overline{X}'$. As ι is also fully faithful, there exists a morphism $\overline{g} \colon \overline{X} \to Y$ in $\mathcal{C}$ such that $\iota(\overline{g}) \simeq \overline{g}'$ and hence $p(\overline{g}) \simeq g$. We can now use the already proven criterion to deduce that $\overline{g}$ is p-cartesian from $\overline{g}'$ being p'-cartesian. This finishes the proof that p is a cartesian fibration.

Finally, let $f \colon X \to Z$ be a p-cartesian morphism in $\mathcal{C}$. We want to show that $\iota(f)$ is p'-cartesian. In $\mathcal{C}'$ we can factor $\iota(f)$ as $\iota(f) = \psi' \circ \varphi'$, where ψ' is p'-cartesian and φ' is a morphism in $\mathcal{C}'_{p(X)}$, as depicted in the following commutative diagram

$$\begin{tikzcd} \iota(X) \arrow[r, "\varphi'"] \arrow[dr, "\iota(f)"'] & Y' \arrow[d, "\psi'"] \\ & \iota(Z) \end{tikzcd}$$

lying over the following commutative diagram in $\mathcal{D}$.

$$\begin{tikzcd} p(X) \arrow[r, "\mathrm{id}_{p(X)}"] \arrow[dr, "p(f)"'] & p(X) \arrow[d, "p(f)"] \\ & p(Z) \end{tikzcd}$$

Using the assumptions on ι, we can find an object Y in $\mathcal{C}$ together with an equivalence $\vartheta \colon Y' \xrightarrow{\simeq} \iota(Y)$, as well as a commuting diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & Y \arrow[dl, "\psi"] \\ & Z & \end{tikzcd}$$

in $\mathcal{C}$ which maps to the following composite commutative diagram in $\mathcal{C}'$.

$$\begin{tikzcd} \iota(X) \arrow[rr, "\iota(\varphi)"] \arrow[dr, dashed, "\varphi'"] \arrow[ddr, "\iota(f)"'] & & \iota(Y) \arrow[ddl, "\iota(\psi)"] \\ & Y' \arrow[ur, dashed, "\vartheta"'] \arrow[d, "\psi'"] & \\ & \iota(Z) & \end{tikzcd}$$

As ϑ is an equivalence and ψ' is p'-cartesian, also $\iota(\psi)$ is p'-cartesian, so that we can conclude that ψ is p-cartesian by the already proven “if”-direction. If follows from [HTT, 2.4.1.7] that φ is also p-cartesian. Furthermore, $p(\varphi)$ is an equivalence as the composition of the two equivalences $\mathrm{id}_{p(X)}$ and $p'(\vartheta)$, so by [HTT, 2.4.1.5] φ itself is an equivalence. Thus $\iota(\varphi)$ is an equivalence and hence by [HTT, 2.4.1.5] p'-cartesian, and so $\iota(f)$ is p'-cartesian by [HTT, 2.4.1.7]. □

C.1.3. Morphisms of cartesian fibrations

Proposition C.1.3.1. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "p"] \arrow[dr, "s"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{E} & \end{tikzcd}$$

be a commutative diagram of ∞-categories such that p, q, and s are cartesian fibrations.

Then p is a morphism of cartesian fibrations over $\mathcal{E}$, i. e. maps s-cartesian morphisms to q-cartesian morphisms. ♡

Proof. Let $f\colon X \to Y$ be an s-cartesian morphism in $\mathcal{C}$. As q is a cartesian fibration, there exists a q-cartesian lift $g\colon Z \to p(Y)$ in $\mathcal{D}$ of $s(f)$. As p is a cartesian fibration, we can further lift g to a p-cartesian morphism $f'\colon X' \to Y$ in $\mathcal{C}$. By [HTT, 2.4.1.3 (3)] f' is even s-cartesian, so by uniqueness of cartesian lifts (see [HTT, 2.4.1.9]) f' and f are equivalent as morphisms in $\mathcal{C}$ and hence $p(f) \simeq p(f') \simeq g$ is q-cartesian because g is. □

C.2. Cocartesian fibrations and products

Let $\mathcal{D}$ be an ∞-category and $F\colon \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$ a functor. Let $\mathcal{O}$ be an ∞-operad. By [HA, 2.4.2.4] the ∞-category $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ of $\mathcal{O}$-monoids in $\mathcal{C}\mathrm{at}_\infty$ can be identified with the ∞-category of $\mathcal{O}$-monoidal ∞-categories. If F preserves products, then we obtain an induced functor on $\mathcal{O}$-monoids, which we can thus interpret as functorially producing $\mathcal{O}$-monoidal ∞-categories out of $\mathcal{O}$-monoids in $\mathcal{D}$. We will be very interested in this situation in this thesis, in particular in Chapter 3. However, it will usually be easier to construct and work with the cocartesian fibration $p\colon \mathcal{C} \to \mathcal{D}$ associated to F rather than with F directly. For this reason we will start this section by describing the property of F preserving products in terms of the cocartesian fibration p (see Definition C.2.0.1), and will then prove some consequences of this property, as well as one result (see Proposition C.2.0.4) that can help deduce that a cocartesian fibration has this property.

Definition C.2.0.1. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration. We say that p has *fibers compatible with products* if $\mathcal{D}$ admits all products and for any set

I and collection of objects Y_i in $\mathcal{D}$ for $i \in I$, the functor

$$\mathcal{C}_{\prod_{i\in I} Y_i} \xrightarrow{\prod_{i\in I}(\mathrm{pr}_i)_!} \prod_{i\in I} \mathcal{C}_{Y_i} \qquad \diamondsuit$$

is an equivalence of ∞-categories, where $\mathrm{pr}_j\colon \prod_{i\in I} Y_i \to Y_j$ is the projection and $(\mathrm{pr}_j)_!$ is the functor induced by pr_j on fibers.

Remark C.2.0.2. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration that is classified by a functor $F : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$. Then p has fibers compatible with products if and only if $\mathcal{D}$ admits all products and F preserves products. $\diamondsuit$

If $p\colon \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration whose fibers are compatible with products, then we will see in the next proposition that $\mathcal{C}$ admits all products as well, and p preserves them. In fact we can say more and also describe concretely how to construct products in $\mathcal{C}$.

Proposition C.2.0.3. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration whose fibers are compatible with products in the sense of Definition C.2.0.1.*

Let I be a set and $(X_i)_{i\in I}$ a collection of objects in $\mathcal{C}$. As fibers of p are compatible with products, we obtain the following equivalence.

$$\mathcal{C}_{\prod_{i\in I} p(X_i)} \xrightarrow{\prod_{i\in I}(\mathrm{pr}_i)_!} \prod_{i\in I} \mathcal{C}_{p(X_i)}$$

There thus exists an object X in $\mathcal{C}$ lying over $\prod_{i\in I} p(X_i)$ together with p-cocartesian morphisms $\overline{\mathrm{pr}_i}\colon X \to X_i$ lying over the projections

$$\mathrm{pr}_i\colon \prod_{i\in I} p(X_i) \to p(X_i)$$

in $\mathcal{D}$.

Then the morphisms $\overline{\mathrm{pr}_i}$ exhibit X as a product of the collection of objects X_i for $i \in I$ in $\mathcal{C}$. In particular, $\mathcal{C}$ admits all products and p preserves products. $\heartsuit$

Proof. We use notation as in the statement. By [HTT, 4.4.1] we need to prove for every object Z of $\mathcal{C}$ that the map

$$\mathrm{Map}_{\mathcal{C}}(Z, X) \xrightarrow{\prod_{i\in I}(\overline{\mathrm{pr}_i}\circ -)} \prod_{i\in I} \mathrm{Map}_{\mathcal{C}}(Z, X_i)$$

is an equivalence. This map fits into the following commutative square as the left vertical map, with the horizontal maps being induced by p.

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(Z,X) \arrow[r] \arrow[d, "\prod_{i\in I}(\overline{\mathrm{pr}_i}\circ -)"'] & \mathrm{Map}_{\mathcal{D}}\Big(p(Z), \prod_{i\in I} p(X_i)\Big) \arrow[d, "\prod_{i\in I}(\mathrm{pr}_i\circ -)"] \\
\prod_{i\in I}\mathrm{Map}_{\mathcal{C}}(Z,X_i) \arrow[r] & \prod_{i\in I}\mathrm{Map}_{\mathcal{D}}(p(Z),p(X_i))
\end{tikzcd} \tag{$*$}$$

As by definition the projections pr_i exhibit $\prod_{i\in I} p(X_i)$ as a product of $(X_i)_{i\in I}$, it follows by [HTT, 4.4.1] that the right vertical map is an equivalence. Let $f\colon p(Z) \to \prod_{i\in I} p(X_i)$ be a morphism. We can extend diagram $(*)$ to a morphism of fiber sequences by taking the fiber of the top horizontal map over f and of the lower horizontal map over $(\mathrm{pr}_i \circ f)_{i\in I}$. By the five lemma it will then suffice to show that for every such f the induced map on fibers is an equivalence.

To identify this induced map on fibers, we let $\overline{f}\colon Z \to f_!Z$ be a p-cocartesian lift of f, let

$$\overline{\mathrm{pr}_j}'\colon f_!Z \to \mathrm{pr}_{j!}(f_!Z)$$

be a p-cocartesian lift of

$$\mathrm{pr}_j\colon \prod_{i\in I} p(X_i) \to p(X_j)$$

and ponder the following diagram.

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}_{\prod_{i\in I} p(X_i)}}(f_!Z,X) \arrow[r, "-\circ\overline{f}"] \arrow[d, "pr_!"] & \mathrm{Map}_{\mathcal{C}}(Z,X) \arrow[r, "p"] \arrow[d, "\overline{\mathrm{pr}_j}\circ -"] & \mathrm{Map}_{\mathcal{D}}\Big(p(Z), \prod_{i\in I} p(X_i)\Big) \arrow[d, "\mathrm{pr}_j\circ -"] \\
\mathrm{Map}_{\mathcal{C}_{p(X_j)}}\Big(\mathrm{pr}_{j!}(f_!Z), X_j\Big) \arrow[r, "-\circ\left(\overline{\mathrm{pr}_j}'\circ\overline{f}\right)"] & \mathrm{Map}_{\mathcal{C}}(Z,X_j) \arrow[r, "p"] & \mathrm{Map}_{\mathcal{D}}(p(Z),p(X_j))
\end{tikzcd} \tag{$**$}$$

The top and bottom rows come with homotopies of the composition to const_f and $\mathrm{const}_{\mathrm{pr}_j\circ f}$, respectively. For the top horizontal sequence this homotopy is indicated in the following diagram, the case for the lower horizontal diagram is analogous.

$$
\begin{tikzcd}
 & \mathrm{Map}_{\mathcal{C}_{\prod_{i\in I} p(X_i)}}(f_!Z, X) \arrow[dl, "-\circ\overline{f}"'] \arrow[d] \arrow[dd, bend left=60, "\mathrm{const}_{\mathrm{id}}"] \arrow[ddd, bend left=80, "\mathrm{const}_f"] \\
\mathrm{Map}_{\mathcal{C}}(Z, X) \arrow[ddr, bend right, "p"'] & \mathrm{Map}_{\mathcal{C}}(f_!Z, X) \arrow[l, "-\circ\overline{f}"] \arrow[d, "p"] \\
 & \mathrm{Map}_{\mathcal{D}}\Big(\prod_{i\in I} p(X_i), \prod_{i\in I} p(X_i)\Big) \arrow[d, "-\circ f"] \\
 & \mathrm{Map}_{\mathcal{D}}\Big(p(Z), \prod_{i\in I} p(X_i)\Big)
\end{tikzcd}
$$

By [HTT, 2.4.4.2 and the discussion preceding it], this homotopy upgrades the top row of diagram $(**)$ into a fiber sequence, and analogously for the bottom row.

Unpacking the various definitions we can also upgrade the vertical morphisms in diagram $(**)$ into a morphism of fiber sequences. For example commutativity of the left square essentially boils down to the functor

$$\mathrm{pr}_j \colon \mathcal{C}_{\prod p(X_i)} \to \mathcal{C}_{X_j}$$

by definition sending a morphism $g\colon f_!Z \to X$ to the essentially unique morphism $\mathrm{pr}_{i!}g$ that fits in a commutative diagram

$$
\begin{tikzcd}
f_!Z \arrow[r] \arrow[d, "g"'] & \mathrm{pr}_{i!}f_!Z \arrow[d, "\mathrm{pr}_{i!}g"] \\
X \arrow[r] & \mathrm{pr}_{i!}X
\end{tikzcd}
$$

where the horizontal morphisms are p-cocartesian lifts of pr_i, see [HTT, 5.2.1].

We have thus shown that the induced morphism on fibers (which we have to show is an equivalence) can be identified with the morphism

$$\prod_{i\in I}(\mathrm{pr}_{i!})\colon\ \mathrm{Map}_{\mathcal{C}_{\prod_{i\in I} p(X_i)}}(f_!Z, X) \to \prod_{i\in I} \mathrm{Map}_{\mathcal{C}_{p(X_j)}}\Big(\mathrm{pr}_{j!}(f_!Z), X_j\Big)$$

But that this is an equivalence follows immediately from

$$\prod_{i\in I}(\mathrm{pr}_{i!})\colon \mathcal{C}_{\prod_{i\in I} p(X_i)} \to \prod_{i\in I} \mathcal{C}_{p(X_j)}$$

being an equivalence and mapping spaces in products of ∞-categories being equivalent to the respective product of mapping spaces. $\square$

The following proposition will be key to show that some cocartesian fibrations we are interested in have fibers that are compatible with products.

Proposition C.2.0.4. *Let*

$$\begin{tikzcd} \mathcal{C}' \arrow[rr, "F"] \arrow[dr, "p'"'] & & \mathcal{C} \arrow[dl, "p"] \\ & \mathcal{D} & \end{tikzcd}$$

be a morphism of cocartesian fibrations over $\mathcal{D}$ and assume that p' and p have fibers that are compatible with products in the sense of Definition C.2.0.1.

If F is also a cocartesian fibration, then its fibers are also compatible with products. ♡

Proof. Let I be a set and $(X_i)_{i\in I}$ a collection of objects in $\mathcal{C}$. Proposition C.2.0.3 provides us with an object X in $\mathcal{C}_{\prod_{i\in I} p(X_i)}$ together with, for every element j of I, p-cocartesian lifts $\overline{\mathrm{pr}_i}\colon X \to X_j$ of the projections $\mathrm{pr}_j\colon \prod_{i\in I} p(X_i) \to p(X_j)$, such that the collection of morphisms $(\overline{\mathrm{pr}_i})_{i\in I}$ exhibits X as the product of $(X_i)_{i\in I}$ in $\mathcal{C}$.

As F is a morphism of cocartesian fibrations, we obtain a commutative square as depicted as the right hand square in the following diagram.

$$\begin{tikzcd} \mathcal{C}'_X \arrow[r] \arrow[d] & \mathcal{C}'_{\prod_{i\in I} p(X_i)} \arrow[r, "F_{\prod_{i\in I} p(X_i)}"] \arrow[d, "\prod_{i\in I}\left((\mathrm{pr}_i)_!^{p'}\right)"'] & \mathcal{C}_{\prod_{i\in I} p(X_i)} \arrow[d, "\prod_{i\in I}\left((\mathrm{pr}_i)_!^{p}\right)"] \\ \prod\limits_{i\in I}\mathcal{C}'_{X_i} \arrow[r] & \prod\limits_{i\in I}\mathcal{C}'_{p(X_i)} \arrow[r, "\prod_{i\in I} F_{p(X_i)}"] & \prod\limits_{i\in I}\mathcal{C}_{p(X_i)} \end{tikzcd} \tag{$*$}$$

Taking fibers in the horizontal direction, over X in the top line, and over $(X_i)_{i\in I}$ in the bottom line, we obtain the induced commutative square depicted on the left. As by assumption both p' and p have fibers that are compatible with products, the middle and right vertical functors are equivalences, and hence so is the induced left vertical functor. We are not quite done however, as a priori this functor is the induced functor constructed from p'-cocartesian morphisms, whereas we need to show that the functor

$$\prod_{i\in I}(\overline{\mathrm{pr}_i})_!^F : \mathcal{C}'_X \to \prod_{i\in I}\mathcal{C}'_{X_i} \tag{$**$}$$

is an equivalence, which is constructed from F-cocartesian morphisms.

So let Y be an object in $\mathcal{C}'_X$ and let $\overline{\mathrm{pr}_i}'\colon X \to \overline{\mathrm{pr}_i}_!(X)$ be an F-cocartesian lift of $\overline{\mathrm{pr}_i}$. As $\overline{\mathrm{pr}_i}'$ maps under F to the p-cocartesian morphism $\overline{\mathrm{pr}_i}$, we can conclude by [HTT, 2.4.1.3 (3)] that $\overline{\mathrm{pr}_i}'$ is in fact also an p'-cocartesian lift of pr_i. We can thus identify the functor $(**)$ with the left vertical functor in diagram $(*)$. □

If $p\colon \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration whose fibers are compatible with products, then by Proposition C.2.0.3 $\mathcal{C}$ admits products and p preserves products, so we obtain an induced symmetric monoidal functor $p^\times : \mathcal{C}^\times \to \mathcal{D}^\times$ with respect to the cartesian symmetric monoidal structures, see [HA, 2.4.1.8]. It will be useful for us to know that $p^\times$ is again a cocartesian fibration, so we will show this as Proposition C.2.0.6 below, after the following technical prerequisite.

Proposition C.2.0.5. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration with fibers compatible with products in the sense of Definition C.2.0.1. Then products of p-cocartesian morphisms are again p-cocartesian.* ♡

Proof. Let I be a set and let $f_i\colon C_i \to C'_i$ be a p-cocartesian morphism in $\mathcal{C}$ for every element i of I. We have to show that the product

$$f \coloneqq \prod_{i\in I} f_i\colon \prod_{i\in I} C_i \to \prod_{i\in I} C'_i$$

is p-cocartesian. By Proposition C.2.0.3, p preserves products, so f lies over the morphism $\prod_{i\in I} p(f_i)$. We can then factor f as indicated in the following diagram

$$\begin{tikzcd} & \varphi_!(\prod_{i\in I} C_i) \arrow[d, dashed, "\psi"] \\ \prod\limits_{i\in I} C_i \arrow[ur, "\varphi"] \arrow[r, "f"'] & \prod\limits_{i\in I} C'_i \end{tikzcd}$$

where φ is a p-cocartesian lift of $\prod_{i\in I} p(f_i)$ and ψ lies over $\mathrm{id}_{\prod_{i\in I} p(C'_i)}$. It then suffices to show that ψ is an equivalence.

Let i be an element of I, and let $\overline{\mathrm{pr}_i}\colon \varphi_!(\prod_{i\in I} C_i) \to C''_i$ be a p-cocartesian lift of $\mathrm{pr}_i\colon \prod_{i\in I} p(C'_i) \to p(C'_i)$. It then follows from Proposition C.2.0.3 that the collection $(\overline{\mathrm{pr}_i})_{i\in I}$ exhibits $\varphi_!(\prod_{i\in I} C_i)$ as a product $\prod_{i\in I} C''_i$. Furthermore, ψ induces morphisms $\psi_j\colon C''_j \to C'_j$ for every element j of I as in the following diagram, and ψ can thus be identified with the product $\prod_{i\in I} \psi_i$.

$$\begin{tikzcd} \prod\limits_{i\in I} C''_i \arrow[r, "\psi"] \arrow[d, "\mathrm{pr}_j"] & \prod\limits_{i\in I} C'_i \arrow[d, "\mathrm{pr}_j"] \\ C''_j \arrow[r, dashed, "\psi_j"] & C'_j \end{tikzcd}$$

The following commuting diagram depicts the situation:

$$\begin{tikzcd} \prod_{i\in I} C_i \arrow[r, "f"] \arrow[d, "\mathrm{pr}_j"] \arrow[rr, bend left, "\varphi"] & \prod_{i\in I} C_i' \arrow[d, "\mathrm{pr}_j"] & \prod_{i\in I} C_i'' \arrow[l, "\prod_{i\in I}\psi_i"'] \arrow[d, "\mathrm{pr}_j"] \\ C_j \arrow[r, "f_j"'] & C_j' & C_j'' \arrow[l, "\psi_j"] \end{tikzcd}$$

In the outer commuting diagram, all morphisms except possibly ψ_j are p-cocartesian, so by [HTT, 2.4.1.7] also ψ_j is p-cocartesian. It then follows from [HTT, 2.4.1.5] and $p(\psi_j) = \mathrm{id}_{p(C_j)}$ that ψ_j is even an equivalence. Hence $\psi = \prod_{i\in I}\psi_i$ is an equivalence, and thus f is p-cocartesian as it is equivalent to the p-cocartesian morphism φ. □

Proposition C.2.0.6. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration whose fibers are compatible with products in the sense of Definition C.2.0.1. Let*

$$p^\times : \mathcal{C}^\times \to \mathcal{D}^\times$$

be the induced symmetric monoidal functor between the respective cartesian symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{D}$ as in [HA, 2.4.1.8] (using that $\mathcal{C}$ has all products and p preserves products by Proposition C.2.0.3).

Then $p^\times$ is also a cocartesian fibration. ♡

Proof. We will apply [GHN15, 9.6][2] to the commutative triangle

$$\begin{tikzcd} \mathcal{C}^\times \arrow[rr, "p^\times"] \arrow[dr, "q"'] & & \mathcal{D}^\times \arrow[dl, "r"] \\ & \mathsf{Fin}_* & \end{tikzcd}$$

where q and r are the cocartesian fibrations that are part of the structure of a symmetric monoidal ∞-category. In this situation (the dual version of) [GHN15, 9.6] states that $p^\times$ is a cocartesian fibration if the following hold:

(a) q and r are cocartesian fibrations.

(b) $p^\times$ sends q-cocartesian morphisms to r-cocartesian morphisms.

(c) For each object $\langle n\rangle$ in Fin_*, the induced functor on fibers

$$p^\times_{\langle n\rangle} : \mathcal{C}^\times_{\langle n\rangle} \to \mathcal{D}^\times_{\langle n\rangle}$$

is a cocartesian fibration.

[2][GHN17] is the published version of [GHN15], but does not contain [GHN15, 9.6].

(d) Let $n, m \geq 0$, let $f_1, \dots, f_n$ and $g_1, \dots, g_m$ be morphisms in $\mathcal{C}$ (with $f_i \colon X_i \to X_i'$ and $g_i \colon Y_i \to Y_i'$), and let φ and ψ be morphisms in $\mathcal{C}^\times$ such that the following square in $\mathcal{C}^\times$ commutes[3]

$$\begin{tikzcd} X_1 \oplus \cdots \oplus X_n \arrow[r, "\varphi"] \arrow[d, "f_1 \oplus \cdots \oplus f_n"'] & Y_1 \oplus \cdots \oplus Y_m \arrow[d, "g_1 \oplus \cdots \oplus g_m"] \\ X_1' \oplus \cdots \oplus X_n' \arrow[r, "\psi"'] & Y_1' \oplus \cdots \oplus Y_m' \end{tikzcd} \tag{$*$}$$

and lies over a commuting square of the following form in Fin_*, with $\alpha \colon \langle n \rangle \to \langle m \rangle$ some morphism.

$$\begin{tikzcd} \langle n \rangle \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & \langle m \rangle \arrow[d, "\mathrm{id}"] \\ \langle n \rangle \arrow[r, "\alpha"'] & \langle m \rangle \end{tikzcd}$$

Assume that φ and ψ are q-cocartesian and $f_1 \oplus \cdots \oplus f_n$ is $(p^\times)_{\langle n \rangle}$-cocartesian. Then $g_1 \oplus \cdots \oplus g_m$ is $(p^\times)_{\langle m \rangle}$-cocartesian.

Condition (a) holds by definition, and (b) holds as $p^\times$ is a symmetric monoidal functor from $\mathcal{C}^\times$ to $\mathcal{D}^\times$ (see the definition in [HA, 2.1.3.7]). The functor $p^\times_{\langle n \rangle}$ can be identified with $p^{\times n} \colon \mathcal{C}^{\times n} \to \mathcal{D}^{\times n}$, so (c) follows from the fact that products of cocartesian fibrations are again cocartesian fibrations (which follows from [HTT, 2.4.2.3]).

So now suppose we are in the situation of condition (d). We have to show that $g_1 \oplus \cdots \oplus g_m$ is $p^\times_{\langle m \rangle}$-cocartesian. Unpacking the data of the commutative square $(*)$ we see that it corresponds to the data of a commutative square

$$\begin{tikzcd} \prod\limits_{\alpha(i)=j} X_i \arrow[r, "\varphi_j"] \arrow[d, "\prod_{\alpha(i)=j} f_i"'] & Y_j \arrow[d, "g_j"] \\ \prod\limits_{\alpha(i)=j} X_i' \arrow[r, "\psi_j"'] & Y_j' \end{tikzcd}$$

in $\mathcal{C}$ for every $1 \leq j \leq m$. That φ and ψ are q-cocartesian implies that φ_j and ψ_j are equivalences, so we can conclude that g_j is equivalent to $\prod_{\alpha(i)=j} f_i$ in $\mathcal{C}$. As $f_1 \oplus \cdots \oplus f_n$ is $p^\times_{\langle n \rangle}$-cocartesian, it follows from the identification $p^\times_{\langle n \rangle} \simeq p^{\times n}$ in combination with [HTT, 3.1.2.1] that f_i is p-cocartesian for

[3] We are using the notation from [HA, 2.1.1.15]: For $f_1, \dots, f_n \colon \mathcal{C} \to \mathcal{C}$ we denote by $f_1 \oplus \cdots \oplus f_n$ the morphism in $\mathcal{C}_{\langle n \rangle}$ which under the equivalence $\mathcal{C}_{\langle n \rangle} \simeq \mathcal{C}^n$ corresponds to the tuple $(f_1, \dots, f_n)$.

each $1 \leq i \leq n$. Applying Proposition C.2.0.5 we can then conclude that $\prod_{\alpha(i)=j} f_i$ is also p-cocartesian, so g_j is equivalent to a p-cocartesian morphism and thus p-cocartesian as well. Applying the equivalence $p^{\times}_{\langle m \rangle} \simeq p^{\times m}$ and [HTT, 3.1.2.1] again we conclude that $g_1 \oplus \cdots \oplus g_m$ is $p^{\times}_{\langle m \rangle}$-cocartesian. $\square$

Appendix D.

More ∞-category theory

This appendix is really a continuation of Appendix A and collects some facts about more basic concepts of ∞-category theory: Undercategories in Section D.1 and adjunctions in Section D.2.

D.1. Undercategories

In this section we discuss undercategories. [HTT, 1.2.9.5] gives a definition in terms of quasicategories, so we start in Section D.1.1 by providing a model independent construction that can be carried out in $\mathcal{C}\mathrm{at}_\infty$. We then show in Section D.1.2 that the property of a functor being (fully) faithful or a monomorphism is preserved by passing to induced functors on undercategories. Finally, in Section D.1.3 we describe mapping spaces in an overcategory $\mathcal{C}_{X/}$ as pullbacks of mapping spaces in $\mathcal{C}$.

D.1.1. Model independent construction

Proposition D.1.1.1. *Let $\mathcal{C}$ be an ∞-category and X an object of $\mathcal{C}$. Let $\mathtt{C}$ be a quasicategory representing $\mathcal{C}$ and $\mathtt{X}$ an object of $\mathtt{C}$ representing X.*

Then the undercategory $\mathtt{C}_{\mathtt{X}/}$ defined as in [HTT, 1.2.9.5], together with its projection functor $\mathtt{C}_{\mathtt{X}/} \to \mathtt{C}$ represent the functor

$$\mathrm{ev}_1 \circ \mathrm{pr}_1 \colon \operatorname{Fun}([1], \mathcal{C}) \times_{\mathcal{C}} \{X\} \to \mathcal{C}$$

in $\mathcal{C}\mathrm{at}_\infty$, where the pullback is taken with respect to the functor ev_0 and the inclusion of $\{X\}$ into $\mathcal{C}$. ♡

Proof. The inclusion of $\{0\}$ into $[1]$ is a cofibration of simplicial sets, so the functor

$$\mathrm{ev}_0 \colon \mathsf{sSet}([1], \mathtt{C}) \to \mathtt{C}$$

is a Kan fibration by [Hov99, 4.2.8 and 4.2.2]. In particular, using [HTT, 3.3.1.4 and 2.4.2.4], the pullback (along morphisms like in the statement) $\mathsf{sSet}([1], \mathtt{C}) \times_{\mathtt{C}} \{\mathtt{X}\}$ is a homotopy pullback in the Joyal model structure, and thus represents $\operatorname{Fun}([1], \mathcal{C}) \times_{\mathcal{C}} \{X\}$.

The claim now follows from checking that $\mathsf{sSet}([1], \mathtt{C})$ satisfied the defining universal property of $\mathtt{C}_{\mathtt{X}/}$ (see [HTT, 1.2.9.5 and 1.2.9.2]). □

D.1.2. Undercategories, faithful functors, fully faithful functors, and monomorphisms

Proposition D.1.2.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a monomorphism (faithful functor, fully faithful functor) in $\mathcal{C}\mathrm{at}_\infty$ and X an object of $\mathcal{C}$. Then the induced functor on undercategories $\mathcal{C}_{X/} \to \mathcal{D}_{F(X)/}$ is a monomorphism (faithful functor, fully faithful functor) as well.* ♡

Proof. Using the description of undercategories from Proposition D.1.1.1, this follows immediately from Proposition B.5.1.1, Proposition B.3.0.1, and Proposition B.5.3.1. □

D.1.3. Mapping spaces in undercategories

In this section we show that mapping spaces in undercategories can be calculated through the expected pullback diagram. Before we can show this, we need the following small result on how initial objects interact with functors which are retractions.

Proposition D.1.3.1. *Let $\iota\colon \mathcal{C} \to \mathcal{D}$ and $r\colon \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories and assume that $r \circ \iota$ is homotopic to the identity functor.*

Let X be an initial object of $\mathcal{D}$. As X is initial, there is an essentially unique morphism $f\colon X \to \iota r X$ in $\mathcal{D}$. Assume that $rf\colon rX \to r\iota r X$ is an equivalence. Then rX is an initial object of $\mathcal{C}$. ♡

Proof. Let Y be an object of $\mathcal{C}$ and consider the following commutative diagram of mapping spaces.

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(rX, Y) \arrow[d, "\iota"] & \\
\mathrm{Map}_{\mathcal{D}}(\iota r X, \iota Y) \arrow[r, "f^*"] \arrow[d, "r"] & \mathrm{Map}_{\mathcal{D}}(X, \iota Y) \arrow[d, "r"] \\
\mathrm{Map}_{\mathcal{C}}(r\iota r X, r\iota Y) \arrow[r, "r(f)^*"'] & \mathrm{Map}_{\mathcal{C}}(rX, r\iota Y)
\end{tikzcd}$$

The left vertical composite is homotopic to the identity by the assumption that $r\iota \simeq \mathrm{id}_{\mathcal{C}}$ and the bottom horizontal functor is an equivalence as $r(f)$ is an equivalence by assumption. As the mapping space in the middle right is contractible by the assumption that X is initial, it thus follows that the top left mapping space $\mathrm{Map}_{\mathcal{C}}(rX, Y)$ is also contractible[1], which is what we need to show. □

Proposition D.1.3.2. *Let $\mathcal{C}$ be an ∞-category, X an object of $\mathcal{C}$, and $f\colon X \to Y$ and $g\colon X \to Z$ morphisms in $\mathcal{C}$. Let $p\colon \mathcal{C}_{X/} \to \mathcal{C}$ be the projection functor.*

[1] As a retract of a contractible space.

Then the commutative diagram in $\mathcal{S}$

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}_{X/}}(f,g) \arrow[r] \arrow[d, "p"'] & \{g\} \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(Y,Z) \arrow[r, "f^*"'] & \mathrm{Map}_{\mathcal{C}}(X,Z)
\end{tikzcd}$$

is a pullback diagram. ♡

Proof. Note that there is a degenerate commutative triangle

$$\begin{tikzcd}
& X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "f"] & \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

in $\mathcal{C}$ that we interpret as a morphism $\mathrm{id}_X \to f$ in $\mathcal{C}_{X/}$, which we will call f'.

By [HTT, 2.1.2.2], $p\colon \mathcal{C}_{X/} \to \mathcal{C}$ is a left fibration, and hence by (the dual of) [HTT, 2.4.2.4] a cocartesian fibration such that every morphism of $\mathcal{C}_{X/}$ is p-cocartesian. Applying (the dual of) [HTT, 2.4.4.3] to the p-cocartesian morphism $f'\colon \mathrm{id}_X \to f$, we obtain the following pullback diagram in $\mathcal{S}$.

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}_{X/}}(f,g) \arrow[r, "f'^*"] \arrow[d, "p"'] & \mathrm{Map}_{\mathcal{C}_{X/}}(\mathrm{id}_X,g) \arrow[d, "p"] \\
\mathrm{Map}_{\mathcal{C}}(Y,Z) \arrow[r, "f^*"'] & \mathrm{Map}_{\mathcal{C}}(X,Z)
\end{tikzcd}$$

Note that

$$\begin{tikzcd}
& X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "g"] & \\
X \arrow[rr, "g"'] & & Z
\end{tikzcd}$$

is a point in $\mathrm{Map}_{\mathcal{C}_{X/}}(\mathrm{id}_X, g)$ that maps to g under p, so it suffices to show that the mapping space $\mathrm{Map}_{\mathcal{C}_{X/}}(\mathrm{id}_X, g)$ is contractible, i. e. that id_X is an initial object in $\mathcal{C}_{X/}$.

We provide a quick proof for this fact here in the setting of quasicategories. So let $\mathtt{C}$ be a quasicategory and X an object of $\mathtt{C}$. To show that id_X is an initial object of $\mathtt{C}_{X/}$ it suffices by Proposition D.1.3.1[2] to provide a retraction r of the inclusion $\mathtt{C}_{X/} \to \{i\} \star \mathtt{C}_{X/}$ that sends the unique morphism $i \to \mathrm{id}_X$ in $\{i\} \star \mathtt{C}_{X/}$ to an equivalence.

Using the universal property of $\mathtt{C}_{X/}$ (see [HTT, 1.2.9.2]) it suffices for this to give a morphism[3]

$$\varphi\colon (\{x\} \star \{i\}) \star \mathtt{C}_{X/} \to \mathtt{C}$$

[2]The idea for this argument is from the proof of [HTT, 1.2.12.5].
[3]We are using associativity of the join operation $\star$, see [HTT, 1.2.8].

such that the restriction of φ to $\{x\} \star \mathtt{C}_{X/} \to \mathtt{C}$ is adjoint to the identity of $\mathtt{C}_{X/}$ (this corresponds to r being a retraction of the inclusion) and such that the unique 2-simplex

$$\begin{tikzcd} x \arrow[d] \arrow[dr] & \\ & \mathrm{id}_X \\ i \arrow[ur] & \end{tikzcd}$$

in $(\{x\} \star \{i\}) \star \mathtt{C}_{X/}$ is mapped by φ to the degenerate 2-simplex

$$\begin{tikzcd} X \arrow[d, "\mathrm{id}_X"'] \arrow[dr, "\mathrm{id}_X"] & \\ & X \\ X \arrow[ur, "\mathrm{id}_X"'] & \end{tikzcd}$$

which covers the condition of the unique morphism $i \to \mathrm{id}_X$ being sent to an equivalence.

We can define such a morphism as follows: Let $q\colon \{x\} \star \{i\} \to \{x\}$ be the unique morphism. Then we take the composite

$$(\{x\} \star \{i\}) \star \mathtt{C}_{X/} \xrightarrow{q \star \mathrm{id}_{\mathtt{C}_{X/}}} \{x\} \star \mathtt{C}_{X/} \longrightarrow \mathtt{C}$$

where the second morphism is adjoint to $\mathrm{id}_{\mathtt{C}_{X/}}$. $\square$

D.2. Adjunctions

In this section we discuss adjunctions of ∞-categories. In Section D.2.1 we briefly recall the two equivalent descriptions of adjunctions that are explicitly given in [HTT] and prove that they are equivalent to a third characterization. In Section D.2.2 we discuss the interaction of adjunctions with $\mathrm{Fun}(\mathcal{C}, -)$ for some ∞-category $\mathcal{C}$.

D.2.1. Equivalent characterizations of adjoints

There are several ways to define adjunctions of ∞-categories. The definition used in [HTT] describes adjunctions as cocartesian and cartesian fibrations over [1] (see [HTT, 5.2.2.1]). Lurie also shows that adjunctions are equivalently given by pairs of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ together with a unit transformation $u\colon \mathrm{id}_{\mathcal{C}} \to G \circ F$ satisfying the usual property for mapping spaces (see [HTT, 5.2.2.7 and 5.2.2.8]). We will use both descriptions and refer to [HTT, 5.2.2] for full definitions and how to translate between the two descriptions. We will also need a related third description, which we prove in the next proposition.

Proposition D.2.1.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories, and $\eta\colon \mathrm{id}_{\mathcal{C}} \to G \circ F$ a natural transformation. Then the following are equivalent.*

(1) There exists a natural transformation $\epsilon\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ and the composite natural transformations

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$$

and

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$$

are homotopic to id_F and id_G.

(2) η is a unit transformation for (F, G) in the sense of [HTT, 5.2.2.7]. ♡

Proof. Let us first assume (2). The proof of (1) is really an extension of what is shown in the proof of [HTT, 5.2.2.8], so we will assume the reader is familiar with that proof and sketch the additions that need to be made.

In [HTT, 5.2.2.8], assuming (2), an adjunction $q\colon \mathcal{M} \to [1]$ in the sense of [HTT, 5.2.2.1] associated to F and G is constructed from η. Let

$$\Phi\colon [1] \times \mathcal{C} \to \mathcal{M}$$

be the pointwise (in $\mathcal{C}$) q-cocartesian natural transformation from the inclusion[4] of $\mathcal{C}$ into $\mathcal{M}$ to F exhibiting F as associated to q and similarly Ψ for G.

It is clear from unpacking the definitions, that the unit transformation extracted from q in the other direction of [HTT, 5.2.2.8] can be identified with η. One can extract a natural transformation $\epsilon\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ in a completely analogous manner, as we will also explain in more detail now.

Both natural transformations η and ϵ are obtained are by combining [HTT, 3.1.2.1][5] and [HTT, 2.4.1.4] to lift find fillers in certain diagrams of natural transformations. For example, for ϵ we consider the following diagram of functors $\mathcal{D} \to \mathcal{M}$

$$\begin{tikzcd} & \mathrm{id}_{\mathcal{D}} \\ G \arrow[ur, "\Psi"] \arrow[dr, "\Phi G"'] & \\ & FG \arrow[uu, dashed, "\epsilon"'] \end{tikzcd}$$

where a filler for the dashed arrow and the triangle can be found as the bottom left arrow is cocartesian.

[4]We identify $\mathcal{C}$ with $\mathcal{M}_0$ and $\mathcal{D}$ with $\mathcal{M}_1$.

[5]That induced functors $q_*\colon \operatorname{Fun}(\mathcal{J}, \mathcal{M}) \to \operatorname{Fun}(\mathcal{J}, [1])$ are again (co)cartesian fibrations and natural transformations are q_*-(co)cartesian if and only if they are pointwise q-(co)cartesian.

To show that

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$$

is homotopic to the identity, we can now ponder the following diagram of functors $\mathcal{C} \to \mathcal{M}$.

$$\begin{tikzcd}
 & F \\
GF \arrow[ur, "\Psi F"] \arrow[r, "\Phi GF"] \arrow[dr, "\Psi F"] & FGF \arrow[u, dashed, "\epsilon F"'] \\
\mathrm{id}_{\mathcal{C}} \arrow[u, dashed, "\eta"] \arrow[r, "\Phi"'] & F \arrow[u, dashed, "F\eta"']
\end{tikzcd}$$

The dashed arrow on the left comes with a filler for the triangle at the bottom left and uses that ΨF is q_*-cartesian. The dashed arrow on the bottom right then comes with a filler for the lower square and uses that Φ is q_*-cocartesian. The dashed arrow on the upper right comes with a filler for the upper triangle and uses that ΦGF is q_*-cocartesian. We can thus conclude that $\epsilon F \circ F\eta$ is a filler in the following diagram.

$$\begin{tikzcd}
GF \arrow[r, "\Psi F"] & F \\
\mathrm{id}_{\mathcal{C}} \arrow[u, "\eta"] \arrow[r, "\Phi"'] & F \arrow[u, dashed, "\epsilon F \circ F\eta"']
\end{tikzcd}$$

But by definition of η (see the lower left triangle in the previous diagram), one such filler is id_F, so it follows that $\epsilon F \circ F\eta \simeq \mathrm{id}_F$. The other case is completely analogous. This shows (1).

We now assume (1) and show that η is a unit transformation for (F, G). For this we have to show that for every object C in $\mathcal{C}$ and object D in $\mathcal{D}$, the composition

$$\mathrm{Map}_{\mathcal{D}}(F(C), D) \xrightarrow{G} \mathrm{Map}_{\mathcal{C}}(GF(C), G(D)) \xrightarrow{(\eta_C)^*} \mathrm{Map}_{\mathcal{C}}(C, G(D))$$

is an equivalence. Using ϵ we can define a map in the opposite direction as

$$\mathrm{Map}_{\mathcal{C}}(C, G(D)) \xrightarrow{F} \mathrm{Map}_{\mathcal{D}}(F(C), FG(D)) \xrightarrow{(\epsilon_D)_*} \mathrm{Map}_{\mathcal{D}}(F(C), D)$$

and it follows immediately from (1) that these two maps are inverse equivalences. □

D.2.2. Adjunctions and Fun

In this short section we show that whenever $\mathcal{C}$ is an ∞-category, the functor $\mathrm{Fun}(\mathcal{C}, -)$ preserves adjunctions in a manner made precise in the next proposition.

Proposition D.2.2.1. *Let $p\colon \mathcal{M} \to [1]$ be a cartesian and cocartesian functor, and $F\colon \mathcal{M}_0 \to \mathcal{M}_1$ the corresponding left adjoint, $G\colon \mathcal{M}_1 \to \mathcal{M}_0$ the corresponding right adjoint, and $u\colon \mathrm{id}_{\mathcal{M}_0} \to G \circ F$ the corresponding unit transformation.*

Let $\mathcal{C}$ be an ∞-category. Then the functor $p'\colon \mathcal{M}' \to [1]$ that is defined by the following pullback diagram

$$\begin{tikzcd}
\mathcal{M}' \arrow[r] \arrow[d, "p'"'] & \mathrm{Fun}(\mathcal{C}, \mathcal{M}) \arrow[d, "p_*"] \\
{[1]} \arrow[r, "\mathrm{const}"'] & \mathrm{Fun}(\mathcal{C}, [1])
\end{tikzcd}$$

is also a cartesian and cocartesian fibration and hence defines an adjunction. Furthermore, the fibers $\mathcal{M}'_0$ and $\mathcal{M}'_1$ can be identified with $\mathrm{Fun}(\mathcal{C}, \mathcal{M}_0)$ *and* $\mathrm{Fun}(\mathcal{C}, \mathcal{M}_1)$*, and under this identification the encoded left adjoint can be identified with F_*, the encoded right adjoint with G_*, and the corresponding unit transformation with u_*.* ♡

Proof. That p' is again a cartesian and cocartesian fibration follows from [HTT, 3.1.2.1] and Proposition C.1.1.1. Using composability of pullback diagrams and $\mathrm{Fun}(\mathcal{C}, -)$ preserving pullbacks we obtain the following chain of equivalences with which we can identify $\mathcal{M}'_i$ as stated.

$$\begin{aligned}
\mathcal{M}'_i &\simeq \mathrm{Fun}(\mathcal{C}, \mathcal{M}) \times_{\mathrm{Fun}(\mathcal{C},[1])} \{\mathrm{const}_i\} \\
&\simeq \mathrm{Fun}(\mathcal{C}, \mathcal{M}) \times_{\mathrm{Fun}(\mathcal{C},[1])} \mathrm{Fun}(\mathcal{C}, \{i\}) \\
&\simeq \mathrm{Fun}\big(\mathcal{C}, \mathcal{M} \times_{[1]} \{i\}\big) \\
&\simeq \mathrm{Fun}(\mathcal{C}, \mathcal{M}_i)
\end{aligned}$$

Let the commuting diagram

$$\begin{tikzcd}
\mathcal{M}_0 \times [1] \arrow[rr, "F'"] \arrow[dr, "\mathrm{pr}_2"'] & & \mathcal{M} \arrow[dl, "p"] \\
& {[1]} &
\end{tikzcd}$$

exhibit F as the left adjoint to p (see [HA, 5.2.1.1 and 5.2.2.1]). We can then construct a diagram exhibiting F_* as the left adjoint to p' as indicated in the following diagram

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{C},\mathcal{M}_0)\times[1] \arrow[rr] \arrow[dr, dashed, "(F_*)'"] \arrow[ddr, "\mathrm{pr}_2"'] & & \operatorname{Fun}(\mathcal{C},\mathcal{M}_0\times[1]) \arrow[d, "F'_*"] \arrow[dd, bend left, "(\mathrm{pr}_2)_*"] \\
& \mathcal{M}' \arrow[r] \arrow[d, "p'"] & \operatorname{Fun}(\mathcal{C},\mathcal{M}) \arrow[d, "p_*"] \\
& {[1]} \arrow[r, "\mathrm{const}"'] & \operatorname{Fun}(\mathcal{C},[1])
\end{tikzcd}$$

where the top horizontal functor is the composition

$$\operatorname{Fun}(\mathcal{C},\mathcal{M}_0)\times[1] \xrightarrow{\mathrm{id}\times\mathrm{const}} \operatorname{Fun}(\mathcal{C},\mathcal{M}_0)\times\operatorname{Fun}(\mathcal{C},[1]) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{C},\mathcal{M}_0\times[1])$$

That $(F_*)'$ as constructed in the above diagram indeed exhibits F_* as the left adjoint associated to p' follows from the description of cocartesian morphisms in [HTT, 3.1.2.1] and Proposition C.1.1.1.

The statements regarding G_* and u_* can be proven analogously. □

Appendix E.

∞-operads and algebras

This appendix collects various results concerning ∞-operads and their ∞-categories of algebras.

We begin in Section E.1 with generic facts on (morphisms of) ∞-operads. For most of the remaining sections we then turn towards ∞-categories of algebras. In Section E.2 we will look into the relationship between ∞-categories of algebras and base changes of ∞-operads, and in Section E.3 we show that passing from morphisms of ∞-operads to functors between the respective ∞-categories of algebras preserves various properties.

If $\mathcal{O}$ is an ∞-operad and $\mathcal{C}$ is a symmetric monoidal ∞-category, then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ inherits an induced symmetric monoidal structure, which will be discussed in Section E.4. If $\mathcal{O}'$ is another ∞-operad, then the symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ allows us to take $\mathcal{O}'$-algebras in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. In Section E.5 we will show that there is another way to describe $\mathcal{O}'$-algebras in $\mathcal{O}$-algebras in $\mathcal{C}$, namely as $\mathcal{O} \otimes \mathcal{O}'$-algebras in $\mathcal{C}$. In Section E.6 we then discuss the commutative ∞-operad Comm and show that the tensor product of ∞-operads of any ∞-operad $\mathcal{O}$ with Comm is equivalent to Comm again.

In Section E.7 we discuss colimits of algebras as well as free algebras, and in particular when they are preserved by induced functors on algebra ∞-categories. Finally, in Section E.8 we discuss relative tensor products and when monoidal functors preserve them. We also show that pushouts of commutative algebras are given by relative tensor products.

E.1. ∞-operads

In this section we collect three statements relating to properties of morphisms of ∞-operads or helpful for showing that a functor is a morphism of ∞-operads or a symmetric monoidal functor. Concretely, Section E.1.1 helps showing that a morphism of ∞-operads between symmetric monoidal ∞-categories is symmetric monoidal, Section E.1.2 is about consequences of a morphism of ∞-operads being conservative, and Section E.1.3 discusses functors that are pullbacks of a morphism of ∞-operads along a cocartesian fibration of ∞-operads and vice versa.

E.1.1. Symmetric monoidal functors

By definition[1], a morphism of ∞-operads between symmetric monoidal ∞-categories is symmetric monoidal if it is a morphism of cocartesian fibrations, so preserves all cocartesian morphisms[2]. In the following proposition, we show that it suffices to check cocartesian lifts of two select morphisms in Fin_*: The multiplication $\mu\colon \langle 2\rangle \to \langle 1\rangle$ and unit $\epsilon\colon \langle 0\rangle \to \langle 1\rangle$. This is an analogue of [HA, 2.1.2.9] which similarly reduces the amount of inert morphisms that need to be checked to verify a functor over Fin_* is a morphism of ∞-operads.

Proposition E.1.1.1. *Let*

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "p_{\mathcal{C}}"'] & & \mathcal{D}^{\otimes} \arrow[dl, "p_{\mathcal{D}}"] \\ & \mathsf{Fin}_* & \end{tikzcd}$$

be a commutative diagram of morphisms of ∞-operads, and assume that $p_{\mathcal{C}}$ and $p_{\mathcal{D}}$ exhibit $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ as symmetric monoidal ∞-categories. Then the following two conditions are equivalent.

(1) $F^{\otimes}$ is symmetric monoidal, i. e. maps $p_{\mathcal{C}}$-cocartesian morphisms to $p_{\mathcal{D}}$-cocartesian morphisms.

(2) $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian lifts of the active morphism[3] $\mu\colon \langle 2\rangle \to \langle 1\rangle$ and $p_{\mathcal{C}}$-cocartesian lifts of the unique morphism $\epsilon\colon \langle 0\rangle \to \langle 1\rangle$ to $p_{\mathcal{D}}$-cocartesian morphisms. ♡

Proof. It is clear that (1) implies (2), so it remains to show the converse direction. Morphisms in Fin_* are generated (by composition) by morphisms of the following forms (compare [HA, 2.1.2.2]).

(A) Inert morphisms[4].

(B) For every $n \geq 1$ the morphism $\mu_n\colon \langle n+1\rangle \to \langle n\rangle$ that sends an element i of $\langle n+1\rangle^{\circ}$ to i if $i \leq n$, and to n otherwise[5].

(C) For every $n \geq 0$ the inclusion $\epsilon_n\colon \langle n\rangle \to \langle n+1\rangle$ (i. e. sending i to i).

As the collection of cocartesian morphisms is closed under composition [HTT, 2.4.1.7] and cocartesian lifts with fixed source object are unique up to equivalence [HTT, 2.4.1.9], it suffices to prove that $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian lifts of morphisms of type (A), (B), and (C) to $p_{\mathcal{D}}$-cocartesian morphisms. By

[1]See [HA, 2.1.3.7].
[2]With respect to the respective canonical cocartesian fibrations of ∞-operads to Fin_*.
[3]So this is the morphism that sends 1 and 2 to 1.
[4]Note that in particular all isomorphisms are inert.
[5]So n is the unique element of the target that has two preimages, n and $n+1$.

assumption we already know that $F^{\otimes}$ is a morphism of ∞-operads and hence preserves inert morphisms, so this covers type (A).

We now show that $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian lifts of morphisms of type (B) to $p_{\mathcal{D}}$-cocartesian morphisms. So let $n \geq 1$, let μ_n be the morphism of Fin_* defined in (B), and let $f\colon X \to Y$ be a $p_{\mathcal{C}}$-cocartesian lift of μ_n. As $p_{\mathcal{D}}$ is a cocartesian fibration, we can lift μ_n to a $p_{\mathcal{D}}$-cocartesian morphism $\overline{f}\colon F^{\otimes}(X) \to (\mu_n)_!(F^{\otimes}(X))$, and obtain an induced morphism g lying over $\mathrm{id}_{\langle n \rangle}$, such that there is a commutative diagram as follows.

$$\begin{tikzcd}
 & (\mu_n)_!(F^{\otimes}(X)) \arrow[d, dashed, "g"] \\
F^{\otimes}(X) \arrow[ur, "\overline{f}"] \arrow[r, "F^{\otimes}(f)"'] & F^{\otimes}(Y)
\end{tikzcd}$$

By [HTT, 2.4.1.7 and 2.4.1.5], $F^{\otimes}(f)$ is $p_{\mathcal{D}}$-cocartesian if and only if g is an equivalence, so we prove the latter.

Let us first consider $\rho^j_!(g)$ for $1 \leq j < n$. This is the induced morphism indicated in the following diagram, where $\overline{r}$ and r are $p_{\mathcal{D}}$-cocartesian lifts of ρ^j.

$$\begin{tikzcd}
 & (\mu_n)_!(F^{\otimes}(X)) \arrow[r, "\overline{r}"] \arrow[d, dashed, "g"] & (\rho^j \circ \mu_n)_!(F^{\otimes}(X)) \arrow[d, dashed, "\rho^j_!(g)"] \\
F^{\otimes}(X) \arrow[ur, "\overline{f}"] \arrow[r, "F^{\otimes}(f)"'] & F^{\otimes}(Y) \arrow[r, "r"'] & \rho^j_!(F^{\otimes}(Y))
\end{tikzcd}$$

But note that for $1 \leq j < n$ the composition $\rho^j \circ \mu_n$ is equal to ρ^j. The morphism $\rho^j_!(g)$ is thus also equivalent to the morphism

$$g_j\colon \rho^j_!(F^{\otimes}(X)) \to \rho^j_!\big(F^{\otimes}(Y)\big)$$

induced by $r \circ F^{\otimes}(f)$. Now let

$$\begin{tikzcd}
 & Y \arrow[r, "s"] \arrow[d, dashed, "\mathrm{id}"] & \rho^j_!(Y) \arrow[d, dashed, "\rho^j_!(\mathrm{id})"] \\
X \arrow[ur, "f"] \arrow[r, "f"'] & Y \arrow[r, "s"'] & \rho^j_!(Y)
\end{tikzcd}$$

be the diagram constructed completely analogously from f in $\mathcal{C}^{\otimes}$, with s a $p_{\mathcal{C}}$-cocartesian lift of ρ^j. In this case we can use f itself as a $p_{\mathcal{C}}$-cocartesian lift of μ_n, and the identity morphism can play the role of g. In particular, the morphism $f_j\colon \rho^j_!(Y) \to \rho^j_!(Y)$ induced by $s \circ f$ is an equivalence. As $F^{\otimes}$ preserves inert morphisms $F^{\otimes}(s)$ can be identified with r, and $F^{\otimes}(s \circ f)$ with $\overline{r} \circ \overline{f}$. This implies that $F^{\otimes}(f_j) \simeq g_j$, and as $F^{\otimes}$ preserves equivalences, g_j must be an equivalence.

Let us now consider $\rho^n(g)$. In this case $\rho^n \circ \mu_n$ is not ρ^n, but $\mu \circ \rho^{n,n+1}$, where $\rho^{n,n+1}\colon \langle n+1\rangle \to \langle 2\rangle$ maps i to $*$ if $i < n$, maps n to 1, and maps $n+1$ to 2. We can argue completely analogously to the previous case, but have to additionally use that $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian lifts of μ to $p_{\mathcal{D}}$-cocartesian morphisms, which is the case by assumption (2).

As the functor

$$\mathcal{D}_{\langle n\rangle} \xrightarrow{\prod_{1\le j\le n} \rho^j_!} \mathcal{D}_{\langle 1\rangle}^{\times n}$$

is an equivalence and we showed that $\rho^j_!(g)$ is an equivalence for every $1 \le j \le n$, we can conclude that g is an equivalence. Thus we have shown that $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian lifts of morphisms of type (B) to $p_{\mathcal{D}}$-cocartesian morphisms.

The case of morphisms of type (C) is similar, in this case we will need to use the assumption regarding ϵ. □

E.1.2. Conservative morphisms of ∞-operads

In the following proposition we record a very useful consequence of a morphism of ∞-operads being conservative.

Proposition E.1.2.1. *Let*

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "p_{\mathcal{C}}"'] & & \mathcal{D}^{\otimes} \arrow[dl, "p_{\mathcal{D}}"] \\ & \mathsf{Fin}_* & \end{tikzcd}$$

be a commutative diagram of morphisms of ∞-operads, and assume that $F^{\otimes}$ is a conservative functor, i. e. reflects equivalences. Then the following hold.

(1) A morphism f in $\mathcal{C}^{\otimes}$ is inert if and only if $F^{\otimes}(f)$ is inert.

(2) Assume that $p_{\mathcal{C}}$ and $p_{\mathcal{D}}$ exhibit $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ as symmetric monoidal ∞-categories, and that $F^{\otimes}$ is symmetric monoidal. Then a morphism f in $\mathcal{C}^{\otimes}$ is $p_{\mathcal{C}}$-cocartesian if and only if $F^{\otimes}(f)$ is $p_{\mathcal{D}}$-cocartesian. ♡

Proof. In both cases the "only if"-direction is handled directly by the assumption that $F^{\otimes}$ is a morphism of ∞-operads, and that $F^{\otimes}$ is even symmetric monoidal in the case of (2).

We will prove the "if"-direction of both (1) and (2) at the same time. So let $f\colon X \to Y$ be a morphism in $\mathcal{C}^{\otimes}$ that lies over a morphism φ in Fin_* and is mapped by $F^{\otimes}$ to a $p_{\mathcal{D}}$-cocartesian morphism in $\mathcal{D}^{\otimes}$. For (1) assume additionally that φ is inert. We have to show that f is $p_{\mathcal{D}}$-cocartesian.

We can factor f as indicated in the following commutative diagram in $\mathcal{C}^{\otimes}$

$$\begin{tikzcd} & \varphi_! X \arrow[d, dashed, "f''"] \\ X \arrow[ur, "f'"] \arrow[r, "f"'] & Y \end{tikzcd}$$

such that f' is $p_{\mathcal{C}}$-cocartesian and f'' lies over an identity morphism in Fin_*. Both $F^{\otimes}(f')$ and $F^{\otimes}(f)$ are $p_{\mathcal{D}}$-cocartesian morphisms, so by [HTT, 2.4.1.7 and 2.4.1.5] $F^{\otimes}(f'')$ is an equivalence. As $F^{\otimes}$ is conservative, it follows that f'' is also an equivalence, which by [HTT, 2.4.1.7 and 2.4.1.5] implies that f is $p_{\mathcal{C}}$-cocartesian. □

E.1.3. Base changes of cocartesian fibrations of ∞-operads

By Proposition C.1.1.1 a pullback of a cocartesian fibration along any functor is again a cocartesian fibration. The next proposition can be considered an upgrade of this statement to the situation in which both functors are morphisms of ∞-operads.

Proposition E.1.3.1. *Let*

$$\begin{tikzcd} \mathcal{C}'^{\otimes} \arrow[r, "q"] \arrow[d, "p'"'] & \mathcal{C}^{\otimes} \arrow[d, "p"] & \\ \mathcal{O}'^{\otimes} \arrow[r, "r"'] & \mathcal{O}^{\otimes} \arrow[r, "p_{\mathcal{O}}"'] & \mathsf{Fin}_* \end{tikzcd}$$

be a commutative diagram in Cat_{∞} such that the square is a pullback square, $p_{\mathcal{O}}$ and r are morphisms of ∞-operads, and p is a cocartesian fibration of ∞-operads.

Then p' is a cocartesian fibration of ∞-operads and q is a morphism of ∞-operads. Furthermore, a morphism f in $\mathcal{C}'^{\otimes}$ is inert if and only if $q(f)$ and $p'(f)$ are inert. ♡

Proof. By Proposition C.1.1.1 p' is a cocartesian fibration, and the description of p'-cocartesian morphisms also implies that if $n \geq 0$ and X_i are objects in $\mathcal{O}'$ for $1 \leq i \leq n$, and $f_i \colon X_1 \oplus \cdots \oplus X_n \to X_i$ are the canonical inert morphisms in $\mathcal{O}'^{\otimes}$, then the induced functor on fibers

$$\mathcal{C}'^{\otimes}_{X_1 \oplus \cdots \oplus X_n} \xrightarrow{\prod_{1 \leq i \leq n} f_{i!}} \mathcal{C}'^{\otimes}_{X_i} \qquad (*)$$

can be identified with the following functor that is induced on the fibers of p.

$$\mathcal{C}^{\otimes}_{r(X_1 \oplus \cdots \oplus X_n)} \xrightarrow{\prod_{1 \leq i \leq n} r(f_i)_!} \mathcal{C}^{\otimes}_{r(X_i)}$$

As r is a morphism of ∞-operads we can for each $1 \leq i \leq n$ identify $r(f_i)$ with the inert morphism $r(X_1) \oplus \cdots \oplus r(X_n) \to r(X_i)$. As p is a cocartesian fibration of ∞-operads, it thus follows that $(*)$ is an equivalence, so p' is a cocartesian fibration of ∞-operads[6].

Let f be a morphism in $\mathcal{C}'^{\otimes}$. It remains to show that f is inert if and only if $q(f)$ and $p'(f)$ are inert. Denote the compositions from the four ∞-categories in the square to Fin_* by p with subscript the name of the underlying ∞-category.

Assume that f is inert. Then f is by definition $p_{\mathcal{C}'}$-cocartesian, and as p' preserves inert morphisms, $p'(f)$ is inert, so $p_{\mathcal{O}'}$-cocartesian. It follows from [HTT, 2.4.1.3 (3)] that f is p'-cocartesian. By Proposition C.1.1.1 we then obtain that $q(f)$ is p-cocartesian. Furthermore, $p(q(f)) = r(p'(f))$ is inert, i. e. $p_{\mathcal{O}}$-cocartesian, as r is a morphism of ∞-operads. We can again use [HTT, 2.4.1.3 (3)] to conclude that $q(f)$ is $p_{\mathcal{C}}$-cocartesian, so inert.

Now assume that $q(f)$ and $p'(f)$ are inert. Again, as r is a morphism of ∞-operads, $p(q(f)) = r(p'(f))$ is inert, so by [HTT, 2.4.1.3 (3)] $q(f)$ is p-cocartesian, which by Proposition C.1.1.1 implies that f is p'-cocartesian, from which we can deduce with another application of [HTT, 2.4.1.3 (3)] that f is $p_{\mathcal{C}'}$-cocartesian, so inert. □

E.2. Alg and base change

This section concerns the interaction of Alg with base changes, with the upshot being the following. Given a commutative diagram

$$\begin{tikzcd} & \mathcal{C}'^{\otimes} \arrow[r, "F^{\otimes}"] \arrow[d, "p'"'] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ \mathcal{O}''^{\otimes} \arrow[r, "\alpha"'] & \mathcal{O}'^{\otimes} \arrow[r, "\beta"'] & \mathcal{O}^{\otimes} \end{tikzcd}$$

of ∞-operads such that the square is a pullback diagram in $\mathcal{C}\mathrm{at}_\infty$, we will obtain an induced pullback diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{C}') \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}''}(p)"] \\ \{\beta \circ \alpha\} \arrow[r] & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{O}) \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories of algebras.

[6] See [HA, 2.1.2.13 and 2.1.2.12].

Construction E.2.0.1. Let

$$\begin{tikzcd} & \mathcal{C}'^{\otimes} \arrow[r, "F^{\otimes}"] \arrow[d, "p'"'] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ \mathcal{O}''^{\otimes} \arrow[r, "\alpha"'] & \mathcal{O}'^{\otimes} \arrow[r, "\beta"'] & \mathcal{O}^{\otimes} \end{tikzcd}$$

be a commutative diagram of ∞-operads such that the square is a pullback diagram in $\mathcal{C}\mathrm{at}_{\infty}$.

Applying $\mathrm{Fun}(\mathcal{O}''^{\otimes}, -)$ to the pullback square we obtain the pullback on the right in the following diagram, with the left square a pullback square as well, by definition.

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{O}'^{\otimes}}(\mathcal{O}''^{\otimes}, \mathcal{C}'^{\otimes}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}'^{\otimes}) \arrow[r, "F^{\otimes}_{*}"] \arrow[d, "p'_{*}"'] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}) \arrow[d, "p_{*}"] \\ \{\alpha\} \arrow[r] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{O}'^{\otimes}) \arrow[r, "\beta_{*}"'] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{O}^{\otimes}) \end{tikzcd}$$

Comparing the combined outer pullback square [HTT, 4.4.2.1] to the pullback square

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}) \arrow[d, "p_{*}"] \\ \{\beta \circ \alpha\} \arrow[r] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{O}^{\otimes}) \end{tikzcd}$$

we obtain a canonical equivalence

$$\mathrm{Fun}_{\mathcal{O}'^{\otimes}}\left(\mathcal{O}''^{\otimes}, \mathcal{C}'^{\otimes}\right) \simeq \mathrm{Fun}_{\mathcal{O}^{\otimes}}\left(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}\right)$$

of ∞-categories. ♢

Proposition E.2.0.2. *In the situation of Construction E.2.0.1 the equivalence*

$$\mathrm{Fun}_{\mathcal{O}'^{\otimes}}\left(\mathcal{O}''^{\otimes}, \mathcal{C}'^{\otimes}\right) \simeq \mathrm{Fun}_{\mathcal{O}^{\otimes}}\left(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}\right)$$

restricts to an equivalence on the full subcategories of algebras as follows.

$$\mathrm{Alg}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{C}') \simeq \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{C}) \qquad ♡$$

Proof. Unpacking the definitions the statement boils down to the following: Let

$$\begin{tikzcd} \mathcal{O}''^{\otimes} \arrow[rr, "A"] \arrow[dr, "\alpha"'] & & \mathcal{C}'^{\otimes} \arrow[dl, "p'"] \\ & \mathcal{O}'^{\otimes} & \end{tikzcd}$$

be a commuting diagram and let f be an inert morphism in $\mathcal{O}''^{\otimes}$. Denote by $q\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ the unique morphism of ∞-operads. We have to show that $A(f)$ is $q\beta p'$-cocartesian if and only if $F^{\otimes}(A(f))$ is qp-cocartesian.

As α is a morphism of ∞-operads, it preserves inert morphisms, so the morphism $\alpha(f) = p'(A(f))$ is $q\beta$-cocartesian. Then [HTT, 2.4.1.3 (3)] implies that $A(f)$ is $q\beta p'$-cocartesian if and only if $A(f)$ is p'-cocartesian. By Proposition C.1.1.1 $A(f)$ is p'-cocartesian if and only if $F^{\otimes}(A(f))$ is p-cocartesian. But as $\beta \circ \alpha$ preserves inert morphisms, $\beta(\alpha(f))$ is q-cocartesian, so again by [HTT, 2.4.1.3 (3)] $F^{\otimes}(A(f))$ is p-cocartesian if and only if $F^{\otimes}(A(f))$ is qp-cocartesian. $\square$

Proposition E.2.0.3 ([HA, 2.1.3.1])**.** *Let $\gamma\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ and $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be morphisms of ∞-operads. Then the pullback diagram of ∞-categories*

$$\begin{tikzcd}
\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[d, "p_*"] \\
\{\gamma\} \arrow[r] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes})
\end{tikzcd}$$

induces on full subcategories a pullback diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}'}(p)"] \\
\{\gamma\} \arrow[r] & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O})
\end{tikzcd}$$

of ∞-categories[7]. ♡

Proof. There is a commutative cube in $\mathcal{C}\mathrm{at}_{\infty}$

$$\begin{tikzcd}
& \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \arrow[dl] \arrow[dd] \\
\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[rr] \arrow[dd] & & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[dd] & \\
& \{\gamma\} \arrow[rr] \arrow[dl] & & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}) \arrow[dl] \\
\{\gamma\} \arrow[rr] & & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}) &
\end{tikzcd}$$

with all functors from the back to the front inclusions of full subcategories. One can use Proposition B.5.2.1 to show that the top and bottom squares are

[7] We are using the the definition given in [HA, 2.1.3.1] for $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ as a full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$. The alternative description as the pullback given in this statement is also mentioned in [HA, 2.1.3.1].

pullback squares as follows: For the top square, consider the induced diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[dr, dashed, "\theta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{D} \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \arrow[d] \\
& \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes) \arrow[r] & \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)
\end{tikzcd}$$

where $\mathcal{D}$ is constructed as a pullback of the square. The right vertical functor is fully faithful, so by Proposition B.5.2.1 the left vertical functor is fully faithful as well. As we also know that the functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$ is also fully faithful, it follows that the induced functor θ is fully faithful too. To show that θ is an equivalence it thus suffices to show essential surjectivity [HTT, 1.2.10]. As $\mathcal{D} \to \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$ is fully faithful, an object in $\mathcal{D}$ can be thought of as an object in $\mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$, i. e. a commutative triangle

$$\begin{tikzcd}
\mathcal{O}'^\otimes \arrow[rr, "A"] \arrow[dr, "\gamma"'] & & \mathcal{C}^\otimes \arrow[dl, "p"] \\
& \mathcal{O}^\otimes &
\end{tikzcd} \tag{E.1}$$

such that the corresponding object in $\mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$, i. e. A, lies in $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$, i. e. A must be a morphism of ∞-operads. But this is precisely the condition for an object of $\mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$ to lie in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$, so θ is essentially surjective and hence an equivalence, which implies that the top square of the cube is a pullback diagram. That the bottom square is a pullback diagram can be proven analogously.

As the top and front of the cube are pullback diagrams, the composite of those two squares is a pullback diagram as well by [HTT, 4.4.2.1]. This composite is equivalent to the composite formed by the back and bottom squares, so using that the bottom square is a pullback and the other direction of [HTT, 4.4.2.1] we can conclude that the back square is a pullback as well. ☐

Remark E.2.0.4. Combining Proposition E.2.0.2 and Proposition E.2.0.3 in the situation of Construction E.2.0.1, we obtain the following pullback diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{C}') \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}''}(p)"] \\
\{\beta \circ \alpha\} \arrow[r] & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{O})
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$. Tracing through the definitions is is not difficult to see that this square is also natural in $\mathcal{C}$ (with $\mathcal{O}$,$\mathcal{O}'$, and $\mathcal{O}''$ staying fixed and $\mathcal{C}'$ changing with $\mathcal{C}$ as a pullback). ♢

E.3. Properties preserved by Alg

In this section we show that passing to ∞-categories of algebras preserves several properties of functors. Specifically, we will discuss pullbacks in Section E.3.1, cocartesian fibrations in Section E.3.2, adjoints in Section E.3.3, the property of a functor being conservative in Section E.3.4, and fully faithfulness in Section E.3.5.

E.3.1. Pullbacks

Proposition E.3.1.1. *Let*

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r, "F^{\otimes}"] \arrow[d, "G^{\otimes}"'] & \mathcal{D}^{\otimes} \arrow[d, "H^{\otimes}"] & \mathcal{O}'^{\otimes} \arrow[d, "\alpha^{\otimes}"] \\ \mathcal{E}^{\otimes} \arrow[r, "K^{\otimes}"'] & \mathcal{F}^{\otimes} \arrow[r, "p_{\mathcal{F}}"'] & \mathcal{O}^{\otimes} \end{tikzcd}$$

be a commutative diagram of ∞-operads such that the square is a pullback diagram in $\mathcal{C}\mathrm{at}_{\infty}$*. Assume furthermore that a morphism f in $\mathcal{C}^{\otimes}$ is inert if and only if $F^{\otimes}(f)$ and $G^{\otimes}(f)$ are inert.*

Then the induced commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(F)"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(G)"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(H)"] \\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{E}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(K)"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{F}) \end{tikzcd}$$

is a pullback diagram in $\mathcal{C}\mathrm{at}_{\infty}$. ♡

Proof. As $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$ preserves pullbacks and limits commute with each other, we first obtain an induced pullback square as follows.

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r, "F_*^{\otimes}"] \arrow[d, "G_*^{\otimes}"'] & \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}) \arrow[d, "H_*^{\otimes}"] \\ \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{E}^{\otimes}) \arrow[r, "K_*^{\otimes}"'] & \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{F}^{\otimes}) \end{tikzcd}$$

Let $\mathcal{P}$ be defined to be the pullback in the square in the following diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[dd, bend right, "\iota"'] \arrow[ddr, bend right, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(G)"] \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(F)"] & & \\
& \mathcal{P} \arrow[r] \arrow[d] \arrow[dl, "\psi"] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(H)"] \\
\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{E}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(K)"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{F})
\end{tikzcd}$$

where φ and ψ are the induced functors, and ι is the canonical fully faithful inclusion. It suffices to show that φ is an equivalence. By Proposition B.5.3.1, ψ is fully faithful with essential image spanned by those functors $A\colon \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ over $\mathcal{O}^{\otimes}$ whose compositions with $F^{\otimes}$ and $G^{\otimes}$ send inert morphisms to inert morphisms. But by the assumptions on inert morphisms in $\mathcal{C}^{\otimes}$, this means that the essential image of ψ is exactly the essential image of ι. It now follows from Proposition B.4.4.1 and Proposition B.4.3.1 that φ is an equivalence. □

E.3.2. Cocartesian fibrations

Proposition E.3.2.1. *Let*

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q"] \\
& \mathcal{O}^{\otimes} &
\end{tikzcd}$$

be a commuting diagram of maps of ∞-operads, and let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be another map of ∞-operads.

If $F^{\otimes}$ is a cocartesian fibration, then the induced functor

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(F)\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$$

is a cocartesian fibration as well. ♡

Proof. Consider the following commutative diagram induced by $F^{\otimes}$

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[drr, bend left, "\iota_{\mathcal{C}}"] \arrow[dr, "\varphi"] \arrow[ddr, bend right, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(F)"'] & & & \\
& \mathcal{E} \arrow[r, "\iota_{\mathcal{E}}"] \arrow[d, "p"] & \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r] \arrow[d, "F^{\otimes}_{*}"] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[d, "F^{\otimes}_{*}"] \\
& \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[r, "\iota_{\mathcal{D}}"'] & \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}) \arrow[r] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes})
\end{tikzcd}$$

where $\mathcal{E}$ is defined to be the pullback of the middle square, ι_C, ι_D, and the two right horizontal functors are the canonical ones, and φ is the induced functor into the pullback.

By [HTT, 3.1.2.1], the right vertical morphism $F^{\otimes}_{*} = \mathrm{Fun}(\mathrm{id}_{\mathcal{O}'^{\otimes}}, F^{\otimes})$ is a cocartesian fibration, so as both squares are pullback squares we can apply Proposition C.1.1.1 to conclude that p is also a cocartesian fibration. As cocartesian fibrations are closed under composition [HTT, 2.4.2.3 (3)], it thus suffices to show that φ is a cocartesian fibration.

By definition, $\iota_{\mathcal{D}}$ and $\iota_{\mathcal{C}}$ are inclusions of full subcategories and hence fully faithful, and as a pullback of $\iota_{\mathcal{D}}$, Proposition B.5.2.1 implies that $\iota_{\mathcal{E}}$ is fully faithful as well. It follows that φ is also fully faithful, so by Proposition C.1.2.1 it suffices to show that for any object A in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ and p-cocartesian morphism $\theta\colon \varphi(A) \to B'$ in $\mathcal{E}$ there exists an object B in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ such that $\varphi(B)$ is equivalent to B'.

Unpacking definitions, this means the following. Assume we have given a morphism $\theta\colon A \to B$ in $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$, which we can think of as a natural transformation between two commuting triangles as in the following diagram.

$$\begin{tikzcd}
\mathcal{O}'^{\otimes} \arrow[rr, bend left, "A"] \arrow[rr, bend right, "B"'] \arrow[rr, phantom, "\Downarrow\theta"] \arrow[dr, "\alpha"'] & & \mathcal{C}^{\otimes} \arrow[dl, "p"] \\
& \mathcal{O}^{\otimes} &
\end{tikzcd}$$

We furthermore assume that:

(a) A preserves inert morphisms. This corresponds to A lying in the full subcategory $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$.

(b) $F^{\otimes} \circ B\colon \mathcal{O}'^{\otimes} \to \mathcal{D}^{\otimes}$ preserves inert morphisms. This corresponds to B lying in the full subcategory $\mathcal{E}$ of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$.

(c) For every object O in $\mathcal{O}'^{\otimes}$, the morphism $\theta_O\colon A(O) \to B(O)$ in $\mathcal{C}$ is $F^{\otimes}$-cocartesian. This corresponds to θ (considered as a morphism in $\mathcal{E}$) being p-cocartesian.

We then have to show that B preserves inert morphisms.

In the following we let $q_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ and $q_{\mathcal{D}}\colon \mathcal{D}^{\otimes} \to \mathsf{Fin}_*$ be the canonical maps of ∞-operads. Let $f\colon U \to V$ be an inert morphism in $\mathcal{O}^{\otimes}$. We have to show that $B(f)$ is $q_{\mathcal{C}}$-cocartesian. The natural transformation θ induces the following commuting square in $\mathcal{C}^{\otimes}$.

$$\begin{tikzcd}
A(U) \arrow[r, "\theta_U"] \arrow[d, "A(f)"'] & B(U) \arrow[d, "B(f)"] \\
A(V) \arrow[r, "\theta_V"'] & B(V)
\end{tikzcd}$$

By (a), A preserves inert morphisms, so $A(f)$ is inert, hence $q_{\mathcal{C}}$-cocartesian. As $F^{\otimes}$ is a map of ∞-operads it also preserves inert morphism, and thus $F^{\otimes}(A(f))$ is $q_{\mathcal{D}}$-cocartesian. It then follows from [HTT, 2.4.1.3 (3)] that $A(f)$ is $F^{\otimes}$-cocartesian. By (c) both θ_U and θ_V are also $F^{\otimes}$-cocartesian, so it follows from [HTT, 2.4.1.7] that $B(f)$ is $F^{\otimes}$-cocartesian as well. Finally, $F^{\otimes}(B(f))$ is $q_{\mathcal{D}}$-cocartesian by (b), so by applying [HTT, 2.4.1.3 (3)] in the other direction we can conclude that $B(f)$ is $q_{\mathcal{C}}$-cocartesian. ☐

E.3.3. Adjoints

Proposition E.3.3.1. *Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, $p_{\mathcal{D}}\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$, as well as $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be maps of ∞-operads and let furthermore $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ and $G^{\otimes}\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ be maps of ∞-operads over $\mathcal{O}^{\otimes}$. Let $u\colon \mathrm{id}_{\mathcal{C}^{\otimes}} \to G^{\otimes} \circ F^{\otimes}$ be a natural transformation exhibiting $F^{\otimes}$ as left adjoint to $G^{\otimes}$ and assume that $p_{\mathcal{C}}$ maps u to the identity natural transformation of $p_{\mathcal{C}}$ (in other words, u is a unit for an adjunction between $F^{\otimes}$ and $G^{\otimes}$ relative to $\mathcal{O}^{\otimes}$ in the sense of [HA, 7.3.2.3]).*

Then the induced natural transformation

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(u)\colon \mathrm{id}_{\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})} \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(G \circ F)$$

exhibits $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(F)$ *as left adjoint to* $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(G)$. ♡

Proof. Applying $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$ we obtain two commuting triangles as indicated in the following diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[rr, shift left, "F^{\otimes}_*"] \arrow[dr, "p_{\mathcal{C}*}"'] & & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}) \arrow[ll, shift left, "G^{\otimes}_*"] \arrow[dl, "p_{\mathcal{D}*}"] \\
& \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}) &
\end{tikzcd}$$

as well as a natural transformation $u_*\colon \mathrm{id}_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})} \to G^{\otimes}_* \circ F^{\otimes}_*$. By Proposition D.2.2.1, u_* exhibits $F^{\otimes}_*$ as left adjoint to $G^{\otimes}_*$. As $p_{\mathcal{C}*}$ maps u_* to the identity natural transformation of $p_{\mathcal{C}*}$, this makes u_* into the unit for an adjunction between $F_*^{\otimes}$ and $G_*^{\otimes}$ relative to $\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes})$ in the sense of [HA, 7.3.2.3]. Taking the pullback of this adjunction along $\{\alpha\} \to \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes})$ and applying [HA, 7.3.2.5] yields an induced adjunction between the ∞-categories $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ and $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes})$. The claim now follows by restricting the relevant functors and natural transformation to the full subcategories of ∞-operad maps [HA, 2.1.2.7]. □

E.3.4. Reflecting equivalences

Proposition E.3.4.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories and $F\colon \mathcal{C} \to \mathcal{D}$ a symmetric monoidal functor. Let $\mathcal{O}$ be an ∞-operad.*

Assume that F is conservative, i. e. reflects equivalences. Then $\mathrm{Alg}_{\mathcal{O}}(F)$ is conservative as well. ♡

Proof. There is a commutative diagram as follows for every object X in $\mathcal{O}$.

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(F)"] \arrow[d, "\mathrm{ev}_X"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) \arrow[d, "\mathrm{ev}_X"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

Now suppose that φ is a morphism in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ such that $\mathrm{Alg}_{\mathcal{O}}(F)(\varphi)$ is an equivalence. By [HA, 3.2.2.6] ev_X preserves equivalences, so the morphism

$$\mathrm{ev}_X(\mathrm{Alg}_{\mathcal{O}}(F)(\varphi)) = F(\mathrm{ev}_X(\varphi))$$

is an equivalence for every object X of $\mathcal{O}$. As F is conservative, this implies that $\mathrm{ev}_X(\varphi)$ is an equivalence for every object X of $\mathcal{O}$, which by another application of [HA, 3.2.2.6] implies that φ is an equivalence. □

E.3.5. Fully faithfulness

Proposition E.3.5.1. *Let*

$$\begin{tikzcd} & \mathcal{C}'^{\otimes} \arrow[rr, "\iota^{\otimes}"] \arrow[dr, "p_{\mathcal{C}'}"'] & & \mathcal{C}^{\otimes} \arrow[dl, "p_{\mathcal{C}}"] \\ \mathcal{O}'^{\otimes} \arrow[rr, "\alpha^{\otimes}"'] & & \mathcal{O}^{\otimes} & \end{tikzcd}$$

be a commutative diagram of ∞-operads and assume that ι is fully faithful.

Then the functor

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\iota)\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}') \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$$

is fully faithful. Furthermore, an object A of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ lies in the essential image of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\iota)$ if and only if for every object X of $\mathcal{O}'$ the evaluation $\mathrm{ev}_X(A)$ of A at X lies in the essential image of ι. ♡

Proof. Combining Propositions B.3.0.1, B.5.1.1 and B.5.3.1 we obtain that

$$\iota^{\otimes}_{*}\colon \operatorname{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}'^{\otimes}) \to \operatorname{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$$

is fully faithful with essential image spanned precisely by those functors $F^{\otimes}\colon \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ over $\mathcal{O}^{\otimes}$ for which $F^{\otimes}(X)$ lies in the essential image of $\iota^{\otimes}$ for every object X of $\mathcal{O}'^{\otimes}$. There is a commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}'^{\otimes}) \arrow[r, "\iota^{\otimes}_{*}"] & \operatorname{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \\
\operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[u] \arrow[r, "\operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\iota)"'] & \operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}') \arrow[u]
\end{tikzcd}$$

where the the vertical functors are the canonical inclusions and thus by definition fully faithful, so it follows that $\operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\iota)$ is also fully faithful, with essential image spanned by those algebras whose associated functors $F^{\otimes}\colon \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ are such that $F^{\otimes}(X)$ lies in the essential image of $\iota^{\otimes}$ for every object X of $\mathcal{O}'^{\otimes}$.

As $F^{\otimes}$ and $\alpha^{\otimes}$ are morphisms of ∞-operads, we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{O}'^{\otimes}_{\langle n\rangle} \arrow[r, "F^{\otimes}_{\langle n\rangle}"] \arrow[d, no head, "\simeq"'] & \mathcal{C}^{\otimes}_{\langle n\rangle} \arrow[d, no head, "\simeq"] & \mathcal{C}'^{\otimes}_{\langle n\rangle} \arrow[l, "\alpha^{\otimes}_{\langle n\rangle}"'] \arrow[d, no head, "\simeq"] \\
\mathcal{O}'^{\times n} \arrow[r, "F^{\times n}"'] & \mathcal{C}^{\times n} & \mathcal{C}'^{\times n} \arrow[l, "\alpha^{\times n}"]
\end{tikzcd}$$

for every $n \geq 0$ that shows that $F^{\otimes}(X)$ lying in the essential image of $\iota^{\otimes}$ for every object X of $\mathcal{O}'^{\otimes}$ is equivalent to $F(X)$ lying in the essential image of ι for every object X of $\mathcal{O}$. □

E.4. Induced ∞-operad structures on Alg

Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $\mathcal{O}$ an ∞-operad. Then the tensor product on $\mathcal{C}$ induces a symmetric monoidal structure on $\operatorname{Alg}_{\mathcal{O}}(\mathcal{C})$ such that the forgetful functor $\operatorname{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ can be upgraded to a symmetric monoidal functor. In the setting of quasicategories, this structure is constructed in [HA, 3.2.4.1, 3.2.4.2, and 3.2.4.3]. However, it is not immediately obvious from the definition that this construction does not depend on the choice of representatives (or in other words, whether it is invariant under categorical equivalences). In Section E.4.1 we will give a description of the construction that can be performed entirely in $\mathfrak{Cat}_{\infty}$, i. e. without the help of models like quasicategories, and show that it agrees with the one given by Lurie. Apart from the aesthetic gain from being able to work as model independently as possible, the reformulated description will also be helpful in some results we will prove later.

In Section E.4.2 we will then collect a number of properties that the induced ∞-operad structure has, deducing most of them from the results of [HA, 3.2.4]. It would also be possible to prove these statements without referring back to the quasicategorical model. However, we need to show agreement of the two approaches anyway, as throughout the text we will need to make use of several other results from [HA] using the induced ∞-operad structure on algebras, so giving an independent, more model-independent proof of the statements discussed in Section E.4.2 would not save us from having to go through the comparison in Section E.4.1.

E.4.1. The quasicategorical model

In this section we discuss Lurie's quasicategorical model for induced ∞-operad structures on ∞-categories of algebras, and compare it to a more model-independent definition.

We will make use of the following convention during our discussion.

Convention E.4.1.1. In contrast with the rest of the text, wherever we explicitly invoke this convention every notion should be taken to refer to the respective quasicategorical notion as defined in [HTT] and [HA]. So for example the claim that a diagram of quasicategories commutes means that it is a strictly commuting diagram of simplicial sets, and an ∞-operad is a map of simplicial sets $\mathtt{O}^{\otimes} \to \mathsf{Fin}_*$ where $\mathtt{O}$ is a quasicategory and such that the map satisfies some properties, rather than a morphism $\mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ in $\mathcal{C}\mathrm{at}_\infty$ satisfying some properties. ♢

We start by reviewing the construction given in [HA, 3.2.4.1].

Definition E.4.1.2 ([HA, 3.2.4.1])**.** We make use of Convention E.4.1.1 in this construction. Let $\mathtt{p}_{\mathtt{O}} \colon \mathtt{O}^{\otimes} \to \mathsf{Fin}_*$, $\mathtt{p}_{\mathtt{O}'} \colon \mathtt{O}'^{\otimes} \to \mathsf{Fin}_*$, and $\mathtt{p}_{\mathtt{O}''} \colon \mathtt{O}''^{\otimes} \to \mathsf{Fin}_*$ be ∞-operads, and let $\mathtt{q} \colon \mathtt{C}^{\otimes} \to \mathtt{O}''^{\otimes}$ be a fibration of ∞-operads, i. e. a map of ∞-operads where $\mathtt{q}$ is also a categorical fibration of quasicategories (see [HA, 2.1.2.10]). Let $\mathtt{f} \colon \mathtt{O}^{\otimes} \times \mathtt{O}'^{\otimes} \to \mathtt{O}''^{\otimes}$ be a bifunctor of ∞-operads, i. e. a functor of quasicategories such that the diagram

$$\begin{tikzcd}
\mathtt{O}^{\otimes} \times \mathtt{O}'^{\otimes} \arrow[r, "\mathtt{f}"] \arrow[d, "\mathtt{p}_{\mathtt{O}} \times \mathtt{p}_{\mathtt{O}'}"'] & \mathtt{O}''^{\otimes} \arrow[d, "\mathtt{p}_{\mathtt{O}''}"] \\
\mathsf{Fin}_* \times \mathsf{Fin}_* \arrow[r, "\wedge"'] & \mathsf{Fin}_*
\end{tikzcd}$$

commutes and such that $\mathtt{f}$ sends pairs of inert morphisms to inert morphisms, see [HA, 2.2.5.3].

Define $\widetilde{\Phi}$ to be the functor $\mathsf{sSet}_{/\mathtt{O}^\otimes} \to \mathsf{Set}$ that sends $\mathtt{g}\colon \mathtt{K} \to \mathtt{O}^\otimes$ to the set of commutative diagrams as indicated below.

$$\begin{tikzcd} \mathtt{K} \times \mathtt{O}'^\otimes \arrow[r, "\mathtt{F}"] \arrow[d, "\mathtt{g}\times \mathrm{id}_{\mathtt{O}'^\otimes}"'] & \mathtt{C}^\otimes \arrow[d, "\mathtt{r}"] \\ \mathtt{O}^\otimes \times \mathtt{O}'^\otimes \arrow[r, "\mathtt{f}"'] & \mathtt{O}''^\otimes \end{tikzcd} \tag{E.2}$$

Furthermore, define $\Phi\colon \mathsf{sSet}_{/\mathtt{O}^\otimes} \to \mathsf{Set}$ to be the functor which sends a map $\mathtt{g}\colon \mathtt{K} \to \mathtt{O}^\otimes$ to the subset of $\widetilde{\Phi}(\mathtt{g})$ of commutative diagrams (E.2) which have the property that $\mathtt{F}(\mathrm{id}_\mathtt{k}, \alpha)$ is inert for every vertex $\mathtt{k}$ of $\mathtt{K}$ and every inert morphism α in $\mathtt{O}''^\otimes$.

We say that an object $\mathtt{r}$ in $\mathsf{sSet}_{/\mathtt{O}^\otimes}$ is *a quasicategorical model (a quasicategorical pre-model) for the ∞-operad structure on algebras* with respect to $\mathtt{f}$, $\mathtt{q}$, etc. as introduced above, if there exists a natural bijection of functors $\mathsf{sSet}_{/\mathtt{O}^\otimes} \to \mathsf{Set}$ between $\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^\otimes}}(-, \mathtt{r})$ and Φ (between $\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^\otimes}}(-, \mathtt{r})$ and $\widetilde{\Phi}$).

Note that the Yoneda lemma implies that if a quasicategorical (pre-)model for the ∞-operad structure on algebras exists, then it is unique up to isomorphism in $\mathsf{sSet}_{/\mathtt{O}^\otimes}$. We will give a more concrete construction of a quasicategorical (pre-)model for the ∞-operad structure on algebras below. $\diamondsuit$

Remark E.4.1.3. In this remark we make use of Convention E.4.1.1, and assume that we are in the situation of Definition E.4.1.2. Let

$$\widetilde{\mathtt{r}}\colon \widetilde{\mathrm{Alg}}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^\otimes \to \mathtt{O}^\otimes$$

be a quasicategorical pre-model for the ∞-operad structure on algebras, and let ϕ be a natural bijection $\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^\otimes}}(-, \widetilde{\mathtt{r}}) \cong \widetilde{\Phi}$. We then define a sub-simplicial set $\mathrm{Alg}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^\otimes$ of $\widetilde{\mathrm{Alg}}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^\otimes$ as the sub-simplicial set spanned by those vertices $\mathtt{A}$ which correspond under ϕ to maps

$$\mathtt{O}'^\otimes \cong \{\mathtt{A}\} \times \mathtt{O}'^\otimes \xrightarrow{\mathtt{F}} \mathtt{C}^\otimes$$

that preserve inert morphisms.

Let $\mathtt{r}\colon \mathrm{Alg}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^\otimes \to \mathtt{O}^\otimes$ be the restriction of $\widetilde{\mathtt{r}}$ to $\mathrm{Alg}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^\otimes$. As the condition defining the natural subset Φ of $\widetilde{\Phi}$ can be checked vertex-wise (in $\mathtt{K}$, where we use the notation from (E.2)), it is clear that ϕ restricts to a bijection between $\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^\otimes}}(-, \mathtt{r})$ and Φ. We conclude that $\mathtt{r}$ is a quasicategorical model for the ∞-operad structure on algebras. $\diamondsuit$

Proposition E.4.1.4. *In this proposition Convention E.4.1.1 applies. Assume we are in the situation of Definition E.4.1.2. Let $\widetilde{\mathtt{r}}$ be the functor*

$$\widetilde{\mathtt{r}}\colon \mathrm{Fun}\big(\mathtt{O}'^\otimes, \mathtt{C}^\otimes\big) \times_{\mathrm{Fun}(\mathtt{O}'^\otimes, \mathtt{O}''^\otimes)} \mathtt{O}^\otimes \xrightarrow{\mathrm{pr}_2} \mathtt{O}^\otimes$$

where the functor $\mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}) \to \mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes})$ *that is part of the pullback is* q_*, *and the functor* $\mathtt{O}^{\otimes} \to \mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes})$ *is the adjoint functor to* $\mathtt{f}$*. Let* $\mathtt{r}$ *be the restriction of* $\widetilde{\mathtt{r}}$ *to* $\mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes})' \times_{\mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes})} \mathtt{O}^{\otimes}$, *where* $\mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes})'$ *is the sub-simplicial set of* $\mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes})$ *spanned by the vertices which are functors* $\mathtt{O}'^{\otimes} \to \mathtt{C}^{\otimes}$ *that preserve inert morphisms.*

Then $\widetilde{\mathtt{r}}$ *is a quasicategorical pre-model for the* ∞*-operad structure on algebras and* $\mathtt{r}$ *is a quasicategorical model for the* ∞*-operad structure on algebras.* ♡

Proof. Let $\mathtt{g}\colon \mathtt{K} \to \mathtt{O}^{\otimes}$ be an object in $\mathsf{sSet}_{/\mathtt{O}^{\otimes}}$. There is a chain of bijections which are natural in $\mathtt{g}$ as follows.

$$\begin{aligned}
&\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^{\otimes}}}(\mathtt{g}, \widetilde{\mathtt{r}})\\
&\cong \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{K}, \mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}) \times_{\mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes})} \mathtt{O}^{\otimes}\big) \times_{\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K}, \mathtt{O}^{\otimes})} \{\mathtt{g}\}\\
&\cong \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{K} \times \mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big) \times_{\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K} \times \mathtt{O}'^{\times}, \mathtt{O}''^{\otimes})} \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{K}, \mathtt{O}^{\otimes}\big) \times_{\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K}, \mathtt{O}^{\otimes})} \{\mathtt{g}\}\\
&\cong \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{K} \times \mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big) \times_{\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K} \times \mathtt{O}'^{\times}, \mathtt{O}''^{\otimes})} \{\mathtt{f} \circ (\mathtt{g} \times \mathrm{id}_{\mathtt{O}'^{\otimes}})\}\\
&\cong \widetilde{\Phi}(\mathtt{g})
\end{aligned}$$

This shows the claim about $\widetilde{\mathtt{r}}$. The claim for $\mathtt{r}$ follows using Remark E.4.1.3 after noting that for a vertex $\mathtt{A}$ of $\mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}) \times_{\mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes})} \mathtt{O}^{\otimes}$ considered as a functor

$$\mathtt{a}\colon \{\mathtt{A}\} \to \mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}) \times_{\mathrm{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes})} \mathtt{O}^{\otimes}$$

the composition of the chain of bijections above with the projection to

$$\mathrm{Mor}_{\mathsf{sSet}}\big(\{\mathtt{A}\} \times \mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big) \cong \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big)$$

sends $\mathtt{a}$ to $\mathrm{pr}_1(\mathtt{A})$. □

We can now state the construction of the induced ∞-operad structure on ∞-categories of algebras without referring to quasicategories.

Proposition E.4.1.5. *Let* $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$, $p_{\mathcal{O}'}\colon \mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$, *as well as* $p_{\mathcal{O}''}\colon \mathcal{O}''^{\otimes} \to \mathsf{Fin}_*$ *be* ∞*-operads and let* $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}''^{\otimes}$ *be a morphism of* ∞*-operads. Let* $f\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ *be a bifunctor of* ∞*-operads.*

Let $\mathsf{p}_{\mathtt{O}}\colon \mathtt{O}^{\otimes} \to \mathsf{Fin}_*$, $\mathsf{p}_{\mathtt{O}'}\colon \mathtt{O}'^{\otimes} \to \mathsf{Fin}_*$, *and* $\mathsf{p}_{\mathtt{O}''}\colon \mathtt{O}''^{\otimes} \to \mathsf{Fin}_*$ *be functors of quasicategories which represent* $p_{\mathcal{O}}$, $p_{\mathcal{O}'}$, *and* $p_{\mathcal{O}''}$, *respectively. Let* $\mathtt{q}\colon \mathtt{C}^{\otimes} \to \mathtt{O}''^{\otimes}$ *be a categorical fibration of quasicategories representing* q *and let* $\mathtt{f}\colon \mathtt{O}^{\otimes} \times \mathtt{O}'^{\otimes} \to \mathtt{O}''^{\otimes}$ *be a functor of quasicategories representing* f.

Define $\mathrm{Alg}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})$, $\widetilde{\mathrm{Alg}}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})$, $\widetilde{\mathtt{r}}$, $\mathtt{r}$, $\mathtt{s}'$, *and* $\mathtt{s}$ *via the following diagram, where the two squares are to be pullback diagrams,* $\widehat{\mathtt{f}}$ *is adjoint to* $\mathtt{f}$, *and* $\mathtt{i}_{\mathrm{Fun}}$ *is the inclusion of the full sub-simplicial set spanned by those vertices which* *correspond to functors that preserve inert morphisms.*

$$\begin{tikzcd}
\mathrm{Alg}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes} \arrow[r, "\mathtt{i}_{\mathrm{Alg}}"] \arrow[d, "\mathtt{s}"] \arrow[rr, bend left, "\mathtt{r}"] & \widetilde{\mathrm{Alg}}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes} \arrow[r, "\widetilde{\mathtt{r}}"] \arrow[d, "\mathtt{s}'"] & \mathtt{O}^{\otimes} \arrow[d, "\widehat{\mathtt{f}}"] \\
\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big)' \arrow[r, "\mathtt{i}_{\mathrm{Fun}}"'] & \mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big) \arrow[r, "\mathtt{q}_*"'] & \mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes}\big)
\end{tikzcd} \tag{E.3}$$

Then the above diagram represents the following diagram in $\mathcal{C}\mathrm{at}_\infty$, were both squares are pullback diagrams as well, $\widehat{f}$ is adjoint to f, and ι_{Fun} is the inclusion of the full subcategory spanned by those functors that preserve inert morphisms.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\iota_{\mathrm{Alg}}"] \arrow[d, "s"] \arrow[rr, bend left, "r"] & \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\widetilde{r}"] \arrow[d, "s'"] & \mathcal{O}^{\otimes} \arrow[d, "\widehat{f}"] \\
\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})' \arrow[r, "\iota_{\mathrm{Fun}}"'] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r, "q_*"'] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})
\end{tikzcd}$$

♡

Proof. What we have to show is that both squares in diagram (E.3) are homotopy pullback diagrams with respect to the Joyal model structure. We begin by showing that $\mathtt{q}_*$ and $\mathtt{i}_{\mathrm{Fun}}$ are categorical fibrations.

By assumption $\mathtt{q}\colon \mathtt{C}^{\otimes} \to \mathtt{O}''^{\otimes}$ is a categorical fibration of quasicategories. A map of simplicial sets is a categorical fibration if and only if it has the right lifting property with respect to maps of simplicial sets which are monomorphisms as well as categorical equivalences (see [HTT, 2.2.5.1]). By adjoining the lifting problems we need to solve to show that $\mathtt{q}_*$ is a categorical fibration we are reduced to showing that if $\mathtt{j}$ is a map of simplicial sets which is a monomorphism as well as a categorical equivalence, then $\mathtt{j} \times \mathrm{id}_{\mathtt{O}'^{\otimes}}$ is so as well. That $\mathtt{j} \times \mathrm{id}_{\mathtt{O}'^{\otimes}}$ is again a monomorphism is clear, and that it is also a categorical equivalence is [HTT, 2.2.5.4].

We next argue that $\mathtt{i}_{\mathrm{Fun}}$ is also a categorical fibration. As $\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big)$ is a quasicategory by [HTT, 1.2.7.3 (1)], we can apply [HTT, 2.4.6.5] so that it suffices to show that $\mathtt{i}_{\mathrm{Fun}}$ is an inner fibration and that for any natural equivalence $\varphi\colon \mathtt{g} \to \mathtt{g}'$ of functors $\mathtt{O}'^{\otimes} \to \mathtt{C}^{\otimes}$ such that $\mathtt{g}$ preserves inert morphisms it follows that $\mathtt{g}'$ preserves inert morphisms as well. The latter property follows immediately from the fact that cocartesian morphisms are closed under equivalences. It remains to show that $\mathtt{i}_{\mathrm{Fun}}$ is an inner fibration. But note that every horn inclusion $\Lambda^n_i \subseteq \Delta^n$ for $0 < i < n$ is an isomorphism on 0-simplices, and as $\mathtt{i}_{\mathrm{Fun}}$ is the inclusion of a full sub-simplicial set lifting positive dimensional simplices is always possible, so $\mathtt{i}_{\mathrm{Fun}}$ is an inner fibration.

We have now shown that $\mathtt{q}_*$ and $\mathtt{i}_{\mathrm{Fun}}$ are both categorical fibrations. By assumption $\mathtt{O}^{\otimes}$ is a quasicategory and $\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes}\big)$ is a quasicategory by [HTT, 1.2.7.3 (1)], so it follows from [HTT, A.2.4.4, variant (i) and A.2.4.5] that the right square in diagram (E.3) is a homotopy pullback square with respect to the Joyal model structure. As a pullback of the categorical fibration $\mathtt{q}_*$ is the functor $\widetilde{\mathtt{r}}$ a categorical fibration as well, so as $\mathtt{O}^{\otimes}$ is a quasicategory, $\widetilde{\mathrm{Alg}}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes}$ is also a quasicategory [HTT, 2.4.6.1]. It was already mentioned that $\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big)$ is a quasicategory, so we can apply [HTT, A.2.4.4, variant (i) and A.2.4.5] again to conclude that the left square in diagram (E.3) is also a homotopy pullback square with respect to the Joyal model structure. □

E.4.2. Properties of the induced ∞-operad structure

In Proposition E.4.1.5 we gave a construction of the induced ∞-operad structure on ∞-categories of algebras that could be formulated without referring back to quasicategorical models. In this section we collect the properties of this construction.

Remark E.4.2.1. In the situation of Proposition E.4.1.5, it follows from Proposition B.5.2.1 that as ι_{Fun} is a fully faithful functor, so is ι_{Alg}. We can thus identify

$$\iota_{\mathrm{Alg}} \colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})} \mathcal{O}^{\otimes}$$

with the inclusion of the full subcategory spanned by those objects whose projection to the first factor is a functor $\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ that preserves inert morphisms. ♢

Remark E.4.2.2. Let $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{O}''$ be ∞-operads, let

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "q_{\mathcal{C}}"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q_{\mathcal{D}}"] \\ & \mathcal{O}''^{\otimes} & \end{tikzcd}$$

be a commutative diagram of ∞-operads, and let $f \colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads. Then the functor indicated as the right vertical functor in the following diagram induces a functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}$ on algebras that makes the diagram commute

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\iota^{\mathcal{C}}_{\mathrm{Alg}}"] \arrow[d, dashed, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}"'] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})} \mathcal{O}^{\otimes} \arrow[d, "(F^{\otimes})_* \times_{\mathrm{id}} \mathrm{id}"] \\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D})^{\otimes} \arrow[r, "\iota^{\mathcal{D}}_{\mathrm{Alg}}"'] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})} \mathcal{O}^{\otimes} \end{tikzcd}$$

where $\iota^{\mathcal{C}}_{\mathrm{Alg}}$ and $\iota^{\mathcal{D}}_{\mathrm{Alg}}$ are as in Remark E.4.2.1. This follows immediately from the description in Remark E.4.2.1, as F preserves inert morphisms as a morphism of ∞-operads. From the definition it is also clear that $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}$ is compatible with the projections to $\mathcal{O}^{\otimes}$. ♢

In light of Proposition E.4.1.4 and Proposition E.4.1.5, all the properties listed in [HA, 3.2.4.2 and 3.2.4.3] apply to $r\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$. We restate them as the proposition below for easier reference.

Proposition E.4.2.3 ([HA, 3.2.4.2 and 3.2.4.3])**.** *Let $\mathcal{O}$, $\mathcal{O}'$, as well as $\mathcal{O}''$ be ∞-operads, let $q_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}''^{\otimes}$ be a morphism of ∞-operads, and let $f\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads.*

Let

$$\iota^{\mathcal{C}}_{\mathrm{Alg}}\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})} \mathcal{O}^{\otimes}$$

be as in Proposition E.4.1.5 and Remark E.4.2.1 and denote by $r_{\mathcal{C}}$ the composition $\mathrm{pr}_2 \circ \iota^{\mathcal{C}}_{\mathrm{Alg}}$. Then the following hold:

(0) Let X be an object of $\mathcal{O}$. Then $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}_{X}$ can be identified with $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})$, where the latter ∞-category of algebras is taken with respect to the following morphism of ∞-operads.

$$f_X\colon \mathcal{O}'^{\otimes} \simeq \{X\} \times \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \xrightarrow{f} \mathcal{O}''^{\otimes}$$

This identification is compatible with the respective inclusions into the following ∞-categories.

$$\begin{aligned} &\big(\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})} \mathcal{O}^{\otimes}\big) \times_{\mathcal{O}^{\otimes}} \{X\} \\ \simeq\ &\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})} \{f_X\} \end{aligned}$$

(1) The functor $r_{\mathcal{C}}$ is a morphism of ∞-operads.

(2) A morphism α in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}$ lying over an inert morphism in $\mathcal{O}^{\otimes}$ is inert if and only if for every object X of $\mathcal{O}'$, the morphism

$$\mathrm{ev}_X\big(\mathrm{pr}_1(\iota^{\mathcal{C}}_{\mathrm{Alg}}(\alpha))\big)$$

in $\mathcal{C}^{\otimes}$ is inert.

(3) If $q_{\mathcal{C}}$ is a cocartesian fibration of ∞-operads, then so is $r_{\mathcal{C}}$.

(4) Assume that $q_{\mathcal{C}}$ is a cocartesian fibration of ∞-operads. Then a morphism α in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}$ is $r_{\mathcal{C}}$-cocartesian if and only if for every object X of $\mathcal{O}'$, the morphism obtained by evaluating at X, i. e.

$$\mathrm{ev}_X\big(\mathrm{pr}_1(\iota^{\mathcal{C}}_{\mathrm{Alg}}(\alpha))\big)$$

is $q_{\mathcal{C}}$-cocartesian.

(5) *Let X be an object of $\mathcal{O}'$. Then the functor $\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota^{\mathcal{C}}_{\mathrm{Alg}}$ is a morphism of ∞-operads and fits into a commutative diagram*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota^{\mathcal{C}}_{\mathrm{Alg}}"] \arrow[d, "r_{\mathcal{C}}"'] & \mathcal{C}^{\otimes} \arrow[d, "q_{\mathcal{C}}"] \\ \mathcal{O}^{\otimes} \arrow[r] & \mathcal{O}''^{\otimes} \end{tikzcd}$$

where the bottom horizontal functor is the following composition.

$$\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \times \{X\} \xrightarrow{f} \mathcal{O}''^{\otimes}$$

Furthermore, if $q_{\mathcal{C}}$ is a cocartesian fibration of ∞-operads, then the composition $\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota^{\mathcal{C}}_{\mathrm{Alg}}$ sends $r_{\mathcal{C}}$-cocartesian morphisms to $q_{\mathcal{C}}$-cocartesian morphisms.

We can also consider how the above properties behave under induced functors as in Remark E.4.2.2. So let

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "q_{\mathcal{C}}"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q_{\mathcal{D}}"] \\ & \mathcal{O}''^{\otimes} & \end{tikzcd}$$

be a commutative diagram of ∞-categories, and let $\iota^{\mathcal{D}}_{\mathrm{Alg}}$ and $r_{\mathcal{D}}$ be defined analogously to $\iota^{\mathcal{C}}_{\mathrm{Alg}}$ and $r_{\mathcal{C}}$. Then the following hold.

(6) *Let X be an object of $\mathcal{O}$. Then there is a commutative diagram*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}_X \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}_X"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)"] \\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D})^{\otimes}_X \arrow[r, dash, "\simeq"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D}) \end{tikzcd}$$

where the horizontal functors are the equivalences from (0).

(7) *The functor*

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes} \colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D})^{\otimes}$$

is a morphism of ∞-operads.

(8) *If $q_{\mathcal{C}}$ and $q_{\mathcal{D}}$ are cocartesian fibrations of ∞-operads, and F is an $\mathcal{O}''$-monoidal functor, then the functor*

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes} \colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D})^{\otimes}$$

is $\mathcal{O}$-monoidal.

(9) Let X be an object of $\mathcal{O}'$. Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}"] \arrow[d, "\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota^{\mathcal{C}}_{\mathrm{Alg}}"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D})^{\otimes} \arrow[d, "\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota^{\mathcal{D}}_{\mathrm{Alg}}"] \\
\mathcal{C}^{\otimes} \arrow[r, "F^{\otimes}"'] & \mathcal{D}^{\otimes}
\end{tikzcd}$$

of ∞-operads. ♡

Proof. Claims (0) to (4) are just restatements of [HA, 3.2.4.2 and 3.2.4.3] which applies in this form due to Proposition E.4.1.4 and Proposition E.4.1.5.

Claim (5) follows directly from (2) and (4). Claim (6) follows immediately from Remark E.4.2.2 and (0). Combining that F is a morphism of ∞-operads with the description of inert morphisms in (2) implies (7), and if F is a $\mathcal{O}''$-monoidal, then combining this with (4) implies (8). Finally, (9) is immediate from the definitions. □

Remark E.4.2.4. Let $\mathcal{O}_L$, $\mathcal{O}'_L$, $\mathcal{O}_R$, and $\mathcal{O}'$ be ∞-operads, let $q_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}'^{\otimes}$ be a morphism of ∞-operads, let

$$f\colon \mathcal{O}_L^{\otimes} \times \mathcal{O}_R^{\otimes} \to \mathcal{O}'^{\otimes}$$

be a bifunctor of ∞-operads, and let $\alpha^{\otimes}\colon \mathcal{O}'_L \to \mathcal{O}_L$ be a morphism of ∞-operads.

We obtain another bifunctor of ∞-categories f' as the following composition.

$$f'\colon \mathcal{O}'^{\otimes}_L \times \mathcal{O}_R^{\otimes} \xrightarrow{\alpha^{\otimes}\times\mathrm{id}} \mathcal{O}_L^{\otimes} \times \mathcal{O}_R^{\otimes} \xrightarrow{f} \mathcal{O}'^{\otimes}$$

We obtain a pullback diagram as follows

$$\begin{tikzcd}
\mathrm{Fun}\big(\mathcal{O}_R^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathcal{O}_R^{\otimes}, \mathcal{O}'^{\otimes}\right)} \mathcal{O}'^{\otimes}_L \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{id}\times_{\mathrm{id}}\alpha^{\otimes}"'] & \mathcal{O}'^{\otimes}_L \arrow[d, "\alpha^{\otimes}"] \\
\mathrm{Fun}\big(\mathcal{O}_R^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathcal{O}_R^{\otimes}, \mathcal{O}'^{\otimes}\right)} \mathcal{O}^{\otimes}_L \arrow[r, "\mathrm{pr}_2"'] & \mathcal{O}^{\otimes}_L
\end{tikzcd}$$

where the pullbacks on the left are take with respect to the morphisms as in Proposition E.4.1.5, on the top with respect to f' and the bottom with respect to f.

It is clear from the definition of ι_{Alg} (see Remark E.4.2.1) that an object lies in the essential image of the functor ι_{Alg} associated to the bifunctor f' if and only if $\mathrm{id} \times_{\mathrm{id}} \alpha^{\otimes}$ maps that object to the essential image of the functor ι_{Alg} associated to the bifunctor f.

It thus follows from Proposition B.5.3.1, Proposition B.4.4.1, and Proposition B.4.3.1 that the above pullback diagram induces another pullback diagram as follows, where the $\mathrm{Alg}_{\mathcal{O}_R/\mathcal{O}'}(\mathcal{C})^{\otimes}$ at the top left is the one with respect to the bifunctor f' and the one at the bottom left is with respect to the bifunctor f.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}_R/\mathcal{O}'}(\mathcal{C})^{\otimes} \arrow[r, "\mathrm{pr}_2\circ\iota_{\mathrm{Alg}}"] \arrow[d] & \mathcal{O}'^{\otimes}_L \arrow[d, "\alpha^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}_R/\mathcal{O}'}(\mathcal{C})^{\otimes} \arrow[r, "\mathrm{pr}_2\circ\iota_{\mathrm{Alg}}"'] & \mathcal{O}^{\otimes}_L
\end{tikzcd}$$

By Proposition E.4.2.3 (7), the horizontal functors are morphisms of ∞-operads, $\alpha^{\otimes}$ is by assumption a morphism of ∞-operads, and it then follows from Proposition E.4.2.3 (2) that the left vertical functor is also a morphism of ∞-operads. $\diamondsuit$

E.5. Iterating Alg

Proposition E.4.2.3 allows us to "iterate" passing to the ∞-category of algebras. In this section we show that there is an alternative description of algebras of algebras: There is an equivalence of ∞-categories between the ∞-category of $\mathcal{O}$-algebras in $\mathcal{O}'$-algebras $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}))$ and the ∞-category of $\mathcal{O}\otimes\mathcal{O}'$-algebras $\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C})$. This equivalence goes through an intermediate step, the ∞-category $\mathrm{BiFunc}(\mathcal{O},\mathcal{O}',\mathcal{C})$ of bifunctors of ∞-operads.

Proposition E.5.0.1. *Let $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$, $p'_{\mathcal{O}}\colon \mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$, as well as $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be ∞-operads.*

Then there is a commutative diagram as follows[8]

$$\begin{tikzcd}
\mathrm{BiFunc}(\mathcal{O},\mathcal{O}';\mathcal{C}) \arrow[r] \arrow[d, "\Phi_2"', "\simeq"] & \mathrm{Fun}(\mathcal{O}^{\otimes}\times\mathcal{O}'^{\otimes},\mathcal{C}^{\otimes}) \arrow[d, "\widehat{(-)}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})) \arrow[r] & \mathrm{Fun}(\mathcal{O}^{\otimes},\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathcal{C}^{\otimes}))
\end{tikzcd}$$

where $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ carries the ∞-operad structure from Proposition E.4.2.3, see [HA, 3.2.4.4] [9]*, the horizontal functors are the canonical ones, and the functor $\widehat{(-)}$ sends a functor G to its adjoint $\widehat{G}$. The functor Φ_2 is an equivalence.* $\heartsuit$

Proof. We consider the following diagram, in which the outer square corresponds to the square from the statement. We will explain the individual

[8] See [HA, 2.2.5.3] for a definition of BiFunc.

[9] There is a bifunctor of ∞-operads $\mathsf{Fin}_*\times\mathcal{O}'^{\otimes} \xrightarrow{\mathrm{id}\times p_{\mathcal{O}'}} \mathsf{Fin}_*\times\mathsf{Fin}_* \xrightarrow{-\wedge-} \mathsf{Fin}_*$ and it is with respect to this bifunctor that we apply Proposition E.4.2.3.

functors in the text below.

$$
\begin{tikzcd}
\mathrm{BiFunc}(\mathcal{O},\mathcal{O}';\mathcal{C}) \arrow[r, dashed, "\Phi_2", "\simeq"'] \arrow[dddd, "j"'] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})) \arrow[d, "i"] \\
& \mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{O}^{\otimes}, \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}) \arrow[d, "(\iota_{\mathrm{Alg}})_*"] \\
& \mathcal{E} := \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes},\mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathsf{Fin}_*)} \mathsf{Fin}_*\big) \arrow[dd, bend left=60, "P_3"] \\
& \\
\mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r, "V", "\simeq"'] \arrow[d, "P_1"'] & \mathrm{Fun}_{\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathsf{Fin}_*)}(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes},\mathcal{C}^{\otimes})) \arrow[uu, "W", "\simeq"'] \arrow[d, "P_2"] \\
\mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r, shift left, "\widehat{(-)}"] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes},\mathcal{C}^{\otimes})) \arrow[l, shift left, "\widetilde{(-)}", "\simeq"']
\end{tikzcd}
$$

Functors P_1, P_2, and P_3 are constructed from the relevant projection and forgetful functors: P_1 forgets that the functor was over Fin_*, and similarly for P_2. The functor P_3 additionally postcomposes with the projection to the first factor. Functors $\widetilde{(-)}$ and $\widehat{(-)}$ send functors to their respective adjoints, both are equivalences.

We use notation from Proposition E.4.2.3, so ι_{Alg} is the inclusion of the full subcategory $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ of $\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathsf{Fin}_*)} \mathsf{Fin}_*$ of those objects whose projection to the first factor is a functor $\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ preserving inert morphisms. It follows from Proposition B.3.0.1 that $(\iota_{\mathrm{Alg}})_*$ is also fully faithful, and applying Proposition B.5.3.1 and Remark B.5.1.2 we can further conclude that the functor ι_{Alg_*} in the diagram is fully faithful, with essential image spanned by precisely those objects of $\mathcal{E}$ which are mapped by P_3 to functors

$$\mathcal{O}^{\otimes} \to \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)$$

which evaluated at every object of $\mathcal{O}^{\otimes}$ yield a functor $\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ that preserves inert morphisms.

The functor i is the canonical inclusion of the full subcategory of those functors $\mathcal{O}^{\otimes} \to \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ over Fin_* which send inert morphisms to inert morphisms. Using Proposition C.1.2.1, Proposition C.1.1.1, and [HTT, 3.1.2.1] we can reformulate this condition: i is the inclusion of the full subcategory of objects who are mapped by $P_3 \circ (\iota_{\mathrm{Alg}})_*$ to functors

$$\mathcal{O}^{\otimes} \to \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)$$

which send an inert morphism in $\mathcal{O}^{\otimes}$ to a natural transformation for which every component is an inert morphism in $\mathcal{C}^{\otimes}$.

The above discussion can be summarized as follows: The composition $(\iota_{\mathrm{Alg}})_* \circ i$ is fully faithful, and an object E of $\mathcal{E}$ is in the essential image of $(\iota_{\mathrm{Alg}})_* \circ i$ precisely when $\widetilde{P_3(E)}$ is a functor $\mathcal{O}^\otimes \times \mathcal{O}'^\otimes \to \mathcal{C}^\otimes$ that preserves inert morphisms separately in each variable. As identity morphism in $\mathcal{O}^\otimes$ and $\mathcal{O}'^\otimes$ are inert [HTT, 2.4.1.5] and cocartesian morphisms are closed under composition [HTT, 2.4.1.7], this condition is equivalent to the functor sending pairs of inert morphisms to inert morphisms in $\mathcal{C}^\otimes$.

The functor W is an equivalence and constructed using compatibility of Fun with pullbacks, the $\times$ − Fun-adjunction, as well as the pasting law for pullbacks [HTT, 4.4.2.1]; It is the following composition.

$$\begin{aligned}
&\mathrm{Fun}_{\mathrm{Fun}(\mathcal{O}'^\otimes,\mathsf{Fin}_*)}\big(\mathcal{O}^\otimes, \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)\big)\\
\simeq\ &\mathrm{Fun}\big(\mathcal{O}^\otimes, \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)\big) \times_{\mathrm{Fun}(\mathcal{O}^\otimes,\mathrm{Fun}(\mathcal{O}'^\otimes,\mathsf{Fin}_*))} \{(\widehat{p_{\mathcal{O}} \wedge p_{\mathcal{O}'}})\}\\
\simeq\ &\mathrm{Fun}\big(\mathcal{O}^\otimes, \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)\big) \times_{\mathrm{Fun}(\mathcal{O}^\otimes,\mathrm{Fun}(\mathcal{O}'^\otimes,\mathsf{Fin}_*))} \mathrm{Fun}\big(\mathcal{O}^\otimes, \mathsf{Fin}_*\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times_{\mathrm{Fun}(\mathcal{O}^\otimes,\mathsf{Fin}_*)} \{(\widehat{p_{\mathcal{O}} \wedge p_{\mathcal{O}'}})\}\\
\simeq\ &\mathrm{Fun}\big(\mathcal{O}^\otimes, \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes) \times_{\mathrm{Fun}(\mathcal{O}'^\otimes,\mathsf{Fin}_*)} \mathsf{Fin}_*\big) \times_{\mathrm{Fun}(\mathcal{O}^\otimes,\mathsf{Fin}_*)} \{(\widehat{p_{\mathcal{O}} \wedge p_{\mathcal{O}'}})\}\\
\simeq\ &\mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}^\otimes, \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes) \times_{\mathrm{Fun}(\mathcal{O}'^\otimes,\mathsf{Fin}_*)} \mathsf{Fin}_*\big)
\end{aligned}$$

It is clear that W defined like this satisfies $P_3 \circ W \simeq P_2$.

The equivalence V is defined using quite similar manipulations, as indicated below.

$$\begin{aligned}
&\mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}^\otimes \times \mathcal{O}'^\otimes, \mathcal{C}^\otimes\big)\\
\simeq\ &\mathrm{Fun}\big(\mathcal{O}^\otimes \times \mathcal{O}'^\otimes, \mathcal{C}^\otimes\big) \times_{\mathrm{Fun}(\mathcal{O}^\otimes\times\mathcal{O}'^\otimes,\mathsf{Fin}_*)} \{p_{\mathcal{O}} \wedge p_{\mathcal{O}'}\}\\
\simeq\ &\mathrm{Fun}\big(\mathcal{O}^\otimes, \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)\big) \times_{\mathrm{Fun}(\mathcal{O}^\otimes,\mathrm{Fun}(\mathcal{O}'^\otimes,\mathsf{Fin}_*))} \{\widetilde{p_{\mathcal{O}} \wedge p_{\mathcal{O}'}}\}\\
\simeq\ &\mathrm{Fun}_{\mathrm{Fun}(\mathcal{O}'^\otimes,\mathsf{Fin}_*)}\big(\mathcal{O}^\otimes, \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)\big)
\end{aligned}$$

It is clear that then $P_2 \circ V \simeq \widehat{(-)} \circ P_1$.

The description obtained above of the essential image of the fully faithful functor $(\iota_{\mathrm{Alg}})_* \circ i$ now implies that the composition $V^{-1} \circ W^{-1} \circ (\iota_{\mathrm{Alg}})_* \circ i$ is fully faithful with essential image spanned by those functors $\mathcal{O}^\otimes \times \mathcal{O}'^\otimes \to \mathcal{C}^\otimes$ that map pairs of inert morphisms to inert morphisms. But this is by definition [HA, 2.2.5.3] precisely the essential image of the fully faithful functor j. This shows that an induced functor Φ_2 making the diagram commute exists and that Φ_2 is an equivalence. □

Proposition E.5.0.2. *Let $p_{\mathcal{O}}\colon \mathcal{O}^\otimes \to \mathsf{Fin}_*$, $p'_{\mathcal{O}}\colon \mathcal{O}'^\otimes \to \mathsf{Fin}_*$, as well as $p_{\mathcal{C}}\colon \mathcal{C}^\otimes \to \mathsf{Fin}_*$ be ∞-operads, and let $F\colon \mathcal{O}^\otimes \times \mathcal{O}'^\otimes \to \mathcal{O}''^\otimes$ be a bifunctor of ∞-operads (see [HA, 2.2.5.3]). Then there exists a commutative diagram as* *follows*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \arrow[r] \arrow[d, "\Phi_1"'] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}) \arrow[d, "F^*"] \\ \mathrm{BiFunc}(\mathcal{O}, \mathcal{O}'; \mathcal{C}) \arrow[r] & \mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \end{tikzcd}$$

where the horizontal functors are the canonical ones.

By definition [HA, 2.2.5.3] F exhibits $\mathcal{O}''^{\otimes}$ as a tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ if and only if Φ_1 is an equivalence for every ∞-operad $\mathcal{C}$. ♡

Proof. The existence of the induced dashed functor Φ_1 on full subcategories in the following diagram follows immediately from the fact that F maps pairs of inert morphisms to inert morphisms.

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \arrow[r, dashed, "\Phi_1", "\simeq"'] \arrow[d] & \mathrm{BiFunc}(\mathcal{O}, \mathcal{O}'; \mathcal{O}'') \arrow[d] \\ \mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r, "F^*"'] & \mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes}) \end{tikzcd}$$

□

E.6. The commutative ∞-operad

Let $\mathcal{O}$ be an ∞-operad. In the next proposition we show that the ∞-operad Comm has the property that the tensor product of $\mathcal{O}$ and Comm is given by Comm again.

Proposition E.6.0.1. *Let $p_{\mathcal{O}} \colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be a reduced*[10] *∞-operad and denote the essentially unique object in $\mathcal{O}$ by $\mathfrak{o}$.*

Then the bifunctor of ∞-operads[11]

$$\alpha \colon \mathcal{O}^{\otimes} \times \mathsf{Comm}^{\otimes} \xrightarrow{p_{\mathcal{O}} \times \mathrm{id}} \mathsf{Comm}^{\otimes} \times \mathsf{Comm}^{\otimes} \xrightarrow{- \wedge -} \mathsf{Comm}^{\otimes}$$

exhibits Comm as a tensor product of $\mathcal{O}$ and Comm.

Let $p_{\mathcal{C}} \colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. By applying Proposition E.4.2.3 to the bifunctor of ∞-operads $- \wedge -$ we obtain an induced ∞-operad $\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C})^{\otimes}$, and the forgetful functor $\mathrm{ev}_{\langle 1 \rangle} \colon \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \to \mathcal{C}$ can by Proposition E.4.2.3 (5) be upgraded to a morphism of ∞-operads.

[10] See [HA, 2.3.4.1] for a definition. It means that $\mathcal{O}$ is a unital ∞-operad and that the underlying ∞-category $\mathcal{O}$ is a contractible ∞-groupoid.

[11] See [HA, 2.2.5.1] for $- \wedge -$.

Then there is a commutative diagram as follows.

$$\begin{tikzcd} & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C})) \arrow[dl, "\mathrm{ev}_{\mathfrak{o}}"', "\simeq"] \arrow[dr, "\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\langle 1\rangle})"] & \\ \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[rr, "p_{\mathcal{O}}^*"'] & & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \end{tikzcd} \tag{E.4}$$

Furthermore, the forgetful functor $\mathrm{ev}_{\mathfrak{o}}$ *is an equivalence. In particular, if* $p_{\mathcal{O}} = \mathrm{id}_{\mathsf{Fin}_*}$*, then* $\mathrm{Alg}_{\mathsf{Comm}}(\mathrm{ev}_{\langle 1\rangle})$ *is homotopic to* $\mathrm{ev}_{\langle 1\rangle}$ *and an equivalence.* ♡

Proof. Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. What we have to show for the first part of the claim is that the functor

$$\Phi_1\colon\ \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \to \mathrm{BiFunc}(\mathcal{O}, \mathsf{Comm}; \mathcal{C})$$

from Proposition E.5.0.2 is an equivalence. Note that by Proposition E.5.0.1, the functor

$$\Phi_2\colon\ \mathrm{BiFunc}(\mathcal{O}, \mathsf{Comm}; \mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}))$$

is an equivalence. We consider the following diagram of commutative squares that summarizes the situation.

$$\begin{tikzcd} \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[r] \arrow[d, "\Phi_1"'] & \mathrm{Fun}(\mathsf{Fin}_*, \mathcal{C}^{\otimes}) \arrow[d, "(\alpha)^*"] \\ \mathrm{BiFunc}(\mathcal{O}, \mathsf{Comm}; \mathcal{C}) \arrow[r] \arrow[d, "\Phi_2"', "\simeq"] & \mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathsf{Fin}_*, \mathcal{C}^{\otimes}) \arrow[d, "\widehat{(-)}"] \\ \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C})) \arrow[r] \arrow[d, "\mathrm{ev}_{\mathfrak{o}}"'] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathsf{Fin}_*, \mathcal{C}^{\otimes})) \arrow[d, "\mathrm{ev}_{\mathfrak{o}}"] \\ \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[r] & \mathrm{Fun}(\mathsf{Fin}_*, \mathcal{C}^{\otimes}) \end{tikzcd} \tag{$*$}$$

Define $\Phi'_{\mathcal{C}}$ to be the left vertical composition $\Phi'_{\mathcal{C}} \coloneqq \mathrm{ev}_{\mathfrak{o}} \circ \Phi_2 \circ \Phi_1$. As the ∞-operad $\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C})^{\otimes}$ is cocartesian by [HA, 3.2.4.10], we can apply [HA, 2.4.3.9], which states that the forgetful functor $\mathrm{ev}_{\mathfrak{o}}$ is an equivalence. To show that α exhibits Comm as a tensor product of $\mathcal{O}$ and Comm it thus suffices to show that $\Phi'_{\mathcal{C}}$ is an equivalence.

Using naturality of $\widehat{(-)}$ we can identify the right vertical composition with precomposition with the following functor.

$$\mathsf{Fin}_* \xrightarrow{\mathrm{const}_{\mathfrak{o}} \times \mathrm{id}_{\mathsf{Fin}_*}} \mathcal{O}^{\otimes} \times \mathsf{Fin}_* \xrightarrow{p_{\mathcal{O}} \times \mathrm{id}_{\mathsf{Fin}_*}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{-\wedge -} \mathsf{Fin}_*$$

This functor is naturally equivalent to $\mathrm{id}_{\mathsf{Fin}_*}$, so we conclude that the vertical composition on the right in diagram $(*)$ is naturally equivalent to the identity.

Diagram $(*)$ is natural in $\mathcal{C}$[12], so it follows that the morphism of ∞-operads $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ induces a commutative cubes as follows.

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[rr] \arrow[dl] \arrow[dd, "\Phi'_{\mathcal{C}}" near start] & & \mathrm{Fun}(\mathsf{Fin}_*, \mathcal{C}^{\otimes}) \arrow[dl] \arrow[dd, "\mathrm{id}"] \\
\mathrm{Alg}_{\mathsf{Comm}}(\mathsf{Comm}) \arrow[rr, crossing over] \arrow[dd, "\Phi'_{\mathsf{Comm}}"'] & & \mathrm{Fun}(\mathsf{Fin}_*, \mathsf{Fin}_*) & \\
& \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[rr] \arrow[dl] & & \mathrm{Fun}(\mathsf{Fin}_*, \mathcal{C}^{\otimes}) \arrow[dl] \\
\mathrm{Alg}_{\mathsf{Comm}}(\mathsf{Comm}) \arrow[rr] & & \mathrm{Fun}(\mathsf{Fin}_*, \mathsf{Fin}_*) \arrow[from=uu, crossing over, "\mathrm{id}" near start] &
\end{tikzcd}$$

Note that the functor $\mathrm{Alg}_{\mathsf{Comm}}(\mathsf{Comm}) \to \mathrm{Fun}(\mathsf{Fin}_*, \mathsf{Fin}_*)$ can be identified with the inclusion of $\{\mathrm{id}_{\mathsf{Fin}_*}\}$, from which it also follows that Φ'_{Comm} can be identified with the identity. Passing to the induced functors from $\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C})$ into the pullbacks of the top and bottom squares we conclude that there is a commutative squares as indicated below

$$\begin{tikzcd}
\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[r] \arrow[d, "\Phi'_{\mathcal{C}}"'] & \mathrm{Fun}_{\mathsf{Fin}_*}(\mathsf{Fin}_*, \mathcal{C}^{\otimes}) \arrow[d, "\mathrm{id}"] \\
\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[r] & \mathrm{Fun}_{\mathsf{Fin}_*}(\mathsf{Fin}_*, \mathcal{C}^{\otimes})
\end{tikzcd}$$

where the horizontal functors are the canonical inclusions. As both horizontal functors are by definition the same inclusion of a full subcategory, it follows[13] that $\Phi'_{\mathcal{C}}$ is homotopic to the identity functor and hence an equivalence.

It remains to show that there exists a commutative diagram (E.4). For this we can proceed very analogously. As we now know that Φ_1 in diagram $(*)$ is an equivalence, it suffices to construct a homotopy between $p^*_{\mathcal{O}} \circ \mathrm{ev}_{\mathsf{o}} \circ \Phi_2 \circ \Phi_1$ and $\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\langle 1\rangle}) \circ \Phi_2 \circ \Phi_1$. Completely analogously to the arguments above, this time using that the compositions

$$\mathcal{O}^{\otimes} \xrightarrow{p_{\mathcal{O}}} \mathsf{Fin}_* \xrightarrow{\mathrm{const}_{\mathsf{o}} \times \mathrm{id}_{\mathsf{Fin}_*}} \mathcal{O}^{\otimes} \times \mathsf{Fin}_* \xrightarrow{p_{\mathcal{O}} \times \mathrm{id}_{\mathsf{Fin}_*}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{- \wedge -} \mathsf{Fin}_*$$

and

$$\mathcal{O}^{\otimes} \xrightarrow{\mathrm{id}_{\mathcal{O}^{\otimes}} \times \mathrm{const}_{\langle 1\rangle}} \mathcal{O}^{\otimes} \times \mathsf{Fin}_* \xrightarrow{p_{\mathcal{O}} \times \mathrm{id}_{\mathsf{Fin}_*}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{- \wedge -} \mathsf{Fin}_*$$

[12] One can check that the two squares involving Φ_1 and Φ_2 are natural in $\mathcal{C}$ by going through their definitions. This is also discussed in Remark F.3.0.4 below.

[13] See Proposition B.4.4.1 and Proposition B.4.3.1.

are both naturally equivalent to $p_{\mathcal{O}}$, one can obtain commutative diagrams

$$\begin{tikzcd} \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathsf{Fin}_*}(\mathsf{Fin}_*, \mathcal{C}^{\otimes}) \arrow[d, "p_{\mathcal{O}}^*"] \\ \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r] & \mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes}) \end{tikzcd}$$

for both $p_{\mathcal{O}}^* \circ \mathrm{ev}_{\mathfrak{o}} \circ \Phi_2 \circ \Phi_1$ as well as $\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\langle 1 \rangle}) \circ \Phi_2 \circ \Phi_1$ as the left vertical functor. We thus obtain a homotopy between $p_{\mathcal{O}}^* \circ \mathrm{ev}_{\mathfrak{o}} \circ \Phi_2 \circ \Phi_1$ and $\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\langle 1 \rangle}) \circ \Phi_2 \circ \Phi_1$ by using that the bottom horizontal functor is the inclusion of a full subcategory and applying Proposition B.4.4.1 and Proposition B.4.3.1. □

E.7. Colimits and free algebras

In this section we discuss (operadic) colimits and free algebras, as well as compatibility of functors

$$\mathrm{Alg}_{\mathcal{O}}(F)\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$$

induced by a symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, with free algebras and colimits.

We start in Section E.7.1 by discussing operadic colimits, which will be an ingredient for the later sections. In Section E.7.2 we then discuss free algebras, and in analogy we will also briefly show that induced functors on ∞-categories of left modules preserve free *modules* in Section E.7.4. In Section E.7.3 we provide a result for $\mathrm{Alg}_{\mathcal{O}}(F)$ preserving small colimits.

E.7.1. Operadic colimits

In this section we discuss some helpful results regarding operadic colimit diagrams. Section E.7.1.1 covers a criterion that simplifies checking whether certain types of diagrams in a symmetric monoidal ∞-category are operadic colimit diagrams, and Section E.7.1.2 applies this to show that colimit-preserving symmetric monoidal functors also preserve operadic colimits. Both statements as well as their proofs are essentially taken from [GH15, A.2.9][14].

E.7.1.1. A criterion for operadic colimits

We record the following proposition whose proof is essentially given in the proof of [GH15, A.2.9].

[14] The paper [GH15] is however concerned with the theory of non-symmetric ∞-operads, rather than the symmetric ∞-operads used in [HA], which is why we do not merely cite [GH15, A.2.9].

Proposition E.7.1.1 ([GH15, A.2.9]). *Let $q\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category that is compatible with small colimits in the sense of [HA, 3.1.1.18] and let $p\colon K^{\triangleright} \to \mathcal{C}^{\otimes}$ be a diagram such that $q \circ p$ is the constant functor with value $\langle i \rangle$. Let $m\colon \langle i\rangle \to \langle 1\rangle$ be the unique active morphism.*

Then the following two conditions are equivalent.

(1) p is an operadic q-colimit diagram[15].

(2) The composition

$$K^{\triangleright} \xrightarrow{p} \mathcal{C}^{\otimes}_{\langle i\rangle} \xrightarrow{m_!} \mathcal{C} \tag{E.5}$$

is a colimit diagram. ♡

Proof. By [HA, 3.1.1.16] the condition (1) is equivalent to the following condition.

(3) For every object Y of $\mathcal{C}^{\otimes}$ the composition

$$K^{\triangleright} \xrightarrow{p} \mathcal{C}^{\otimes}_{\langle i\rangle} \xrightarrow{-\oplus Y} \mathcal{C}^{\otimes}_{\langle i\rangle \oplus q(Y)} \xrightarrow{m'_!} \mathcal{C} \tag{$*$}$$

is a colimit diagram, where $m'\colon \langle i\rangle \oplus q(Y) \to \langle 1\rangle$ is the unique active morphism.

Note that given an object Y of $\mathcal{C}^{\otimes}$, we can write the unique active morphism

$$m'\colon \langle i\rangle \oplus q(Y) \to \langle 1\rangle$$

as the composition

$$m' = \mu \circ (m \oplus m'')$$

with $m''\colon q(Y) \to \langle 1\rangle$ and $\mu\colon \langle 2\rangle \to \langle 1\rangle$ the unique active morphisms. By [HA, 2.2.4.8], we can identify $(m \oplus m'')_!$ with $m_! \oplus m''_!$, so that the composition in $(*)$ can be identified with

$$K^{\triangleright} \xrightarrow{p} \mathcal{C}^{\otimes}_{\langle i\rangle} \xrightarrow{-\oplus Y} \mathcal{C}^{\otimes}_{\langle i\rangle \oplus q(Y)} \xrightarrow{m_! \oplus m''_!} \mathcal{C}^{\otimes}_{\langle 2\rangle} \xrightarrow{\mu_!} \mathcal{C}$$

which can be further identified, using the functoriality of $\oplus$, with the composition

$$K^{\triangleright} \xrightarrow{p} \mathcal{C}^{\otimes}_{\langle i\rangle} \xrightarrow{m_!} \mathcal{C}^{\otimes}_{\langle 1\rangle} \xrightarrow{-\oplus m''_!(Y)} \mathcal{C}^{\otimes}_{\langle 2\rangle} \xrightarrow{\mu_!} \mathcal{C}$$

which finally can be identified with the following composition.

$$K^{\triangleright} \xrightarrow{p} \mathcal{C}^{\otimes}_{\langle i\rangle} \xrightarrow{m_!} \mathcal{C} \xrightarrow{-\otimes m''_!(Y)} \mathcal{C} \tag{$**$}$$

As we assumed that the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits, $(**)$ is a colimit diagram for all objects Y of $\mathcal{C}^{\otimes}$ if and only if (E.5) is a colimit diagram[16]. □

[15] See [HA, 3.1.1.2] for the definition.

[16] The composition (E.5) can be identified with $(**)$ in the special case of $Y = \mathbb{1}_{\mathcal{C}}$.

E.7.1.2. Symmetric monoidal functors and operadic colimits

The following statement is given in [GH15, A.2.9] with the same proof as given below.

Proposition E.7.1.2 ([GH15, A.2.9]). *Let $q\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ and $q'\colon \mathcal{C}'^{\otimes} \to \mathsf{Fin}_*$ be symmetric monoidal ∞-categories that are compatible with small colimits in the sense of [HA, 3.1.1.18] and let $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{C}'^{\otimes}$ be a symmetric monoidal functor such that F preserves colimits.*

Let $p\colon K^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ be a operadic q-colimit diagram. Then $F^{\otimes} \circ p$ is a operadic q'-colimit diagram. ♡

Proof. Let $p_0 = q \circ p$ and let r_0 be the constant functor $K^{\triangleright} \to \mathsf{Fin}_*$ with image $p_0(\infty)$[17]. Then there is a unique natural transformation $\alpha_0\colon p_0 \to r_0$. By [HTT, 3.1.2.1] we can lift this natural transformation to a natural transformation $\alpha\colon p \to r$ of functors $K^{\triangleright} \to \mathcal{C}^{\otimes}$ such that for each object k of K the morphism $\alpha_k\colon p(k) \to r(k)$ is q-cocartesian.

Note that by construction of α_0 the functor α factors through $\mathcal{C}^{\otimes}_{\mathrm{act}}$. Furthermore, α_∞ is q-cocartesian and lies over the equivalence $\mathrm{id}_{p_0(\infty)}$ and is thus an equivalence by [HTT, 2.4.1.5]. Hence all the assumptions for [HA, 3.1.1.15 (2)] are satisfied and we can conclude that as p is an operadic q-colimit diagram, so is r.

As $F^{\otimes}$ maps q-cocartesian morphisms to q'-cocartesian morphisms and preserves equivalences, we can apply [HA, 3.1.1.15 (2)] also to $F^{\otimes} \circ \alpha$ to conclude that $F \circ p$ is an operadic q'-colimit diagram if and only if $F \circ r$ is, so it now suffices to show that $F \circ r$ is an operadic q'-colimit diagram.

Let $m\colon p_0(\infty) \to \langle 1\rangle$ be the unique active morphism. Then by Proposition E.7.1.1 the composite

$$K^{\triangleright} \xrightarrow{r} \mathcal{C}^{\otimes}_{p_0(\infty)} \xrightarrow{m_!} \mathcal{C} \tag{$*$}$$

is a colimit diagram, and it suffices to show that

$$K^{\triangleright} \xrightarrow{r} \mathcal{C}^{\otimes}_{p_0(\infty)} \xrightarrow{F^{\otimes}} \mathcal{C}'^{\otimes}_{p_0(\infty)} \xrightarrow{m_!} \mathcal{C}' \tag{$**$}$$

is a colimit diagram.

But as F is symmetric monoidal, composition $(**)$ can be identified with

$$K^{\triangleright} \xrightarrow{r} \mathcal{C}^{\otimes}_{p_0(\infty)} \xrightarrow{m_!} \mathcal{C} \xrightarrow{F} \mathcal{C}'$$

so that this is a colimit diagram follows from $(*)$ being a colimit diagram and F preserving colimit diagrams by assumption. □

E.7.2. Free algebras

In this section we discuss free algebras; existence of free algebras in Section E.7.2.1 and compatibility of induced functors on ∞-categories of algebras with free algebras in Section E.7.2.2.

[17] ∞ denotes the cone point of $K^{\triangleright}$.

E.7.2.1. Detection of free algebras

Let $\mathcal{C}$ be a symmetric monoidal ∞-category, $\mathcal{O}$ an ∞-operad, and X an object of the underlying ∞-category of $\mathcal{O}$. We can then ask whether the forgetful functor

$$\mathrm{ev}_X\colon\ \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$$

has a left adjoint, i. e. a free algebra functor[18]. In a more general setting, [HA, 3.1.3.4] shows existence of a free algebra functor, under some assumptions. However, those assumptions, requiring existence of certain operadic colimit diagrams, are not a priori easy to verify[19]. In the next proposition we thus provide easier to check conditions for $\mathcal{C}$ in the case that $\mathcal{O}$ is either Assoc or $\mathbb{E}_0$ that imply the existence of free algebras, and discuss descriptions of the free algebra generated by a an object of $\mathcal{C}$.

Proposition E.7.2.1 ([HA, 4.1.1.18 and 4.1.1.19])**.** *Let $q\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category. Let $\mathcal{O}$ be either Assoc or $\mathbb{E}_0$. Furthermore, assume the following.*

- *If $\mathcal{O} = \mathsf{Assoc}$, assume that $\mathcal{C}$ admits countable coproducts and that the tensor product preserves countable coproducts in each variable.*
- *If $\mathcal{O} = \mathbb{E}_0$, assume that $\mathcal{C}$ admits finite coproducts and that the tensor product preserves finite coproducts in each variable.*

Then the forgetful functor

$$\mathrm{ev}_{\langle 1\rangle}\colon\ \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$$

admits a left adjoint $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}}}$ *and for every object X of $\mathcal{C}$, the unit*

$$X \to \mathrm{ev}_{\langle 1\rangle}\Big(\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}}}(X)\Big)$$

of the adjunction exhibits $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}}}(X)$ *as a q-free $\mathcal{O}$-algebra generated by X*[20]*.*

Let X be an object of $\mathcal{C}$, let A be an object of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$*, and let furthermore $f\colon X \to \mathrm{ev}_{\langle 1\rangle}(A)$ be a morphism in $\mathcal{C}$. Then the following are equivalent.*

(1) f exhibits A as a q-free $\mathcal{O}$-algebra generated by X.

(2) The morphism

$$\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}}}(X) \to A$$

that is adjoint to f is an equivalence in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$*.*

---

[18]See [HA, 3.1].

[19]Unless much stronger assumptions are available, such as the symmetric monoidal structure on $\mathcal{C}$ being compatible with small colimits. See [HA, 3.1.3.5].

[20]See [HA, 3.1.3.1 and 3.1.3.12] for a definition.

(3) • *If $\mathcal{O} = \mathsf{Assoc}$: The composition*

$$\coprod_{n\geq 0} X^{\otimes n} \xrightarrow{\amalg_{n\geq 0} f^{\otimes n}} \coprod_{n\geq 0} \mathrm{ev}_{\langle 1\rangle}(A)^{\otimes n} \to \mathrm{ev}_{\langle 1\rangle}(A)$$

is an equivalence, where the morphisms $\mathrm{ev}_{\langle 1\rangle}(A)^{\otimes n} \to \mathrm{ev}_{\langle 1\rangle}(A)$ are those associated to the evaluation of A at an active morphism $\langle n\rangle \to \langle 1\rangle$ in $\mathsf{Assoc}^{\otimes}$[21].

• *If $\mathcal{O} = \mathbb{E}_0$: The composition*

$$\mathbb{1} \amalg X \xrightarrow{\mathrm{id}_{\mathbb{1}} \amalg f} \mathbb{1} \amalg \mathrm{ev}_{\langle 1\rangle}(A) \xrightarrow{i \amalg \mathrm{id}} \mathrm{ev}_{\langle 1\rangle}(A)$$

is an equivalence, where i is the morphism associated to the evaluation of A at the unique morphism $\langle 0\rangle \to \langle 1\rangle$ in $(\mathbb{E}_0)^{\otimes}$. ♡

Proof. For $\mathcal{O} = \mathsf{Assoc}$, this is precisely [HA, 4.1.1.18][22], albeit under stronger assumptions regarding what colimits $\mathcal{C}$ needs to be admit and its tensor product needs to be compatible with. That countable coproducts suffice is remarked in [HA, 4.4.1.19]. This follows by tracing through the proof of [HA, 4.1.1.18], where one is ultimately led to [HA, 3.1.3.4], where one needs to ensure that one can construct certain operadic q-colimit diagrams. One then notes that in the specific situation we need to apply this the diagram category is equivalent to $\coprod_{n\geq 0} \mathcal{P}(n)$, where $\mathcal{P}(n)$ are the spaces defined in [HA, 3.1.3.9]. For Assoc these spaces can easily seen to be contractible[23], so colimits indexed by this diagram category are countable coproducts.

The proof for $\mathcal{O} = \mathbb{E}_0$ is completely analogous; the relevant $\mathcal{P}(n)$ are empty for $n > 1$ rather than contractible. □

E.7.2.2. Symmetric monoidal functors and free algebras

Given a symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, an ∞-operad $\mathcal{O}$, and an object X of the underlying ∞-category of $\mathcal{O}$, the induced functor on ∞-categories of algebras

$$\mathrm{Alg}_{\mathcal{O}}(F)\colon\ \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$$

is compatible with the respective forgetful functors ev_X. The next proposition gives conditions for $\mathcal{C}$, $\mathcal{D}$, and F such that $\mathrm{Alg}_{\mathcal{O}}(F)$ is also compatible with the respective free algebra functors.

Proposition E.7.2.2. *Let $\alpha^{\otimes}\colon \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ be a morphism of ∞-operads, $q_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ and $q_{\mathcal{D}}\colon \mathcal{D}^{\otimes} \to \mathsf{Fin}_*$ symmetric monoidal ∞-categories, and $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a symmetric monoidal functor.*

Assume one of the following sets of assumptions.

[21] Which active morphism is chosen does not change whether the composition is an equivalence or not.

[22] The proof can be found above the statement.

[23] See [HA, above 4.1.1.18].

(1) • *$\mathcal{C}$ and $\mathcal{D}$ admit small colimits.*
- *The tensor product functors of $\mathcal{C}$ and $\mathcal{D}$ preserve small colimits separately in each variable.*
- *$F\colon \mathcal{C} \to \mathcal{D}$ preserves small colimits.*

(2) • *$\mathcal{O}^{\otimes} = \mathsf{Triv}^{\otimes}$ and $\mathcal{O}'^{\otimes} = \mathsf{Assoc}$.*
- *$\mathcal{C}$ and $\mathcal{D}$ admit countable coproducts.*
- *The tensor product functors of $\mathcal{C}$ and $\mathcal{D}$ preserve countable coproducts separately in each variable.*
- *$F\colon \mathcal{C} \to \mathcal{D}$ preserves countable coproducts.*

(3) • *$\mathcal{O}^{\otimes} = \mathsf{Triv}^{\otimes}$ and $\mathcal{O}'^{\otimes} = \mathbb{E}_0$.*
- *$\mathcal{C}$ and $\mathcal{D}$ admit finite coproducts.*
- *The tensor product functors of $\mathcal{C}$ and $\mathcal{D}$ preserve finite coproducts separately in each variable.*
- *$F\colon \mathcal{C} \to \mathcal{D}$ preserves finite coproducts.*

Then the following commutative diagram induced by F (where the two horizontal functors are the forgetful functors given by precomposition with α)

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \arrow[r, "U_{\mathcal{C}}"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}'}(F)"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(F)"] \\ \mathrm{Alg}_{\mathcal{O}'}(\mathcal{D}) \arrow[r, "U_{\mathcal{D}}"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) \end{tikzcd} \tag{E.6}$$

is left adjointable[24], i. e. $U_{\mathcal{C}}$ and $U_{\mathcal{D}}$ have left adjoints $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}$ and $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{D})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})}$, and the associated push-pull transformation

$$\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{D})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})} \circ \mathrm{Alg}_{\mathcal{O}}(F) \to \mathrm{Alg}_{\mathcal{O}'}(F) \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}$$

is a natural equivalence. ♡

Proof. By [HA, 3.1.3.5] in case (1) and Proposition E.7.2.1 in cases (2) and (3), the left adjoints exist and for A an object of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ the unit

$$\eta^{\mathcal{C}}_A\colon A \to U_{\mathcal{C}}\left(\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(A)\right)$$

of the adjunction exhibits $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(A)$ as the free $\mathcal{O}'$-algebra generated by A, and completely analogously for the other adjunction, whose unit we denote by $\eta^{\mathcal{D}}$.

[24] See [HTT, 7.3.1.1] for the definition.

Let A be an object in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. We have to show[25] that the morphism

$$\Big(\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{D})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})} \circ \mathrm{Alg}_{\mathcal{O}}(F)\Big)(A) \to \Big(\mathrm{Alg}_{\mathcal{O}'}(F) \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}\Big)(A)$$

that is adjoint to the following composition[26]

$$\begin{aligned}
&(\mathrm{Alg}_{\mathcal{O}}(F))(A)\\
\xrightarrow{\mathrm{Alg}_{\mathcal{O}}(F)(\eta^{\mathcal{C}}_A)}\; &\Big(\mathrm{Alg}_{\mathcal{O}}(F) \circ U_{\mathcal{C}} \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}\Big)(A)\\
\xrightarrow{\simeq}\; &\Big(U_{\mathcal{D}} \circ \mathrm{Alg}_{\mathcal{O}'}(F) \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}\Big)(A)
\end{aligned}$$

is an equivalence (see [HTT, beginning of 7.3.1]).

But by definition of $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{D})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})}$ (see [HA, 3.1.3.5 and 3.1.3.1] in case (1) and see Proposition E.7.2.1 in cases (2) and (3)), the former morphism is an equivalence if and only if the latter morphism exhibits

$$\Big(\mathrm{Alg}_{\mathcal{O}'}(F) \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}\Big)(A)$$

as a $q_{\mathcal{D}}$-free $\mathcal{O}'$-algebra generated by $(\mathrm{Alg}_{\mathcal{O}}(F))(A)$ – so this is what we need to show.

Similarly, by definition of $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}$, the morphism

$$\eta^{\mathcal{C}}_A \colon A \to \Big(U_{\mathcal{C}} \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}\Big)(A)$$

exhibits $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(A)$ as a $q_{\mathcal{C}}$-free $\mathcal{O}'$-algebra generated by A.

Proof in case (1): Unpacking the definitions of free algebras (see [HA, 3.1.3.1]) one sees that the claim boils down to showing that $F^{\otimes}$ preserves certain operadic colimit diagrams, so the claim follows from Proposition E.7.1.2.

Proof in cases (2) and (3): In these cases we can use the criteria from Proposition E.7.2.1 and thus the claim follows from F being symmetric monoidal and preserving countable/finite colimits. □

E.7.3. Induced functors on Alg and colimits

In the following proposition we show that a colimit preserving symmetric monoidal functor induces a colimit preserving functor on ∞-categories of algebras.

[25] By Proposition A.3.2.1 a natural transformation is a natural equivalence if and only if it is a pointwise equivalence.

[26] The equivalence used is to be the one obtained from the equivalence $\mathrm{Alg}_{\mathcal{O}}(F) \circ U_{\mathcal{C}} \simeq U_{\mathcal{D}} \circ \mathrm{Alg}_{\mathcal{O}'}(F)$ encoded in the commutative diagram (E.6).

Proposition E.7.3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories and $F\colon \mathcal{C} \to \mathcal{D}$ a symmetric monoidal functor. Assume that $\mathcal{C}$ and $\mathcal{D}$ admit all small colimits, that the tensor product functors of $\mathcal{C}$ and $\mathcal{D}$ preserve small colimits separately in each variable, and that F preserves small colimits.*

Let $\mathcal{O}$ be an ∞-operad. Then $\mathrm{Alg}_{\mathcal{O}}(F)$ preserves small colimits as well. ♡

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})"] \arrow[d, "U_{\mathcal{C}}"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) \arrow[d, "U_{\mathcal{D}}"] \\
\mathrm{Fun}(\mathcal{O}, \mathcal{C}) \arrow[r, "F_*"'] & \mathrm{Fun}(\mathcal{O}, \mathcal{D})
\end{tikzcd} \tag{$*$}$$

where $U_{\mathcal{C}}$ and $U_{\mathcal{D}}$ are the forgetful functors.

To show that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})$ preserves colimits it suffices by combining [HTT, 4.2.3.12] with [HA, 1.3.3.10 (2)] to show that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})$ preserves sifted colimits as well as coproducts.

By [HA, 3.2.3.1][27] together with [HTT, 5.1.2.3 (2)], the two vertical functors in diagram $(*)$ detects sifted colimits. As F preserves all small colimits by assumption, we obtain with [HTT, 5.1.2.3 (2)] that the bottom horizontal functor in diagram $(*)$ preserves all small, so in particular all sifted, colimits. We can thus conclude that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})$ preserves sifted colimits.

It then follows from the proof of [HA, 3.2.3.3][27] that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})$ also preserves coproducts if the composition with the left adjoint $\mathrm{Free}_{\mathcal{C}}$ of $U_{\mathcal{C}}$ does. But by Proposition E.7.2.2 there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})"] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) \\
\mathrm{Fun}(\mathcal{O}, \mathcal{C}) \arrow[r, "F_*"'] \arrow[u, "\mathrm{Free}_{\mathcal{C}}"] & \mathrm{Fun}(\mathcal{O}, \mathcal{D}) \arrow[u, "\mathrm{Free}_{\mathcal{D}}"']
\end{tikzcd}$$

where $\mathrm{Free}_{\mathcal{D}}$ is the left adjoint of $U_{\mathcal{D}}$. That the composition from the bottom left to the top right in this diagram preserves coproducts now follows immediately from F_* preserving small colimits as mentioned above and $\mathrm{Free}_{\mathcal{D}}$ preserving colimits as a left adjoint [HTT, 5.2.3.5]. □

E.7.4. Free modules

Similarly to Proposition E.7.2.2, which dealt with compatibility of induced functors on ∞-categories of algebras with free algebras, the next propositions discusses compatibility of induced functors on ∞-categories of left modules with free modules.

[27] Which is applicable to our situation by Proposition E.2.0.2.

Proposition E.7.4.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor of monoidal ∞-categories and let R be an (associative) algebra in $\mathcal{C}$. Then the commutative diagram*

$$\begin{tikzcd} \mathrm{LMod}_R(\mathcal{C}) \arrow[r, "\mathrm{ev}_\mathfrak{m}"] \arrow[d, "\mathrm{LMod}_R(F)"'] & \mathcal{C} \arrow[d, "F"] \\ \mathrm{LMod}_{F(R)}(\mathcal{D}) \arrow[r, "\mathrm{ev}_\mathfrak{m}"'] & \mathcal{D} \end{tikzcd} \tag{E.7}$$

induced by F is left adjointable in the sense of [HTT, 7.3.1.1], i. e. the associated push-pull transformation

$$\mathrm{Free}_\mathcal{D} \circ F \to \mathrm{LMod}_R(F) \circ \mathrm{Free}_\mathcal{C}$$

is an equivalence, where $\mathrm{Free}_\mathcal{C}$ and $\mathrm{Free}_\mathcal{D}$ are the free module functors for $\mathcal{C}$ and $\mathcal{D}$, respectively (see [HA, 4.2.4.8]).

In other words, F preserves free left-R-modules. The analogous statement is true for right-R-modules. ♡

Proof. Let X be an object of $\mathcal{C}$. By Proposition A.3.2.1 it suffices to show that the push-pull morphism

$$(\mathrm{Free}_\mathcal{D} \circ F)(X) \to (\mathrm{LMod}_R(F) \circ \mathrm{Free}_\mathcal{C})(X)$$

is an equivalence, and as $\mathrm{ev}_\mathfrak{m}$ is conservative by [HA, 4.2.3.3], we actually only need to show that $\mathrm{ev}_\mathfrak{m}$ of that morphism is an equivalence.

Consider the following commutative diagram that will be explained below.

$$\begin{tikzcd} F(R) \otimes F(X) \arrow[r] \arrow[dd] & F(R) \otimes (\mathrm{ev}_\mathfrak{m} \circ \mathrm{Free}_\mathcal{D} \circ F)(X) \arrow[r] \arrow[d] & (\mathrm{ev}_\mathfrak{m} \circ \mathrm{Free}_\mathcal{D} \circ F)(X) \arrow[d] \\ & F(R) \otimes (\mathrm{ev}_\mathfrak{m} \circ \mathrm{LMod}_R(F) \circ \mathrm{Free}_\mathcal{C})(X) \arrow[r] \arrow[d] & (\mathrm{ev}_\mathfrak{m} \circ \mathrm{LMod}_R(F) \circ \mathrm{Free}_\mathcal{C})(X) \arrow[d] \\ F(R) \otimes F(X) \arrow[r] \arrow[d] & F(R) \otimes F((\mathrm{ev}_\mathfrak{m} \circ \mathrm{Free}_\mathcal{C})(X)) \arrow[d] & F((\mathrm{ev}_\mathfrak{m} \circ \mathrm{Free}_\mathcal{C})(X)) \arrow[d] \\ F(R \otimes X) \arrow[r] & F(R \otimes (\mathrm{ev}_\mathfrak{m} \circ \mathrm{Free}_\mathcal{C})(X)) \arrow[r] & F((\mathrm{ev}_\mathfrak{m}) \circ \mathrm{Free}_\mathcal{C}(X)) \end{tikzcd}$$

The left horizontal morphisms are induced by the units of the adjunctions $\mathrm{Free}_\mathcal{D} \dashv \mathrm{ev}_\mathfrak{m}$ and $\mathrm{Free}_\mathcal{C} \dashv \mathrm{ev}_\mathfrak{m}$, and the right horizontal morphisms are (induced by) the action morphism of the respective modules. The top vertical morphisms on the left and the bottom vertical morphism on the right are the identity morphisms, and the bottom vertical morphism in the left and middle column are the equivalences arising from monoidality of F. In the middle and right column, the top vertical morphism is induced by the push-pull-transformation, and the middle vertical morphisms arise are the equivalences that arise from commutativity of (E.7).

The composition of the top two horizontal morphisms is an equivalence by the definition of free modules [HA, 4.2.4.1], and so is the composition of the bottom two horizontal morphisms. The two left vertical morphisms as well as the bottom and middle vertical morphism on the right are equivalences as well, so it follows that the vertical morphism at the top right is an equivalence, which is what needed to be shown. □

E.8. Relative tensor products

Let C be a monoidal category and R, S, and T associative algebras in C. If M is an R-S-bimodule and N an S-T-bimodule, then we can form the relative tensor product of M with N over S, denoted by $M \otimes_S N$, which yields an R-T-bimodule.

This construction is generalized to the ∞-categorical setting in [HA, 4.4][28], and can be (very) roughly summarized as follows. If $\mathcal{C}$ is a monoidal ∞-category that is compatible with $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits, R an associative algebra in $\mathcal{C}$, M a right-R-module, and N a left-R-module, then there exists a simplicial object in $\mathcal{C}$ denoted by $\mathrm{Bar}_R(M, N)$ that is given in level n by[29] $M \otimes R^{\otimes n} \otimes N$ and has structure morphisms constructed from the unit morphism of R, the multiplication of R, and the action of R on M and N. The relative tensor product $M \otimes_R N$ is then the geometric realization of $\mathrm{Bar}_R(M, N)$. See [HA, 4.4.2.8].

In this section we will record some properties of relative tensor products that we will need.

Proposition E.8.0.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal ∞-categories and $F\colon \mathcal{C} \to \mathcal{D}$ a monoidal functor. Assume that $\mathcal{C}$ and $\mathcal{D}$ admit $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits, their tensor product functors commute with $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits in each variable separately, and F preserves $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits.*

Then F preserves relative tensor products. ♡

Remark E.8.0.2. Let us clarify what the statement of Proposition E.8.0.1 actually means at a more concrete or technical level. Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Assoc}^{\otimes}$ and $p_{\mathcal{D}}\colon \mathcal{D}^{\otimes} \to \mathsf{Assoc}^{\otimes}$ be the cocartesian fibrations of ∞-operads that exhibit $\mathcal{C}$ and $\mathcal{D}$ as monoidal ∞-categories. Suppose we have given a morphism

[28] Unfortunately there seems to be a mistake in the definition of $\mathsf{Tens}^{\otimes}$ in [HA, 4.4.1.1]. For morphisms one should additionally require for any element j of $\langle n' \rangle^{\circ}$ such that $c'_{-}(j) \neq c'_{+}(j)$ that the preimage of j under α is non-empty. One can think of it like this: Any nontrivial step from $c'_{-}(j)$ to $c'_{+}(j)$ needs to come from a step in the preimages.

The same mistake occurs in the description [HA, 4.3.1.5] of the ∞-operad encoding bimodules. Here one needs to make the same correction. Without this correction algebras over this operad would not consist of triples (R, M, S) with R and S associative algebras and M an R-S-bimodule, but such triples together with an additional unit morphism $\mathbb{1} \to M$ for M, encoded by the morphism from the unique object $\emptyset$ over $\langle 0 \rangle$ to $\mathfrak{m}$.

[29] That this is really how the bar construction looks like in level n can be seen by digging through and unpacking the definition [HA, 4.4.2.7], but it is a bit tedious.

of generalized ∞-operads φ fitting into the following commutative diagram

$$\begin{tikzcd}
 & \mathcal{C}^{\otimes} \arrow[r, "F^{\otimes}"] \arrow[dr, "p_{\mathcal{C}}"'] & \mathcal{D}^{\otimes} \arrow[d, "p_{\mathcal{D}}"] \\
\mathsf{Tens}^{\otimes}_{\succ} \arrow[ur, "\varphi"] \arrow[r] & \mathsf{Assoc}^{\otimes} &
\end{tikzcd}$$

where the bottom horizontal functor is the forgetful functor. Then the statement of Proposition E.8.0.1 is that if φ is an operadic $p_{\mathcal{C}}$-colimit diagram, then $F^{\otimes} \circ \varphi$ is an operadic $p_{\mathcal{D}}$-colimit diagram, see [HA, 4.4.2.3].

From this the various other, perhaps more concrete, formulations of what it means for a monoidal functor to preserve relative tensor products follow. For example we then have a commutative square

$$\begin{tikzcd}
\mathrm{BiMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BiMod}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{BiMod}(\mathcal{D}) \times_{\mathrm{Alg}(\mathcal{D})} \mathrm{BiMod}(\mathcal{D}) \arrow[d] \\
\mathrm{BiMod}(\mathcal{C}) \arrow[r] & \mathrm{BiMod}(\mathcal{D})
\end{tikzcd}$$

where the horizontal functors are those induced by F and the vertical functors are the relative tensor product functors of [HA, 4.4.2.11]. ♢

Proof of Proposition E.8.0.1. We will make use the notation and setup from Remark E.8.0.2. Let the restriction of φ to $\mathsf{Tens}^{\otimes}_{[2]}$ correspond to a quintuple (R, M, S, N, T), with R, S, and T associative algebras in $\mathcal{C}$, with M an R,S-bialgebra, and N an S,T-bialgebra, see [HA, 4.4.2.2]. Similarly, let the restriction to $\mathsf{Tens}^{\otimes}_{[1]}$ correspond to a triple (R', X, T') with R' and T' associative algebras and X an R',T'-bialgebra.

By [HA, 4.4.2.8], the morphisms $R \to R'$ and $T \to T'$ induced by φ are equivalences[30] and the comparison morphism

$$|\mathrm{Bar}_S(M, N)| \to \mathrm{ev}_{\mathfrak{m}}(X)$$

is an equivalence. What we have to show is that the morphisms $F(R) \to F(R')$ and $F(T) \to F(T')$ induced by $F^{\otimes} \circ \varphi$ are equivalences and that the comparison morphism

$$\left|\mathrm{Bar}_{F(S)}(F(M), F(N))\right| \to F(\mathrm{ev}_{\mathfrak{m}}(X)) \tag{$*$}$$

is an equivalence.

The former is clear because these morphisms are just given by F applied to the analogous morphisms $R \to R'$ and $T \to T'$ in $\mathcal{C}$.

[30] Condition (i) boils down to this, as Assoc is reduced, see [HA, 4.4.2.6].

As $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian morphisms to $p_{\mathcal{D}}$-cocartesian morphisms, it follows from the definition that

$$\mathrm{Bar}_{F(S)}(F(M), F(N)) \simeq F \circ \mathrm{Bar}_S(M, N)$$

see [HA, 4.4.2.7]. That $(*)$ is an equivalence now follows from combining this with F preserving $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits by assumption. $\square$

Proposition E.8.0.3. *Let $\mathcal{C}$ be a cocartesian symmetric monoidal structure*[31] *such that the underlying ∞-category of $\mathcal{C}$ admits $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits as well as pushouts*

Then the tensor product of $\mathcal{C}$ is compatible with $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits as well as pushouts in the sense of [HA, 3.1.1.18].

Let R, S, and T be associative algebras in $\mathcal{C}$. Let $f\colon M \to M'$ be a morphism in $\mathrm{BiMod}_{R,S}(\mathcal{C})$ and $g\colon N \to N'$ a morphism in $\mathrm{BiMod}_{S,T}(\mathcal{C})$. We obtain a commutative diagram

$$\begin{tikzcd}
M \otimes_S N \arrow[r, "\mathrm{id}_M \otimes_{\mathrm{id}_S} g"] \arrow[d, "f \otimes_{\mathrm{id}_S} \mathrm{id}_N"'] & M \otimes_S N' \arrow[d, "f \otimes_{\mathrm{id}_S} \mathrm{id}_{N'}"] \\
M' \otimes_S N \arrow[r, "\mathrm{id}_{M'} \otimes_{\mathrm{id}_S} g"'] & M' \otimes_S N'
\end{tikzcd}$$

in $\mathrm{BiMod}_{R,T}(\mathcal{C})$. Then this diagram is a pushout square. $\heartsuit$

Proof. We first show that the symmetric monoidal structure on $\mathcal{C}$ is compatible with pushouts and $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits. So let X be an object of $\mathcal{C}$. Let $\mathcal{I}$ be either $\mathbf{\Delta}^{\mathrm{op}}$ or $\Lambda_0^2 = (\bullet \leftarrow \bullet \rightarrow \bullet)$ and $F\colon \mathcal{I} \to \mathcal{C}$ a functor. It suffices to show that the canonical comparison morphism

$$\mathrm{colim}(X \amalg F) \to X \amalg \mathrm{colim}\, F$$

is an equivalence. As colimits commute with colimits [HTT, 5.5.2.3] this morphism factors as an equivalence $\mathrm{colim}(X \amalg F) \simeq (\mathrm{colim}\,\mathrm{const}_X) \amalg (\mathrm{colim}\, F)$ and the canonical morphism $(\mathrm{colim}\,\mathrm{const}_X) \amalg (\mathrm{colim}\, F) \to X \amalg \mathrm{colim}\, F$. It thus suffices to show that $(\mathrm{colim}\,\mathrm{const}_X) \to X$ is an equivalence, which follows from [HTT, 4.4.4.10], as $\mathcal{I}$ is weakly contractible[32].

We can now apply [HA, 4.3.3.9] to conclude that pushouts are detected by the forgetful functor $\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{BiMod}_{R,T}(\mathcal{C}) \to \mathcal{C}$, so combining this with the description of relative tensor products from [HA, 4.4.2.8] it suffices to show

[31] See [HA, 2.4.0.1] for a definition.
[32] This means that the ∞-groupoid completion of $\mathcal{I}$ is contractible as a space.

that the commutative diagram

$$\begin{tikzcd}
\lvert \mathrm{Bar}_S(M,N) \rvert \arrow[r, "\lvert \mathrm{Bar}_{\mathrm{id}_S}(\mathrm{id}_M,g) \rvert"] \arrow[d, "\lvert \mathrm{Bar}_{\mathrm{id}_S}(f,\mathrm{id}_N) \rvert"'] & \lvert \mathrm{Bar}_S(M,N') \rvert \arrow[d, "\lvert \mathrm{Bar}_{\mathrm{id}_S}(f,\mathrm{id}_{N'}) \rvert"] \\
\lvert \mathrm{Bar}_S(M',N) \rvert \arrow[r, "\lvert \mathrm{Bar}_{\mathrm{id}_S}(\mathrm{id}_{M'},g) \rvert"'] & \lvert \mathrm{Bar}_S(M',N') \rvert
\end{tikzcd}$$

is a pushout square in $\mathcal{C}$.

Using compatibility of colimits with colimits again it suffices to show for every $n \geq 0$ that the commutative square

$$\begin{tikzcd}
M \amalg (\coprod_{i=1}^n S) \amalg N \arrow[r, "\mathrm{id}_M \amalg \mathrm{id} \amalg g"] \arrow[d, "f \amalg \mathrm{id} \amalg \mathrm{id}_N"'] & M \amalg (\coprod_{i=1}^n S) \amalg N' \arrow[d, "f \amalg \mathrm{id} \amalg \mathrm{id}_{N'}"] \\
M' \amalg (\coprod_{i=1}^n S) \amalg N \arrow[r, "\mathrm{id}_{M'} \amalg \mathrm{id} \amalg g"'] & M' \amalg (\coprod_{i=1}^n S) \amalg N'
\end{tikzcd}$$

is a pushout square in $\mathcal{C}$, which yet again follows from colimits commuting with colimits, as this is evidently a coproduct of pushout diagrams. □

Construction E.8.0.4. Let $\mathcal{C}$ be a symmetric monoidal ∞-category, and assume that the underlying ∞-category admits $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits, and that the tensor product functor preserves $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits separately in each variable.

Let $f\colon R \to S$ and $g\colon R \to T$ be morphisms in $\mathrm{CAlg}(\mathcal{C})$. We can upgrade f and g to morphisms in right-R-modules and left-R-modules in $\mathrm{CAlg}(\mathcal{C})$, as we now explain for g, the case for f being completely analogous.

By [HA, 3.2.4.7] the induced symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is cocartesian, so by [HA, 2.4.3.9] the forgetful functor

$$\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \to \mathrm{CAlg}(\mathcal{C})$$

is an equivalence, and so we can upgrade g to a morphism $\overline{g}$ in $\mathrm{Alg}(\mathrm{CAlg}(\mathcal{C}))$ with $\mathrm{ev}_{\mathfrak{a}}(\overline{g}) \simeq g$.

By applying the section $\mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \to \mathrm{LMod}(\mathrm{CAlg}(\mathcal{C}))$ discussed in [HA, 4.2.1.17] we obtain a morphism $\widetilde{g}\colon (R,R) \to (T,T)$ in $\mathrm{LMod}(\mathrm{CAlg}(\mathcal{C}))$ together with equivalences $\mathrm{ev}_{\mathfrak{a}}(\widetilde{g}) \simeq \overline{g}$ and $\mathrm{ev}_{\mathfrak{m}}(\widetilde{g}) \simeq g$. The forgetful functor $\mathrm{LMod}(\mathrm{CAlg}(\mathcal{C})) \to \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C}))$ is a cartesian fibration by [HA, 4.2.3.2] and a cartesian lift of $\overline{g}$ with target (T,T) lies over an equivalence in $\mathrm{CAlg}(\mathcal{C})$. This cartesian lift can be interpreted as the restriction of the T-action on T to R along $\overline{g}$. We obtain an induced morphism of left-R-modules $g'\colon R \to T$ with $\mathrm{ev}_{\mathfrak{m}}(g') \simeq g$.

By [HA, 3.2.3.2] the ∞-category $\mathrm{CAlg}(\mathcal{C})$ admits $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits, and as the forgetful functor $\mathrm{ev}_{\langle 1 \rangle}\colon\ \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ is both symmetric monoidal by

Proposition E.4.2.3 (5) as well as preserves and detects $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits by [HA, 3.2.3.2], it follows that the induced tensor product on $\mathrm{CAlg}(\mathcal{C})$ is compatible with $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits as well.

We thus obtain a commutative diagram in $\mathrm{CAlg}(\mathcal{C})$ as follows

$$\begin{tikzcd}
R \arrow[rr, "g"] \arrow[dr, dash, "\simeq"] \arrow[dd, "f"'] & & T \arrow[d, dash, "\simeq"] \\
& R \otimes_R R \arrow[r, "\mathrm{id}_R \otimes_{\mathrm{id}_R} g'"] \arrow[d, "f' \otimes_{\mathrm{id}_R} \mathrm{id}_R"'] & R \otimes_R T \arrow[d, "f' \otimes_{\mathrm{id}_R} \mathrm{id}_T"] \\
S \arrow[r, dash, "\simeq"'] & S \otimes_R R \arrow[r, "\mathrm{id}_S \otimes_{\mathrm{id}_R} g'"'] & S \otimes_R T
\end{tikzcd} \tag{E.8}$$

where the unlabeled equivalences are the unitality equivalences of the relative tensor product discussed in [HA, 4.4.3.16], see also [HA, 4.4.3.18]. ♢

Proposition E.8.0.5. *Assume that we are in the situation of Construction E.8.0.4, and that* $\mathcal{C}$ *additionally admits small colimits and that the tensor product preserves small colimits separately in each variable.*

Then the commutative square

$$\begin{tikzcd}
R \arrow[r, "g"] \arrow[d, "f"'] & T \arrow[d] \\
S \arrow[r] & S \otimes_R T
\end{tikzcd}$$

from (E.8) *is a pushout square in* $\mathrm{CAlg}(\mathcal{C})$. ♡

Proof. It suffices to show that the smaller square on the lower right in diagram (E.8) is a pushout square.

Note that by [HA, 3.2.3.3] $\mathrm{CAlg}(\mathcal{C})$ again admits small colimits. We can thus apply Proposition E.8.0.3, which shows the claim. □

Appendix F.

Cartesian symmetric monoidal ∞-categories

In this appendix we collect some results relating to cartesian symmetric monoidal ∞-categories. In Section F.1 we discuss how cocartesian fibrations whose fibers are compatible with products in the sense of Definition C.2.0.1 interact with cartesian symmetric monoidal structures. The short section Section F.2 describes limits in ∞-categories of monoids. The main part of this section is Section F.3, in which we discuss how to relate $\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C})$, $\mathrm{Mon}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C})$, $\mathrm{Mon}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}))$, and $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}))$, where $\mathcal{C}$ is an ∞-category admitting finite products that is equipped with the cartesian symmetric monoidal structure, and $\mathcal{O}$ and $\mathcal{O}'$ are ∞-operads.

F.1. Cocartesian fibrations and cartesian symmetric monoidal structures

Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration whose fibers are compatible with products in the sense of Definition C.2.0.1, and let $\pi_{\mathcal{D}}\colon \mathcal{D}^{\times} \to \mathcal{D}$ be the cartesian structure on the cartesian symmetric monoidal structure on $\mathcal{C}$ (see [HA, 2.4.1]). By Proposition C.2.0.3, p preserves products. The goal of this section is to show that the induced functor $p^{\times}: \mathcal{C}^{\times} \to \mathcal{D}^{\times}$ can be obtained as a pullback of p along $\pi_{\mathcal{D}}$. Before we can prove this, we first show the following statement regarding how cocartesian morphisms interact with weak cartesian structures.

Proposition F.1.0.1. *Let $q\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category and $\pi\colon \mathcal{C}^{\otimes} \to \mathcal{D}$ a weak cartesian structure[1] on $\mathcal{C}^{\otimes}$.*

Let $C \simeq C_1 \oplus \cdots \oplus C_n$ be an object of $\mathcal{C}_{\langle n\rangle}$ with C_i an object of $\mathcal{C}$ for $1 \leq i \leq n$. Let $\varphi\colon \langle n\rangle \to \langle m\rangle$ be a morphism in Fin_* and let $f: C \to C'$ be a q-cocartesian lift of φ.

[1] See [HA, 2.4.1.1] for a definition

Then there exists a commutative diagram

$$\begin{tikzcd}
\pi(C) \arrow[r, "\pi(f)"] \arrow[d, "\simeq"'] & \pi(C') \arrow[d, "\simeq"] \\
\prod\limits_{1\leq i\leq n} \pi(C_i) \arrow[r] & \prod\limits_{\substack{1\leq i\leq n,\\ \varphi(i)\neq *}} \pi(C_i)
\end{tikzcd}$$

where the bottom horizontal morphism is the projection to the subproduct, the right vertical morphism is an equivalence, and the left vertical morphism is induced by the canonical morphisms $\pi(C) \to \pi(C_i)$ (which are induced by inert morphisms lying over ρ^i), and thus an equivalence as π is a lax cartesian structure. ♡

Proof. We first consider the case in which φ is inert. Then we can identify f with the following canonical projection morphism.

$$\bigoplus_{1\leq i\leq n} C_i \to \bigoplus_{\substack{1\leq i\leq n,\\ \varphi(i)\neq *}} C_i$$

Let

$$g_j\colon \bigoplus_{\substack{1\leq i\leq n,\\ \varphi(i)\neq *}} C_i \to C_j$$

be the canonical projection morphism for $1 \leq j \leq n$ with $\varphi(j) \neq *$ and define h_j similarly to be the projection $\bigoplus_{1\leq i\leq n} C_i \to C_j$ for $1 \leq j \leq n$. That π is lax cartesian means that the morphism

$$\pi\left(\bigoplus_{1\leq i\leq n} C_i\right) \xrightarrow{\prod_{1\leq i\leq n} h_i} \prod_{1\leq i\leq n} \pi(C_i)$$

is an equivalence, and similarly for $\bigoplus_{1\leq i\leq n, \varphi(i)\neq *} C_i$. The claim now follows from the fact that for $1 \leq i \leq n$ with $\varphi(i) \neq *$ the composition $g_i \circ f$ can be identified with h_i.

Let us now consider the general case. Let $g\colon C' \to C''$ be a q-cocartesian lift of the active morphism $\langle m\rangle \to \langle 1\rangle$. As π is a weak cartesian structure, $\pi(g)$ is an equivalence. It thus suffices to consider the case where $m = 1$. We can factor φ as a composition $\varphi = \alpha\circ\beta$ where β is inert and α active (see [HA, 2.1.2.2]). Lifting β and α to a commuting triangle $f \simeq g \circ h$ of q-cocartesian morphisms, with h a lift of β and g a lift of α, we can again use the fact that π is a weak cartesian structure (and that $m = 1$) to conclude that $\pi(g)$ is an equivalence. We are thus reduced to the case of inert morphisms, which we have already proven. □

Proposition F.1.0.2. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration whose fibers are compatible with products in the sense of Definition C.2.0.1, and let $\pi_{\mathcal{C}}\colon \mathcal{C}^{\times} \to \mathcal{C}$ and $\pi_{\mathcal{D}}\colon \mathcal{D}^{\times} \to \mathcal{D}$ be the cartesian structures on the cartesian symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, respectively (see [HA, 2.4.1.5 (5)]).*

Then the square induced via [HA, 2.4.1.8 and 2.4.1.6] by the product preserving functor p (see Proposition C.2.0.3)

$$\begin{tikzcd} \mathcal{C}^{\times} \arrow[r, "\pi_{\mathcal{C}}"] \arrow[d, "p^{\times}"'] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D}^{\times} \arrow[r, "\pi_{\mathcal{D}}"'] & \mathcal{D} \end{tikzcd}$$

is a pullback in $\mathcal{C}\mathrm{at}_{\infty}$. ♡

Proof. Consider the following commutative diagram, where the square is a pullback square.

$$\begin{tikzcd} \mathcal{C}^{\times} \arrow[dr, dashed, "\theta^{\otimes}"] \arrow[drr, bend left, "\pi_{\mathcal{C}}"] \arrow[ddr, bend right, "p^{\times}"'] & & \\ & \mathcal{C}^{\otimes} \arrow[r, "\pi"] \arrow[d, "p^{\otimes}"'] & \mathcal{C} \arrow[d, "p"] \\ & \mathcal{D}^{\times} \arrow[r, "\pi_{\mathcal{D}}"'] & \mathcal{D} \end{tikzcd}$$

It suffices to show that $\theta^{\otimes}$ is an equivalence.

The ∞-category $\mathcal{D}^{\times}$ comes with a cocartesian fibration, which we will denote by $q\colon \mathcal{D}^{\times} \to \mathsf{Fin}_*$, that makes $\mathcal{D}^{\times}$ into a symmetric monoidal ∞-category with underlying ∞-category $\mathcal{D}$ (see [HA, 2.4.1.5 (4)]). With this we can now state the three claims through which the proof will proceed:

(A) $p^{\otimes}$ is a cocartesian fibration of ∞-operads[2].

It follows from (A) that the functor $q \circ p^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ upgrades $\mathcal{C}^{\otimes}$ to a symmetric monoidal ∞-category. Note that by construction $p^{\times}\colon \mathcal{C}^{\times} \to \mathcal{D}^{\times}$ arises as a symmetric monoidal functor between symmetric monoidal ∞-categories, so in particular $p^{\times}$ can be lifted to a functor over Fin_*. It then follows that also $\theta^{\otimes}$ can be lifted to a functor over Fin_*. This gives meaning to the next claim.

(B) The functor $\theta^{\otimes}$ can be upgraded to a symmetric monoidal functor.

(C) The functor $\theta = \theta^{\otimes}_{\langle 1 \rangle}$ is an equivalence.

[2] See [HA, 2.1.2.13] for the definition.

Once we have proven these three claims, the statement follows immediately from [HA, 2.1.3.8], which states that as a symmetric monoidal functor (by (B)), $\theta^{\otimes}$ is already an equivalence if θ is is an equivalence (which it is by (C)).

Proof of (A): As p is a cocartesian fibration we can conclude by Proposition C.1.1.1 that $p^{\otimes}$ is also a cocartesian fibration. We will use [HA, 2.1.2.12] to show that $p^{\otimes}$ is even a cocartesian fibration of ∞-operads. So let

$$D \simeq D_1 \oplus \cdots \oplus D_n$$

be an object in $\mathcal{D}^{\times}_{\langle n\rangle}$ with D_i objects in $\mathcal{D}$ for $1 \le i \le n$, and let $f^i\colon D \to D_i$ for $1 \le i \le n$ be the canonical inert morphisms. We have to show that the induced morphism on fibers

$$\mathcal{C}^{\otimes}_{D} \xrightarrow{\prod_{1\le i\le n} f^i_!} \prod_{1\le i\le n} \mathcal{C}^{\otimes}_{D_i} \tag{F.1}$$

is an equivalence of ∞-categories. The fiber of $p^{\otimes}$ over some object D' can be identified with the fiber of p over $\pi_{\mathcal{D}}(D')$, and it follows from the description of $p^{\otimes}$-cocartesian morphisms in Proposition C.1.1.1 that this identification is compatible with the respective induced morphisms on fibers. We can thus identify functor (F.1) with the following functor.

$$\mathcal{C}_{\pi_{\mathcal{D}}(D)} \xrightarrow{\prod_{1\le i\le n} \pi_{\mathcal{D}}(f^i)_!} \prod_{1\le i\le n} \mathcal{C}_{\pi_{\mathcal{D}}(D_i)} \tag{F.2}$$

As $\pi_{\mathcal{D}}$ is a lax cartesian structure[3] we can identify $\pi_{\mathcal{D}}(D)$ with the product $\prod_{1\le i\le n} \pi_{\mathcal{D}}(D_i)$ and the morphisms $\pi_{\mathcal{D}}(f^j)\colon \pi_{\mathcal{D}}(D) \to \pi_{\mathcal{D}}(D_j)$ for $1 \le j \le n$ with the projection pr_j. We can thus identify functor (F.2) with the following functor.

$$\mathcal{C}_{\prod_{1\le i\le n} \pi_{\mathcal{D}}(D_i)} \xrightarrow{\prod_{1\le i\le n} \mathrm{pr}_{i!}} \prod_{1\le i\le n} \mathcal{C}_{\pi_{\mathcal{D}}(D_i)}$$

But the cocartesian fibration p has by assumption fibers compatible with products, and this means exactly that functors of this form are equivalences.

Proof of (B): Let f be a $q \circ p^{\otimes} \circ \theta^{\otimes}$-cocartesian morphism in $\mathcal{C}^{\times}$. Then we have to show that $\theta^{\otimes}(f)$ is $q \circ p^{\otimes}$-cocartesian. As $p^{\times}$ is symmetric monoidal, the morphism $p^{\times}(f) = p^{\otimes}(\theta^{\otimes}(f))$ is q-cocartesian, so by [HTT, 2.4.1.3 (3)] it suffices to show that $\theta^{\otimes}(f)$ is $p^{\otimes}$-cocartesian. Applying Proposition C.1.1.1 we are further reduced to showing that $\pi(\theta^{\otimes}(f)) = \pi_{\mathcal{C}}(f)$ is p-cocartesian. As $\pi_{\mathcal{C}}$ is a weak cartesian structure, Proposition F.1.0.1 shows that $\pi_{\mathcal{C}}(f)$ is a projection from a product to a factor, and by the description of products in $\mathcal{C}$ given in Proposition C.2.0.3, projection morphisms in $\mathcal{C}$ are p-cocartesian.

[3]See [HA, 2.4.1.1].

Proof of (C): Consider the commuting diagram

$$\begin{tikzcd}
\mathcal{C}^{\times}_{\langle 1\rangle} \arrow[r] \arrow[d, "\theta"'] & \mathcal{C}^{\times} \arrow[d, "\theta^{\otimes}"'] \arrow[dr, bend left, "\pi_{\mathcal{C}}"] & \\
\mathcal{C}^{\otimes}_{\langle 1\rangle} \arrow[r] \arrow[d] & \mathcal{C}^{\otimes} \arrow[r, "\pi"] \arrow[d, "p^{\otimes}"'] & \mathcal{C} \arrow[d, "p"] \\
\mathcal{D}^{\times}_{\langle 1\rangle} \arrow[r] \arrow[d] & \mathcal{D}^{\times} \arrow[r, "\pi_{\mathcal{D}}"'] \arrow[d, "q"'] & \mathcal{D} \\
\{\langle 1\rangle\} \arrow[r] & \mathsf{Fin}_* &
\end{tikzcd}$$

where the horizontal functors on the left are all the respective inclusions, and the vertical functors on the left are the functors induced by vertical functors in the middle. All squares in the diagram are pullback squares. As $\pi_{\mathcal{D}}$ is a cartesian structure, the composition $\mathcal{D}^{\times}_{\langle 1\rangle} \to \mathcal{D}$ in the third row is an equivalence. As the two squares in the middle row are pullbacks (and hence so is the outer commuting rectangle in the middle row) it follows that the composition $\mathcal{C}^{\otimes}_{\langle 1\rangle} \to \mathcal{C}$ in the second row is an equivalence as well. As $\pi_{\mathcal{C}}$ is a cartesian structure, the composition $\mathcal{C}^{\times}_{\langle 1\rangle} \to \mathcal{C}$ at the top is also an equivalence. It follows that θ must also be an equivalence. □

Remark F.1.0.3. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration whose fibers are compatible with products in the sense of Definition C.2.0.1. Then combining Proposition F.1.0.2 with Proposition C.1.1.1 we obtain another, independent, proof of Proposition C.2.0.6. ♢

F.2. Monoids and limits

In this short section we briefly discuss limits in ∞-categories of monoids.

Proposition F.2.0.1. *Let $\mathcal{O}$ be an ∞-operad and $\mathcal{C}$ an ∞-category.*

Let $\mathcal{I}$ be a small ∞-category and assume that $\mathcal{C}$ admits $\mathcal{I}$-indexed limits. Then $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ (for a definition see [HA, 2.4.2.1]) admits $\mathcal{I}$-indexed limits as well, and they are preserved and detected by the inclusion functor

$$\iota\colon \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\big)$$

as well as the composition

$$\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\iota} \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\big) \xrightarrow{j^*} \mathrm{Fun}(\mathcal{O}, \mathcal{C})$$

where $j\colon \mathcal{O} = \mathcal{O}^{\otimes}_{\langle 1\rangle} \to \mathcal{O}^{\otimes}$ is the inclusion. ♡

Proof. As ι is the inclusion of a full subcategory, it follows from [HTT, 1.2.13.7] that to show that $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ admits $\mathcal{I}$-indexed limits and that ι preserves and detects them it suffices to show that $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ is closed under $\mathcal{I}$-indexed limits in $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C})$. But this follows immediately from the definition [HA, 2.4.2.1] in combination with the fact that limits in functor categories are computed pointwise [HTT, 5.1.2.3], and that limits commute with limits [HTT, 5.5.2.3].

For the composition $j^* \circ \iota$, note that there is a commutative diagram as follows

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[rr, "\simeq"] \arrow[d] & & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \arrow[d, "\iota"] \\
\mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{O}^{\otimes}, \mathcal{C}^{\times}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}^{\times}) \arrow[r, "\pi_*"] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}) \arrow[d, "j^*"] \\
\mathrm{Fun}(\mathcal{O}, \mathcal{C}) \arrow[rr, no head, "\mathrm{id}"'] & & \mathrm{Fun}(\mathcal{O}, \mathcal{C})
\end{tikzcd}$$

where the unlabeled functors are the obvious forgetful functors or inclusions, and the top horizontal functor is an equivalence by [HA, 2.4.2.5]. That $j^* \circ \iota$ preserves and detects $\mathcal{I}$-indexed limits now follows from [HA, 3.2.2.4] in combination with [HTT, 5.1.2.3] and Proposition E.2.0.2. □

F.3. Cartesian symmetric monoidal ∞-categories and iterating Mon and Alg

Let $\mathcal{C}$ be an ∞-category admitting finite products and let $\mathcal{O}$ and $\mathcal{O}'$ be two ∞-operads. Then $\mathcal{C}$ can be upgraded to a symmetric monoidal ∞-category with the cartesian symmetric monoidal structure $\mathcal{C}^{\times}$ (see [HA, 2.4.1.5]). We can then consider the ∞-category of $\mathcal{O} \otimes \mathcal{O}'$-algebras in $\mathcal{C}^{\times}$, denoted by $\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C})$. By [HA, 2.4.2.5] this ∞-category is equivalent to an ∞-category that can be constructed without invoking $\mathcal{C}^{\times}$, namely the ∞-category of $\mathcal{O} \otimes \mathcal{O}'$-monoids $\mathrm{Mon}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C})$.

On the other hand, the cartesian symmetric monoidal structure $\mathcal{C}^{\times}$ induces a symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$, and there is an equivalence

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}))$$

as we saw in Section E.5. One would expect that the induced symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ is again cartesian so that we can identify $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ with $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\times}$ and hence with $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})^{\times}$, so that we ultimately obtain further equivalences such as

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}))$$

in $\mathcal{C}\mathrm{at}_\infty$.

In this section we will show that this is indeed the case, and describe the steps involved in these types of equivalences in detail, as we will need to know not only that such equivalences exist but also concrete descriptions of the corresponding objects under those equivalences.

Construction F.3.0.1. Let $p_{\mathcal{O}'}\colon \mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad, let furthermore $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to N\mathsf{Fin}_*$ be a symmetric monoidal ∞-category, and let $\pi\colon \mathcal{C}^{\otimes} \to \mathcal{C}$ be a cartesian structure[4].

There is a bifunctor of ∞-operads

$$f\colon \mathsf{Fin}_* \times \mathcal{O}'^{\otimes} \xrightarrow{\mathrm{id}_{\mathsf{Fin}_*} \times p_{\mathcal{O}'}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{\wedge} \mathsf{Fin}_*$$

where $\wedge$ is the bifunctor of ∞-operads defined in [HA, 2.2.5.1].

Consider the functor[5][6]

$$q\colon \ \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \xrightarrow{\iota_{\mathrm{Alg}}} \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \xrightarrow{\mathrm{pr}_2} \mathsf{Fin}_* \tag{F.3}$$

defined as in Proposition E.4.1.5, which by Proposition E.4.1.5 and [HA, 3.2.4.2 and 3.2.4.3 (3)] defines a symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$.

Finally, define $\widetilde{\pi}'$ as the following composition.

$$\widetilde{\pi}'\colon \ \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \xrightarrow{\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}} \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \xrightarrow{\pi_*} \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\big) \qquad \diamondsuit$$

Proposition F.3.0.2. *In the situation of Construction F.3.0.1, the functor $\widetilde{\pi}'$ factors through $\mathrm{Mon}_{\mathcal{O}'^{\otimes}}(\mathcal{C})$, i. e. there exists a functor $\widetilde{\pi}$ fitting into a commuting diagram*

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \arrow[rr, "\widetilde{\pi}"] \arrow[dr, "\widetilde{\pi}'"'] & & \mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}) \arrow[dl] \\
& \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}) &
\end{tikzcd}$$

where the functor $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C})$ *is the canonical inclusion*[7].

Furthermore, $\widetilde{\pi}$ is a cartesian structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$. $\heartsuit$

---

[4]See [HA, 2.4.1.1] for the definition.

[5]We write $\mathrm{Alg}_{\mathcal{O}'}$ instead of $\mathrm{Alg}_{\mathcal{O}'/\mathsf{Fin}_*}$.

[6]One should be careful not to confuse the functor $\mathsf{Fin}_* \to \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*)$ appearing in the pullback with the inclusion of the constant functors. Instead this functor is the one adjoint to the composition

$$\mathsf{Fin}_* \times \mathcal{O}'^{\otimes} \xrightarrow{\mathrm{id} \times p_{\mathcal{O}'}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{\wedge} \mathsf{Fin}_*$$

In particular, this means that the functors $\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ one obtains from objects of $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ by projecting to the first factor are generally *not* functors over Fin_*, so even though they preserve inert morphisms we can not interpret them as maps of ∞-operads.

[7]$\mathrm{Mon}_{\mathcal{O}'^{\otimes}}(\mathcal{C})$ is defined as a full subcategory of $\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\big)$, see [HA, 2.4.2.1]

Proof. Let A be an object of $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$, lying over $\langle n\rangle$, i. e. $q(A) = \langle n\rangle$. What we have to show is that the functor

$$\widetilde{\pi}'(A) = \pi \circ (\mathrm{pr}_1(\iota_{\mathrm{Alg}}(A)))\colon \mathcal{O}'^{\otimes} \xrightarrow{\mathrm{pr}_1(\iota_{\mathrm{Alg}}(A))} \mathcal{C}^{\otimes} \xrightarrow{\pi} \mathcal{C}$$

is an $\mathcal{O}'$-monoid. For ease of notation we will write $A' \coloneqq \mathrm{pr}_1(\iota_{\mathrm{Alg}}(A))$.

So let $X \simeq X_1 \oplus \cdots \oplus X_m$ be an object of $\mathcal{O}'^{\otimes}_{\langle m\rangle}$, with X_i objects of $\mathcal{O}'$ for $1 \leq i \leq m$. For $1 \leq i \leq m$, let $g_i\colon X \to X_i$ be an inert morphism lying over $\rho^i\colon \langle m\rangle \to \langle 1\rangle$. We have to show that then

$$\pi(A'(X)) \xrightarrow{\prod_{1\leq i\leq m} \pi\left(A'(g_i)\right)} \prod_{1\leq i\leq m} \pi(A'(X_i)) \tag{$*$}$$

is an equivalence in $\mathcal{C}$.

By definition, $A'\colon \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ preserves inert morphisms, so the morphisms $A'(g_i)$ are inert morphisms in $\mathcal{C}^{\otimes}$. Furthermore, for $1 \leq i \leq m$ we have

$$\begin{aligned} p_{\mathcal{C}}(A'(g_i)) &= p_{\mathcal{C}}(\mathrm{pr}_1(\iota_{\mathrm{Alg}}(A))(g_i)) \\ &= ((p_{\mathcal{C}_*} \circ \mathrm{pr}_1)(\iota_{\mathrm{Alg}}(A)))(g_i) \\ &= \left(\left(\widehat{f} \circ \mathrm{pr}_2\right)(\iota_{\mathrm{Alg}}(A))\right)(g_i) \\ &= f\left(\mathrm{id}_{\left(\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}\right)(A)}, g_i\right) \\ &= f\left(\mathrm{id}_{q(A)}, g_i\right) \\ &= f\left(\mathrm{id}_{\langle n\rangle}, g_i\right) \\ &= \mathrm{id}_{\langle n\rangle} \wedge p_{\mathcal{O}'}(g_i) \\ &= \mathrm{id}_{\langle n\rangle} \wedge \rho^i \end{aligned}$$

where $\widehat{f}\colon \mathsf{Fin}_* \to \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*)$ is the adjoint of f and thus the functor occurring in the pullback in (F.3). So for $1 \leq i \leq m$ the morphism $A'(g_i)$ in $\mathcal{C}^{\otimes}$ is a $p_{\mathcal{C}}$-cocartesian lift of $\mathrm{id}_{\langle n\rangle} \wedge \rho^i$.

Let Y_i be an object in $\mathcal{C}$ for each element i in $(\langle n\rangle \wedge \langle m\rangle)^{\circ}$ such that there is an equivalence

$$A'(X) \simeq \bigoplus_{i\in(\langle n\rangle\wedge\langle m\rangle)^{\circ}} Y_i$$

in $\mathcal{C}^{\otimes}_{\langle n\rangle\wedge\langle m\rangle}$. Applying Proposition F.1.0.1 we have an identification

$$\pi(A'(X)) \simeq \prod_{i\in(\langle n\rangle\wedge\langle m\rangle)^{\circ}} \pi(Y_i)$$

such that for each $1 \leq j \leq m$ the morphism $\pi(A'(g_j))$ corresponds to the following projection to the subfactor.

$$\prod_{i\in(\langle n\rangle\wedge\langle m\rangle)^{\circ}} \pi(Y_i) \to \prod_{\substack{i\in(\langle n\rangle\wedge\langle m\rangle)^{\circ},\\ \left(\mathrm{id}_{\langle n\rangle} \wedge\rho^j\right)(i)\neq *}} \pi(Y_i)$$

As $\langle m\rangle^\circ$ can be written as the disjoint union $\bigcup_{1\leq j\leq m}\left\{\, i\in\langle m\rangle^\circ \;\middle|\; \rho^j(i)\neq *\,\right\}$ it follows that we also have a decomposition of $(\langle n\rangle\wedge\langle m\rangle)^\circ$ as a disjoint union as follows

$$(\langle n\rangle\wedge\langle m\rangle)^\circ = \bigcup_{1\leq j\leq m}\left\{\, i\in(\langle n\rangle\wedge\langle m\rangle)^\circ \;\middle|\; \big(\mathrm{id}_{\langle n\rangle}\wedge\rho^i\big)(i)\neq *\,\right\}$$

which implies that the morphism $(*)$ is an equivalence, and $\widetilde{\pi}'$ thus factors over $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$.

It remains to show that $\widetilde{\pi}$ is a cartesian structure. We start by showing that $\widetilde{\pi}$ is a lax cartesian structure. So let A_i be objects of $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ for $1\leq i\leq n$, and let $g_i\colon A\coloneqq A_1\oplus\cdots\oplus A_n\to A_i$ be an inert lift of ρ^i for each $1\leq i\leq n$. We have to show that

$$\widetilde{\pi}(A)\xrightarrow{\prod_{1\leq i\leq n}\widetilde{\pi}(g_i)}\prod_{1\leq i\leq n}\widetilde{\pi}(A_i) \tag{$**$}$$

is an equivalence in $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$. As the inclusion $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})\to\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathcal{C})$ is fully faithful and equivalences in functor categories are detected pointwise (see Proposition A.3.2.1), it suffices to check that for every $m\geq 0$ and every object X of $\mathcal{O}'^{\otimes}_{\langle m\rangle}$ evaluation at X of morphism $(**)$ is an equivalence in $\mathcal{C}$. As by Proposition F.2.0.1 the inclusion $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})\to\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathcal{C})$ preserves products, and as products in functor categories are detected pointwise [HA, 5.1.2.3] we can thus identify the evaluation at X of the morphism $(**)$ with the morphism

$$\widetilde{\pi}'(A)(X)\xrightarrow{\prod_{1\leq i\leq n}\left(\widetilde{\pi}'(g_i)(X)\right)}\prod_{1\leq i\leq n}\widetilde{\pi}'(A_i)(X)$$

in $\mathcal{C}$, which by using the definition of $\widetilde{\pi}$ is the following morphism

$$\pi((\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(A)(X))\xrightarrow{\prod_{1\leq i\leq n}(\pi(h_i))}\prod_{1\leq i\leq n}\pi((\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(A_i)(X)) \tag{$***$}$$

where we use the notation $h_i\coloneqq(\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(g_i)(X)$.

Let $1\leq j\leq n$. By assumption, g_i is q-cocartesian, which by [HA, 3.2.4.3 (4)] implies that h_j is $p_{\mathcal{C}}$-cocartesian. Unwrapping the definition completely analogously to when we showed that $\widetilde{\pi}'$ factors over monoids we find that $p_{\mathcal{C}}(h_i)=\rho^i\wedge\mathrm{id}_{\langle m\rangle}$. That $(***)$ is an equivalence can now be shown completely analogously to before.

We next need to show that $\widetilde{\pi}$ is in fact a weak cartesian structure. So assume that $g\colon A\to A'$ is a q-cocartesian morphism lying over the active morphism $\alpha\colon\langle n\rangle\to\langle 1\rangle$. We have to show that $\widetilde{\pi}(g)$ is an equivalence in $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$. Similarly to before it suffices to check that for each $m\geq 0$ and object $X\in\mathcal{O}'^{\otimes}_{\langle m\rangle}$ the morphism $\pi(h)$ is an equivalence, where $h\coloneqq(\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(g)(X)$. Also analogously to the case above, we find that h is a $p_{\mathcal{C}}$-cocartesian lift of $\alpha\wedge\mathrm{id}_{\langle m\rangle}$, which is an active morphism as α is active. That $\pi(h)$ is an equivalence now follows from Proposition F.1.0.1.

Finally, it remains to show that the weak cartesian structure $\widetilde{\pi}$ is a cartesian structure. Consider the following commutative diagram, where the two top squares and the square on the right are pullback squares.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}_{\langle 1\rangle} \arrow[r, "k"] \arrow[d, "j"'] & \mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r] \arrow[d, "r"] & \{\langle 1\rangle\} \arrow[d] \\
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \arrow[r, "\iota_{\mathrm{Alg}}"'] \arrow[dd, "\widetilde{\pi}"'] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \mathsf{Fin}_* \arrow[d, "\widehat{f}"] \\
 & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r, "p_{\mathcal{C}*}"'] \arrow[d, "\pi_*"] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*) \\
\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}) \arrow[r] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}) &
\end{tikzcd}$$

The ∞-category of functors $\mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ over Fin_* is to be taken with respect to $p_{\mathcal{O}'}$ and $p_{\mathcal{C}}$ – this description uses that as $\langle 1\rangle \wedge -$ is naturally isomorphic to the identity functor on Fin_* we can identify $\widehat{f}(\langle 1\rangle)$ with $p_{\mathcal{O}'}$.

What we need to show is that $\widetilde{\pi} \circ j$ is an equivalence. As ι_{Alg} is the inclusion of the full subcategory of objects A such that $\mathrm{pr}_1(A)$ preserves inert morphisms, we can apply Proposition B.5.2.1 to conclude that k is the inclusion of the full subcategory of objects A such that $(\mathrm{pr}_1 \circ r)(A)$ preserves inert morphisms. This implies that the composite $\widetilde{\pi} \circ j$ can be identified with the functor $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \to \mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$ that is an equivalence by [HA, 2.4.2.5]. □

Proposition F.3.0.3. *Let $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ and $p'_{\mathcal{O}}\colon \mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$ be ∞-operads, let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category, and let $\pi : \mathcal{C}^{\otimes} \to \mathcal{C}$ be a cartesian structure. Let $F\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads (see [HA, 2.2.5.3]).*

Then there is a commutative diagram as follows such Ψ, Φ_2, Φ_3 and Ψ' are equivalences. If F exhibits $\mathcal{O}''^{\otimes}$ as a tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$, then

Φ_1 *is an equivalence as well.*

$$
\begin{tikzcd}
\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}) \arrow[r] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}) \\
\mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \arrow[r] \arrow[u, "\Psi", "\simeq"'] \arrow[d, "\Phi_1"'] & \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}) \arrow[u, "\pi_*"'] \arrow[d, "F^*"] \\
\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\otimes}) \arrow[r] \arrow[d, "\Phi_2"', "\simeq"] & \mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[d, "\widehat{(-)}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})) \arrow[r] \arrow[d, "\Phi_3"', "\simeq"] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})) \arrow[d, "(\pi_*)_*"] \\
\mathrm{Mon}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})) \arrow[r] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C})) \\
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})) \arrow[r] \arrow[u, "\Psi'", "\simeq"'] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C})^{\times}\Big) \arrow[u, "\big(\pi_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C})}\big)_*"']
\end{tikzcd}
\tag{F.4}
$$

The symmetric monoidal ∞-category $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$ *appearing on the bottom left carries the cartesian symmetric monoidal structure* $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})^{\times}$ *(see [HA, 2.4.1.5]) and* $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ *appearing on the left in the middle row carries the symmetric monoidal structure from Construction F.3.0.1. The horizontal functors are all the respective canonical functors that combine the various inclusions and forgetful functors or projections. The functor* $\widehat{-}$ *sends a functor* G *to its adjoint, which we denote by* $\widehat{G}$*.* ♡

Proof. The existence of equivalences Ψ and Ψ' making the topmost and bottommost square of (F.4) commute is shown in [HA, 2.4.2.5].

Construction of Φ_1 and Φ_2 fitting into the diagram was handled in Proposition E.5.0.2 and Proposition E.5.0.1.

We are left to construct Φ_3. Proposition F.3.0.2 provides us with a cartesian structure

$$\widetilde{\pi}\colon\ \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \to \mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$$

Applying [HA, 2.4.2.5] we obtain Composition with $\widetilde{\pi}$ then induces an equivalence Φ_3 as in the following commuting diagram by

$$
\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})) \arrow[r, dashed, "\Phi_3", "\simeq"'] \arrow[d] & \mathrm{Mon}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})) \arrow[d] \\
\mathrm{Fun}_{\mathsf{Fin}_*}\Big(\mathcal{O}^{\otimes}, \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}\Big) \arrow[r, "\widetilde{\pi}_* \circ \mathrm{pr}"'] \arrow[d] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})) \arrow[d] \\
\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})) \arrow[r, "(\pi_*)_*"'] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}))
\end{tikzcd}
$$

where pr denotes the forgetful functor

$$\mathrm{Fun}_{\mathsf{Fin}_*}\Big(\mathcal{O}^{\otimes}, \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}\Big) \to \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}\Big)$$

and the vertical functors are the canonical functors constructed the various forgetful functors, inclusions, and projections. The bottom square commutes by definition of $\widetilde{\pi}$, see Construction F.3.0.1. □

Remark F.3.0.4. The right column of (F.4) is covariantly functorial in $\mathcal{C}^{\otimes}$ (together with its cartesian structure) and contravariantly functorial in F.

Let

$$\begin{tikzcd} \mathcal{O} \times \mathcal{O}' \arrow[r, "F"] \arrow[d, "\alpha^{\otimes}\times\beta^{\otimes}"'] & \mathcal{O}'' \arrow[d, "\gamma^{\otimes}"] \\ \mathcal{U} \times \mathcal{U}' \arrow[r, "G"'] & \mathcal{U}'' \end{tikzcd}$$

be a commutative diagram of functors over Fin_* with $\alpha^{\otimes}$, $\beta^{\otimes}$, and $\gamma^{\otimes}$ morphisms of ∞-operads, and F and G bifunctors of ∞-operads.

Let

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r, "H^{\otimes}"] \arrow[d, "\pi_{\mathcal{C}}"'] & \mathcal{D}^{\otimes} \arrow[d, "\pi_{\mathcal{D}}"] \\ \mathcal{C} \arrow[r, "H"'] & \mathcal{D} \end{tikzcd}$$

be a commutative diagram of ∞-categories with $H^{\otimes}$ a symmetric monoidal functor of symmetric monoidal ∞-categories and $\pi_{\mathcal{C}}$ and $\pi_{\mathcal{D}}$ cartesian structures.

Then the induced commutative diagram on the right column of (F.4) restricts to a commutative diagram as follows.

$$\begin{tikzcd} \mathrm{Mon}_{\mathcal{U}''}(\mathcal{C}) \arrow[r, "\mathrm{Mon}_{\gamma}(H)"] & \mathrm{Mon}_{\mathcal{O}''}(\mathcal{D}) \\ \mathrm{Alg}_{\mathcal{U}''}(\mathcal{C}) \arrow[r, "\mathrm{Alg}_{\gamma}(H)"] \arrow[u, "\Psi", "\simeq"'] \arrow[d, "\Phi_1"'] & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{D}) \arrow[u, "\simeq", "\Psi"'] \arrow[d, "\Phi_1"] \\ \mathrm{BiFunc}(\mathcal{U}^{\otimes},\mathcal{U}'^{\otimes};\mathcal{C}^{\otimes}) \arrow[r, "\mathrm{BiFunc}(\alpha^{\otimes},\beta^{\otimes};H^{\otimes})"] \arrow[d, "\Phi_2"', "\simeq"] & \mathrm{BiFunc}(\mathcal{O}^{\otimes},\mathcal{O}'^{\otimes};\mathcal{D}^{\otimes}) \arrow[d, "\simeq"', "\Phi_2"] \\ \mathrm{Alg}_{\mathcal{U}}(\mathrm{Alg}_{\mathcal{U}'}(\mathcal{C})) \arrow[r, "\mathrm{Alg}_{\alpha}(\mathrm{Alg}_{\beta}(H))"] \arrow[d, "\Phi_3"', "\simeq"] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{D})) \arrow[d, "\simeq"', "\Phi_3"] \\ \mathrm{Mon}_{\mathcal{U}}(\mathrm{Mon}_{\mathcal{U}'}(\mathcal{C})) \arrow[r, "\mathrm{Mon}_{\alpha}(\mathrm{Mon}_{\beta}(H))"] & \mathrm{Mon}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{D})) \\ \mathrm{Alg}_{\mathcal{U}}(\mathrm{Mon}_{\mathcal{U}'}(\mathcal{C})) \arrow[r, "\mathrm{Alg}_{\alpha}(\mathrm{Mon}_{\beta}(H))"] \arrow[u, "\Psi'", "\simeq"'] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{D})) \arrow[u, "\simeq", "\Psi'"'] \end{tikzcd}$$

One could argue for this by considering the individual constructions, or one could use that the first, third, fourth, and fifth horizontal functor in (F.4) are monomorphisms[8][9] and apply the uniqueness part of Proposition B.4.3.1. This also implies compatibility with compositions.

Additionally, note that construction of Φ_1 and Φ_2 does not need the assumption that $\mathcal{C}$ carries a cartesian symmetric monoidal structure[10], so if we only consider the part of the above diagram involving Φ_1 and Φ_2, then we can drop this assumption. $\diamondsuit$

[8] That we only need those horizontal functors to be monomorphisms is because they are the "targets" in the diagram.

[9] The first and third horizontal functors are by definition fully faithful, so monomorphisms by Proposition B.4.4.1. The third and fourth horizontal functors are equivalent, so the fourth one is a monomorphism as well. Finally, the fifth horizontal functor is a monomorphism by a combination of the definitions, Proposition B.4.4.1, Proposition B.5.1.1, and Proposition B.1.2.1.

[10] See Proposition E.5.0.2 and Proposition E.5.0.1.

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