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PhD thesis

On the Hochschild homology of hypersurfaces as a mixed complex

Malte Sander Leip

Advisor: Lars Hesselholt

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Abstract

In this thesis we describe Hochschild homology over k of quotients of polynomial algebras $k[x_1, \ldots, x_n]/f$ for certain polynomials f in $n \leq 2$ variables, as an object of the ∞-category of mixed complexes $\mathcal{M}$ixed, where k is a commutative ring in which 2 is invertible.

In 1992, the Buenos Aires Cyclic Homology Group [BACH] constructed, for any n and any commutative ring k, a quasiisomorphism between the standard Hochschild complex over k of $k[x_1, \ldots, x_n]/f$ and a quite small chain complex, under the assumption that f is monic with respect to a chosen monomial order. This result was improved upon by Larsen in 1995 [Lar95] by taking the mixed structure into account as well, though only considering polynomials f in $n = 2$ variables that are monic with respect to one of the variables.

Assuming a conjectural description of Hochschild homology of polynomial rings, we extend these previous results by constructing, for a large subset of the polynomials f considered in [BACH], a strict mixed structure on the chain complex described in [BACH] and showing that it represents the Hochschild homology over k of $k[x_1, \ldots, x_n]/f$ as an object in the ∞-category of mixed complexes. We also verify the conjecture in some cases, leading to unconditional results for $n \leq 2$ variables, as long as 2 is invertible in k.

The results of this thesis do not rely on the two aforementioned prior results, but instead use the modern approach to Hochschild homology based on ∞-categorical methods. Along the way, to be able to state and prove our result in this setting, we prove some results that may be of independent interest.

Resumé

I denne afhandling beskriver vi Hochschild homologi over k for kvotienter af polynomiumsalgebraer $k[x_1, \dots, x_n]/f$ for visse polynomier f i $n \leq 2$ variable, som et objekt i ∞-kategorien $\mathcal{M}\mathrm{ixed}$ af såkaldte blandede komplekser, for k en kommutativ ring, hvori 2 er invertibel.

I 1992 konstruerede Buenos Aires Cyclic Homology gruppen [BACH] en kvasiisomorfi mellem standardhochschildkomplekset over k af $k[x_1, \dots, x_n]/f$ og et lille kædekompleks, under antagelsen, at f er monisk med hensyn til en valgt monomisk ordning, men for alle n og alle kommutative ring k. Denne resultat blev forbedret af Larsen i 1995 [Lar95], som også betragtede den blandede struktur, dog kun for polynomier f i $n = 2$ variabler som er monisk med hensyn til én af de to variable.

Under antagelsen af en formodete beskrivelse af Hochschild homologi af polynomiumsalgebraer generaliserer vi disse tidligere resultater ved at konstruere, for en stor delmængde af de polynomier f studeret i [BACH], en strengt blandet struktur på kædekomplekset beskrevet i [BACH] og at vise, at det repræsenterer Hochschild homologi over k af $k[x_1, \dots, x_n]/f$ som objekt i ∞-kategorien af blandede komplekser. Vi også verificerer formodningen i nogle tilfælde, og får dermed ubetingede resultater for $n \leq 2$ variable, forudsat, at 2 er invertibel i k.

Resultaterne i denne afhandling afhænger ikke af de to førnævnte arbejder, men bruger derimod den moderne tilgang til Hochschild homologi baseret på ∞-kategoriske metoder. Undervejs til at kunne beskrive og bevise vores resultat i denne ramme beviser vi nogle resultater som kan have selvstændig interesse.

Contents

Chapter 1.

Introduction

In this thesis we evaluate Hochschild homology over a commutative ring k of quotients of polynomial algebras $k[x_1, \ldots, x_n]/f$ for certain polynomials f, as an object of the ∞-category of mixed complexes $\mathcal{M}\mathrm{ixed}$, assuming a conjectural description of Hochschild homology of polynomial algebras. We do this by giving an explicit, and quite small, strict mixed complex representing $\mathrm{HH}(k[x_1, \ldots, x_n]/(f))$. We verify the conjecture in some cases, leading to unconditional results in the case of $n \leq 2$ variables as long as 2 is invertible in k. This result improves upon prior work by Larsen [Lar95] where stronger conditions on f are imposed[1], and by the Buenos Aires Cyclic Homology Group [BACH], where only the underlying chain complex was considered. The results of this thesis do not rely on the two aforementioned prior results, but use a different approach, employing the modern framework for Hochschild homology in the setting of ∞-categories.

The motivation for calculating Hochschild homology as a mixed complex stems from its usefulness to calculations of algebraic K-theory. The modern framework for topological cyclic homology by Nikolaus–Scholze [NikSch] opened up the possibility of obtaining calculations of algebraic K-theory using trace methods with only Hochschild homology as a mixed complex as input, via a method developed by Speirs [Spe18; Spe20; Spe21] and Hesselholt–Nikolaus [HN20]. In this modern setting, Hochschild homology is a functor of ∞-categories

$$\mathrm{HH}_{\mathbb{T}}\colon \mathrm{Alg}\big(\mathcal{D}(k)\big) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

assigning to each associative algebra in the derived category of k an object of $\mathcal{D}(k)$ equipped with an action by the circle group $\mathbb{T}$. The ∞-category $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ is equivalent to the underlying ∞-category $\mathcal{M}\mathrm{ixed}$ of a model category Mixed of strict mixed complexes[2], and we denote the composition of $\mathrm{HH}_{\mathbb{T}}$ with this equivalence by $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$. We now formulate the main result of this thesis, and will explain the meaning of the conditions on f and the notation used in the formula for d later in this introduction.

Theorem A. *Let k be a commutative ring in which 2 is invertible[3], $n \leq 2$ a positive integer, and $\preceq$ a monomial order (for n variables). Let f be a monic (with respect to $\preceq$) polynomial in n variables, and assume that furthermore the following property holds for any $\vec{i} \in \mathbb{Z}_{\geq 0}^n$ such that the coefficient of the monomial $x^{\vec{i}}$ in f is non-zero: If $1 \leq j \leq n$*

[1] But no assumption is made on invertibility of 2 in k.

[2] A strict mixed complex is a chain complex with an additional operator d increasing degree by 1 and satisfying $\partial \mathrm{d} + \mathrm{d}\partial = 0$ and $\mathrm{d}^2 = 0$.

[3] The assumption that 2 is invertible in k is not needed when $n \leq 1$.

and $\deg_{\preceq}(f)_j \neq 0$*, then* $\vec{i}_j \leq \deg_{\preceq}(f)_j$*. In other words, we require that every monomial appearing in* f *divides the leading monomial, after replacing by* 1 *those variables that do not appear in the leading monomial of* f*.*

Then there is an equivalence[4]

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[x_1,\ldots,x_n]/f\big) \simeq \gamma_{\mathsf{Mixed}}\big(k[x_1,\ldots,x_n]/f \otimes \Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n) \otimes \Gamma(t)\big)$$

in $\mathcal{M}\mathrm{ixed}$*, where*

$$k[x_1,\ldots,x_n]/f \otimes \Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n) \otimes \Gamma(t)$$

is a strict mixed complex with underlying $\mathbb{Z}$*-graded* k*-module*[5] *as indicated, with* x_i *of degree* 0*,* $\mathrm{d}\,x_i$ *of degree* 1 *and* t *of degree* 2*. The boundary operator* ∂ *is defined by extending the following formulas*[6] *by* k*-linearity and the Leibniz rule, where* $P \in k[x_1,\ldots,x_n]/f$*,* $1 \leq i \leq n$*, and* $m \geq 0$*.*

$$\partial(P) = 0, \qquad \partial(\mathrm{d}\,x_i) = 0, \qquad \partial\Big(t^{[m]}\Big) = -p(\mathrm{d}\,f)t^{[m-1]}$$

The differential d *is defined by extending by* k*-linearity the following formula for a polynomial* $P \in k[x_1,\ldots,x_n]$*,* $\vec{\epsilon} \in \{0,1\}^n$*, and* $m \geq 0$*.*

$$\mathrm{d}\Big(p(P)\,\mathrm{d}\,x^{\vec{\epsilon}}t^{[m]}\Big) := \left(p\Big(\mathrm{d}\Big(r_f^0(P)\Big)\Big) + mp\Big(q_f^1\Big(\mathrm{d}\,f \cdot r_f^0(P)\Big)\Big)\right)\mathrm{d}\,x^{\vec{\epsilon}}t^{[m]} \qquad \heartsuit$$

A proof of Theorem A can be found on Page 543. Most of the steps in the proof of Theorem A do not require the assumption that $n \leq 2$ and that 2 is invertible in k. We however need Conjecture D to hold for f. Conjecture D will be formulated and verified for $n \leq 2$ as long as 2 is invertible in k in Section 7.5.

Let us now give an overview over the remainder of this chapter. We begin in Section 1.1 by describing our motivation for studying Hochschild homology as a mixed complex, which arises from its relevance in the methods used in calculations of algebraic K-theory groups in [Spe20], [Spe21], and [HN20].

In Section 1.2 we explain how $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$, the main object of study, as well as the ∞-category $\mathcal{M}\mathrm{ixed}$ and 1-category Mixed are defined.

We will then turn towards describing the proof of Theorem A, which splits up naturally into two main steps. We describe the first main step in Section 1.3, which involves writing the quotient $k[x_1,\ldots,x_n]/f$ as a relative tensor product $k[x_1,\ldots,x_n] \otimes_{k[t]} k$, and then using that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserves relative tensor products. This yields a strict mixed complex X_f of medium size representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$. Finding a smaller sub-mixed-complex such that the inclusion into X_f is a quasiisomorphism is the content of the second main step in the proof of Theorem A and will be described in Section 1.4. Along

[4] γ_{Mixed} is a functor from the category of strict mixed complexes with cofibrant underlying chain complex to $\mathcal{M}\mathrm{ixed}$ and will be discussed in Section 1.2.2.

[5] We will use the commutative $\mathbb{Z}$-graded k-algebra structure to write elements and describe ∂, but we warn that d does *not* satisfy the Leibniz rule, so this is not an algebra in strict mixed complexes.

[6] We denote by p the quotient morphism $p\colon k[x_1,\ldots,x_n] \to k[x_1,\ldots,x_n]/f$.

the way we will introduce the definitions of concepts and notation used in the formulation of Theorem A.

In Section 1.5 we then give an overview over the content of the individual chapters and appendices of this thesis, and in Section 1.6 we describe some directions for future work and questions left open by this thesis.

1.1. Motivation

The project that eventually became this thesis started with the goal of determining the structure of the algebraic K-theory groups $\mathrm{K}_*(k[x_1, \ldots, x_n]/(x_1 \cdots x_n), (x_1, \ldots, x_n))$ for k a perfect field of positive characteristic, with the polynomial $x_1 \cdots x_n$ geometrically corresponding to the union of the coordinate hyperplanes. A method recently made possible by the Nikolaus–Scholze framework for topological cyclic homology [NikSch], and used by Speirs in the case of truncated polynomial algebras [Spe20][7] and the union of coordinate axes [Spe21][8], and by Hesselholt–Nikolaus for cuspidal curves [HN20], makes attacking such questions significantly easier.

In all these cases, what is determined are algebraic K-theory groups

$$\mathrm{K}_*\big(k[x_1, \ldots, x_n]/(f_1, \ldots, f_m), (x_1, \ldots, x_n)\big)$$

for k a perfect field of positive characteristic, n and m positive integers, and $f_1, \ldots, f_m$ specific polynomials in n variables with coefficients in $\mathbb{Z}$. This is done by employing trace methods, and the input ultimately required for this method circles around $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m))$, though there are variations between [Spe21], [Spe20], and [HN20] in what precisely is used as input. The following table is an overview.

	n	$(f_1, \ldots, f_m)$	Input used
[Spe21]	$n \geq 1$	$(x_i x_j)_{i \neq j}$	$\mathrm{B}^{\mathrm{cyc}}(\Pi)$[9] as an object of $\mathcal{S}_*^{\mathrm{B}\mathbb{T}}$
[Spe20]	1	(x_1^a), for $a \geq 1$ an integer	Homotopy groups of $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1]/(x_1^a))$ together with Connes' operator
[HN20]	2	$x_1^a - x_2^b$ for $a, b \geq 2$ relatively prime	$\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1, x_2]/(x_1^a - x_2^b))$ as an object of $\mathcal{D}(\mathbb{Z})^{\mathrm{B}\mathbb{T}}$

In [Spe21], Speirs uses that $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1, \ldots, x_n]/(x_i x_j)_{i \neq j})$ is the $\mathbb{Z}$-linearization of a space with $\mathbb{T}$-action $\mathrm{B}^{\mathrm{cyc}}(\Pi)$, and manages to even determine the $\mathbb{T}$-equivariant homotopy type of $\mathrm{B}^{\mathrm{cyc}}(\Pi)$, rather than only its $\mathbb{Z}$-linearization. In general we would however expect that it will be easier to only determine $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m))$ itself, which is all that is required.

[7] The relevant K-theory groups had first been evaluated by Hesselholt–Madsen [HM97], but the calculation was significantly simplified by Speirs.

[8] Generalizing results by Hesselholt [Hes07] from the two-dimensional case.

[9] $\mathrm{B}^{\mathrm{cyc}}(\Pi)$ denotes the cyclic bar construction of the pointed monoid

$$\Pi = \{0, 1, x_1, x_1^2, \ldots, x_2, x_2^2, \ldots, x_n, x_n^2, \ldots\}.$$

In contrast, in [Spe20] Speirs manages to get by with even less information than $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1]/(x_1^a))$ as an object of $\mathcal{D}(\mathbb{Z})^{\mathrm{B}\,\mathbb{T}}$, only using its homology as well as Connes' operator (induced by the circle action), and extracting e. g. the homotopy groups of the $\mathbb{T}$-fixed points using the fixed points spectral sequence. In this particular case, this is made feasible due to $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1]/(x_1^a))$ decomposing into pieces whose homology is concentrated in only two successive degrees, making the relevant spectral sequences easy enough to evaluate. In more complicated cases we can however not expect to (in general) be able to fully evaluate those spectral sequences without additional information.

Thus, in order to expand the results of [Spe20], [Spe21], and [HN20] to similar algebras, it seems reasonable to start by evaluating the relevant Hochschild homology $\mathrm{HH}_{\mathbb{T}}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m))$ as an object of $\mathcal{D}(\mathbb{Z})^{\mathrm{B}\,\mathbb{T}}$.

1.2. Hochschild homology as a mixed complex

1.2.1. Hochschild homology as an object with circle action

Having motivated our interested in $\mathrm{HH}_{\mathbb{T}}$, we will now give an idea of how it is defined. As $\mathrm{HH}_{\mathbb{T}}$ is a special case of the cyclic bar construction, we begin in somewhat greater generality.

Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category. Then the *cyclic bar construction* for $\mathcal{C}$ is a functor

$$\mathrm{B}^{\mathrm{cyc}}\colon \operatorname{Alg}(\mathcal{C}) \to \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$$

that associates to every associative algebra R in $\mathcal{C}$ an object with $\mathbb{T}$-action $\mathrm{B}^{\mathrm{cyc}}(R)$ in $\mathcal{C}$. To construct the underlying object in $\mathcal{C}$ of $\mathrm{B}^{\mathrm{cyc}}(R)$, one proceeds in two steps. One first constructs out of R a simplicial object $\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)$ in $\mathcal{C}$ such that $\mathrm{B}^{\mathrm{cyc}}_{n}(R)$ is given by $R^{\otimes(n+1)}$ and the structure morphisms $d_i\colon R^{\otimes n} \to R^{\otimes(n-1)}$ and $s_i\colon R^{\otimes n} \to R^{\otimes(n+1)}$ can be described as follows.

1. If $i \leq n-2$, then d_i is $\mathrm{id}_R^{\otimes i} \otimes \mu \otimes \mathrm{id}_R^{\otimes(n-2-i)}$, where $\mu\colon R \otimes R \to R$ is the multiplication morphism.

2. d_{n-1} is the postcomposition of the symmetry isomorphism that brings the last tensor factor to the front with $\mu \otimes \mathrm{id}_R^{\otimes(n-2)}$.

3. s_i is $\mathrm{id}_R^{i+1} \otimes \iota \otimes \mathrm{id}_R^{\otimes(n-i-1)}$, where $\iota\colon \mathbb{1}_{\mathcal{C}} \to R$ is the unit morphism.

Defining a simplicial object in $\mathcal{C}$, i. e. a functor $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{C}$, also requires data for higher morphisms; for a full definition of the functor $\mathrm{B}^{\mathrm{cyc}}_{\bullet}\colon \operatorname{Alg}(\mathcal{C}) \to \operatorname{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{C})$ see Section 6.1.2. The underlying object of $\mathrm{B}^{\mathrm{cyc}}(R)$ is then given by the geometric realization[10] of $\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)$. The circle action on $\mathrm{B}^{\mathrm{cyc}}(R)$ is constructed by first using cyclic permutations of the tensor factors to upgrade $\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)$ to a *cyclic object* in $\mathcal{C}$, i. e. lift the functor $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ to a functor to $\operatorname{Fun}(\boldsymbol{\Lambda}^{\mathrm{op}}, \mathcal{C})$, where $\boldsymbol{\Lambda}$ is Connes' cyclic category. The additional structure encoded by $\boldsymbol{\Lambda}$ equips the geometric realization of a cyclic object with the action

[10] So the underlying object of $\mathrm{B}^{\mathrm{cyc}}(R)$ is $\left|\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)\right| \coloneqq \operatorname{colim}_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)$.

of the circle group, so that composing $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ with the geometric realization functor for cyclic objects yields a functor $\mathrm{B}^{\mathrm{cyc}}\colon \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$. For a more detailed account of the construction of $\mathrm{B}^{\mathrm{cyc}}$ we refer to Chapter 6.

In the special case $\mathcal{C} = \mathcal{S}\mathrm{p}$, the ∞-category of spectra, the functor $\mathrm{B}^{\mathrm{cyc}}$ is denoted by THH, and if $\mathcal{C}$ is $\mathcal{D}(k)$, the derived ∞-category of a commutative ring k, we denote the functor $\mathrm{B}^{\mathrm{cyc}}$ by $\mathrm{HH}_{\mathbb{T}}(-/k)$ and call $\mathrm{HH}_{\mathbb{T}}(R/k)$ the Hochschild homology of R over k. We will from now on fix a commutative ring k and just write $\mathrm{HH}_{\mathbb{T}}(-)$ instead of $\mathrm{HH}_{\mathbb{T}}(-/k)$.

1.2.2. Mixed complexes

Our goal is to determine $\mathrm{HH}_{\mathbb{T}}(R)$ for specific k-algebras R. However it is somewhat difficult to write down and manipulate objects of $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ directly, so we use strict mixed complexes instead. The situation can be summarized by the following diagram.

$$\begin{tikzcd} & \mathsf{Mixed}_{\mathrm{cof}} \arrow[d, "\gamma_{\mathsf{Mixed}}"] \\ \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[r, dash, "\simeq"] & \mathcal{M}\mathrm{ixed} \end{tikzcd} \tag{1.1}$$

The horizontal functor is an equivalence between $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ and the ∞-*category of mixed complexes*, which the functor γ_{Mixed} exhibits as the underlying ∞-category of the 1-category with weak equivalences $\mathsf{Mixed}_{\mathrm{cof}}$ of *strict mixed complexes* (with cofibrant underlying chain complexes)[11].

We begin explaining diagram (1.1) with the 1-category Mixed. A strict mixed complex consists of an underlying chain complex of k-modules X (with boundary operator ∂ *decreasing* degree) together with an additional operator d, that we sometimes call the differential, *increasing* degree by 1, and satisfying the following identities.

$$\mathrm{d} \circ \mathrm{d} = 0 \qquad \text{and} \qquad \mathrm{d} \circ \partial + \partial \circ \mathrm{d} = 0$$

A morphism of strict mixed complexes is a morphism of underlying chain complexes that commutes with the respective differentials d. The strict mixed complexes and their morphisms define a 1-category Mixed.

There is also another description of Mixed: It is isomorphic to the category of left modules in $\mathsf{Ch}(k)$ over the differential graded algebra $\mathrm{D} = k[\mathrm{d}]/(\mathrm{d}^2)$, where d is of chain degree 1. Under this isomorphism $\mathsf{Mixed} \cong \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$, the action by the element d of D corresponds to the differential d. This suggests a definition of the ∞-category of mixed complexes as $\mathcal{M}\mathrm{ixed} \coloneqq \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$. The symmetric monoidal functor[12] $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ exhibiting $\mathcal{D}(k)$ as the underlying ∞-category of $\mathsf{Ch}(k)$ then induces a functor

$$\gamma_{\mathsf{Mixed}}\colon \mathsf{Mixed}_{\mathrm{cof}} \to \mathcal{M}\mathrm{ixed}$$

[11] The reason why we do not just say that $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ is exhibited as the underlying ∞-category of $\mathsf{Mixed}_{\mathrm{cof}}$ by the composition is that, while both $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ and $\mathcal{M}\mathrm{ixed}$ carry symmetric monoidal structures, the equivalence is only shown to be $\mathbb{E}_1$-monoidal. We should thus be careful to distinguish $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ and $\mathcal{M}\mathrm{ixed}$ whenever $\mathbb{E}_2$-monoidal structures may become relevant.

[12] The superscript cof refers to the subcategory of cofibrant objects.

where $\mathsf{Mixed}_{\mathrm{cof}}$ refers to the subcategory of Mixed spanned by those strict mixed complexes whose underlying chain complex is cofibrant with respect to the projective model structure[13].

We can make $\mathsf{Mixed}_{\mathrm{cof}}$ into a category with weak equivalences, where a morphism is a weak equivalence if and only if the underlying morphism of chain complexes is a quasiisomorphism, and it turns out that γ_{Mixed} then exhibits $\mathcal{M}\mathrm{ixed}$ as the ∞-category obtained from $\mathsf{Mixed}_{\mathrm{cof}}$ by inverting weak equivalences. We will discuss both Mixed as well as $\mathcal{M}\mathrm{ixed}$ in greater detail in Chapter 4.

The equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ is the composition of two different equivalences. There first is an equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k))$, where $k \boxtimes \mathbb{T}$ is the *k-linear circle*. The remaining equivalence $\mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \simeq \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) = \mathcal{M}\mathrm{ixed}$ is then induced by an equivalence $k \boxtimes \mathbb{T} \simeq \mathrm{D}$ in $\mathrm{Alg}(\mathcal{D}(k))$. We discuss these equivalences in detail in Chapter 5.

1.3. The first step in the proof of the main result

As mentioned before, we define $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ to be the composition of $\mathrm{HH}_{\mathbb{T}}$ with a specific equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ sketched above. Theorem A then sets the task before us to define a strict mixed complex that is mapped by γ_{Mixed} to an object in $\mathcal{M}\mathrm{ixed}$ that is equivalent to $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$.

The proof of Theorem A proceeds in two main steps. The idea of the first main step is to use that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is compatible with relative tensor products and that the quotient $k[x_1, \ldots, x_n]/f$ can be written as a relative tensor product of polynomial algebras[14].

Before going into more detail about why $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is compatible with relative tensor products, let us first describe the monoidal structure on Mixed. Given strict mixed complexes X and Y, we define the underlying chain complex of $X\otimes Y$ to be the tensor product in $\mathsf{Ch}(k)$ of the underlying chain complexes. The differential d is then defined using the Leibniz rule, so $\mathrm{d}(x \otimes y) = \mathrm{d}(x) \otimes y + (-1)^{\deg_{\mathsf{Ch}}(x)} x \otimes \mathrm{d}(y)$. Taking the perspective that a strict mixed complex is a left-D-module as described above, this symmetric monoidal structure arises from a bialgebra structure on D, where the comultiplication maps d to $\mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}$. Chapter 3 constructs monoidal structures on ∞-categories of left modules over bialgebras in a functorial way, so that we can upgrade $\gamma_{\mathsf{Mixed}}\colon \mathsf{Mixed}_{\mathrm{cof}} \to \mathcal{M}\mathrm{ixed}$ to a monoidal functor.

That $\mathrm{HH}_{\mathbb{T}}$ is a symmetric monoidal functor essentially follows from the fact that $\boldsymbol{\Delta}^{\mathrm{op}}$ is sifted and the tensor product in $\mathcal{D}(k)$ preserves colimits separately in each variable; we roughly obtain equivalences

$$\left|R^{\bullet+1}\right| \otimes \left|S^{\bullet+1}\right| \simeq \left|R^{\bullet+1} \otimes S^{\bullet+1}\right| \simeq \left|(R \otimes S)^{\bullet+1}\right|$$

that should make plausible that $\mathrm{HH}_{\mathbb{T}}$ is symmetric monoidal. $\mathrm{HH}_{\mathbb{T}}$ also preserves sifted colimits, and hence preserves relative tensor products. For more details see Chapter 6.

[13] See Fact 4.1.3.1 for a definition.
[14] This idea was suggested by Thomas Nikolaus.

To then deduce that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ also preserves relative tensor products it remains to show that $\mathcal{D}(k)^{\mathrm{B}\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ preserves relative tensor products. As an equivalence, it is clear that this functor preserves sifted colimits, but that it is $\mathbb{E}_1$-monoidal is not obvious, relying on a longer argument[15] carried out in Section 5.1, showing that D and $k \boxtimes \mathbb{T}$ are equivalent not only as associative algebras in $\mathcal{D}(k)$, but as $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebras[16] .

The quotient $k[x_1, \ldots, x_n]/f$ is isomorphic to the relative tensor product

$$k[x_1, \ldots, x_n] \otimes_{k[t]} k$$

in $\mathrm{Alg}(\mathrm{LMod}_k(\mathsf{Ab}))$, where t acts by multiplication with f on $k[x_1, \ldots, x_n]$ and by multiplication with 0 on k. Under the assumptions made for f in Theorem A, this ordinary relative tensor product calculates the *derived* one, so that we obtain an equivalence

$$k[x_1, \ldots, x_n]/f \simeq k[x_1, \ldots, x_n] \otimes_{k[t]} k$$

in $\mathrm{Alg}(\mathcal{D}(k))$ as well, inducing an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[x_1, \ldots, x_n]/f\big) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[x_1, \ldots, x_n]\big) \otimes_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t])} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

in $\mathcal{M}\mathrm{ixed}$.

To proceed we require a description of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n])$ and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$ as modules over $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t])$ in $\mathcal{M}\mathrm{ixed}$. The following conjecture provides such a description in terms of the mixed complexes of de Rham forms.

Conjecture D. *Let $n \geq$ be an integer and f an element of $k[x_1, \ldots, x_n]$. Denote by $F\colon k[t] \to k[x_1, \ldots, x_n]$ the morphism of commutative k-algebras that maps t to f and by $G\colon k[t] \to k$ the morphism of commutative k-algebras that maps t to 0. Then there exists a commutative diagram*

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, dash, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k/k}\right) \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[t]\big) \arrow[u, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(G)"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] \arrow[r, dash, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[t]/k}\right) \arrow[u, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{G/k}\right)"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{F/k}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[x_1, \ldots, x_n]\big) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[x_1,\ldots,x_n]/k}\right)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ *such that the horizontal morphisms are equivalences.*

We will often refer to the existence of such a commutative diagram for a specific f as "Conjecture D holds for f". ♧

[15]The strategy for this argument was suggested by Achim Krause.

[16]I. e. as commutative and coassociative bialgebras.

Conjecture D will be discussed in Section 7.5, where we will also show that it holds if $n \leq 1$ or $n = 2$ and 2 is invertible in k.

Assuming that Conjecture D holds for f, we then obtain an equivalence[17]

$$\begin{aligned} &\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[x_1, \ldots, x_n]\big) \otimes_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t])} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \\ \simeq\ &\gamma_{\mathsf{Mixed}}\big(k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)\big) \otimes_{\gamma_{\mathsf{Mixed}}(k[t]\otimes\Lambda(\mathrm{d}\, t))} \gamma_{\mathsf{Mixed}}(k) \end{aligned}$$

where x_i and t are in degree 0, $\mathrm{d}\, x_i$ and $\mathrm{d}\, t$ are in degree 1, and t acts by multiplication with f on $k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)$ and trivially on k. As alluded to by the naming, the differential of the respective mixed complexes maps x_i to $\mathrm{d}\, x_i$ and t to $\mathrm{d}\, t$, and is defined on the other elements by k-linearity and the Leibniz rule, while all three underlying chain complexes have zero boundary operator.

To obtain a strict mixed complex that represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[x_1, \ldots, x_n]/f\big)$ we thus have to calculate the derived tensor product of $k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)$ over $k[t] \otimes \Lambda(\mathrm{d}\, t)$ with k in Mixed. To do so, we need to replace k with a sufficiently cofibrant replacement as a module over $k[t] \otimes \Lambda(\mathrm{d}\, t)$ in Mixed. Such a replacement is given by a strict complex A whose underlying graded k-module is given by the tensor product[18]

$$k[t] \otimes \Lambda(\mathrm{d}\, t) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$$

where t is of degree 0, $\mathrm{d}\, t$ and s are of degree 1, and $\mathrm{d}\, s$ is of degree 2. The boundary operator ∂ and differential d are k-linear and satisfy the Leibniz rule, and are thus determined by the following formulas.

$$\begin{aligned} &\partial(t) = 0, \qquad \partial(\mathrm{d}\, t) = 0, \qquad \partial(s) = t, \qquad \partial\Big(\mathrm{d}\, s^{[m]}\Big) = -\,\mathrm{d}\, t\, \mathrm{d}\, s^{[m-1]} \\ &\mathrm{d}(t) = \mathrm{d}\, t, \qquad \mathrm{d}(\mathrm{d}\, t) = 0, \qquad \mathrm{d}(s) = \mathrm{d}\, s^{[1]}, \qquad \mathrm{d}\Big(\mathrm{d}\, s^{[m]}\Big) = 0 \end{aligned}$$

There is an obvious morphism of algebras in Mixed from $k[t] \otimes \Lambda(\mathrm{d}\, t)$ to A that maps t to t. In Section 8.2 it is shown that this makes A into a sufficiently cofibrant replacement for k as a left-$(k[t] \otimes \Lambda(\mathrm{d}\, t))$-module to calculate the derived relative tensor product discussed above as the ordinary relative tensor product

$$\begin{aligned} &\big(k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)\big) \otimes_{k[t]\otimes\Lambda(\mathrm{d}\, t)} \big(k[t] \otimes \Lambda(\mathrm{d}\, t) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)\big) \\ \cong\ &k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s) =: X_f \end{aligned}$$

in Mixed. We thus obtain an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[x_1, \ldots, x_n]/f\big) \simeq \gamma_{\mathsf{Mixed}}\big(k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)\big)$$

in $\mathcal{M}\mathrm{ixed}$. The boundary operator ∂ and differential d satisfy the Leibniz rule on X_f, and $\partial(s) = f$.

[17] The notation Λ is used for the exterior algebra, see Section 2.3 (29).

[18] The notation Γ is used for the divided power algebra, see Section 2.3 (30).

1.4. The second step in the proof of the main result

With the strict mixed complex X_f as above we already have a reasonably small strict model for $\mathrm{HH}_{\mathrm{Mixed}}(k[x_1, \ldots, x_n]/f)$, but we still want to identify a smaller, quasiisomorphic, sub-mixed-complex. In particular, X_f is given by $k[x_1, \ldots, x_n]$ in degree 0, while the homology is $k[x_1, \ldots, x_n]/f$ in degree 0. We will thus try to find a small sub-mixed-complex quasiisomorphic to X_f such that the k-module in degree 0 is isomorphic – as a k-module – to $k[x_1, \ldots, x_n]/f$.

Before we get started with this we first describe one of the assumptions we need to make on f, which is that f needs to be *monic* with respect to a chosen *monomial order*. A monomial order is a well-order $\preceq$ on the set of monomials in $x_1, \ldots, x_n$, or equivalently on $\mathbb{Z}^n_{\geq 0}$, such that $\vec{a} \preceq \vec{b}$ implies $\vec{a} + \vec{c} \preceq \vec{b} + \vec{c}$ for $\vec{a}, \vec{b}, \vec{c} \in \mathbb{Z}^n_{\geq 0}$. From now on we fix a monomial order $\preceq$. We can then define f to be monic (with respect to $\preceq$) if the biggest (with respect to $\preceq$) monomial appearing[19] in f has coefficient 1. The *degree* of f (with respect to $\preceq$), denoted by $\deg_{\preceq}(f)$, is the element of $\mathbb{Z}^n_{\geq 0}$ that is maximal with respect to $\preceq$ such that the coefficient of $x^{\deg_{\preceq}(f)}$ in f is non-zero.

If f is monic, then it is possible to divide polynomials in $x_1, \ldots, x_n$ by f with remainder. Specifically, if P is an element of $k[x_1, \ldots, x_n]$, then there is a unique decomposition of P as $P = q^1_f(P)f + r^0_f(P)$ such that $r^0_f(P)$ is f*-reduced*, meaning that only monomials that are not divisible by the lead monomial of f may appear in $r^0_f(P)$. For more details on these notions for multivariable polynomials see Section 9.1.

One perspective on the just mentioned decomposition is that it means that there is a *unique* f-reduced representative in $k[x_1, \ldots, x_n]$ for every element of $k[x_1, \ldots, x_n]/f$. We can thus define a section ϱ (as morphisms of k-modules) of the quotient morphism $p\colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/f$ by defining $\varrho(p(P))$ to be $r^0_f(P)$. Along ϱ we can thus identify $k[x_1, \ldots, x_n]/f$ as a k-module with the k-submodule of $k[x_1, \ldots, x_n]$ spanned by the reduced polynomials, i. e. $\mathrm{Im}(\varrho)$.

We now start with the sub-graded-k-module $\mathrm{Im}(\varrho)$ of X_f, and discuss what additional generators we need to add to our sub-graded-k-module to satisfy the following three conditions.

(a) It needs to be closed under ∂, to define a subcomplex.

(b) It needs to be closed under d, to define a sub-mixed-complex.

(c) The inclusion into X_f must be a quasiisomorphism.

As we require closedness under d, we first enlarge to the sub-graded-k-module

$$\mathrm{Im}(\varrho) \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)$$

of X_f. Now there are however elements that are multiples of $\mathrm{d}\, f$ and which are cycles but not boundaries, while they *are* boundaries in X_f. In order to achieve (c) we will

[19] That is, having non-zero coefficient.

thus need to add elements whose boundary are the relevant multiples of $\mathrm{d}\, f$. Our first attempt might be to consider the sub-graded-k-module

$$\mathrm{Im}(\varrho) \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes k \cdot \{1, \mathrm{d}\, s^{[1]}\}$$

as $\partial(-\mathrm{d}\, s^{[1]}) = \mathrm{d}\, f$. As we have now created new multiples of both $\mathrm{d}\, f$ as well as $\mathrm{d}\, s^{[1]}$ that will be cycles but not boundaries as needed for (c), we actually keep going and consider the sub-graded-k-module

$$\mathrm{Im}(\varrho) \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(\mathrm{d}\, s)$$

of X_f.

Let us turn towards condition (a) and check whether this could be a subcomplex of X_f. For this, let R be an element of $\mathrm{Im}(\varrho)$. Then we obtain

$$\partial\Big(R\,\mathrm{d}\, s^{[1]}\Big) = -R\,\mathrm{d}\, f = -q^1_f(R\,\mathrm{d}\, f)f - r^0_f(R\,\mathrm{d}\, f)$$

For this to lie in our provisional sub-graded-k-module we need to have $q^1_f(R\,\mathrm{d}\, f) = 0$, but unfortunately this will in general not be the case. To fix this, we should then modify $R\,\mathrm{d}\, s^{[1]}$ by adding another generator whose boundary will be $q^1_f(R\,\mathrm{d}\, f)f$. Such an element is given by $sq^1_f(R\,\mathrm{d}\, f)$, which leads us to the following definition. We define $\mathcal{J}_0$ as the set

$$\mathcal{J}_0 := \left\{ \left(\vec{i}, \vec{\epsilon}, m\right) \in \mathbb{Z}^n_{\geq 0} \times \{0,1\}^n \times \mathbb{Z}_{\geq 0} \;\middle|\; x^{\vec{i}} \text{ is } f\text{-reduced} \right\}$$

and for $(\vec{i}, \vec{\epsilon}, m)$ an element of $\mathcal{J}_0$ we define

$$e_{\vec{i},\vec{\epsilon},m} := x^{\vec{i}}\,\mathrm{d}\, x^{\vec{\epsilon}}\,\mathrm{d}\, s^{[m]} + sq^1_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\,\mathrm{d}\, s^{[m-1]}$$

as an element of X_f. We can then define $X^e_{f,0}$ to be the sub-graded-k-module of X_f spanned by the elements of the form $e_{\vec{i},\vec{\epsilon},m}$ for $(\vec{i}, \vec{\epsilon}, m)$ in $\mathcal{J}_0$.

It turns out that $X^e_{f,0}$ indeed satisfies conditions (a) and (c), but not in general (b). Thus the chain complex $X^e_{f,0}$ *does* represent the underlying object in $\mathcal{D}(k)$ of $\mathrm{HH}_{\mathrm{Mixed}}(k[x_1, \ldots, x_n]/f)$ (this reproves the main result of [BACH] as long as Conjecture D is satisfied for f), but we need to make further assumptions to ensure that $X^e_{f,0}$ is a sub-*mixed*-complex of X_f.

In the formulation of Theorem A we use a sufficient condition for f that is very easy to check and that ensures that $X^e_{f,0}$ is a sub-mixed-complex of X_f. The strict mixed complex used in the statement is then obtained by merely renaming the basis of $X^e_{f,0}$, where the element $e_{\vec{i},\vec{\epsilon},m}$ of $X^e_{f,0}$ corresponds to the element $p(x^{\vec{i}})\,\mathrm{d}\, x^{\vec{\epsilon}} t^{[m]}$ in the strict mixed complex described in Theorem A.

1.5. Overview over the chapters of this thesis

This thesis tries to give a rigorous proof of Theorem A, so it was attempted to include a proof for every needed statement for which no proof could be found in the literature. By necessity this means that many statements and proofs will already have been known to the experts, and some may even have already appeared, spread throughout the literature. This holds particularly with regards to the material contained in the appendices, where we collect various required statements on various aspects of working in an ∞-categorical setting. We hope that this will help fill some gaps in the literature. A reader primarily interested in applying the result and already familiar with Hochschild homology and mixed complexes may thus wish to only read Chapter 9 containing the statement of the result and the notation and notions necessary to understand and apply it, as well as Chapter 10, which contains an example worked out in detail.

The material is ordered linearly; proofs in the appendices only depend on statements occurring earlier in the appendices, and proofs in the main text only depend on statements occurring earlier in the main text or in the appendices.

We now briefly summarize the content of the chapters of this thesis. Each chapter, and most sections and subsections, also begin with an introduction, so we refer there for more details.

In **Chapter 2** we list and explain the notation and conventions that we use, and discuss what we assume the reader is familiar with.

In **Chapter 3** we construct monoidal structures on ∞-categories of left modules over bialgebras. If C is a symmetric monoidal 1-category and A a (associative, coassociative) bialgebra in C, then the category of left-A-modules $\mathrm{LMod}_A(\mathsf{C})$ can be given a monoidal structure again, constructed from the coalgebra structure of A[20]. The underlying object in C of the tensor product of two left-A-modules X and Y is the tensor product in C of the underlying objects, with action of A defined via the composition

$$A \otimes X \otimes Y \xrightarrow{\Delta \otimes \mathrm{id}_X \otimes \mathrm{id}_Y} A \otimes A \otimes X \otimes Y \xrightarrow{\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_X} A \otimes X \otimes A \otimes Y \to X \otimes Y$$

where Δ is the comultiplication, τ is the symmetry isomorphism, and the last morphism is the tensor product of the action morphisms of A on X and Y.

In Chapter 3 we construct such monoidal structures on $\mathrm{LMod}_A(\mathcal{C})$, where $\mathcal{C}$ is now allowed to be an $\mathbb{E}_2$-monoidal ∞-category, and A an $\mathbb{E}_1, \mathbb{E}_1$-bialgebra in $\mathcal{C}$. Our construction will be functorial in both A as well as $\mathcal{C}$ and thus allow us to compare Mixed, $\mathcal{M}\mathrm{ixed}$, and $\mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k))$, which are all monoidal ∞-categories arising via this construction.

In **Chapter 4** we define the 1-category Mixed and ∞-category $\mathcal{M}\mathrm{ixed}$. Beyond what was already mentioned in Section 1.2.2, we also discuss model structures on Mixed and $\mathrm{Alg}(\mathsf{Mixed})$, show that $\mathcal{M}\mathrm{ixed}$ and $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ are the respective underlying ∞-categories, and put the classical notion of strongly homotopy linear morphisms of strict mixed complexes into this context. That every algebra in $\mathcal{M}\mathrm{ixed}$ has a strict model will play a role in Chapter 7, when we discuss $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of polynomial algebras as an object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$.

[20]This monoidal structure should not be confused with the monoidal structure one can define using relative tensor products over A if A is commutative.

In **Chapter 5** we construct a monoidal equivalence between $\mathcal{D}(k)^{\mathrm{B}\mathbb{T}}$ and $\mathcal{M}\mathrm{ixed}$, as discussed in Section 1.2.2 above.

In **Chapter 6** we define Hochschild homology, both in its modern incarnation as a symmetric monoidal functor of ∞-categories

$$\mathrm{HH}_{\mathbb{T}}\colon \mathrm{Alg}(\mathcal{D}(k)) \to \mathcal{D}(k)^{\mathrm{B}\mathbb{T}}$$

as well as the classical model for Hochschild homology given by the standard Hochschild complex. In particular, we discuss how the standard Hochschild complex represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ as a mixed complex (by [Hoy18]) as well as HH of commutative rings as an object of $\mathrm{CAlg}(\mathcal{D}(k))$.

In **Chapter 7** we show that the mixed complex of de Rham forms is a model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of polynomial algebras in at most 2 variables as an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. Important input for this will be the comparison results discussed in Chapter 6 as well as the strictification result for algebras in $\mathcal{M}\mathrm{ixed}$ from Chapter 4. We also discuss compatibility with morphisms of polynomial algebras, by formulating Conjecture C and Conjecture D, and proving them in some cases.

In **Chapter 8** we perform the first step of the proof of Theorem A that we discussed in Section 1.3 above. The main result of Chapter 8 will be applicable in more generality, providing a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(y_1, \ldots, y_n))$ for R a commutative algebra in $\mathsf{Ch}(k)$, and $y_1, \ldots, y_n$ elements of R in degree 0, providing that the requirements of Proposition 8.3.0.1 are met, and we in particular are given a strict model of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ with sufficient structure.

Finally, we put everything together in **Chapter 9**. This chapter introduces the necessary notions for multivariable polynomials and carries out the second step of the proof of Theorem A that we discussed in Section 1.4 above.

For actual applications, we expect that the user of Theorem A will likely need to further simplify the resulting strict mixed complex. In **Chapter 10** we thus discuss the example $f = x_1^2 - x_2 x_3$ in detail[21], identifying an even smaller strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}[x_1, x_2, x_3]/f)$ than the one given by Theorem A (conditional on Conjecture D holding for f). We take care to not only prove the end result, but to describe the steps in the order and manner that one would take them when trying to come up with such a simplification, and hope that this example will help the reader to similarly simplify the result of Theorem A for other concrete polynomials.

The appendices contain various material relating to working with various notions in an ∞-categorical setting that do not have a very strong thematic relation to the main content of this thesis, apart from being needed in it.

Appendix A and **Appendix D** contain some statements on basic notions of ∞-category theory, such as mapping spaces, undercategories, and adjunctions. The reason this material is split up into two appendices is in order to conserve linearity of the material in the appendices, as some material from Appendix A is needed in the intermediate appendices, from where Appendix D needs some results.

[21] As this is an example in three variables, Theorem A only holds for f conditional on Conjecture D. However, it is an interesting example with which we can demonstrate the combinatorial notions used to formulate the result of Theorem A, and how the result can be further manipulated.

In **Appendix B** we discuss the notions of (fully) faithful functors of ∞-categories as well as monomorphisms in $\mathcal{C}\mathrm{at}_\infty$.

Appendix C collects a number of statements involving (co)cartesian fibrations. In particular, we discuss for functors of ∞-categories $F\colon \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ the property of the cocartesian fibration classified by F that corresponds to $\mathcal{C}$ having all products and F preserving them.

In **Appendix E** we discuss various statements that relate to ∞-operads and their ∞-categories of algebras, such as the induced ∞-operad structures on ∞-categories of algebras, free algebras, and relative tensor products.

Appendix F discusses cartesian symmetric monoidal ∞-categories. If $\mathcal{C}$ is a cartesian symmetric monoidal ∞-category and $\mathcal{O}$ an ∞-operad, then the ∞-categories of $\mathcal{O}$-algebras and $\mathcal{O}$-monoids in $\mathcal{C}$ are equivalent. A large part of Appendix F is concerned with iterating this, i. e. applying $\mathrm{Alg}_{\mathcal{O}'}$ or $\mathrm{Mon}_{\mathcal{O}'}$ to $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ or $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ and comparing the resulting ∞-categories. The reason is that we not only need to know that there exist some equivalences between the various ∞-categories, but require concrete descriptions of specific equivalences.

1.6. Future directions

In this section we present some questions left open by this thesis and directions for future work. The most obvious open problem is the conjecture our main result depends on.

(1) Conjecture D is proven in Chapter 7 only for $n \leq 2$ variables, in the case $n = 2$ requiring an assumption on k. Showing this conjecture for polynomials in more variables would extend Theorem A.

The next possibility for future work we would like to mention is the application to calculations of algebraic K-theory.

(2) Let k be a perfect field of positive characteristic, n a positive integer, and f a polynomial in n variables satisfying the conditions of Theorem A. One can then try to determine the structure of $\mathrm{K}(k[x_1, \ldots, x_n]/f, (x_1, \ldots, x_n))$ using the techniques of [Spe20], [Spe21], and [HN20], using the strict mixed complex representing $\mathrm{HH}_{\mathrm{Mixed}}(k[x_1, \ldots, x_n]/f)$ as the starting point.

    The project that became this thesis was in fact started with the goal of determining the structure of

$$\mathrm{K}_*\big(k[x_1, \ldots, x_n]/(x_1 \cdots x_n), (x_1, \ldots, x_n)\big)$$

    i. e. of the K-theory groups of the union of hyperplanes. Another first test case to apply this to might be the cone $x_1^2 = x_2 x_3$, i. e. trying to determine the structure of $\mathrm{K}(k[x_1, \ldots, x_n]/(x_1^2 - x_2 x_3), (x_1, x_2, x_3))$. To obtain new unconditional results both of these would require first extending the validity of Theorem A by proving Conjecture D for the three-variable case.

There are also a number of questions directly left open in this thesis.

(3) In [Spe20] and [HN20] it is important that THH and $\mathrm{HH}_{\mathbb{T}}$ have a compatible decomposition as a sum, which arises from a grading on the polynomial ring with respect to which the polynomial divided out is homogeneous.

Before tackling (2) it will therefore be important to upgrade Theorem A to take into account such a grading.

(4) In Chapter 5 we show that there is an $\mathbb{E}_1$-monoidal equivalence between $\mathcal{D}(k)^{\mathrm{B}\mathbb{T}}$ and $\mathcal{M}\mathrm{ixed}$. Does there exist an $\mathbb{E}_2$-monoidal equivalence? One can also add some additional conditions, such as asking for a commutative triangle

$$\begin{tikzcd} \mathcal{D}(k)^{\mathrm{B}\mathbb{T}} \arrow[rr, no head, "\simeq"] \arrow[dr] & & \mathcal{M}\mathrm{ixed} \arrow[dl] \\ & \mathcal{D}(k) & \end{tikzcd}$$

of $\mathbb{E}_2$-monoidal functors, with the horizontal one being an equivalence, and where the two other functors are the forgetful ones.

(5) Theorem A is shown in Proposition 9.5.2.3, where, apart from Conjecture D needing to hold for f, the condition is actually that f needs to be monic and satisfy $\mathrm{logdim}_f(\mathrm{d}\,f) \leq 1$, rather then the condition used in the the formulation of Theorem A above, which implies $\mathrm{logdim}_f(\mathrm{d}\,f) \leq 1$ by Corollary 9.4.2.6. This leaves the question whether Corollary 9.4.2.6 is sharp. To be more precise, suppose $f \neq 1$ is a polynomial that is monic with respect to a monomial ordering $\preceq$ and such that $\mathrm{logdim}_f(\mathrm{d}\,f) \leq 1$. Then does it hold for every $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ such that the coefficient of the monomial $x^{\vec{i}}$ in f is non-zero that if $1 \leq j \leq n$ and $\deg_{\preceq}(f)_j \neq 0$, then $\vec{i}_j \leq \deg_{\preceq}(f)_j$?

(6) A related question to (5) is what kind of values $\mathrm{logdim}_f(\mathrm{d}\,f)$ can take. In particular, is there a monic polynomial f such that $\mathrm{logdim}_f(\mathrm{d}\,f)$ is finite, but bigger than 1?

(7) Is there a class of monic polynomials f with $\mathrm{logdim}_f(\mathrm{d}\,f) > 1$ and for which $X^e_{f,0}$ is not a sub-mixed-complex of X_f, but there is some other, intermediate sub-mixed-complex that is also equivalent to X_f? For example it may be that there exists such a sub-mixed-complex for some f in which the power of f is bounded[22,23], unlike in X.

It is possible that $\mathrm{logdim}_f(\mathrm{d}\,f)$ has already been studied (if so, likely under a different name), so perhaps there already exist answers to (5) and (6) in the literature.

[22] In the sense that it is generated as a graded k-module by elements $c_{\vec{i},l,\vec{\epsilon},m}$ and $e_{\vec{i},l,\vec{\epsilon},m}$ as in Definition 9.2.3.2 such that l is smaller than or equal to some $l_{\max}$.

[23] For example the power of f might be bounded by $n \cdot (\mathrm{logdim}_f(\mathrm{d}\,f) - 1)$ (as long as $\mathrm{logdim}_f(\mathrm{d}\,f) \geq 1$), which would yield the correct bound 0 for $\mathrm{logdim}_f(\mathrm{d}\,f) = 1$. This would of course not be helpful if it turned out that there exist no monic polynomials f with $1 < \mathrm{logdim}_f(\mathrm{d}\,f) < \infty$.

1.7. Acknowledgments

There are many people who helped make this thesis possible. This project was suggested by my advisor, Lars Hesselholt, and he and my co-advisor, Jesper Grodal, helped keep me on track. The idea to approach Hochschild homology of quotients by writing quotients as relative tensor products was suggested to me by Thomas Nikolaus, and a sketch of the proof strategy used to show formality of the k-linear circle as an $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebra in Section 5.1 was suggested to me by Achim Krause.

In a earlier version there was a mistake in the proof of what is now Proposition 7.2.2.2, where it was claimed that ϵ is natural with respect to all morphisms of k-algebras, rather than only those that map variables to variables. This incorrect result was then used in what amounted to a proof of Conjecture B, Conjecture C, and Conjecture D for polynomial algebras in arbitrary many variables. The mistake was pointed out by Thomas Nikolaus.

I had useful mathematical discussions that left their marks on this thesis with many people, among them David Bauer, Elden Elmanto, Aras Ergus, Jesper Grodal, Lars Hesselholt, Kaif Hilman, Joshua Hunt, Achim Krause, Markus Land, Jonas McCandless, Thomas Nikolaus, Riccardo Pengo, Philipp Schmitt, Martin Speirs, and Robin Sroka, and received helpful feedback on earlier drafts from David Bauer, Aras Ergus, Jesper Grodal, Lars Hesselholt, and Martin Speirs. I am sure to have missed someone who should have been listed above, for which I apologize.

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Chapter 2.

Notation and conventions

2.1. Prerequisites

We will work extensively in ∞-categorical settings, and thus reading this thesis will likely require a solid foundation in the theory of ∞-categories and higher algebra as developed in [HTT] and [HA]. We will however try to give references for any major statements that we use, and we refrain from using statements that are well-known to experts without giving a proof ourselves if no citable reference could be found in the literature – many such statements are thus collected in the appendices.

We assume that the reader is familiar with the basics of (homological) algebra, as well as the theory of model categories, for which we use [Hov99] and [HTT, A.2] as our main references. Wherever terminology differs between [Hov99] and [HTT, A.2], we follow the terminology of [HTT, A.2].

In contrast, it is not strictly necessary to have prior exposure to Hochschild homology or related concepts, as all the necessary definitions will be provided.

2.2. On how this thesis is structured

To make it easy to reference parts of this thesis we make liberal use of section subdivisions and encapsulate a large part of the material in various environments such as remarks, constructions, propositions, proofs, and similar.

To mark the end of such an environment we use several different symbols, which appear on the end of the last line of the respective environment, i. e. rightmost on the page. A square $\square$ is used to denote the end of a proof, as is usual. For statements that come with a proof we use a heart $\heartsuit$, and for statements that could come with a proof (facts, conjectures, etc.) but do not we use a club $\clubsuit$. Other environments, such as definitions, constructions, etc. are ended with a diamond $\diamondsuit$. The author first saw the idea to use card suits for environment end markers in Tashi Walde's Master's thesis.

The only types of mathematical statements with proof that we distinguish in the text are corollaries (for statements whose proof is a direct specialization of previous results) and propositions (for everything else). The only exception is Theorem A, which is stated in the introduction.

2.3. Various notations and conventions

In this section we state various conventions and notation that will be used throughout the thesis.

(1) We fix a commutative ring k for the entire thesis. If X and Y are k-modules, then $X \otimes Y$ refers to the tensor product over k unless something else is explicitly stated.

(2) With regards to ∞-categories, we try to work as model independently as possible, so by an ∞-category we mean an object in the $(\infty, 2)$-category of ∞-categories $\mathcal{C}\mathrm{at}_\infty$, not a representative in a specific model, such as quasicategories[1]. In particular, if we e. g. talk about a pullback of ∞-categories, then this refers to a pullback in the ∞-category of ∞-categories, not to a (categorical) pullback of quasicategories (simplicial sets).

(3) We denote by $\mathcal{C}\mathrm{at}_\infty$ the ∞-category of ∞-categories. If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then there exists an ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$, denoted by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. We will thus also consider $\mathcal{C}\mathrm{at}_\infty$ as an $(\infty, 2)$-category, though we will not require a general theory of $(\infty, 2)$-categories.

(4) We denote by Cat the $(\infty, 2)$-category[2] of 1-categories[3], as a full subcategory of $\mathcal{C}\mathrm{at}_\infty$. We will thus not use any notation to indicate the inclusion[4] of Cat into $\mathcal{C}\mathrm{at}_\infty$; if C is a 1-category, then C is in particular an ∞-category.

(5) We use different fonts to visually distinguish between 1-categories, ∞-categories, quasicategories, and other kinds of objects. Named 1-categories (like Ring rather than C) use the same font as unnamed 1-categories, for named ∞-categories we use a different calligraphic font than for unnamed ∞-categories.

We illustrate this with the following table.

Type of object	Font description	Examples
1-category	sans-serif	C, D, E
Named 1-category	sans-serif	Cat, Ring, $\mathsf{Ch}(k)$, Mixed, sSet
∞-category	calligraphic	$\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$
Named ∞-category	calligraphic	$\mathcal{C}\mathrm{at}_\infty$, $\mathcal{D}(k)$, $\mathcal{M}\mathrm{ixed}$, $\mathcal{S}$
Quasicategories[5]	typewriter	$\mathtt{C}$, $\mathtt{D}$, $\mathtt{E}$, $\mathtt{f}$, $\mathtt{p}$
Other	serif and Greek	C, D, E, Φ, Ψ, α, β, γ, a, b, c

[1] For the implications for (co)cartesian fibrations see the introduction to Appendix C.
[2] By [HTT, 2.3.4.8] Cat is actually a $(2, 2)$-category.
[3] For us, 1-categories are ∞-categories with discrete mapping spaces, compare [HTT, 2.3.4.1, 2.3.4.5, and 2.3.4.18].
[4] If we model ∞-categories by quasicategories, then this inclusion is given by the nerve construction, see [HTT, 1.1.2.6].
[5] Including morphisms.

(6) The following table collects notation for some named 1-categories.

Notation	Description / ∞-category of	Reference
Set	sets	
Fin	finite sets	
sSet	simplicial sets	[HTT, A.2.7]
Top	nice[6]topological spaces	[Hov99, 2.4.21]
Ab	abelian groups	
$\mathsf{Ch}(k)$	chain complexes of k-modules	Definition 4.1.1.1
PoSet	partially ordered sets	Definition 6.1.1.2
$\mathbb{Z}\mathsf{PoSet}$	partially ordered sets with $\mathbb{Z}$-action	Definition 6.1.1.2
Mixed	strict mixed complexes	Definition 4.2.1.2

(7) The following table collects notation for some named ∞-categories.

Notation	Description / ∞-category of	Reference
$\mathcal{S}$	spaces	[HTT, 1.2.16]
$\mathcal{S}\mathrm{p}$	spectra	[HA, 1.4.3]
$\mathcal{D}(k)$	derived category of k	Proposition 4.3.2.1 (1)
$\mathcal{P}\mathrm{r}$	presentable ∞-categories, as a full subcategory of $\mathcal{C}\mathrm{at}_\infty$	[HTT, 5.5.0.1]
$\mathcal{P}\mathrm{r}^{\mathrm{L}}$	presentable ∞-categories, morphisms are functors preserving all small colimits, as a subcategory of $\mathcal{P}\mathrm{r}$	[HTT, 5.5.3.1]
$\mathcal{M}\mathrm{ixed}$	mixed complexes	Notation 4.4.0.2

(8) We generally follow the notation used in [HA] for ∞-operads that we use, though with a different font to be consistent with (4) and (5).

Notation	Notation in [HA]	Name	Reference
Comm or Fin_*	Comm or $\mathcal{F}\mathrm{in}_*$	commutative ∞-operad	[HA, 2.1.1.18]
Assoc	Assoc	associative ∞-operad	[HA, 4.1.1.3]
Triv	$\mathcal{T}\mathrm{riv}$	trivial ∞-operad	[HA, 2.1.1.20]
LM	$\mathcal{LM}$	∞-operad of left modules	[HA, 4.2.1.7]
$\mathbb{E}_n$	$\mathbb{E}_n$	∞-operad of little n-cubes	[HA, 5.1.0.3 and 5.1.1.6]

By [HA, 5.1.0.7] there is an equivalence of ∞-operads $\mathbb{E}_1 \simeq \mathsf{Assoc}$. We will identify

[6] It is not really relevant for us if one takes k-spaces, compactly generated topological spaces, or another variant. What is important for us is that geometric realization and the singular simplicial set functor define a Quillen equivalence as follows.

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, shift left, "|-|"] \arrow[r, phantom, "\perp"] & \mathsf{Top} \arrow[l, shift left, "\mathrm{Sing}"] \end{tikzcd}$$

these two ∞-operads along this equivalence and use $\mathbb{E}_1$ and Assoc as interchangeable notation. The ∞-operad $\mathbb{E}_\infty$ is by definition equal to Comm.

(9) We sometimes use parenthesis to cover multiple cases at the same time to avoid repetitious language. For example we might write

> X is adjective$_1$ (adjective$_2$, adjective$_3$) if it satisfies property$_1$ (property$_2$, property$_3$).

which is to be interpreted as

> X is adjective$_1$ if it satisfies property$_1$. Furthermore, X is adjective$_2$ if it satisfies property$_2$. Finally, X is adjective$_3$ if it satisfies property$_3$.

A variant version of this convention is

> X is (adverb) adjective if it satisfies property$_1$ (property$_2$).

which is to be read as follows.

> X is adjective if it satisfies property$_1$. Furthermore, X is adverb adjective if it satisfies property$_2$.

(10) If $\mathcal{C}$ is an ∞-category, then we use the notation

$$\mathrm{Map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$

for the mapping space functor. Similarly, if C is a (Ab-enriched, or $\mathrm{LMod}_k(\mathsf{Ab})$-enriched) 1-category, then we denote by $\mathrm{Mor}_{\mathsf{C}}$ (by $\mathrm{Hom}_{\mathsf{C}}$) the morphism set functor (Hom functor) with codomain Set (Ab and $\mathrm{LMod}_k(\mathsf{Ab})$, respectively). If C is an object of $\mathcal{C}$, then we use use $\mathrm{Aut}_{\mathcal{C}}(C)$ as the notation for the automorphism space of C, i. e. the subspace of $\mathrm{Map}_{\mathcal{C}}(C, C)$ spanned by equivalences $C \to C$.

(11) We use $-$ as notation for an unnamed argument in order to describe functions (and functors etc.) without introducing unnecessary notation. For example, instead of defining the function that maps a real number to its square by

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^2$$

and then using f in some place where a function $\mathbb{R} \to \mathbb{R}$ is expected, we would just use the following notation.

$$-^2$$

If there is more than one argument we may subscript $-$, such as in the following example.

$$(-_1 + -_2)^2\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Finally, we also use $\bullet$ in a similar manner for “inner” functions. For example

$$\bullet^{-}\colon \mathbb{Z}_{\geq 1} \to \mathrm{Mor}_{\mathsf{Set}}(\mathbb{R}, \mathbb{R})$$

would refer to the map that sends n to the map that sends x to x^n.

(12) Let C be a model category with class of weak equivalences W. Then we denote by $\mathrm{Ho}_W(\mathsf{C})$ the homotopy category of C in the model-category sense. If $\mathcal{C}$ is an ∞-category, then we denote by $\mathrm{Ho}(\mathcal{C})$ the homotopy category of $\mathcal{C}$ as defined in [HTT, 1.2.3]. For the relationship between these two definitions, see Proposition A.1.0.1.

(13) Let C be a model category. Then we denote by $\mathsf{C}^{\mathrm{cof}}$ (by $\mathsf{C}^{\mathrm{fib}}$) the full subcategory of cofibrant (fibrant) objects of C. The model categories we consider admit functorial (co)fibrant replacement functors, which we will denote as follows.

$$-^{\mathrm{cof}}\colon \mathsf{C} \to \mathsf{C}^{\mathrm{cof}} \qquad \text{and} \qquad -^{\mathrm{fib}}\colon \mathsf{C} \to \mathsf{C}^{\mathrm{fib}}$$

(14) Let $\mathcal{C}$ be an ∞-category admitting products. If X and Y are objects of $\mathcal{C}$ and $X \times Y$ a product object of X and Y, then we denote by $\mathrm{pr}_1\colon X \times Y \to X$ and $\mathrm{pr}_2\colon X \times Y \to Y$ the morphisms that exhibit $X \times Y$ as a product of X and Y.

If $f_1\colon X \to Y_1$ and $f_2\colon X \to Y_2$ are two morphisms in $\mathcal{C}$, then we denote by

$$f_1 \times f_2\colon X \to Y_1 \times Y_2$$

the induced morphism determined by equivalences $\mathrm{pr}_i \circ (f_1 \times f_2) \simeq f_i$.

If $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ are two morphisms in $\mathcal{C}$, then we will also denote by

$$f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$$

the induced morphism between the products, which is determined by equivalences $\mathrm{pr}_i \circ (f_1 \times f_2) \simeq f_i \circ \mathrm{pr}_i$. While this could in principle lead to confusion, we will always make clear in the context which of the two interpretations are intended.

Analogous notation is used for products over more factors, possibly indexed by a set.

(15) We say that a functor of ∞-categories *detects* something[7] if it both preserves and reflects it.

(16) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and $\mathcal{E}$ another ∞-category. Then we sometimes denote by F_* the induced functor

$$\mathrm{Fun}(\mathcal{E}, F)\colon\ \mathrm{Fun}(\mathcal{E}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{D})$$

and by F^* the following induced functor.

$$\mathrm{Fun}(F, \mathcal{E})\colon\ \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

We also use this notation in variant cases, such as induced functors on subcategories of functor categories, or ∞-categories of functors over another ∞-category.

[7] For example equivalences or colimits.

(17) Let $p\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. We will often just say that $\mathcal{O}$ is an ∞-operad, dropping the $\otimes$ superscript, or even that $F\colon \mathcal{O} \to \mathcal{O}'$ is a morphism of ∞-operads when $\mathcal{O}'$ is another ∞-operad[8]. If we are referring to $\mathcal{O}^{\otimes}$ as an ∞-category, for example talking about an object of $\mathcal{O}^{\otimes}$, then we will however never drop the superscript. To make this convention consistent, the total ∞-category of a functor to Fin_* that we think of as an ∞-operad will always be denoted by a notation that includes a superscript $\otimes$. We hope that this will not lead to confusion in practice, but will instead make many terms more concise and readable.

(18) Consistent with (17), if $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{O}''$ are ∞-operads, then we use the notation $\mathrm{BiFunc}(\mathcal{O}, \mathcal{O}'; \mathcal{O}'')$ for the ∞-category of bifunctors of ∞-operads that is denoted by $\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{O}''^{\otimes})$ in [HA] – see [HA, 2.2.5.3].

(19) If $\mathcal{O}$ and $\mathcal{C}$ are ∞-operads, then we denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ the ∞-category of ∞-operad morphism from $\mathcal{O}$ to $\mathcal{C}$ [9]. If $\mathcal{O} = \mathsf{Assoc}$ we will also write $\mathrm{Alg}(\mathcal{C})$ instead, and if $\mathcal{O} = \mathsf{Comm}$ we will also write $\mathrm{CAlg}(\mathcal{C})$.

Similarly, if $\mathcal{O} = \mathsf{Assoc}$ we will just say "monoidal" and if $\mathcal{O} = \mathsf{Comm}$ we will say "symmetric monoidal" instead of "$\mathcal{O}$-monoidal".

(20) For $n \geq 1$ an integer and $1 \leq i \leq n$ we denote by $\rho^i\colon \langle n\rangle \to \langle 1\rangle$ the morphism of Fin_* defined in [HA, 2.0.0.2], i. e. given by the following formula.

$$\rho^i(j) \coloneqq \begin{cases} 1 & \text{if } i = j \\ * & \text{otherwise} \end{cases}$$

(21) Let $\mathcal{O}$ be an ∞-operad. Then we use $\oplus$ as notation for the operation defined and discussed in [HA, 2.1.1.15 and 2.2.4.6]. In particular, if X_i is an object in $\mathcal{O}$ for $1 \leq i \leq n$, then $X = X_1 \oplus \cdots \oplus X_n$ will be an object in $\mathcal{O}^{\otimes}_{\langle n\rangle}$, coming with inert morphisms $X \to X_i$ in $\mathcal{O}^{\otimes}$ lying over ρ^i, or equivalently equivalences $\rho^i_!(X) \simeq X_i$.

If we introduce an object $X \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ as $X \simeq X_1 \oplus \cdots \oplus X_n$ for X_i objects of $\mathcal{O}$, then we implicitly assume that X comes with inert morphisms $X \to X_i$ lying over ρ^i.

(22) If $p\colon \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration and $f\colon X \to Y$ a morphism in $\mathcal{D}$, then we usually denote the induced morphism on fibers[10] (see [HTT, 5.2.1]) by $f_!\colon \mathcal{C}_X \to \mathcal{C}_Y$ if the cocartesian fibration p is clear from context, and otherwise as $f^p_!$.

(23) Let $\mathcal{C}$ be an ∞-category. A *subcategory* of $\mathcal{C}$ is an ∞-category $\mathcal{C}'$ together with a monomorphism[11] $\iota\colon \mathcal{C}' \to \mathcal{C}$ in $\mathfrak{Cat}_{\infty}$. Up to equivalence a subcategory of $\mathcal{C}$ is given by specifying a replete subcategory of $\mathrm{Ho}\,\mathcal{C}$, see Section B.6.

[8] Where we of course already use this convention, so implicitly we introduced a functor $\mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$ exhibiting $\mathcal{O}'^{\otimes}$ as an ∞-operad, and F is actually to be a functor $\mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ over Fin_*.

[9] See [HA, 2.1.2.7] We will also use the related notation introduced in [HA, 2.1.3.1].

[10] The notation $\mathcal{C}_X$ refers to the fiber of p over X, i. e. to the pullback object $\{X\} \times_{\mathcal{D}} \mathcal{D}$ of p along the inclusion of $\{X\}$.

[11] See Appendix B for more on monomorphisms in $\mathfrak{Cat}_{\infty}$.

(24) Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be ∞-categories. Then we denote by

$$\widehat{-}\colon \operatorname{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \xrightarrow{\simeq} \operatorname{Fun}\big(\mathcal{C}, \operatorname{Fun}(\mathcal{D}, \mathcal{E})\big)$$

and

$$\check{-}\colon \operatorname{Fun}\big(\mathcal{C}, \operatorname{Fun}(\mathcal{D}, \mathcal{E})\big) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

the equivalences arising from the $\times$-Fun-adjunction[12]. We will use the same notation for the equivalences

$$\widehat{-}\colon \operatorname{Fun}(\mathcal{D} \times \mathcal{C}, \mathcal{E}) \xrightarrow{\simeq} \operatorname{Fun}\big(\mathcal{C}, \operatorname{Fun}(\mathcal{D}, \mathcal{E})\big)$$

and

$$\check{-}\colon \operatorname{Fun}\big(\mathcal{C}, \operatorname{Fun}(\mathcal{D}, \mathcal{E})\big) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{D} \times \mathcal{C}, \mathcal{E})$$

and will make clear from context which of the two variants is meant.

(25) Let $\mathcal{C}$ be an ∞-category. We denote by $\mathfrak{CFib}(\mathcal{C})$ the subcategory of $(\mathfrak{Cat}_\infty)_{/\mathcal{C}}$ spanned by the cartesian fibrations and morphisms of cartesian fibrations[13]. Similarly, we denote by $\mathrm{co}\mathfrak{CFib}(\mathcal{C})$ the subcategory of $(\mathfrak{Cat}_\infty)_{/\mathcal{C}}$ spanned by the cocartesian fibrations and morphisms of cocartesian fibrations.

(26) Let $\mathcal{C}$ be an ∞-category. We denote by

$$\operatorname{Gr}\colon \operatorname{Fun}(\mathcal{C}, \mathfrak{Cat}_\infty) \to \mathrm{co}\mathfrak{CFib}(\mathcal{C})$$

the Grothendieck construction that maps a functor $F\colon \mathcal{C} \to \mathfrak{Cat}_\infty$ to the cocartesian fibration classified by F.

(27) Let S be a set. Then an S-graded k-module is an S-tuple of k-modules, or equivalently a functor $S \to \mathrm{LMod}_k(\mathsf{Ab})$ from the discrete category with set of objects S to the category of k-modules.

If $(X_s)_{s \in S}$ is an S-graded k-module, then we can form a k-module $X := \bigoplus_{s \in S} X_s$, but one should not confuse the k-module X with the S-graded k-module $(X_s)_{s \in S}$, for example in the context of (28) directly below.

(28) The category of $\mathbb{Z}$-graded k-modules carries a symmetric monoidal structure defined just like for chain complexes, in which the symmetry isomorphism contains signs – see Definition 4.1.2.1. Commutative algebras in this symmetric monoidal category will then of course involve signs in their commutativity relations, so if x and y are elements of a commutative $\mathbb{Z}$-graded k-algebra A of degrees n and m, then this implies that $x \cdot y = (-1)^{nm} y \cdot x$. In some places in the literature this is referred to as “graded commutativity”. However, as the mentioned symmetric monoidal structure on $\mathbb{Z}$-graded k-modules is the only one we define, there is no other, “non-graded commutativity” one could consider, so we do not use this terminology.

[12] So if $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a functor, then $\widehat{F}\colon \mathcal{C} \to \operatorname{Fun}(\mathcal{D}, \mathcal{E})$ is its adjoint.

[13] See Appendix C for more on (co)cartesian fibrations.

(29) Let M be a $\mathbb{Z}$-graded k-module that is concentrated in odd degrees. Then the tensor algebra $T(M)$ (or $T_k(M)$ if we want to make k explicit) of M is defined as

$$T(M) \coloneqq \bigoplus_{i \geq 0} M^{\otimes i}$$

where the tensor product of $\mathbb{Z}$-graded k-modules is as in (28). One can define a multiplication on $T(M)$ by k-linearly extending the formula

$$(m_1 \otimes \cdots \otimes m_i) \cdot (m'_1 \otimes \cdots \otimes m'_j) \coloneqq m_1 \otimes \cdots \otimes m_i \otimes m'_1 \otimes \cdots \otimes m'_j$$

for $i, j \geq 0$ and $m_1, \ldots, m_i, m'_1, \ldots, m'_j$ elements of M. This makes $T(M)$ into a $\mathbb{Z}$-graded k-algebra, with unit given by the element 1 of $k = M^{\otimes 0}$.

We define the *exterior $\mathbb{Z}$-graded k-algebra generated by M*, denoted by $\Lambda(M)$ or $\Lambda_k(M)$, to be the quotient of $T(M)$ by the two-sided ideal generated by elements of the form $m \cdot m$ for $m \in M$[14].

The composition of the inclusion[15] of M into $T(M)$ with the quotient morphism to $\Lambda(M)$ is an injection, so that we can consider M as a sub-$\mathbb{Z}$-graded-k-module of $\Lambda(M)$, and elements of M generate $\Lambda(M)$ multiplicatively. For m and m' elements of M it holds in $\Lambda(M)$ that

$$m \cdot m' = (m + m') \cdot (m + m') - m' \cdot m - m \cdot m - m' \cdot m' = -m' \cdot m$$

so that $\Lambda(M)$ is in fact a *commutative* $\mathbb{Z}$-graded k-algebra.

Finally, let us note that we will also use the notation $\Lambda(x_1, \ldots, x_n)$ as a shorthand for $\Lambda(k \cdot \{x_1, \ldots, x_n\})$.

(30) For an even integer n we define a commutative $\mathbb{Z}$-graded k-algebra $\Gamma(x)$, called the *divided power $\mathbb{Z}$-graded k-algebra* generated by the variable x in degree n as follows.

The underlying $\mathbb{Z}$-graded k-module is given by

$$\Gamma(x) \coloneqq k \cdot \{1, x^{[1]}, x^{[2]}, \ldots\}$$

with $x^{[i]}$ of degree $i \cdot n$, where we let $x^{[0]} = 1$. A multiplication on $\Gamma(x)$ is defined by k-linearly extending the formula

$$x^{[i]} \cdot x^{[j]} \coloneqq \binom{i+j}{i} x^{[i+j]}$$

[14] This definition differs from the one given in [Lod98, A.1] if 2 is not invertible in k. In those cases the usage of the definition of [Lod98, A.1] is however incorrect with regards to the results we cite from [Lod98] relating to the mixed complex of de Rham forms – the definition we give here is the correct one. In particular, the proof of [Lod98, 3.2.2] implicitly assumes the definition we have given here.

[15] This refers to the inclusion of M as the summand $M^{\otimes 1}$.

for $i, j \geq 0$, which makes $\Gamma(x)$ into a commutative $\mathbb{Z}$-graded k-algebra with multiplicative unit 1.

We furthermore define

$$\Gamma(x_1, \ldots, x_n) \coloneqq \Gamma(x_1) \otimes \cdots \otimes \Gamma(x_n)$$

for all x_i of even degree.

(31) Elements for $\mathbb{Z}^n_{\geq 0}$ are tuples of nonnegative integers $(a_1, \ldots, a_n)$. We will often write such a tuple as $\vec{a}$, and use $\vec{e_i}$ as notation for the tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the single 1 is in the i-th slot. For $\vec{\epsilon} \in \{0, 1\}^n$ we furthermore make the following definition.

$$|\vec{\epsilon}| = \sum_{i=1}^{n} \epsilon_i$$

We use analogous notation for tuples indexed by a set set other than $\{1, \ldots, n\}$ for a natural number n.

(32) For $\vec{a} \in \mathbb{Z}^n_{\geq 0}$ we will write $x^{\vec{a}}$ for the monomial $x_1^{a_1} \cdots x_n^{a_n}$ in the polynomial algebra $k[x_1, \ldots, x_n]$. Vectors in $\mathbb{Z}^n_{\geq 0}$ are added pointwise, and we have e. g. $x^{\vec{a}+\vec{b}} = x^{\vec{a}} \cdot x^{\vec{b}}$.

We use analogous notation for exterior and divided power algebras. Concretely, we will for $\vec{\epsilon} \in \{0, 1\}^n$ use the notation $\mathrm{d}\, x^{\vec{\epsilon}}$ to refer to

$$\mathrm{d}\, x^{\vec{\epsilon}} \coloneqq \mathrm{d}\, x_1^{\epsilon_1} \cdots \mathrm{d}\, x_n^{\epsilon_n}$$

and not to $\mathrm{d}(x^{\vec{\epsilon}})$. One can remember this as the convention that d binds stronger than exponentiation with a vector.

Similarly, for $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ we define

$$x^{[\vec{i}]} \coloneqq x_1^{[i_1]} \cdots x_n^{[i_n]}$$

in the divided power algebra $\Gamma(x_1, \ldots, x_n)$.

(33) If $f \in k[x_1, \ldots, x_n]$ is a polynomial and $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ a vector, then we let $f_{\vec{i}} \in k$ be the coefficient of the monomial $x^{\vec{i}}$ in f, i. e. the unique decomposition of f as a k-linear combination of monomials is as follows.

$$f = \sum_{\vec{i} \in \mathbb{Z}^n_{\geq 0}} f_{\vec{i}} x^{\vec{i}}$$

(34) If $n \geq 0$ is an integer, then we denote by Σ_n the *symmetric group on n elements*; it is the group of bijections of the set $\{1, \ldots, n\}$, also called permutations of $\{1, \ldots, n\}$. It will sometimes be convenient to extend an element σ of Σ_n to a bijection of $\{0, \ldots, n\}$ by setting $\sigma(0) = 0$, which we will do implicitly. If $n' > n$, then there

exists an inclusion of Σ_n into $\Sigma_{n'}$ given by extending an element σ of Σ_n by $\sigma(i) = i$ for $n < i \leq n'$. We also usually not distinguish in notation between σ as an element of Σ_n and its extension as an element of $\Sigma_{n'}$.

Given a permutation σ on n elements and a subset S of $\{1, \ldots, n\}$, we say that σ *preserves the ordering of* S if for every pair of elements $i < i'$ in S it holds that $\sigma(i) < \sigma(i')$. We also use this terminology for other injective maps between totally ordered sets. Let $1 \leq i, j \leq n$. Then there is a unique element of Σ_n that maps i to j and preserves the ordering of $\{1, \ldots, i-1, i+1, \ldots, n\}$. We will call this element $\sigma_{i \to j}$. Note that if $n' > n$, then the extension of $\sigma_{i\to j}$ to a permutation of n' elements is again of the same form, which justifies that n is not part of the notation.

We define $\sigma_{\mathrm{cyc},n}$ to be the element $\sigma_{n\to 1}$ of Σ_n. If n is clear from context we will also denote $\sigma_{\mathrm{cyc},n}$ by σ_{cyc}. We denote by C_n the subgroup of Σ_n generated by $\sigma_{\mathrm{cyc},n}$.

We also need a manner of restricting permutations. Let σ be an element of Σ_n, and S a subset of $\{1, \ldots, n\}$. Denote the set $\sigma(S)$ by S'. Then there are unique order-preserving bijections $\phi\colon \{1, \ldots, |S|\} \to S$ and $\psi\colon S' \to \{1, \ldots, |S|\}$. We define $r_S(\sigma)$ to be the element of $\Sigma_{|S|}$ that is given by the composition $\psi \circ \sigma\big|_S^{S'} \circ \phi$. This defines a map of sets $r_S\colon \Sigma_n \to \Sigma_{|S|}$. Note that in the above situation we have that if σ' is another element of Σ_n, then $r_S(\sigma' \circ \sigma) = r_{S'}(\sigma') \circ r_S(\sigma)$.

We can also add permutations as follows. Let $n, n' \geq 0$. Then there is a group homomorphism $- \amalg -\colon \Sigma_n \times \Sigma_{n'} \to \Sigma_{n+n'}$ given as follows. If σ is an element of Σ_n and σ' an element of $\Sigma_{n'}$, then we define $\sigma \amalg \sigma'$ as follows.

$$\big(\sigma \amalg \sigma'\big)(i) \coloneqq \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq n \\ \sigma'(i-n) + n & \text{if } n+1 \leq i \leq n+n' \end{cases}$$

Note that $r_{\{1,\ldots,n\}} \circ (- \amalg -)$ and $r_{\{n+1,\ldots,n+n'\}} \circ (- \amalg -)$ are the projection to the first and second factor, respectively.

Given a permutation σ on n elements and a subset S of $\{1, \ldots, n\}$, we say that σ *cyclically preserves the ordering of* S if $r_S(\sigma)$ is an element of $C_{|S|}$. This terminology can easily be extended to more general maps. Let $f\colon X \to Y$ be an injective map between any finite totally ordered sets X and Y, and S a subset of X. Then there exist unique order-preserving bijections $\phi\colon \{1, \ldots, |X|\} \to X$ and $\psi\colon \operatorname{Im}(f) \to \{1, \ldots, |X|\}$, making $\sigma \coloneqq \psi \circ f|^{\operatorname{Im}(f)} \circ \phi$ into an element of $\Sigma_{|X|}$. We say that f (cyclically) preserves the ordering of the subset S if σ (cyclically) preserves the ordering of the subset $\phi^{-1}(S)$.

(35) Formulations such as "$\mathcal{C}$ admits all colimits" mean that $\mathcal{C}$ admits all *small* colimits. We never refer to non-small (co)limits with generic formulations. See also Section 2.4 directly below.

2.4. Size issues

In Section 2.3 (4) we defined Cat as the 1-category[16] of all 1-categories. Taken directly as stated Cat would be an object of itself and we would run into the usual set-theoretic paradoxes, so we need to be more careful in defining Cat.

The usual way to deal with this issue is to postulate the existence of Grothendieck universes $\mathcal{U}_1 \in \mathcal{U}_2 \in \mathcal{U}_3$ (and possibly more if required), which are sets whose elements satisfy the usual axioms of set theory. Sets that are elements of $\mathcal{U}_i$ are called $\mathcal{U}_i$*-small*. We can then perform all the usual operations of set theory with $\mathcal{U}_i$-small sets, but now there exists e. g. a $\mathcal{U}_2$-small set of $\mathcal{U}_1$-small sets (namely $\mathcal{U}_1$).

For $i \geq j$ we could (this is ad hoc notation) define an (i, j)-small 1-category to be a 1-category C whose set of objects is $\mathcal{U}_i$-small and for which $\mathrm{Mor}_{\mathsf{C}}(X, Y)$ is $\mathcal{U}_j$-small for all objects X and Y of C. Let us use $\mathsf{Cat}^{i,j}$ as ad hoc notation for the 1-category of (i, j)-small 1-categories. What we usually consider as 1-categories are $(2, 1)$-small 1-categories, which then form the 1-category $\mathsf{Cat}^{2,1}$, which will however *not* be $(2, 1)$-small itself, though it is $(3, 2)$-small. For a more detailed discussion of Grothendieck universes and size issues in an ∞-categorical context, see [HTT, 1.2.15].

In this thesis we will very often use gadgets such as Cat or $\mathcal{C}\mathrm{at}_\infty$. To be completely rigorous we should thus always keep track of with respect to which universe the various objects we consider are small. In most of the thesis this would however cause significant notational bloat while being completely orthogonal to the rest of the content, so to make the exposition more accessible we will instead stay silent on size issues, while of course still taking care not to use inadmissible arguments. There will be one part of the thesis, Chapter 7, where a size issue is somewhat relevant for the argument, and there we will deal with this issue in an explicit manner.

In particular, we will not decorate $\mathcal{C}\mathrm{at}_\infty$ to keep track of sizes, and might e. g. define an ∞-category as a pullback in $\mathcal{C}\mathrm{at}_\infty$ of a diagram that involves the ∞-category $\mathcal{C}\mathrm{at}_\infty$. While in this notation it would then seem as though the two occurrences of $\mathcal{C}\mathrm{at}_\infty$ refer to the same gadget, a diligent adding of size decorations would distinguish them, and we will be careful not to make any arguments in which is not possible to do so consistently.

[16] We defined Cat as a $(2, 2)$-category, but to make our exposition here easier we only consider the underlying 1-category.

Chapter 3.

Bialgebras and modules over them

Let C be a symmetric monoidal category and A an associative algebra in C. A left module in C over A consists of an object X in C together with a morphism $A \otimes X \to X$ satisfying some properties. If A is commutative, then any left-A-module can naturally be made into a A, A-bimodule, so that we can use the relative tensor product over A to define a monoidal structure on the category of left-A-modules $\mathrm{LMod}_A(\mathsf{C})$.

Now let A be a associative, coassociative bialgebra. Then there is also a way to define a tensor product on $\mathrm{LMod}_A(\mathsf{C})$, and in such a way that the underlying object in C of the tensor product of two left-A-modules X and Y is just given by the tensor product of the two underlying objects. To do this, we need to define an action morphism $A \otimes (X \otimes Y) \to X \otimes Y$, which we do as the composition

$$A \otimes (X \otimes Y) \xrightarrow{\Delta \otimes \mathrm{id}_{X \otimes Y}} (A \otimes A) \otimes (X \otimes Y) \cong (A \otimes X) \otimes (A \otimes Y) \to X \otimes Y$$

where Δ is the comultiplication on A, the middle isomorphism uses associativity and symmetry of the tensor product to swap the two middle tensor factors, and the last morphism is the tensor product of the action morphisms for X and Y. One can then check, that this makes $X \otimes Y$ into a left-A-module.

It is not only possible to construct the monoidal category $\mathrm{LMod}_A(\mathsf{C})$ for individual bialgebras A – this construction enjoys functoriality in both A and C: If $f\colon A \to B$ is a morphism of bialgebras in C, then there is a monoidal functor

$$\mathrm{LMod}_B(\mathsf{C}) \to \mathrm{LMod}_A(\mathsf{C})$$

that preserves the underlying object but restricts the action along f. If A is a bialgebra in C and $F\colon \mathsf{C} \to \mathsf{D}$ is a symmetric monoidal functor, then F induces a monoidal functor

$$\mathrm{LMod}_A(\mathsf{C}) \to \mathrm{LMod}_{F(A)}(\mathsf{D})$$

that sends a left-A-module with underlying object X to a left-$F(A)$-module with underlying object $F(X)$.

To encode this functoriality we can define a category $\mathsf{BiAlgOp}$ as follows. Objects are pairs (C, A) with C a symmetric monoidal category and A an associative and coassociative bialgebra in C. Morphisms from (C, A) to (D, B) are pairs (F, f), where $F\colon \mathsf{C} \to \mathsf{D}$ is a symmetric monoidal functor, and $f\colon B \to F(A)$ is a morphism of bialgebras in D. We can then upgrade the construction of $\mathrm{LMod}_A(\mathsf{C})$ to a functor

$$\mathrm{LMod}\colon \mathsf{BiAlgOp} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathsf{Cat})$$

where $\mathrm{Mon}_{\mathsf{Assoc}}(\mathsf{Cat})$ is the category of monoidal categories.

The goal of this section is to implement this idea for ∞-categories rather than just ordinary categories. In this setting, we want to construct an ∞-category $\mathcal{B}i\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ whose objects can be described as pairs $(\mathcal{C}, A)$, where $\mathcal{C}$ is an $\mathbb{E}_2$-monoidal ∞-category and A an $\mathbb{E}_1, \mathbb{E}_1$-bialgebra in $\mathcal{C}$. We then want to upgrade LMod to a functor

$$\mathcal{B}i\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$$

that can be interpreted as functorially upgrading left module categories over $\mathbb{E}_1$ algebras to $\mathbb{E}_1$-monoidal ∞-categories in the way described above.

We now briefly describe our approach to constructing $\mathcal{B}i\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. Instead of trying to construct $\mathcal{B}i\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ directly, we will first construct an infinity category $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ that can be described as having as objects pairs $(\mathcal{C}, A)$ where $\mathcal{C}$ is a $\mathbb{E}_1$-monoidal infinity category and A is an $\mathbb{E}_1$-algebra in $\mathcal{C}$, and where a morphism from $(\mathcal{C}, A)$ to $(\mathcal{D}, B)$ is given by a pair (F, f) with $F \colon \mathcal{C} \to \mathcal{D}$ an $\mathbb{E}_1$-monoidal functor and $f \colon B \to F(A)$ a morphism in $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D})$. The ∞-category $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ will turn out to have products, with the product of $(\mathcal{C}, A)$ and $(\mathcal{D}, B)$ given by $(\mathcal{C} \times \mathcal{D}, (A, B))$. We can thus consider monoids in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. A monoid in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ roughly consists of an object $(\mathcal{C}, A)$ in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ together with a coherently associative multiplication morphism $(\mathcal{C}, A) \times (\mathcal{C}, A) \to (\mathcal{C}, A)$. Such a morphism corresponds to an $\mathbb{E}_1$-monoidal functor $F \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a morphism $f \colon A \to F(A, A)$ in $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$. By the Eckmann-Hilton argument, $F(A, A)$ is equivalent to $A \otimes A$, so that we can identify f with a morphism $A \to A \otimes A$, which we can interpret as being the comultiplication of a coalgebra structure on A. We will later show that $\mathrm{Mon}_{\mathbb{E}_1}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p})$ indeed implements the discussed idea of what $\mathcal{B}i\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ should be.

Finally, the functor $\mathrm{LMod} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ sending a pair $(\mathcal{C}, A)$ to $\mathrm{LMod}_A(\mathcal{C})$ is product-preserving, so that we obtain an induced functor $\mathcal{B}i\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{C}\mathrm{at}_\infty)$.

Our approach is heavily inspired by [HA, 4.8]. The goal in [HA, 4.8.3] is to functorially encode the fact that the ∞-category of left-A-modules[1] $\mathrm{LMod}_A(\mathcal{C})$ can be upgraded to an ∞-category that is right-tensored over $\mathcal{C}$. The functoriality encoded is however not the same as the one we discussed above: Lurie's construction maps a morphism $A \to B$ of algebras in $\mathcal{C}$ to the functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_B(\mathcal{C})$ that sends a left-A-module X to the left-B-module $B \otimes_A X$. The functor Lurie constructs preserves products as well [HA, 4.8.5.16] and so induces a functor on $\mathbb{E}_1$-monoids. However, due to the covariant functoriality in algebras, this induced functor describes the $\mathbb{E}_1$-monoidal structure induced on $\mathrm{LMod}_A(\mathcal{C})$ by an $\mathbb{E}_2$-algebra A using the relative tensor product over A (see the discussion at the start of this section). Because of this, we will mostly follow the ideas in [HA, 4.8.3 and 4.8.5], making the changes that are needed to make the construction contravariant in algebras.

During preparation of this text, the preprint [Rak20] appeared, in which existence of constructions similar to the ones we discuss below is also claimed in analogy to Lurie's construction, though without proof, see [Rak20, 2.2 and in particular 2.2.6].

[1] Lurie actually considers right modules, but to keep our exposition consistent we will discuss Lurie's results in the analogous form for left modules.

We now give a brief overview of the sections below. In Section 3.1 we will construct $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ as well as the functor $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$. We will also discuss how LMod interacts with presentability. For this we will construct a variant $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ whose objects can be interpreted as pairs $(\mathcal{C}, A)$ with $\mathcal{C}$ a presentable monoidal ∞-category and A an algebra in $\mathcal{C}$, and show that LMod lifts to a functor $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$.

In Section 3.2 we will show that LMod is product-preserving as a functor from $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ to $\mathcal{C}\mathrm{at}_\infty$ and hence induces a symmetric monoidal functor with respect to the respective cartesian symmetric monoidal structures. We will also construct an appropriate symmetric monoidal structure on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ and show that the functor $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ can be upgraded to a symmetric monoidal functor as well.

Bialgebras will be defined in Section 3.3, and in Section 3.4 we will then discuss how LMod induces functors $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ as well as the variant functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}) \to \mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)$, where $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)$ is the ∞-category of presentable $\mathcal{O}$-monoidal ∞-categories. We will furthermore make precise how we can interpret objects of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p})$ as pairs $(\mathcal{C}, A)$, where $\mathcal{C}$ is an $\mathcal{O} \otimes \mathsf{Assoc}$-monoidal ∞-category, and A is an $\mathsf{Assoc}, \mathcal{O}$-bialgebra in $\mathcal{C}$.

3.1. Modules over algebras

In this section we will construct a functor $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ implementing the idea described in the introduction to Chapter 3. To do so we first need to construct the ∞-category $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$, which is to have as objects pairs $(\mathcal{C}, A)$ with $\mathcal{C}$ a monoidal ∞-category and A an associative algebra in $\mathcal{C}$. We can thus interpret $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ as a sort of ∞-category of algebras not only in a single monoidal ∞-category, but a whole collection of them – in this case all of them. The notion that encapsulates the idea of a collection of monoidal ∞-categories is that of a *cocartesian family of monoidal ∞-categories*, which we will define in Section 3.1.1. The process of forming algebras in cocartesian families of monoidal ∞-categories is then defined and studied in Section 3.1.2, and everything is put together to construct $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ and LMod in Section 3.1.3.

3.1.1. Cocartesian families of monoidal ∞-categories

In this section we discuss the notion *cocartesian families of $\mathcal{O}$-monoidal ∞-categories* for ∞-operads $\mathcal{O}$. We start in Section 3.1.1.1 with the definition. In Section 3.1.1.2 we discuss an important example: The *universal* cocartesian family of $\mathcal{O}$-monoidal ∞-categories, which can be though of as the collection of *all* $\mathcal{O}$-monoidal ∞-categories. In particular, this will be the example that we will use to define $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ and LMod as discussed in the introduction to Chapter 3. We end the section with Section 3.1.1.3, in which we discuss the interaction between cocartesian families and products. This will be relevant later, when we want to argue that the functor to be defined $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ is compatible with products.

3.1.1.1. Definition

As we want to form ∞-categories like $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ in which objects are algebras not just in a single monoidal ∞-category, but in a whole collection of monoidal ∞-categories, we first need a definition that encapsulates the idea of combining a collection of monoidal ∞-categories into a single mathematical object.

If $\mathcal{O}$ is an ∞-operad, then by [HA, 2.4.2.4] a cocartesian fibration over $\mathcal{O}^{\otimes}$ is an $\mathcal{O}$-monoidal ∞-category if and only if the associated functor $\mathcal{O}^{\otimes} \to \mathfrak{Cat}_{\infty}$ is an $\mathcal{O}$-monoid. We can thus consider a functor

$$F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$$

for some ∞-category $\mathcal{C}$ as parametrizing a collection of $\mathcal{O}$-monoidal ∞-categories by $\mathcal{C}$. Composing with the inclusion of the full subcategory $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$ into $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty})$, we obtain a functor

$$F'\colon \mathcal{C} \to \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}\big)$$

of which we can take the adjoint $\breve{F'}\colon \mathcal{O}^{\otimes} \times \mathcal{C} \to \mathfrak{Cat}_{\infty}$. By passing to the cocartesian fibration classified by the functor $\breve{F'}$ we then obtain a cocartesian fibration $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$. This cocartesian fibration will have extra properties that correspond to F' factoring over $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$. This leads us to the following proposition and definition.

Proposition 3.1.1.1. *Let $\mathcal{C}$ be an ∞-category, $\mathcal{O}$ an ∞-operad, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian fibration. Then the following are equivalent.*

(1) The functor $F\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty})$ that corresponds to p under the equivalence

$$\mathrm{co}\mathfrak{CFib}\big(\mathcal{O}^{\otimes} \times \mathcal{C}\big) \xleftarrow{\mathrm{Gr}} \mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathcal{C}, \mathfrak{Cat}_{\infty}\big) \xleftarrow{\breve{(-)}} \mathrm{Fun}\Big(\mathcal{C}, \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}\big)\Big)$$

factors through $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$.

(2) For every object X of $\mathcal{C}$ the restriction $p_X\colon \mathcal{D}^{\otimes}_X \to \mathcal{O}^{\otimes}$ is a cocartesian fibration of ∞-operads[2]. ♡

Proof. Let $G := \mathrm{Gr}^{-1}(p)$, let F be as in (1), and let X be an object of $\mathcal{C}$. Naturality of the Grothendieck construction[3] (see [GHN17, A.32]) implies that the cocartesian fibration p_X is classified by the restriction of G to $\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \times \{X\}$. [HA, 2.4.2.4] implies that p_X is a cocartesian fibration of ∞-operads if and only if this restriction is an $\mathcal{O}$-monoid. Using naturality of $\breve{(-)}$ we can reformulate this as follows: The cocartesian fibration p_X is a cocartesian fibration of ∞-operads if and only if $F(X)$ is an $\mathcal{O}$-monoid. As $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$ is defined as the full subcategory of $\mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}\big)$ of $\mathcal{O}$-monoids, this finishes the proof. □

[2] See [HA, 2.1.2.13] for a definition

[3] Precomposing a functor into $\mathfrak{Cat}_{\infty}$ by some functor ι corresponds to taking the base change along ι of the corresponding cocartesian fibration.

Definition 3.1.1.2 ([HA, Definition 4.8.3.1])**.** Let $\mathcal{C}$ be an ∞-category and $\mathcal{O}$ an ∞-operad. A *cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories* is a cocartesian fibration $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ satisfying the conditions in Proposition 3.1.1.1.

We let $\mathrm{co}\mathfrak{C}\mathfrak{Fam}_{\mathcal{O}}(\mathcal{C})$ be the full subcategory of $\mathrm{co}\mathfrak{C}\mathfrak{Fib}(\mathcal{O}^{\otimes} \times \mathcal{C})$ spanned by cocartesian $\mathcal{C}$-families of $\mathcal{O}$-monoidal ∞-categories. ♢

Remark 3.1.1.3 ([HA, 4.8.3.3])**.** Let $\mathcal{C}$ be an ∞-category and $\mathcal{O}$ an ∞-operad. Let ι be the inclusion of $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$ into $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty})$.

Then the equivalences Gr and $\widetilde{(-)}$ as in Proposition 3.1.1.1 restrict as in the following commutative diagram where the right vertical functor is the inclusion, and such that all horizontal functors are equivalences.

$$\begin{tikzcd}
\mathrm{Fun}\Big(\mathcal{C}, \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}\big)\Big) \arrow[r, "\widetilde{(-)}", "\simeq"'] & \mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathcal{C}, \mathfrak{Cat}_{\infty}\big) \arrow[r, "\mathrm{Gr}", "\simeq"'] & \mathrm{co}\mathfrak{C}\mathfrak{Fib}\big(\mathcal{O}^{\otimes} \times \mathcal{C}\big) \\
\mathrm{Fun}\big(\mathcal{C}, \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})\big) \arrow[u, "\iota_*"] \arrow[rr, "\simeq"'] & & \mathrm{co}\mathfrak{C}\mathfrak{Fam}_{\mathcal{O}}(\mathcal{C}) \arrow[u]
\end{tikzcd}$$

Note that $\mathrm{Fun}\big(\mathcal{C}, \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})\big)$ is contravariantly functorial in $\mathcal{C}$ and[4] $\mathcal{O}$, so the construction of $\mathrm{co}\mathfrak{C}\mathfrak{Fam}_{\mathcal{O}}(\mathcal{C})$ must be as well. Using naturality of $\widetilde{(-)}$ and Gr (see [GHN17, A.32] and [Maz19b]) we can describe this functoriality explicitly as follows.

Let $G\colon \mathcal{C}' \to \mathcal{C}$ be a functor of ∞-categories, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$ a functor corresponding under the above equivalence to a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories p. Then the composite functor

$$\mathcal{C}' \xrightarrow{G} \mathcal{C} \xrightarrow{F} \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty}) \xrightarrow{\alpha^*} \mathrm{Mon}_{\mathcal{O}'}(\mathfrak{Cat}_{\infty})$$

corresponds under the above functor to the pullback p' of p along $\alpha \times G$, as in the following diagram.

$$\begin{tikzcd}
\mathcal{D}'^{\otimes} \arrow[r] \arrow[d, "p'"'] & \mathcal{D}^{\otimes} \arrow[d, "p"] \\
\mathcal{O}'^{\otimes} \times \mathcal{C}' \arrow[r, "\alpha \times G"'] & \mathcal{O}^{\otimes} \times \mathcal{C}
\end{tikzcd} \tag{3.1}$$

In particular, the pullback of a cocartesian family of monoidal ∞-categories along a functor of the form $\alpha \times G$ is again a cocartesian family of monoidal ∞-categories. ♢

3.1.1.2. The universal family

In this section we discuss the *universal* cocartesian family of $\mathcal{O}$-monoidal ∞-categories, from which we can obtain every other cocartesian family of $\mathcal{O}$-monoidal ∞-categories by pulling back. This will also be the main example that we will apply later constructions to.

[4] If $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ is a morphism of ∞-operads, then it follows directly from the definition that the functor $\alpha^*\colon \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}) \to \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathfrak{Cat}_{\infty})$ restricts to a functor on monoids.

Definition 3.1.1.4 ([HA, 4.8.3.3])**.** Let $\mathcal{O}$ be an ∞-operad.
We define

$$p^{\mathcal{O}}\colon \widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes} \to \mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$$

to be the cocartesian $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$-family of $\mathcal{O}$-monoidal ∞-categories that under the equivalence in Remark 3.1.1.3 corresponds to the functor $\mathrm{id}_{\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)}$. ♢

Remark 3.1.1.5 ([HA, 4.8.3.3])**.** Let $\mathcal{O}$ be an ∞-operad, let $\mathcal{C}$ be an ∞-category, and let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. Let $F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$ be the functor corresponding to p under the equivalence in Remark 3.1.1.3. Then F factors as $F \simeq \mathrm{id}_{\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)} \circ F$, so by Remark 3.1.1.3 we can conclude that there is a pullback diagram as follows.

$$\begin{tikzcd}
\mathcal{D}^{\otimes} \arrow[r] \arrow[d, "p"'] & \widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes} \arrow[d, "p^{\mathcal{O}}"] \\
\mathcal{O}^{\otimes} \times \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{O}^{\otimes}} \times F"'] & \mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)
\end{tikzcd}$$

♢

3.1.1.3. Compatibility of fibers with products

The property described in the following proposition and definition regarding a cocartesian family of monoidal ∞-categories' interaction with products will be needed later.

Proposition 3.1.1.6. *Let $\mathcal{C}$ be an ∞-category, $\mathcal{O}$ an ∞-operad, $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories, and $F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$ the functor corresponding to p as in Proposition 3.1.1.1. Assume that $\mathcal{C}$ admits all products. Then the following are equivalent.*

(1) F preserves products.

(2) For every object O in $\mathcal{O}^{\otimes}$ the cocartesian fibration

$$p_O\colon \mathcal{D}^{\otimes}_O \coloneqq \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes} \times \mathcal{C}} \left(\{O\} \times \mathcal{C}\right) \xrightarrow{\mathrm{pr}_2} \{O\} \times \mathcal{C} \xrightarrow{\simeq} \mathcal{C}$$

has fibers compatible with products in the sense of Definition C.2.0.1.

(3) For every object O in $\mathcal{O}$ the cocartesian fibration

$$p_O\colon \mathcal{D}^{\otimes}_O \coloneqq \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes} \times \mathcal{C}} \left(\{O\} \times \mathcal{C}\right) \xrightarrow{\mathrm{pr}_2} \{O\} \times \mathcal{C} \xrightarrow{\simeq} \mathcal{C}$$

has fibers compatible with products in the sense of Definition C.2.0.1. ♡

Proof. Proof that (1) implies (2): Let ι denote the inclusion of the full subcategory $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$ into $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_\infty)$, which preserves products by Proposition F.2.0.1. Let O be an object in $\mathcal{O}^{\otimes}$. As limits in in functor categories are computed pointwise by [HTT, 5.1.2.3], the functor $\mathrm{ev}_O\colon \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_\infty) \to \mathfrak{Cat}_\infty$ preserves products as well, and thus the composite $\mathrm{ev}_O \circ \iota \circ F$ preserves products. By using naturality of the Grothendieck construction and $\widetilde{(-)}$ we can conclude that the cocartesian fibration p_O is classified by $\mathrm{ev}_O \circ \iota \circ F$, and hence p_O having fibers compatible with products follows from Remark C.2.0.2.

Proof that (2) implies (3): Clear.

Proof that (3) implies (1): Using notation from above, that p_O has fibers compatible with products for every object O in $\mathcal{O}$ implies by Remark C.2.0.2 that $\mathrm{ev}_O \circ \iota \circ F$ preserves products for every O in $\mathcal{O}$. Combining that products in functor categories are detected pointwise and that the composition

$$\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty) \xrightarrow{\iota} \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_\infty) \to \mathrm{Fun}(\mathcal{O}, \mathfrak{Cat}_\infty)$$

detects products as well by Proposition F.2.0.1 we can conclude that F preserves products. □

Definition 3.1.1.7. Let $\mathcal{C}$ be an ∞-category, $\mathcal{O}$ an ∞-operad, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.

We say that p has the *product-fiber-property* if $\mathcal{C}$ admits all products and satisfies the equivalent conditions in Proposition 3.1.1.6. ♢

The product-fiber-property is preserved by taking the pullback as in Remark 3.1.1.3 of a cocartesian family of monoidal ∞-categories, as long as the functor G preserves products, as we record in the following proposition.

Proposition 3.1.1.8. *In the situation of diagram (3.1) of Remark 3.1.1.3, if p has the product-fiber-property, $\mathcal{C}'$ admits all products, and G preserves products, then p' has the product-fiber property as well.* ♡

Proof. Follows immediately from the definition in terms of condition (2) in Proposition 3.1.1.6 using that (induced maps on) fibers of p' can be identified with (induced maps on) fibers of p by Proposition C.1.1.1. □

Finally, we end this section by noting that the universal cocartesian family of $\mathcal{O}$-monoidal ∞-categories satisfies the product-fiber-property.

Proposition 3.1.1.9. *Let $\mathcal{O}$ be an ∞-operad. Then $p^{\mathcal{O}}$ has the product-fiber-property.* ♡

Proof. Follows immediately from the description Proposition 3.1.1.6 (1), as the functor corresponding to $p^{\mathcal{O}}$ is by definition the identity functor, which preserves products. □

3.1.2. Algebras in cocartesian families

Given a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$, Lurie defines[5] in [HA, Notation 4.8.3.11] an ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ whose objects can be

[5]While the definition is only written down for $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = \mathsf{Assoc}^{\otimes}$ and $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = \mathsf{LM}^{\otimes}$, we present a straightforward generalization.

described as being pairs (X, A) where X is an object of $\mathcal{C}$ (and hence determines a $\mathcal{O}$-monoidal ∞-category $\mathcal{D}_X^{\otimes}$) and A is an object of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_X^{\otimes})$. We will discuss a definition of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ in Section 3.1.2.1. Lurie's definition is not quite written down like the definition we present however, so we next show in Section 3.1.2.2 that the two definitions agree. We will then spend some time discussing various functorialities exhibited by this construction. Fixing $\mathcal{O}' \to \mathcal{O}$, we can vary the cocartesian family of $\mathcal{O}$-monoidal ∞-categories $\mathcal{D}$ by taking pullbacks along functors $\mathcal{C}' \to \mathcal{C}$. In fact, we showed in Remark 3.1.1.5 that every family of $\mathcal{O}$-monoidal ∞-categories can be obtained like this from the universal family of $\mathcal{O}$-monoidal ∞-categories $p^{\mathcal{O}}$. The main message of Section 3.1.2.3 is that we also do not obtain anything new when taking algebras: $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ can be obtained as a pullback of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathfrak{Cat}_\infty))$. More useful is functoriality when varying $\mathcal{O}'$, which we discuss in Section 3.1.2.4, and functoriality that is encoded by the family itself, which will be discussed in Section 3.1.2.5, and in which we we will show that there is a cocartesian fibration $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$. We end this section with Section 3.1.2.6, in which we discuss the interaction of this cocartesian fibration with products in $\mathcal{C}$.

3.1.2.1. Definition

Definition 3.1.2.1. Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. Then we define $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ together with $\mathrm{pr}_{\mathcal{C}}$ and $\mathrm{pr}_{\mathrm{Fun}}$ as the following pullback of ∞-categories.

$$\begin{tikzcd}
\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}"] \arrow[d, "\mathrm{pr}_{\mathcal{C}}"'] & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \arrow[d, "p_*"] \\
\mathcal{C} \arrow[r, "\widehat{(\alpha \times \mathrm{id}_{\mathcal{C}})}"'] & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C}\big)
\end{tikzcd}$$

$\diamondsuit$

Proposition 3.1.2.2. *Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.*

Let A be an object of $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$. Then the following are equivalent.

(1) *The functor $\mathrm{pr}_{\mathrm{Fun}}(A)\colon \mathcal{O}'^{\otimes} \to \mathcal{D}^{\otimes}$ sends inert morphisms to p-cocartesian ones.*

(2) *The functor $A'\colon \mathcal{O}'^{\otimes} \to \mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(A)}$ over $\mathcal{O}^{\otimes}$ which corresponds to A under the equivalence*

$$\begin{aligned}
&\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_{\mathrm{pr}_{\mathcal{C}}(A)} \\
&\simeq \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C})} \mathcal{C} \times_{\mathcal{C}} \{\mathrm{pr}_{\mathcal{C}}(A)\} \\
&\simeq \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C})} \{\mathrm{pr}_{\mathcal{C}}(A)\} \\
&\simeq \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C})} \mathrm{Fun}\Big(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \{\mathrm{pr}_{\mathcal{C}}(A)\}\Big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes})} \{\alpha\}
\end{aligned}$$

$$\simeq \operatorname{Fun}\Big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}\times\mathcal{C}} \big(\mathcal{O}^{\otimes} \times \{\mathrm{pr}_{\mathcal{C}}(A)\}\big)\Big) \times_{\operatorname{Fun}(\mathcal{O}'^{\otimes},\mathcal{O}^{\otimes})} \{\alpha\}$$
$$\simeq \operatorname{Fun}_{\mathcal{O}^{\otimes}}\Big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(A)}\Big)$$

lies in the full subcategory $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_{\mathrm{pr}_{\mathcal{C}}(A)})$ *of* $\mathcal{O}'$*-algebras in the* $\mathcal{O}$*-monoidal* ∞*-category* $\mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(A)}$. ♡

Proof. Let A' be as in (2). The following commutative diagram summarizes the situation, where $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ is the canonical morphism of ∞-operads, ι is inclusion of $\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \times \{\mathrm{pr}_{\mathcal{C}}(A)\}$, and the square in the middle right is a pullback square.

$$\begin{tikzcd}
\mathcal{O}'^{\otimes} \arrow[r, "A'"] \arrow[dr, "\alpha"'] \arrow[rr, bend left, "\mathrm{pr}_{\mathrm{Fun}}(A)"] & \mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(A)} \arrow[r] \arrow[d, "p_{\mathrm{pr}_{\mathcal{C}}(A)}"] & \mathcal{D}^{\otimes} \arrow[d, "p"] \\
& \mathcal{O}^{\otimes} \arrow[r, "\iota"] \arrow[d, "p_{\mathcal{O}}"] & \mathcal{O}^{\otimes} \times \mathcal{C} \\
& \mathsf{Fin}_* &
\end{tikzcd}$$

By definition [HA, 2.1.2.7] A' lies in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_{\mathrm{pr}_{\mathcal{C}}(A)})$ if and only if A' carries inert morphisms to $p_{\mathcal{O}} \circ p_{\mathrm{pr}_{\mathcal{C}}(A)}$-cocartesian ones. As α is a morphism of ∞-operads, it sends inert morphisms to $p_{\mathcal{O}}$-cocartesian ones, so it follows from [HTT, 2.4.1.3 (3)] that A' lies in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_{\mathrm{pr}_{\mathcal{C}}(A)})$ if and only if it carries inert morphisms to $p_{\mathrm{pr}_{\mathcal{C}}(A)}$-cocartesian ones, which by Proposition C.1.1.1 is the case if and only if $\mathrm{pr}_{\mathrm{Fun}}(A)$ carries inert morphisms to p-cocartesian ones. □

Definition 3.1.2.3. Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.

Then we define $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ to be the full subcategory of $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ spanned by those objects satisfying the equivalent conditions in Proposition 3.1.2.2. ♢

Remark 3.1.2.4. In the situation of Definition 3.1.2.3 it follows immediately from Proposition 3.1.2.2 (2) that for any object C of $\mathcal{C}$ the fiber $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_C$ is naturally equivalent to $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_C)$. ♢

3.1.2.2. Comparison with Lurie's definition

Lurie's definition is not phrased quite like Definition 3.1.2.3, so we show below in Proposition 3.1.2.7 that Lurie's definition is equivalent to the one we used.

Definition 3.1.2.5 ([HA, 4.8.3.11]). Let $\mathtt{C}$ be a quasicategory representing an ∞-category $\mathcal{C}$, let $\mathtt{O}$ be a quasicategorical ∞-operad representing an ∞-operad $\mathcal{O}$, let $\mathtt{p}\colon \mathtt{D}^{\otimes} \to \mathtt{O}^{\otimes} \times \mathtt{C}$ be an inner fibration representing a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal

∞-categories, and let $\mathtt{a}\colon \mathtt{O}'^{\otimes} \to \mathtt{O}^{\otimes}$ be a morphism of quasicategorical ∞-operads representing a morphism of ∞-operads $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$.

We define an unnamed property for functors of quasicategories $\mathtt{q}\colon \mathtt{A} \to \mathtt{C}$, which is to hold if there is a natural bijection

$$\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{C}}}(-, \mathtt{q}) \cong \mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^{\otimes}\times\mathtt{C}}}(\mathtt{a} \times -, \mathtt{p})$$

of functors $\mathsf{sSet}_{/\mathtt{C}} \to \mathsf{Set}$. $\diamondsuit$

Remark 3.1.2.6. In the situation of Definition 3.1.2.5, the Yoneda lemma implies that if a $\mathtt{q}$ with the property exists, then it is unique up to canonical isomorphism as an object of $\mathsf{sSet}_{/\mathtt{C}}$. $\diamondsuit$

Proposition 3.1.2.7. *Let $\mathtt{C}$ be a quasicategory representing an ∞-category $\mathcal{C}$, let $\mathtt{O}$ be a quasicategorical ∞-operad representing an ∞-operad $\mathcal{O}$, let $\mathtt{p}\colon \mathtt{D}^{\otimes} \to \mathtt{O}^{\otimes} \times \mathtt{C}$ be an inner fibration of quasicategories representing a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories, and let $\mathtt{a}\colon \mathtt{O}'^{\otimes} \to \mathtt{O}^{\otimes}$ be a morphism of quasicategorical ∞-operads representing a morphism of ∞-operads $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$.*

Define $\mathtt{E}$ and $\mathtt{q}$ via the following categorical pullback square in sSet.

$$\begin{tikzcd}
\mathtt{E} \arrow[r] \arrow[d, "\mathtt{q}"'] & \mathrm{Fun}\left(\mathtt{O}'^{\otimes}, \mathtt{D}^{\otimes}\right) \arrow[d, "\mathtt{p}_*"] \\
\mathtt{C} \arrow[r, "\widehat{(\mathtt{a}\times\mathrm{id}_{\mathtt{C}})}"'] & \mathrm{Fun}\left(\mathtt{O}'^{\otimes}, \mathtt{O}^{\otimes} \times \mathtt{C}\right)
\end{tikzcd}$$

Then the following hold.

(1) *The map $\mathtt{q}$ satisfies the property defined in Definition 3.1.2.5.*

(2) *In particular, if $\mathtt{a} = \mathrm{id}_{\mathsf{Assoc}}$ and $\mathtt{a} = \mathrm{id}_{\mathsf{LM}}$, then $\mathtt{q}$ can be identified with the functors of quasicategories $\widetilde{\mathrm{Alg}}(\mathtt{D}) \to \mathtt{C}$ and $\widetilde{\mathrm{LMod}}(\mathtt{D}) \to \mathtt{C}$ as defined in [HA, 4.8.3.11], respectively.*

(3) *The pullback is a homotopy pullback with respect to the Joyal model structure.*

(4) *The pullback square represents the pullback square of ∞-categories in Definition 3.1.2.1 that defines $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$.*

(5) *If $\mathtt{a} = \mathrm{id}_{\mathsf{Assoc}}$, then $\mathrm{Alg}(\mathtt{D}) \to \mathtt{C}$ as defined in [HA, 4.8.3.11] represents $\mathrm{Alg}_{/\mathsf{Assoc}}(\mathcal{D})$ as defined in Definition 3.1.2.3. If $\mathtt{a} = \mathrm{id}_{\mathsf{LM}}$, then $\mathrm{LMod}(\mathtt{D}) \to \mathtt{C}$ as defined in [HA, 4.8.3.11] represents $\mathrm{Alg}_{/\mathsf{LM}}(\mathcal{D})$ as defined in Definition 3.1.2.3.* $\heartsuit$

Proof. Proof of (1): Let $\mathtt{s}\colon \mathtt{K} \to \mathtt{C}$ be a map of simplicial sets. Then there is a sequence of bijections natural in $\mathtt{s}$ (as an object of $\mathsf{sSet}_{/\mathtt{C}}$) as follows.

$$\begin{aligned}
&\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{C}}}(\mathtt{s}, \mathtt{q}) \\
\cong\ &\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K}, \mathtt{E}) \times_{\mathrm{Mor}_{\mathsf{sSet}}(\mathtt{K},\mathtt{C})} \{\mathtt{s}\}
\end{aligned}$$

$$\begin{aligned}
&\cong \mathrm{Mor}_{\mathrm{sSet}}\Big(\mathtt{K}, \mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{D}^{\otimes}\big)\Big) \times_{\mathrm{Mor}_{\mathrm{sSet}}\left(\mathtt{K}, \mathrm{Fun}\left(\mathtt{O}'^{\otimes}, \mathtt{O}^{\otimes}\times \mathtt{C}\right)\right)} \mathrm{Mor}_{\mathrm{sSet}}(\mathtt{K}, \mathtt{C}) \times_{\mathrm{Mor}_{\mathrm{sSet}}(\mathtt{K},\mathtt{C})} \{\mathtt{s}\}\\
&\cong \mathrm{Mor}_{\mathrm{sSet}}\Big(\mathtt{K}, \mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{D}^{\otimes}\big)\Big) \times_{\mathrm{Mor}_{\mathrm{sSet}}\left(\mathtt{K}, \mathrm{Fun}\left(\mathtt{O}'^{\otimes}, \mathtt{O}^{\otimes}\times \mathtt{C}\right)\right)} \Big\{\widehat{(\mathtt{a}\times \mathtt{s})}\Big\}\\
&\cong \mathrm{Mor}_{\mathrm{sSet}}\big(\mathtt{O}'^{\otimes} \times \mathtt{K}, \mathtt{D}^{\otimes}\big) \times_{\mathrm{Mor}_{\mathrm{sSet}}\left(\mathtt{O}'^{\otimes}\times\mathtt{K}, \mathtt{O}^{\otimes}\times \mathtt{C}\right)} \big\{(\mathtt{a}\times \mathtt{s})\big\}\\
&\cong \mathrm{Mor}_{\mathrm{sSet}_{/\mathtt{O}^{\otimes}\times\mathtt{C}}}(\mathtt{a}\times\mathtt{s}, \mathtt{p})
\end{aligned}$$

Proof of (2): Follows directly from the definition.

Proof of (3): By assumption, $\mathtt{p}$ is a cocartesian fibration in the sense of [HTT, 2.4.2.1], so that by [HTT, 3.1.2.1] the functor of quasicategories $\mathtt{p}_*$ is again a cocartesian fibration in the sense of [HTT, 2.4.2.1]. That the pullback square is a homotopy pullback square in the Joyal model structure follows now by applying [HTT, 3.3.1.4] (to the opposite diagram).

Proof of (4): Follows directly from (3).

Proof of (5): Immediate by unwrapping the definitions of the respective full subcategories. □

3.1.2.3. Functoriality when varying families

We next consider functoriality of ∞-categories of algebras of cocartesian families of monoidal ∞-categories when we vary the cocartesian family. We first discuss functoriality in the ∞-operad factor, for which the following proposition can be considered a generalization of Proposition E.2.0.2.

Remark 3.1.2.8. Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ and $\beta\colon \mathcal{O}''^{\otimes} \to \mathcal{O}'^{\otimes}$ morphisms of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. Assume that the following diagram is a pullback diagram in $\mathcal{C}\mathrm{at}_\infty$.

$$\begin{tikzcd}
\mathcal{D}'^{\otimes} \arrow[r, "G^{\otimes}"] \arrow[d, "p'"'] & \mathcal{D}^{\otimes} \arrow[d, "p"] \\
\mathcal{O}'^{\otimes} \times \mathcal{C} \arrow[r, "\alpha\times\mathrm{id}_{\mathcal{C}}"'] & \mathcal{O}^{\otimes} \times \mathcal{C}
\end{tikzcd} \tag{3.2}$$

By Remark 3.1.1.3 is p' is a cocartesian $\mathcal{C}$-family of $\mathcal{O}'$-monoidal ∞-categories.

Consider the following commutative diagram, where the square on the left is the pullback square of Definition 3.1.2.1 and the square on the right is induced by the pullback square (3.2) by applying $\mathrm{Fun}(\mathcal{O}''^{\otimes}, -)$ and hence also a pullback square.

$$\begin{tikzcd}
\widetilde{\mathrm{Alg}}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{D}') \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}"] \arrow[d, "\mathrm{pr}_{\mathcal{C}}"'] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{D}'^{\otimes}\big) \arrow[r, "G^{\otimes}_*"] \arrow[d, "p'_*"] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{D}^{\otimes}\big) \arrow[d, "p_*"] \\
\mathcal{C} \arrow[r, "\widehat{(\beta\times\mathrm{id}_{\mathcal{C}})}"'] \arrow[rr, bend right, "\widehat{(-)}\left((\alpha\circ\beta)\times\mathrm{id}_{\mathcal{C}}\right)"'] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{O}'^{\otimes}\times\mathcal{C}\big) \arrow[r, "(\alpha\times\mathrm{id}_{\mathcal{C}})_*"'] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{O}^{\otimes}\times\mathcal{C}\big)
\end{tikzcd}$$

By the pasting lemma for pullbacks [HTT, 4.4.2.1], the outer square is a pullback as well, so that we obtain a canonical identification as follows.

$$\widetilde{\mathrm{Alg}}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{D}') \simeq \widetilde{\mathrm{Alg}}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})$$

Furthermore, with the description of p'-cocartesian morphisms from Proposition C.1.1.1 it follows directly from Definition 3.1.2.3 in the form of Proposition 3.1.2.2 (1) that this equivalence restricts to an equivalence of ∞-categories of algebras as follows.

$$\mathrm{Alg}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{D}') \simeq \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}) \qquad \diamondsuit$$

We now turn to functoriality in the ∞-category that parametrizes our cocartesian family of monoidal ∞-categories.

Construction 3.1.2.9. Let $F\colon \mathcal{C}' \to \mathcal{C}$ be a functor of ∞-categories, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. Assume that the following diagram is a pullback diagram in $\mathcal{C}\mathrm{at}_{\infty}$.

$$\begin{tikzcd}
\mathcal{D}'^{\otimes} \arrow[r, "G^{\otimes}"] \arrow[d, "p'"'] & \mathcal{D}^{\otimes} \arrow[d, "p"] \\
\mathcal{O}^{\otimes} \times \mathcal{C}' \arrow[r, "\mathrm{id}_{\mathcal{O}^{\otimes}} \times F"'] & \mathcal{O}^{\otimes} \times \mathcal{C}
\end{tikzcd} \tag{$*$}$$

Remark 3.1.1.3 implies that p' is a cocartesian $\mathcal{C}'$-family of $\mathcal{O}$-monoidal ∞-categories.

Then there is a commutative cube as follows

$$\begin{tikzcd}
& \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[rr, "\mathrm{pr}_{\mathrm{Fun}}"] \arrow[dl, dashed, "F_*"'] \arrow[dd, "\mathrm{pr}_{\mathcal{C}'}"] & & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}'^{\otimes}\big) \arrow[dl, "G_*^{\otimes}"] \arrow[dd, "p'_*"] \\
\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[rr, "\mathrm{pr}_{\mathrm{Fun}}"', crossing over] \arrow[dd, "\mathrm{pr}_{\mathcal{C}}"'] & & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) & \\
& \mathcal{C}' \arrow[rr, "\widehat{(\alpha \times \mathrm{id}_{\mathcal{C}'})}"'] \arrow[dl, "F"'] & & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C}'\big) \arrow[dl, "\big(\mathrm{id}_{\mathcal{O}^{\otimes}} \times F\big)_*"] \\
\mathcal{C} \arrow[rr, "\widehat{(\alpha \times \mathrm{id}_{\mathcal{C}})}"'] & & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C}\big) \arrow[from=uu, crossing over, "p_*"] &
\end{tikzcd}$$

where the front and back squares are the defining pullback squares for $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ and $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}')$, and the dashed functor F_* is the induced one.

The square on the right is obtained by applying $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$ to the pullback square ($*$) and is thus a pullback square as well. As the front square is also a pullback square, it follows that their composition, which we can identify with the composition of the left and back square, is a pullback square as well. As the back square is also a pullback square, if finally follows using the pasting law for pullbacks [HTT, 4.4.2.1] that the square

$$\begin{tikzcd} \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"] \arrow[d, "\mathrm{pr}_{\mathcal{C}'}"'] & \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d, "\mathrm{pr}_{\mathcal{C}}"] \\ \mathcal{C}' \arrow[r, "F"'] & \mathcal{C} \end{tikzcd} \tag{3.3}$$

is a pullback square. ◊

Proposition 3.1.2.10. *Assume we are in the situation of Construction 3.1.2.9. Then the pullback square (3.3) restricts to a pullback square in* $\mathfrak{Cat}_\infty$ *as follows.*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"] \arrow[d, "\mathrm{pr}_{\mathcal{C}'}"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d, "\mathrm{pr}_{\mathcal{C}}"] \\ \mathcal{C}' \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

♡

Proof. It suffices to show that the dashed functor in the following commutative diagram (where the vertical functors are the canonical inclusions) exists and that the square is a pullback square in $\mathfrak{Cat}_\infty$.

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, dashed, "F_*"] \arrow[d] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d] \\ \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"'] & \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \end{tikzcd}$$

Let A be an object in $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}')$. Then by Definition 3.1.2.3 and Proposition 3.1.2.2 (1), A is in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}')$ if and only if $\mathrm{pr}_{\mathrm{Fun}}(A)\colon \mathcal{O}'^{\otimes} \to \mathcal{D}'^{\otimes}$ sends inert morphisms to p'-cocartesian morphisms, which by Proposition C.1.1.1 is the case if and only if $G^{\otimes} \circ \mathrm{pr}_{\mathrm{Fun}}(A) \simeq \mathrm{pr}_{\mathrm{Fun}}(F_*(A))$ sends inert morphisms to p-cocartesian morphisms. Thus A is in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}')$ if and only if $F_*(A)$ is in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$. This shows that in the following commutative diagram of ∞-categories, where the small square is defined as a pullback square, the dashed functor making the outer square commute exists, and that the induced dotted functor is essentially surjective (this uses the description of $\mathcal{E}$ given in Proposition B.5.2.1).

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[drr, dashed, bend left, "F_*"] \arrow[dr, dotted] \arrow[ddr, bend right] & & \\ & \mathcal{E} \arrow[r] \arrow[d, "q"] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d] \\ & \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"'] & \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \end{tikzcd}$$

By Proposition B.5.2.1 the functor q is fully faithful, so the dotted functor is fully faithful as well and hence an equivalence. It follows that the outer square is a pullback square because the inner square is. $\square$

Remark 3.1.2.11. Let $\mathcal{C}$ be an ∞-category, let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads, let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories, and let $F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$ be the functor corresponding to p under the equivalence in Remark 3.1.1.3. By Remark 3.1.1.5 there is a pullback diagram as follows.

$$\begin{tikzcd}
\mathcal{D}^{\otimes} \arrow[r] \arrow[d, "p"'] & \widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[d, "p^{\mathcal{O}}"] \\
\mathcal{O}^{\otimes} \times \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{O}^{\otimes}} \times F"'] & \mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd}$$

Applying Proposition 3.1.2.10 we obtain a pullback diagram of algebra ∞-categories.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[r, "F_*"] \arrow[d, "\mathrm{pr}_{\mathcal{C}}"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}\Big(\widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})\Big) \arrow[d, "\mathrm{pr}_{\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})}"] \\
\mathcal{C} \arrow[r, "F"'] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd}$$

$\diamondsuit$

3.1.2.4. Functoriality when varying the operad

In this section we discuss functoriality of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ when varying $\mathcal{O}'$.

Construction 3.1.2.12. Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ and $\beta\colon \mathcal{O}''^{\otimes} \to \mathcal{O}'^{\otimes}$ morphisms of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.

Then the commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widehat{(\alpha \times \mathrm{id}_{\mathcal{C}})}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}"'] & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C}\big) \arrow[d, "\beta^*"] & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \arrow[l, "p_*"'] \arrow[d, "\beta^*"] \\
\mathcal{C} \arrow[r, "\widehat{(-)}\big((\alpha\circ\beta)\times \mathrm{id}_{\mathcal{C}}\big)"'] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C}\big) & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{D}^{\otimes}\big) \arrow[l, "p_*"]
\end{tikzcd}$$

induces a functor on pullbacks as follows.

$$\beta^*\colon \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \widetilde{\mathrm{Alg}}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}) \qquad \diamondsuit$$

Remark 3.1.2.13. In the situation of Construction 3.1.2.12, if $\gamma\colon \mathcal{O}'''^{\otimes} \to \mathcal{O}''^{\otimes}$ is another morphism of ∞-operads, then it is clear from the definition that the composition $\gamma^* \circ \beta^*$ is equivalent to $(\beta \circ \gamma)^*$. $\diamondsuit$

Proposition 3.1.2.14. *In the situation of Construction 3.1.2.12, the functor*

$$\beta^*\colon \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \widetilde{\mathrm{Alg}}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})$$

restricts to a functor on algebras as follows.

$$\beta^*\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}) \qquad \heartsuit$$

Proof. What we have to show is by Definition 3.1.2.3 in the form of Proposition 3.1.2.2 (1) that the functor

$$\beta^*\colon \operatorname{Fun}\bigl(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\bigr) \to \operatorname{Fun}\bigl(\mathcal{O}''^{\otimes}, \mathcal{D}^{\otimes}\bigr)$$

sends functors that send inert morphisms to p-cocartesian morphisms to functors with the same property. But this follows immediately from the fact that, as it is a morphism of ∞-operads, β preserves inert morphisms. □

Remark 3.1.2.15. Assume we are in the situation of Construction 3.1.2.12, and let C be an object of $\mathcal{C}$. The functor $\beta^*\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})$ is a functor over $\mathcal{C}$ and thus induces a functor as follows.

$$\beta^*_C\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_C \to \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})_C$$

It follows directly from the definition together with Remark 3.1.2.4 that this functor can be identified with the following functor induced by β.

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_C) \to \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}_C) \qquad \diamondsuit$$

Remark 3.1.2.16. Assume we are in the situation of Construction 3.1.2.9 and we are given another morphism of ∞-operads $\beta\colon \mathcal{O}''^{\otimes} \to \mathcal{O}'^{\otimes}$. Then it follows from the respective constructions that β^* and F_* commute in the sense that there is a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"] \arrow[d, "\beta^*"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d, "\beta^*"] \\
\mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}') \arrow[r, "F_*"'] & \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})
\end{tikzcd}$$

$\diamondsuit$

3.1.2.5. Functoriality encoded by families

Let $\mathcal{C}$ be an ∞-category, let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads, and let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. In Section 3.1.2.1 we constructed a functor of ∞-categories

$$\mathrm{pr}_{\mathcal{C}}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$$

and identified the fiber of $\mathrm{pr}_{\mathcal{C}}$ over an object C in $\mathcal{C}$ with $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}_C)$, see Remark 3.1.2.4.

As was explained at the start of Section 3.1.1, we can interpret p as a collection of $\mathcal{O}$-monoidal ∞-categories that is indexed by $\mathcal{C}$. We will show below that $\mathrm{pr}_{\mathcal{C}}$ is again a cocartesian fibration, and thus classified by a functor $\mathcal{C} \to \mathfrak{Cat}_\infty$, which we can then interpret as encoding the functoriality of the construction $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(-)$ that produces the ∞-category of $\mathcal{O}'$-algebras out of an $\mathcal{O}$-monoidal ∞-category.

Proposition 3.1.2.17 ([HA, 4.8.3.13])**.** *Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. Then the following hold.*

(1) $\mathrm{pr}_{\mathcal{C}}\colon \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$ *is a cocartesian fibration and a morphism f is $\mathrm{pr}_{\mathcal{C}}$-cocartesian if and only if $\mathrm{pr}_{\mathrm{Fun}}(f)(X)$ is p-cocartesian for every object X of $\mathcal{O}'^{\otimes}$ (see Definition 3.1.2.1 for this notation).*

(2) $\mathrm{pr}_{\mathcal{C}}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$ *is a cocartesian fibration and a morphism f in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ is $\mathrm{pr}_{\mathcal{C}}$-cocartesian if and only if $\mathrm{pr}_{\mathrm{Fun}}(f)(X)$ is p-cocartesian for every object X of $\mathcal{O}'^{\otimes}$.*

(3) A morphism f in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ is $\mathrm{pr}_{\mathcal{C}}$-cocartesian if and only if $\mathrm{pr}_{\mathrm{Fun}}(f)(X)$ is p-cocartesian for every object X of $\mathcal{O}'$. ♡

Proof. Proof of (1): This is a combination of [HTT, 3.1.2.1] (applying $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$ to a cocartesian fibration) with Proposition C.1.1.1 (taking a pullback of a cocartesian fibration).

Proof of (2): It suffices to verify the assumption needed to apply the dual of Proposition C.1.2.1 to the restriction of $\mathrm{pr}_{\mathcal{C}}\colon \widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$ to $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$. So let A be an object in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ and $f\colon A \to B$ a $\mathrm{pr}_{\mathcal{C}}$-cocartesian morphism in $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$. We have to show that B also lies in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$. By definition this means that we need to show that $\mathrm{pr}_{\mathrm{Fun}}(B)\colon \mathcal{O}'^{\otimes} \to \mathcal{D}^{\otimes}$ sends inert morphisms to p-cocartesian morphisms. So let $\varphi\colon X \to Y$ be an inert morphism in $\mathcal{O}'^{\otimes}$. We obtain a commutative diagram in $\mathcal{D}^{\otimes}$ as follows.

$$\begin{tikzcd}
\mathrm{pr}_{\mathrm{Fun}}(A)(X) \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}(f)(X)"] \arrow[d, "\mathrm{pr}_{\mathrm{Fun}}(A)(\varphi)"'] & \mathrm{pr}_{\mathrm{Fun}}(B)(X) \arrow[d, "\mathrm{pr}_{\mathrm{Fun}}(B)(\varphi)"] \\
\mathrm{pr}_{\mathrm{Fun}}(A)(Y) \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}(f)(Y)"'] & \mathrm{pr}_{\mathrm{Fun}}(B)(Y)
\end{tikzcd} \tag{$*$}$$

As f is $\mathrm{pr}_{\mathcal{C}}$-cocartesian, the top and bottom horizontal morphisms are p-cocartesian by (1). As A lies in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$, the left vertical morphism is p-cocartesian as well. That the right vertical morphism is also p-cocartesian now follows from [HTT, 2.4.1.7].

Proof of (3): Let $f\colon A \to B$ be a morphism in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ and assume that for every object Y in $\mathcal{O}'$ the morphism $\mathrm{pr}_{\mathrm{Fun}}(f)(Y)$ in $\mathcal{D}^{\otimes}$ is p-cocartesian. Let $X \simeq X_1 \oplus \cdots \oplus X_n$ be an object in $\mathcal{O}'^{\otimes}_{\langle n \rangle}$, and denote by $\gamma_i\colon X \to X_i$ the inert morphism in $\mathcal{O}'^{\otimes}$ lying over ρ^i. We have to show that then also $\mathrm{pr}_{\mathrm{Fun}}(f)(X)$ is p-cocartesian.

Let $1 \leq i \leq n$. Consider the following diagram in $\mathcal{D}^{\otimes}$

$$\begin{tikzcd}
 & D \arrow[r, "\Psi"] \arrow[d, dashed, "\Theta"] & D_i \arrow[d, dashed, "\Theta_i"] \\
\mathrm{pr}_{\mathrm{Fun}}(A)(X) \arrow[ur, bend left, "\Phi"] \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}(f)(X)"] \arrow[dr, "\mathrm{pr}_{\mathrm{Fun}}(A)(\gamma_i)"'] & \mathrm{pr}_{\mathrm{Fun}}(B)(X) \arrow[r, "\mathrm{pr}_{\mathrm{Fun}}(B)(\gamma_i)"] & \mathrm{pr}_{\mathrm{Fun}}(B)(X_i) \\
 & \mathrm{pr}_{\mathrm{Fun}}(A)(X_i) \arrow[ur, "\mathrm{pr}_{\mathrm{Fun}}(f)(X_i)"'] &
\end{tikzcd}$$

lying over the following commutative diagram in $\mathcal{O}^{\otimes} \times \mathcal{C}$

$$\begin{tikzcd}
 & \big(\alpha(X), \mathrm{pr}_{\mathcal{C}}(B)\big) \arrow[r, "{(\alpha(\gamma_i),\mathrm{id})}"] \arrow[d, "\mathrm{id}"] & \big(\alpha(X_i), \mathrm{pr}_{\mathcal{C}}(B)\big) \arrow[d, "\mathrm{id}"] \\
\big(\alpha(X), \mathrm{pr}_{\mathcal{C}}(A)\big) \arrow[ur, bend left, "{(\mathrm{id},\mathrm{pr}_{\mathcal{C}}(f))}"] \arrow[r, "{(\mathrm{id},\mathrm{pr}_{\mathcal{C}}(f))}"] \arrow[dr, "{(\alpha(\gamma_i),\mathrm{id})}"'] & \big(\alpha(X), \mathrm{pr}_{\mathcal{C}}(B)\big) \arrow[r, "{(\alpha(\gamma_i),\mathrm{id})}"] & \big(\alpha(X_i), \mathrm{pr}_{\mathcal{C}}(B)\big) \\
 & \big(\alpha(X_i), \mathrm{pr}_{\mathcal{C}}(A)\big) \arrow[ur, "{(\mathrm{id},\mathrm{pr}_{\mathcal{C}}(f))}"'] &
\end{tikzcd}$$

and such that Φ and Ψ are p-cocartesian lifts of $(\mathrm{id}_{\alpha(X)}, \mathrm{pr}_{\mathcal{C}}(f))$ and $(\alpha(\gamma_i), \mathrm{id}_{\mathrm{pr}_{\mathcal{C}}(B)})$, respectively, and such that the dashed morphisms are the canonical fillers. As γ_i is inert and A in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ the morphism $\mathrm{pr}_{\mathrm{Fun}}(A)(\gamma_i)$ is p-cocartesian, and the morphism $\mathrm{pr}_{\mathrm{Fun}}(f)(X_i)$ is p-cocartesian by assumption, as X_i is an object of $\mathcal{O}'$. Considering the outer diagram it thus follows from [HTT, 2.4.1.7] that Θ_i is p-cocartesian, and thus by [HTT, 2.4.1.5] an equivalence.

We now want to conclude that also Θ must be an equivalence. For this, note that as $p_{\mathrm{pr}_{\mathcal{C}}(B)}$ is a cocartesian fibration of ∞-operads, the following functor induced by the inert morphisms $\alpha(\gamma_i)$ on fibers

$$\left(\mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(B)}\right)_{\alpha(X)} \xrightarrow{\prod_{1\leq i\leq n}\left(\alpha(\gamma_i)\right)_!} \prod_{1\leq i\leq n} \left(\mathcal{D}^{\otimes}_{\mathrm{pr}_{\mathcal{C}}(B)}\right)_{\alpha(X_i)}$$

is an equivalence of ∞-categories. By Proposition C.1.1.1 we can identify this functor with the following functor.

$$\mathcal{D}^{\otimes}_{\left(\alpha(X),\mathrm{pr}_{\mathcal{C}}(B)\right)} \xrightarrow{\prod_{1\leq i\leq n}\left(\alpha(\gamma_i),\mathrm{id}\right)_!} \prod_{1\leq i\leq n} \mathcal{D}^{\otimes}_{\left(\alpha(X_i),\mathrm{pr}_{\mathcal{C}}(B)\right)}$$

The morphism Θ lies in $\mathcal{D}^{\otimes}_{(\alpha(X),\mathrm{pr}_{\mathcal{C}}(B))}$, and by definition $\Theta_i \simeq (\alpha(\gamma_i), \mathrm{id})_!(\Theta)$. As we previously concluded that Θ_i is an equivalence for every $1 \leq i \leq n$ we can thus conclude that Θ is an equivalence, and hence p-cocartesian by [HTT, 2.4.1.5]. As Φ is

p-cocartesian by definition we can then use [HTT, 2.4.1.7] to deduce that $\mathrm{pr}_{\mathrm{Fun}}(f)(X)$ is also p-cocartesian. $\square$

Remark 3.1.2.18. Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories. A morphism $g\colon C \to C'$ in $\mathcal{C}$ induces on fibers of p an $\mathcal{O}$-monoidal functor[6] $G\colon \mathcal{D}^{\otimes}_{C} \to \mathcal{D}^{\otimes}_{C'}$. Combining the identifications $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}^{\otimes}_{X}) \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}^{\otimes})_X$ from Remark 3.1.2.4 (for $X = C$ as well as $X = C'$) with Proposition 3.1.2.17, in particular description Proposition 3.1.2.17 (2), we can conclude that we can identify the functor induced by g on fibers of $\mathrm{pr}_{\mathcal{C}}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$ with the functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(G)$. $\diamondsuit$

Definition 3.1.2.19. Let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads. Then we define

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}\colon \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty}) \to \mathfrak{Cat}_{\infty}$$

the be the functor that the cocartesian fibration

$$\mathrm{pr}_{\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}\Big(\widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})\Big) \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$$

is classified by. $\diamondsuit$

Remark 3.1.2.20. Let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads. The functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}$ sends by Remark 3.1.2.4 an $\mathcal{O}$-monoidal ∞-category $\mathcal{C}$ to the ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$, so that we can interpret $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}$ as encoding the full functoriality of the construction of ∞-categories of $\mathcal{O}'$-algebras in $\mathcal{O}$-monoidal ∞-categories.

Now let $\mathcal{C}$ be an ∞-category, $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories, and $F\colon \mathcal{C} \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$ the functor corresponding to p under the equivalence in Remark 3.1.1.3. Then it follows from Remark 3.1.2.11 and naturality of the Grothendieck construction (see [GHN17, A.32] and [Maz19b]) that the cocartesian fibration

$$\mathrm{pr}_{\mathcal{C}}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$$

is classified by the following composition.

$$\mathcal{C} \xrightarrow{F} \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty}) \xrightarrow{\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}} \mathfrak{Cat}_{\infty} \qquad \diamondsuit$$

Proposition 3.1.2.21. *Let $\mathcal{C}$ be an ∞-category, $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ and $\beta\colon \mathcal{O}''^{\otimes} \to \mathcal{O}'^{\otimes}$ a morphism of ∞-operads, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.*

Then the functor

$$\beta^{*}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D})$$

constructed in Construction 3.1.2.12 and Proposition 3.1.2.14, which by construction is a functor over $\mathcal{C}$, is a functor of cocartesian fibrations, i. e. sends $\mathrm{pr}_{\mathcal{C}}$-cocartesian morphisms to $\mathrm{pr}_{\mathcal{C}}$-cocartesian morphisms. $\heartsuit$

[6] This is clear from Proposition 3.1.1.1.

Proof. By definition of β^* there is a commutative diagram as follows.

$$\begin{tikzcd}
\operatorname{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \arrow[rr, "\beta^*"] & & \operatorname{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{D}^{\otimes}\big) \\
\operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[u, "\mathrm{pr}_{\mathrm{Fun}}"] \arrow[rr, "\beta^*"] \arrow[dr, "\mathrm{pr}_{\mathcal{C}}"'] & & \operatorname{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{D}) \arrow[u, "\mathrm{pr}_{\mathrm{Fun}}"'] \arrow[dl, "\mathrm{pr}_{\mathcal{C}}"] \\
& \mathcal{C} &
\end{tikzcd}$$

As the top horizontal functor clearly preserves pointwise p-cocartesian morphisms, criterion Proposition 3.1.2.17 (2) implies that the middle horizontal functor preserves $\mathrm{pr}_{\mathcal{C}}$-cocartesian morphisms. □

3.1.2.6. Algebras in cocartesian families and products

Let $\mathcal{C}$ and $\mathcal{C}'$ be two $\mathcal{O}$-monoidal ∞-categories. Then there is an induced $\mathcal{O}$-monoidal structure on $\mathcal{C} \times \mathcal{C}'$, and it is reasonable to expect that there should be an equivalence as follows.

$$\operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}\big(\mathcal{C} \times \mathcal{C}'\big) \simeq \operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \times \operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}\big(\mathcal{C}'\big)$$

The next proposition shows that this is indeed the case.

Proposition 3.1.2.22. *Let $\mathcal{C}$ be an ∞-category, let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads, and let $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ be a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories that has the product-fiber property from Definition 3.1.1.7. Then the cocartesian fibrations*

$$\mathrm{pr}_{\mathcal{C}}\colon \widetilde{\operatorname{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$$

and

$$\mathrm{pr}_{\mathcal{C}}\colon \operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathcal{C}$$

have fibers compatible with products in the sense of Definition C.2.0.1. ♡

Proof. Let I be a set, let X_i be an object in $\mathcal{C}$ for every element i of I, and let $X := \prod_{i \in I} X_i$. We have to prove that the two functors induced on fibers

$$\widetilde{\operatorname{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_X \xrightarrow{\prod_{i\in I} \mathrm{pr}_{i!}} \prod_{i \in I} \widetilde{\operatorname{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_{X_i} \tag{$*$}$$

and

$$\operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_X \xrightarrow{\prod_{i\in I} \mathrm{pr}_{i!}} \prod_{i \in I} \operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})_{X_i} \tag{$**$}$$

are equivalences.

We start by considering the following commutative triangle induced by the projections $\mathrm{pr}_i\colon X \to X_i$.

$$\begin{tikzcd} \mathcal{D}^{\otimes}_{X} \arrow[rr, "\prod_{i\in I}(pr_{i!})"] \arrow[dr, "p_X"'] & & \mathcal{O}^{\otimes} \times_{\prod_{i\in I}\mathcal{O}^{\otimes}} \prod_{i\in I} \mathcal{D}^{\otimes}_{X_i} \arrow[dl, "\mathrm{pr}_1"] \\ & \mathcal{O}^{\otimes} & \end{tikzcd} \tag{$***$}$$

Both p_X and pr_1 in this diagram are cocartesian fibrations, and the horizontal functor sends p_X-cocartesian morphisms to pr_1-cocartesian morphisms. The statement for p_X and pr_1 follows from p being a cocartesian fibration and applying Proposition C.1.1.1, and in the case of the functor on the right also using that products of cocartesian fibrations are again cocartesian fibrations by [HTT, 3.1.2.1]. This also gives a description of the respective cocartesian morphisms, and with that the statement about the horizontal functor boils down to a statement about p-cocartesian morphisms that holds by [HTT, 2.4.1.7]. By assumption p has the product-fiber property, which precisely means that the horizontal functor in the above diagram is a fiberwise (over $\mathcal{O}^{\otimes}$) equivalence. It now follows from [HTT, 2.4.4.4] that the horizontal functor is itself an equivalence.

We now consider the first of the two functors, $(*)$. Unpacking the definition (Definition 3.1.2.1) of $\widetilde{\mathrm{Alg}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ as a pullback and using Proposition C.1.1.1 we can identify the functor $(*)$ with

$$\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big)_{\mathrm{id}_{\mathcal{O}^{\otimes}} \times \mathrm{const}_X} \xrightarrow{\prod_{i\in I}(\mathrm{id}\times \mathrm{pr}_i)_!} \prod \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big)_{\mathrm{id}_{\mathcal{O}^{\otimes}} \times \mathrm{const}_{X_i}}$$

where the fibers are taken over $\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times \mathcal{C}\big)$, and $\mathrm{id} \times \mathrm{pr}_i$ is the natural transformation of functors $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ from $\mathrm{id}_{\mathcal{O}^{\otimes}} \times \mathrm{const}_X$ to $\mathrm{id}_{\mathcal{O}^{\otimes}} \times \mathrm{const}_{X_i}$ that is given by the identity in the $\mathcal{O}^{\otimes}$ factor and pr_i in the $\mathcal{C}$ factor.

Using that $\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, -\big)$ commutes with pullbacks together with [HTT, 3.1.2.1] we can further identify functor $(*)$ with the functor

$$\mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}_X\big) \xrightarrow{\prod_{i\in I}(\mathrm{pr}_{i!})_*} \prod \mathrm{Fun}_{\mathcal{O}^{\otimes}}\Big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}_{X_i}\Big)$$

and in another step, using composability of pullback diagrams, that $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$ commutes with products, Proposition C.1.1.1 and [HTT, 3.1.2.1] some more, we can further identify this with the following functor.

$$\mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}_X\big) \xrightarrow{\left(\prod_{i\in I}(\mathrm{pr}_{i!})\right)_*} \prod \mathrm{Fun}_{\mathcal{O}^{\otimes}}\left(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes} \times_{\prod_{i\in I}\mathcal{O}^{\otimes}} \prod_{i\in I} \mathcal{D}^{\otimes}_{X_i}\right)$$

This exactly $\mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, -\big)$ applied to the horizontal functor in $(***)$, so this is an equivalence.

Using Proposition 3.1.2.2 one can see that under these equivalences the functor $(**)$ (which is a restriction of $(*)$ on domain and codomain to full subcategories) corresponds to the application of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(-)$ to the horizontal functor in $(***)$, so this functor is also an equivalence. $\square$

Corollary 3.1.2.23. *Let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphism of ∞-operads. Then the cocartesian fibration*

$$\mathrm{pr}_{\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}\left(\widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathfrak{Cat}_\infty)\right) \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$$

has fibers compatible with products in the sense of Definition C.2.0.1. ♡

Proof. Combine Proposition 3.1.2.22 and Proposition 3.1.1.9. □

3.1.3. Functorial construction of ∞-categories of left modules

In Definition 3.1.2.19 we constructed a functor

$$\mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}\colon \mathrm{Mon}_{\mathsf{Assoc}}(\mathfrak{Cat}_\infty) \to \mathfrak{Cat}_\infty$$

that sends an (Assoc-)monoidal[7] ∞-category $\mathcal{C}$ to the ∞-category $\mathrm{Alg}(\mathcal{C}) \coloneqq \mathrm{Alg}_{/\mathsf{Assoc}}(\mathcal{C})$ of Assoc-algebras in $\mathcal{C}$, and can thus be interpreted as encoding the functoriality of the construction $\mathcal{C} \mapsto \mathrm{Alg}(\mathcal{C})$, see Remark 3.1.2.20.

In this section we will similarly construct a functor LMod that can be interpreted as encoding the functoriality of the construction that maps a pair $(\mathcal{C}, A)$ with $\mathcal{C}$ a monoidal ∞-category and A an associative algebra in $\mathcal{C}$, to the ∞-category $\mathrm{LMod}_A(\mathcal{C})$ of left A modules[8]. For functoriality in $\mathcal{C}$, a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ should induce a functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_{FA}(\mathcal{D})$ when A is an associative algebra in $\mathcal{C}$. For functoriality in A, we should be able to form the base change along a morphisms of algebras $f\colon A \to B$ in $\mathcal{C}$, i. e. restricting the action, providing us with a functor $\mathrm{LMod}_B(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C})$.

We already have constructed an ∞-category whose objects can be described as pairs $(\mathcal{C}, A)$ with $\mathcal{C}$ a monoidal ∞-category and A an associative algebra in $\mathcal{C}$, namely

$$\mathcal{A}\mathrm{lg} \coloneqq \mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}\left(\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathfrak{Cat}_\infty)\right)$$

see Remark 3.1.2.20. By taking algebras in $\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathfrak{Cat}_\infty)$ with respect to two other ∞-operads, we will obtain a commutative diagram as follows.

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od} \arrow[r] \arrow[dr] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj} \arrow[r] \arrow[d] & \mathcal{A}\mathrm{lg} \arrow[dl] \\
& \mathrm{Mon}_{\mathsf{Assoc}}(\mathfrak{Cat}_\infty) &
\end{tikzcd} \tag{3.4}$$

Objects in $\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od}$ can be described as tuples $(\mathcal{C}, A, M)$, with $\mathcal{C}$ a monoidal ∞-category, A an associative algebra in $\mathcal{C}$, and M a left module in $\mathcal{C}$ over A. Objects in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj}$ can be described as tuples $(\mathcal{C}, A, X)$, with $\mathcal{C}$ and A as before, but X just an object of $\mathcal{C}$. The functors in diagram (3.4) are the obvious forgetful functors.

[7] We follow e. g. [HA, 4.1.1.10] and call Assoc-monoidal ∞-categories just *monoidal ∞-categories*.
[8] See [HA, 4.2] for this “pointwise” construction of ∞-categories of left modules.

However, $\mathcal{A}\mathrm{lg}$ is not quite the ∞-category needed to encode the functoriality of LMod that we alluded to at the start of this sub-subsection: A morphism from $(\mathcal{C}, A) \to (\mathcal{D}, B)$ consists of a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ and a morphism $F(A) \to B$ of algebras in $\mathcal{D}$. So for our sought-after functoriality of LMod we would like the algebra-part of those morphisms to go in the opposite direction. Luckily, the horizontal functors in diagram (3.4) are morphisms of cocartesian fibrations over $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$, so we can apply the fiberwise $-^{\mathrm{op}}$-construction to fix this. We obtain a commuting triangle

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p} \arrow[rr] \arrow[dr] & & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p} \arrow[dl] \\ & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} & \end{tikzcd} \tag{3.5}$$

that turns out to be a morphism of cocartesian fibrations over $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. Now $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ is the category we are looking for, but the fiber of the cocartesian fibration

$$\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$$

over $(\mathcal{C}, A)$ is $\mathrm{LMod}_A(\mathcal{C})^{\mathrm{op}}$. By passing to the opposite category fiberwise, and converting the morphism of cocartesian fibrations to a natural transformation of functors to $\mathcal{C}\mathrm{at}_\infty$, we obtain a natural transformation that evaluated at $(\mathcal{C}, A)$ is given by the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$.

Let us now give a brief overview of the subsections below. We will start in Section 3.1.3.1 with reviewing the relevant ∞-operads as well as some morphisms between them that we will need. In Section 3.1.3.2 we will then carry out the construction of LMod as a functor from $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ to $\mathcal{C}\mathrm{at}_\infty$ as outlined above. If $\mathcal{C}$ is a presentable monoidal ∞-category and A is an algebra in $\mathcal{C}$, then $\mathrm{LMod}_A(\mathcal{C})$ is also presentable by [HA, 4.2.3.7 (1)]. In Section 3.1.3.3 we will define a variant $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ whose objects can be thought of as as pairs $(\mathcal{C}, A)$ where $\mathcal{C}$ is a presentable monoidal ∞-category and A is an algebra in $\mathcal{C}$, and show that LMod lifts to a functor from $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ to $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

3.1.3.1. Review of the relevant operads

Diagram (3.4) is constructed by taking algebras in $\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ with respect to different ∞-operads, so we begin by discussing the relevant ∞-operads in this section.

Lurie defines in [HA, 4.2.1][9] an ∞-operad LM, which encodes the structure of a left module over an associative algebra: If $\mathcal{C}$ is a symmetric monoidal ∞-category, then we can interpret an LM-algebra in $\mathcal{C}$ as a pair (A, M), where A is an associative algebra in $\mathcal{C}$ and M is a left module over A. Indeed, if $\mathcal{C}$ is a 1-category, then this description holds literally, with the usual classical notions of associative algebras and left modules

[9]Note that our conventions are such that what we denote by $\mathsf{LM}^{\otimes}$ is what Lurie writes as $\mathbf{LM}^{\otimes}$ or $\mathcal{LM}^{\otimes}$ (as we do not notationally distinguish between 1-categories as objects of Cat and $\mathcal{C}\mathrm{at}_\infty$). We also use LM to both denote to $\mathsf{LM}^{\otimes}_{\langle 1 \rangle}$ as well as a shorthand to talk about the ∞-operad $\mathsf{LM}^{\otimes} \to \mathsf{Fin}_*$, which should not be confused with with the different type of object that Lurie denotes by $\mathbf{LM}$ (see [HA, 4.2.1.1]).

over them, see [HA, 4.2.1.4]. The underlying ∞-category of LM is a discrete 1-category with two objects, which we denote by $\mathfrak{a}$ and $\mathfrak{m}$ as in [HA, 4.2.1.1]. In the interpretation of an LM-algebra in $\mathcal{C}$ as a pair (A, M) as before, the underlying object of A is given by evaluation at $\mathfrak{a}$ and the underlying object of M is given by evaluation at $\mathfrak{m}$.

We next fix notation for some morphisms of ∞-operads defined in [HA, 4.2.1] that we will need.

Definition 3.1.3.1. We let

$$\iota_{\mathsf{Assoc}} \colon \mathsf{Assoc}^{\otimes} \to \mathsf{LM}^{\otimes}$$

be the morphism of ∞-operads defined in [HA, 4.2.1.10] and

$$\nu_{\mathsf{Assoc}} \colon \mathsf{LM}^{\otimes} \to \mathsf{Assoc}^{\otimes}$$

the morphism of ∞-operads defined in [HA, 4.2.1.9]. ◊

Continuing with our discussion from before, these to morphisms of ∞-operads can be interpreted as follows: ι_{Assoc} induces a functor $\mathrm{Alg}_{\mathsf{LM}}(\mathcal{C}) \to \mathrm{Alg}_{\mathsf{Assoc}}(\mathcal{C})$ that can be interpreted as mapping (A, M) to A (see [HA, 4.2.1.3]), and ν_{Assoc} induces a functor $\mathrm{Alg}_{\mathsf{Assoc}}(\mathcal{C}) \to \mathrm{Alg}_{\mathsf{LM}}(\mathcal{C})$ that can be interpreted as mapping A to (A, A), with the second A in the pair being A considered as a left module over itself (see [HA, 4.2.1.5]).

We will also need to make use of the trivial ∞-operad Triv defined in [HA, 2.1.1.20], over which algebras are nothing more than objects of the underlying ∞-category. Specifically, the underlying ∞-category of Triv is discrete with a unique object, and for any ∞-operad $\mathcal{O}^{\otimes}$, the functor $\mathrm{Alg}_{\mathsf{Triv}}(\mathcal{O}) \to \mathcal{O}$ induced by evaluation at this object is an equivalence, see [HA, 2.1.3.5].

We can now define an additional morphism of ∞-categories that we will need.

Definition 3.1.3.2. We let

$$\iota_{\mathsf{Triv}} \colon \mathsf{Triv}^{\otimes} \to \mathsf{LM}^{\otimes}$$

be the morphism of ∞-operads that under the equivalence

$$\mathrm{Alg}_{\mathsf{Triv}}(\mathsf{LM}) \xrightarrow{\simeq} \mathsf{LM}^{\otimes}_{\langle 1 \rangle} = \{\mathfrak{a}, \mathfrak{m}\}$$

corresponds to the element $\mathfrak{m}$. ◊

The previous discussion implies that we can interpret the functor induced by ι_{Triv}

$$\mathrm{Alg}_{\mathsf{LM}}(\mathcal{C}) \to \mathrm{Alg}_{\mathsf{Triv}}(\mathcal{C})$$

as mapping (A, M) to the underlying object of M.

3.1.3.2. Construction of LMod

We write $\mathcal{O}^{\otimes} \boxplus \mathcal{O}'^{\otimes}$ for the coproduct of ∞-operads as discussed in [HA, 2.2.3]. We are now ready to construct diagram (3.4): The sequence of morphisms of ∞-operads

$$\begin{tikzcd} \mathsf{Assoc}^{\otimes} \arrow[r, "\iota_1"] \arrow[rrr, bend right, "\mathrm{id}_{\mathsf{Assoc}^{\otimes}}"'] & \mathsf{Assoc}^{\otimes} \boxplus \mathsf{Triv}^{\otimes} \arrow[r, "\iota_{\mathsf{Assoc}} \boxplus \iota_{\mathsf{Triv}}"] & \mathsf{LM}^{\otimes} \arrow[r, "\nu_{\mathsf{Assoc}}"] & \mathsf{Assoc}^{\otimes} \end{tikzcd} \tag{3.6}$$

induces as in Construction 3.1.2.12 and Proposition 3.1.2.14 on algebras in the universal family of Assoc-monoidal ∞-categories p^{Assoc} (see Definition 3.1.1.4) a commutative diagram as follows, where we shorten $\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$ to $\widetilde{\mathrm{Mon}}$.

$$\begin{tikzcd} \mathrm{Alg}_{\mathsf{LM}/\mathsf{Assoc}}\left(\widetilde{\mathrm{Mon}}\right) \arrow[r] \arrow[dr] & \mathrm{Alg}_{\mathsf{Assoc}\boxplus\mathsf{Triv}/\mathsf{Assoc}}\left(\widetilde{\mathrm{Mon}}\right) \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}\left(\widetilde{\mathrm{Mon}}\right) \arrow[dl] \\ & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty}) & \end{tikzcd}$$

The functors to $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$ are the respective functors called $\mathrm{pr}_{\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})}$ in Definition 3.1.2.1, which are cocartesian fibrations by Proposition 3.1.2.17. The above diagram precisely implements the description of diagram (3.4) given in the introduction to Section 3.1.3, as we will see below in Remark 3.1.3.4. This justifies making the following definition.

Definition 3.1.3.3. We define

$$\begin{aligned} \mathcal{A}\mathrm{lg} &:= \mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}\left(\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})\right) \\ \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj} &:= \mathrm{Alg}_{\mathsf{Assoc}\boxplus\mathsf{Triv}/\mathsf{Assoc}}\left(\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})\right) \\ \mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od} &:= \mathrm{Alg}_{\mathsf{LM}/\mathsf{Assoc}}\left(\widetilde{\mathrm{Mon}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})\right) \end{aligned}$$

and denote the respective functors[10] $\mathrm{pr}_{\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})}$ to $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$ by $q_{\mathcal{A}\mathrm{lg}}$, $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj}}$, and $q_{\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od}}$, respectively. Furthermore, we denote the functors induced by the morphisms of ∞-operads in (3.6)

$$\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj} \qquad \text{and} \qquad \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj} \to \mathcal{A}\mathrm{lg}$$

by $U^{\mathcal{L}\mathcal{M}\mathrm{od}}_{\mathcal{O}\mathrm{bj}}$ and $U^{\mathcal{O}\mathrm{bj}}$, respectively. $\diamondsuit$

Remark 3.1.3.4. We can summarize our previous discussions as follows.

Let $G\colon \mathcal{C} \to \mathcal{C}'$ be a monoidal functor of monoidal ∞-categories that we also consider as a morphism of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$. By definition, we can identify the monoidal functor

[10] See Definition 3.1.2.1.

induced by G on fibers of the universal family of Assoc-monoidal ∞-categories p^{Assoc} (see Definition 3.1.1.4) with $G^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{C}'^{\otimes}$ itself.

As $U^{\mathfrak{LMod}}_{\mathfrak{Obj}}$ and $U^{\mathfrak{Obj}}$ are morphisms of cocartesian fibrations over $\mathrm{Mon}_{\mathsf{Assoc}}(\mathfrak{Cat}_\infty)$ by Proposition 3.1.2.21, we obtain an induced commutative diagram as follows.

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathfrak{LMod}_{\mathcal{C}} \arrow[r, "\left(U^{\mathfrak{LMod}}_{\mathfrak{Obj}}\right)_{\mathcal{C}}"] \arrow[d, "G_!"'] & \mathcal{A}\mathrm{lg}\mathfrak{Obj}_{\mathcal{C}} \arrow[r, "\left(U^{\mathfrak{Obj}}\right)_{\mathcal{C}}"] \arrow[d, "G_!"'] & \mathcal{A}\mathrm{lg}_{\mathcal{C}} \arrow[d, "G_!"'] \\
\mathcal{A}\mathrm{lg}\mathfrak{LMod}_{\mathcal{C}'} \arrow[r, "\left(U^{\mathfrak{LMod}}_{\mathfrak{Obj}}\right)_{\mathcal{C}'}"'] & \mathcal{A}\mathrm{lg}\mathfrak{Obj}_{\mathcal{C}'} \arrow[r, "\left(U^{\mathfrak{Obj}}\right)_{\mathcal{C}'}"'] & \mathcal{A}\mathrm{lg}_{\mathcal{C}'}
\end{tikzcd}$$

Using Remark 3.1.2.4, Remark 3.1.2.15, and Remark 3.1.2.18 we can identify this diagram with the following commutative diagram induced by G and the morphisms of ∞-operads in (3.6).

$$\begin{tikzcd}
\mathrm{Alg}_{\mathsf{LM}/\mathsf{Assoc}}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{Alg}_{\mathsf{LM}/\mathsf{Assoc}}(G)"'] & \mathrm{Alg}_{\mathsf{Assoc}\boxplus\mathsf{Triv}/\mathsf{Assoc}}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{Alg}_{\mathsf{Assoc}\boxplus\mathsf{Triv}/\mathsf{Assoc}}(G)"'] & \mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}(G)"'] \\
\mathrm{Alg}_{\mathsf{LM}/\mathsf{Assoc}}(\mathcal{C}') \arrow[r] & \mathrm{Alg}_{\mathsf{Assoc}\boxplus\mathsf{Triv}/\mathsf{Assoc}}(\mathcal{C}') \arrow[r] & \mathrm{Alg}_{\mathsf{Assoc}/\mathsf{Assoc}}(\mathcal{C}')
\end{tikzcd}$$

The ∞-category of algebras over a coproduct of ∞-operads can be identified with the product of the ∞-categories of algebras by [HA, 2.2.3.6][11], and the ∞-category of algebras over Triv is by [HA, 2.1.3.6] equivalent to the underlying ∞-category. Considering also the definition of LMod [HA, 4.2.1.16] we can thus identify the above diagram with the following diagram

$$\begin{tikzcd}
\mathrm{LMod}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{LMod}(G)"'] & \mathrm{Alg}(\mathcal{C}) \times \mathcal{C} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{Alg}(G)\times G"'] & \mathrm{Alg}(\mathcal{C}) \arrow[d, "\mathrm{Alg}(G)"'] \\
\mathrm{LMod}(\mathcal{C}') \arrow[r] & \mathrm{Alg}(\mathcal{C}') \times \mathcal{C}' \arrow[r, "\mathrm{pr}_1"'] & \mathrm{Alg}(\mathcal{C}')
\end{tikzcd}$$

where the left horizontal functors are on the first factor the forgetful functors ι^*_{Assoc} from left modules to algebras from [HA, 4.2.1.13] that send a pair (A, M) with A an associative algebra and M a left module over it to A, and on the second factor the forgetful functors $\mathrm{ev}_{\mathfrak{m}}$ that send a pair (A, M) to M considered as just an object of $\mathcal{C}$ or $\mathcal{C}'$. $\diamondsuit$

Next we fix the variance of morphisms in the fibers.

Definition 3.1.3.5. By applying the functor

$$\begin{aligned}
&\mathrm{co}\mathfrak{CFib}\big(\mathrm{Mon}_{\mathsf{Assoc}}(\mathfrak{Cat}_\infty)\big) \\
&\to \mathrm{Fun}\big(\mathrm{Mon}_{\mathsf{Assoc}}(\mathfrak{Cat}_\infty), \mathfrak{Cat}_\infty\big) \\
&\xrightarrow{(-^{\mathrm{op}})_*} \mathrm{Fun}\big(\mathrm{Mon}_{\mathsf{Assoc}}(\mathfrak{Cat}_\infty), \mathfrak{Cat}_\infty\big) \\
&\to \mathrm{co}\mathfrak{CFib}\big(\mathrm{Mon}_{\mathsf{Assoc}}(\mathfrak{Cat}_\infty)\big)
\end{aligned}$$

[11] To apply this in our situation, combine this with Proposition E.2.0.3 and the fact that pullbacks commute with products.

to the morphisms $U^{\mathcal{LMod}}_{\mathcal{O}\mathrm{bj}}$ and $U^{\mathcal{O}\mathrm{bj}}$ of cocartesian fibrations over $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ we obtain morphisms of cocartesian fibrations $V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ as depicted in the following diagram.

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LMod}\mathcal{O}\mathrm{p} \arrow[r, "V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}"] \arrow[dr, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LMod}\mathcal{O}\mathrm{p}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p} \arrow[r, "V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}"] \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}"] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[dl, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
 & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) &
\end{tikzcd}$$

We define $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LMod}\mathcal{O}\mathrm{p}$, $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}$, $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$, $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LMod}\mathcal{O}\mathrm{p}}$, $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$, and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ as indicated in the diagram. We furthermore define $V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}$ to be the composition $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}} \circ V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$. ♢

Proposition 3.1.3.6. *$V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ from Definition 3.1.3.5 are cocartesian fibrations and $V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ is a morphism of cocartesian fibrations over $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$.*

Furthermore, a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LMod}\mathcal{O}\mathrm{p}$ is $V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}$-cocartesian precisely if it is the composition of a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LMod}\mathcal{O}\mathrm{p}}$-cocartesian morphism with a $(V^{\mathcal{LMod}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian morphism for $\mathcal{C}$ a monoidal ∞-category. The analogous statement holds for $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian morphisms. ♡

Proof. By [GHN15, 9.6][12,13], to show that $V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ are cocartesian fibrations, it suffices to show the following.

(1) $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LMod}\mathcal{O}\mathrm{p}}$, $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ are cocartesian fibrations.

(2) The functor $V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}$ maps $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LMod}\mathcal{O}\mathrm{p}}$-cocartesian morphisms to morphisms that are $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian, and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ maps $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian morphisms to morphisms that are $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian.

(3) Let $\mathcal{C}$ be an object of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then the functor

$$\left(V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}\right)_{\mathcal{C}} : \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LMod}\mathcal{O}\mathrm{p}_{\mathcal{C}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}}$$

induced by $V^{\mathcal{LMod}\mathcal{O}\mathrm{p}}$ on fibers over $\mathcal{C}$ is a cocartesian fibration.

(3') Let $\mathcal{C}$ be an object of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then the functor

$$\left(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\right)_{\mathcal{C}} : \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_{\mathcal{C}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}}$$

induced by $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ on fibers over $\mathcal{C}$ is a cocartesian fibration.

[12] The referenced proposition can be summarized as saying that a morphism of cocartesian fibrations over some ∞-category $\mathcal{C}$ is itself a cocartesian fibration if the restriction to fibers over any object of $\mathcal{C}$ is a cocartesian fibration, and the functor on fibers induced by a morphism in $\mathcal{C}$ preserves those cocartesian morphisms of the fibers.

[13] [GHN17] is the published version of [GHN15], but does not contain [GHN15, 9.6].

(4) Let

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\beta"'] & N \arrow[d, "\gamma"] \\ M' \arrow[r, "\delta"'] & N' \end{tikzcd} \tag{3.7}$$

be a commuting diagram in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}$ lying over the following diagram in $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"'] & \mathcal{D} \arrow[d, "\mathrm{id}"] \\ \mathcal{C} \arrow[r, "\phi"'] & \mathcal{D} \end{tikzcd}$$

Assume that α and δ are $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$-cocartesian and β is $(U^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian. Then γ is $(U^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{D}}$-cocartesian.

(4') Let

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\beta"'] & N \arrow[d, "\gamma"] \\ M' \arrow[r, "\delta"'] & N' \end{tikzcd}$$

be a commuting diagram in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}$ lying over the following diagram in $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"'] & \mathcal{D} \arrow[d, "\mathrm{id}"] \\ \mathcal{C} \arrow[r, "\phi"'] & \mathcal{D} \end{tikzcd}$$

Assume that α and δ are $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian and β is $(U^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian. Then γ is $(U^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{D}}$-cocartesian.

From the proof of [GHN15, 9.6] it also follows that the $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian morphisms will be precisely the compositions of $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian morphisms with $(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian morphisms for $\mathcal{C}$ an object of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. A similar statement holds for $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$. From this it follows that to show that $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ is a morphism of cocartesian fibrations from $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$ to $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ it will suffice to show the following.

(5) $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ sends $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$-cocartesian morphisms to $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$-cocartesian morphisms.

(6) For every object $\mathcal{C}$ of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ the functor $(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ sends $(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian morphisms to $(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian morphisms.

Proof of (1), (2) and (5): Hold by definition.

Proof of (3): Let $\mathcal{C}$ be an object of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. By Remark 3.1.3.4 we can identify the functor $(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ with the opposite of the following forgetful functor.

$$\iota^*_{\mathsf{Assoc}} \colon \mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$$

This forgetful functor is a cartesian fibration by [HA, 4.2.3.2], and thus $(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ is a cocartesian fibration. Furthermore, [HA, 4.2.3.2] also implies that a morphism in $\mathrm{LMod}(\mathcal{C})$ is $(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian if and only if $\mathrm{ev}_{\mathfrak{m}}$ of that morphism is an equivalence.

Proof of (3'): Just as above we can identify the functor $(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ using Remark 3.1.3.4 with the opposite of the left vertical functor in the following pullback diagram.

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{C}) \times \mathcal{C} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{C} \arrow[d] \\ \mathrm{Alg}(\mathcal{C}) \arrow[r] & * \end{tikzcd} \tag{3.8}$$

It follows by Proposition C.1.1.1 and [HTT, 2.4.1.5] that $(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ is a cocartesian fibration and that a morphism in $\mathrm{Alg}(\mathcal{C}) \times \mathcal{C}$ is $(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$-cocartesian if and only if pr_2 of that morphism is an equivalence.

Proof of (6): Follows immediately from the description of the respective cocartesian morphisms given above together with the description of the functor $(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}}$ in Remark 3.1.3.4.

Proof of (4) and (4'): The two proofs are analogous, so we only prove (4).

We use the same notation as in the statement of (4), and by the description of $(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}})_{\mathcal{D}}$-cocartesian morphisms in the proof of (3) we have to show that $\mathrm{ev}_{\mathfrak{m}}(\gamma)$ is an equivalence. Applying $\mathrm{ev}_{\mathfrak{m}}$ to diagram (3.7) we obtain

$$\begin{tikzcd} \mathrm{ev}_{\mathfrak{m}}(M) \arrow[r, "\mathrm{ev}_{\mathfrak{m}}(\alpha)"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}(\beta)"'] & \mathrm{ev}_{\mathfrak{m}}(N) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}(\gamma)"] \\ \mathrm{ev}_{\mathfrak{m}}(M') \arrow[r, "\mathrm{ev}_{\mathfrak{m}}(\delta)"'] & \mathrm{ev}_{\mathfrak{m}}(N') \end{tikzcd} \tag{3.9}$$

where by Proposition 3.1.2.17 the top and bottom horizontal morphisms are p^{Assoc}-cocartesian. Furthermore, the vertical morphism $\mathrm{ev}_{\mathfrak{m}}(\beta)$ is an equivalence, so by [HTT, 2.4.1.5 and 2.4.1.7] we can conclude that $\mathrm{ev}_{\mathfrak{m}}(\gamma)$ is also an equivalence. □

Remark 3.1.3.7. Let $\mathcal{C}$ be a monoidal ∞-category, and let us consider it as an object in $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then using Remark 3.1.3.4 we can identify the diagram

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}_{\mathcal{C}} \arrow[rr, "\left(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\right)_{\mathcal{C}}"] \arrow[dr, "\left(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}\right)_{\mathcal{C}}"'] & & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_{\mathcal{C}} \arrow[dl, "\left(V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\right)_{\mathcal{C}}"] \\ & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}} & \end{tikzcd}$$

with the following diagram.

$$\begin{tikzcd} \mathrm{LMod}(\mathcal{C})^{\mathrm{op}} \arrow[rr, "(\iota^*_{\mathsf{Assoc}})^{\mathrm{op}} \times (\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}"] \arrow[dr, "(\iota^*_{\mathsf{Assoc}})^{\mathrm{op}}"'] & & \mathrm{Alg}(\mathcal{C})^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \arrow[dl, "\mathrm{pr}_1"] \\ & \mathrm{Alg}(\mathcal{C})^{\mathrm{op}} & \end{tikzcd}$$

Let A be an associative algebra in $\mathcal{C}$. Then it follows that we can identify the functor $(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_A = ((V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}})_{\mathcal{C}})_A$, with the following functor.

$$(\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}} \colon\ \mathrm{LMod}_A(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$$

Let us now turn to morphisms in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ and induced functors on fibers. As the functor $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ is a cocartesian fibration, every morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ is the composite of a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphism and a morphism in a fiber. Let $G \colon \mathcal{C} \to \mathcal{C}'$ be a monoidal functor of monoidal ∞-categories, considered as a morphism in $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then by Remark 3.1.3.4 the induced functor on fibers

$$G_! \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}'}$$

can be identified with the functor

$$\mathrm{Alg}(G)^{\mathrm{op}} \colon\ \mathrm{Alg}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Alg}\big(\mathcal{C}'\big)^{\mathrm{op}}$$

which sends an object A of $\mathrm{Alg}(\mathcal{C})$ to an associative algebra $\mathrm{Alg}(G)(A)$ in $\mathcal{C}'$, that has underlying object $G(A)$, and so we will sometimes also write $G(A)$ for $\mathrm{Alg}(G)(A)$. Hence a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ from an object A in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}}$ to an object A' in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}'}$ consists of the composition of a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphism $A \to G(A)$ lying over a monoidal functor $G \colon \mathcal{C} \to \mathcal{C}'$ and a morphism of associative algebras $A' \to G(A)$.

Let us first consider a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphism $\widetilde{G} \colon A \to G(A)$ in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ lying over a monoidal functor $G \colon \mathcal{C} \to \mathcal{C}'$. By the description of cocartesian morphisms with respect to $V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ in Proposition 3.1.3.6, we know that the the functors induced by this morphism on fibers of the cocartesian fibrations $V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ are the restrictions of of the functors induced by G on fibers of the cocartesian fibrations $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}$ and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$. Thus using Remark 3.1.3.4 again we can identify the induced commutative diagram

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}_A \arrow[r, "\left(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\right)_A"] \arrow[d, "\widetilde{G}_!"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_A \arrow[d, "\widetilde{G}_!"] \\ \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}_{G(A)} \arrow[r, "\left(V^{\mathcal{LM}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\right)_{G(A)}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_{G(A)} \end{tikzcd}$$

with the following commutative diagram.

$$\begin{tikzcd}
\mathrm{LMod}_A(\mathcal{C})^{\mathrm{op}} \arrow[r, "(\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}"] \arrow[d, "\mathrm{LMod}(G)^{\mathrm{op}}"'] & \mathcal{C}^{\mathrm{op}} \arrow[d, "G^{\mathrm{op}}"] \\
\mathrm{LMod}_{G(A)}(\mathcal{C}')^{\mathrm{op}} \arrow[r, "(\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}"'] & \mathcal{C}'^{\mathrm{op}}
\end{tikzcd}$$

Let us now consider a morphism $f\colon A' \to A$ of associative algebras in some monoidal ∞-category $\mathcal{C}$, considered as a morphism $\widetilde{f}\colon A \to A'$ in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}} \simeq \mathrm{Alg}(\mathcal{C})^{\mathrm{op}}$. Again using the description of cocartesian morphisms from Proposition 3.1.3.6 together with Remark 3.1.3.4 and [HA, 4.2.3.2] we can identify the commutative diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}_A \arrow[r, "\left(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\right)_A"] \arrow[d, "\widetilde{f}_!"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_A \arrow[d, "\widetilde{f}_!"] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}_{A'} \arrow[r, "\left(V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}\right)_{A'}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}_{A'}
\end{tikzcd}$$

with the following commutative diagram.

$$\begin{tikzcd}
\mathrm{LMod}_A(\mathcal{C})^{\mathrm{op}} \arrow[r, "(\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}"] \arrow[d, "\mathrm{LMod}_f(\mathrm{id}_{\mathcal{C}})^{\mathrm{op}}"'] & \mathcal{C}^{\mathrm{op}} \arrow[d, "\mathrm{id}"] \\
\mathrm{LMod}_{A'}(\mathcal{C})^{\mathrm{op}} \arrow[r, "(\mathrm{ev}_{\mathfrak{m}})^{\mathrm{op}}"'] & \mathcal{C}^{\mathrm{op}}
\end{tikzcd}$$

$\diamondsuit$

Definition 3.1.3.8. By Proposition 3.1.3.6 we have a morphism of cocartesian fibrations over $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ as depicted in the following diagram.

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p} \arrow[rr, "V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}"] \arrow[dr, "V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}"'] & & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p} \arrow[dl, "V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}"] \\
& \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} &
\end{tikzcd}$$

Under the equivalence

$$\mathrm{co}\mathfrak{C}\mathfrak{F}\mathrm{ib}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \xrightarrow[\simeq]{\mathrm{Gr}^{-1}} \mathrm{Fun}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}, \mathfrak{C}\mathrm{at}_\infty) \xrightarrow{(-^{\mathrm{op}})_*} \mathrm{Fun}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}, \mathfrak{C}\mathrm{at}_\infty)$$

the cocartesian fibrations $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}$ and $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ correspond to functors $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathfrak{C}\mathrm{at}_\infty$ that we will denote by LMod and pr, respectively. The morphism of cocartesian fibrations $V^{\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}}_{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ corresponds to a natural transformation from LMod to pr that we will denote by $\mathrm{ev}_{\mathfrak{m}}$. $\diamondsuit$

Remark 3.1.3.9. Let $\mathcal{C}$ be a monoidal ∞-category and A an associative algebra in $\mathcal{C}$. Then Remark 3.1.3.7 shows that the natural transformation $\mathrm{ev}_{\mathfrak{m}}$ as defined in Definition 3.1.3.8 evaluated at A (considered as an object of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}}$) can be identified with the usual forgetful functor[14] $\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$, justifying the notation we chose for the two functors and the natural transformation. Furthermore, Remark 3.1.3.7 shows that LMod, pr, and $\mathrm{ev}_{\mathfrak{m}}$ are similarly compatible with usual notations on morphisms. $\diamondsuit$

3.1.3.3. LMod and colimits

In this section we put together some results from [HA] that imply that the functor LMod interacts well with the property of admitting and being compatible with colimits.

Definition 3.1.3.10 ([HA, 4.8.1.1 and 4.8.3.5] and [HTT, 5.5.3.1])**.** Let $\mathfrak{J}$ be a collection of small ∞-categories and $\mathcal{O}^{\otimes}$ an ∞-operad.

We define an ∞-category $\mathfrak{Cat}_{\infty}(\mathfrak{J})$ together with a monomorphism to $\mathfrak{Cat}_{\infty}$ as the monomorphism that under the construction of Remark B.6.0.1 corresponds to the replete subcategory of $\mathrm{Ho}\,\mathfrak{Cat}_{\infty}$ whose objects are ∞-categories that admit $\mathfrak{J}$-indexed colimits[15] and whose morphisms are represented by those functors that preserve $\mathfrak{J}$-indexed colimits.

We similarly define $\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{J}}(\mathfrak{Cat}_{\infty})$ together with a monomorphism to $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$ as the monomorphism corresponding to the replete subcategory of $\mathrm{Ho}\,\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_{\infty})$ whose objects are the $\mathcal{O}$-monoidal ∞-categories that are compatible with $\mathfrak{J}$-indexed colimits in the sense of [HA, 3.1.1.19 and 3.1.1.18], and whose morphisms are represented by $\mathcal{O}$-monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ such that for every object X of $\mathcal{O}$ the underlying functor of ∞-categories $\mathcal{C}_X \to \mathcal{D}_X$ preserves $\mathfrak{J}$-indexed colimits.

Now let $\mathfrak{J}$ be the collection of all small ∞-categories. We denote by $\mathfrak{Pr}^{\mathrm{L}}$ the full subcategory of $\mathfrak{Cat}_{\infty}(\mathfrak{J})$ spanned by the presentable ∞-categories[16].

We furthermore define $\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{Pr}}(\mathfrak{Cat}_{\infty})$ to be the full subcategory of the ∞-category $\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{J}}(\mathfrak{Cat}_{\infty})$ spanned by $\mathcal{O}$-monoidal ∞-categories which are presentable in the sense of [HA, 3.4.4.1]. $\diamondsuit$

Definition 3.1.3.11. Let $\mathfrak{J}$ be a collection of small ∞-categories. We define $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}$ and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}}$ via the following pullback diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}} \arrow[r] \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}_{\mathsf{Assoc}}^{\mathfrak{J}}(\mathfrak{Cat}_{\infty}) \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathfrak{Cat}_{\infty})
\end{tikzcd}$$

where the lower horizontal functor is the inclusion from Definition 3.1.3.10. We similarly

[14]Here $\mathrm{LMod}_A(\mathcal{C})$ refers to what is defined in [HA, 4.2.1.13].
[15]This means that they must admit $\mathcal{I}$-indexed colimits for every $\mathcal{I}$ in $\mathfrak{J}$.
[16]See [HTT, 5.5]

define $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$ via the following pullback diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \arrow[r] \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)
\end{tikzcd}$$

where the lower horizontal functor is the inclusion from Definition 3.1.3.10. ♢

Proposition 3.1.3.12 ([HA, 4.2.3.5 and 4.2.3.7])**.** *Assume that $\mathfrak{I}$ is a collection of small ∞-categories. Then the restriction of the natural transformation $\mathrm{ev}_{\mathfrak{m}}$ to $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}$ factors through $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})$. Analogously, the restriction to $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ factors through $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The situation is depicted in the following diagram.*

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \arrow[rr, dashed, bend left, "\mathrm{LMod}"] \arrow[rr, dashed, bend right, "\mathrm{pr}"'] \arrow[rr, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] \arrow[d] & & \mathcal{P}\mathrm{r}^{\mathrm{L}} \arrow[d] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}} \arrow[rr, dashed, bend left, "\mathrm{LMod}"] \arrow[rr, dashed, bend right, "\mathrm{pr}"'] \arrow[rr, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] \arrow[d] & & \mathcal{C}\mathrm{at}_\infty(\mathfrak{I}) \arrow[d] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[rr, bend left, "\mathrm{LMod}"] \arrow[rr, bend right, "\mathrm{pr}"'] \arrow[rr, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] & & \mathcal{C}\mathrm{at}_\infty
\end{tikzcd} \tag{3.10}$$

As suggested by the diagram, will denote the induced functors and natural transformations by the same name again. ♡

Proof. Let $E\colon [1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ be the functor encoding the natural transformation $\mathrm{ev}_{\mathfrak{m}}$. By definition the right vertical functors in diagram (3.10) are monomorphisms, so by Proposition B.4.3.1 the composition $E \circ (\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p})$ can be lifted to $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})$ if and only if $\mathrm{Im}(E \circ (\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}))$ is contained in $\mathrm{Im}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}) \to \mathcal{C}\mathrm{at}_\infty)$, and similarly for the lift to $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

In light of Remark 3.1.3.7 and Remark 3.1.3.9, this boils down to the following statements for any ∞-category $\mathcal{I}$, monoidal ∞-category $\mathcal{C}$, associative algebra A in $\mathcal{C}$, monoidal functor $G\colon \mathcal{C} \to \mathcal{D}$, and morphism of associative algebras $g\colon B \to G(A)$.

(1) If the monoidal ∞-category $\mathcal{C}$ is compatible with $\mathcal{I}$-indexed colimits in the sense of [HA, 3.1.1.18], then $\mathrm{LMod}_A(\mathcal{C})$ admits $\mathcal{I}$-indexed colimits.

(2) If $\mathcal{C}$ is a presentable monoidal ∞-category in the sense of [HA, 3.4.4.1], then $\mathrm{LMod}_A(\mathcal{C})$ is presentable.

(3) If the monoidal ∞-category $\mathcal{C}$ is compatible with $\mathcal{I}$-indexed colimits, then the forgetful functor

$$\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$$

preserves $\mathcal{I}$-indexed colimits.

(4) If $\mathcal{C}$ admits and G preserves $\mathcal{I}$-indexed colimits, then the functor induced by G and g

$$\mathrm{LMod}_g(G) \colon \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_B(\mathcal{D})$$

also preserves $\mathcal{I}$-indexed colimits.

Proof of (1): This is [HA, 4.2.3.5 (1)].
Proof of (2): This is [HA, 4.2.3.7 (1)].
Proof of (3): This is [HA, 4.2.3.5 (2)].
Proof of (4): This is a slight generalization of [HA, 4.2.3.7 (2)]. From the natural transformation $\mathrm{ev}_{\mathfrak{m}}$ we obtain a commuting diagram

$$\begin{tikzcd}
\mathrm{LMod}_A(\mathcal{C}) \arrow[r, "\mathrm{LMod}_g(G)"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"'] & \mathrm{LMod}_B(\mathcal{D}) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\
\mathcal{C} \arrow[r, "G"'] & \mathcal{D}
\end{tikzcd}$$

where by assumption the lower horizontal functor preserves $\mathcal{I}$-indexed colimits. It then follows immediately from [HA, 4.2.3.5 (2)] that the top horizontal functor also does so. □

3.2. LMod and monoidality

In this section we will start in Section 3.2.1 by showing $\mathrm{LMod} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ preserves products and can thus be upgraded to a symmetric monoidal functor with respect to the respective cartesian symmetric monoidal structures. Furthermore, this induces a symmetric monoidal structure on the restriction $\mathrm{LMod} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see Proposition 3.1.3.12). This will be shown in Section 3.2.3, after we discuss the relevant symmetric monoidal ∞-categories in Section 3.2.2.

3.2.1. LMod and products

In this short section we show that $\mathrm{LMod} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$ preserves products and can thus be upgraded to a symmetric monoidal functor with respect to the respective cartesian symmetric monoidal structures.

Proposition 3.2.1.1. *The cocartesian fibrations*

$$\begin{aligned} q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{L}\mathcal{M}\mathrm{od}\mathcal{O}\mathrm{p} &\to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \\ q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p} &\to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \\ q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} &\to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \end{aligned}$$

have fibers compatible with products in the sense of Definition C.2.0.1. ♡

Proof. Proposition F.2.0.1 implies that the ∞-category $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ admits products. Combining Remark C.2.0.2 with the fact that $(-)^{\mathrm{op}} \colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ is an equivalence and thus preserves products we are reduced to showing that $q_{\mathcal{A}\mathrm{lg}\mathcal{L}\mathcal{M}\mathrm{od}}$, $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{bj}}$, and $q_{\mathcal{A}\mathrm{lg}}$ have fibers compatible with products. But this follows from combining Proposition 3.1.2.22 with Proposition 3.1.1.9. □

Proposition 3.2.1.2. *The cocartesian fibrations*[17] $V^{\mathcal{L}\mathrm{Mod}\mathcal{O}\mathrm{p}}$ *and* $V^{\mathcal{O}\mathrm{bj}\mathcal{O}\mathrm{p}}$ *from Definition 3.1.3.5 have fibers compatible with products in the sense of Definition C.2.0.1.* ♡

Proof. These cocartesian fibrations are by definition also morphisms of cocartesian fibrations over $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. As those cocartesian fibrations have fibers compatible with products by Proposition 3.2.1.1, the statement follows from Proposition C.2.0.4. □

Proposition 3.2.1.3. *The ∞-category $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ admits all products and the functors*

$$\mathrm{LMod}, \mathrm{pr} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$$

preserve products. ♡

Proof. Follows directly from Proposition 3.2.1.2, Remark C.2.0.2, and the fact that $(-)^{\mathrm{op}}$ is an equivalence and thus preserves products. □

Remark 3.2.1.4. Let $\mathcal{C}$ and $\mathcal{C}'$ be monoidal ∞-categories and A and A' associative algebras in $\mathcal{C}$ and $\mathcal{C}'$, respectively. Then Proposition C.2.0.3 and Proposition 3.2.1.1 imply that the pair (A, A') considered as an object in

$$\Big(\mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}')\Big)^{\mathrm{op}} \simeq \mathrm{Alg}(\mathcal{C} \times \mathcal{C}')^{\mathrm{op}} \simeq \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}\times\mathcal{C}'}$$

is a product in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ of A and A'.

That LMod preserves products by Proposition 3.2.1.3 means in particular that there is an equivalence as follows.

$$\mathrm{LMod}_{(A,A')}(\mathcal{C} \times \mathcal{C}') \simeq \mathrm{LMod}_A(\mathcal{C}) \times \mathrm{LMod}_{A'}(\mathcal{C}')$$ ♢

[17] That they are cocartesian fibrations was shown in Proposition 3.1.3.6.

3.2.2. $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ as a symmetric monoidal ∞-category

To be able to make sense of the claim that $\mathrm{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ should be upgradable to a symmetric monoidal functor, we first need to define symmetric monoidal structures on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and in particular on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$. This is what we will discuss in this section.

We will start in Section 3.2.2.1 by recalling the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, before discussing the symmetric monoidal structure on $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ in Section 3.2.2.2. While we will define $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ directly, showing that this is indeed a symmetric monoidal structure on $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ will require a fair amount of work comparing it to $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\otimes}$, the induced symmetric monoidal structure on algebras in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The reason why we bother to do this rather than just using $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\otimes}$ is that $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ is a better fit when discussing the symmetric monoidal structure on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$, which we do in Section 3.2.2.3.

3.2.2.1. The symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$

In this section we recall the symmetric monoidal structures on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})$ for $\mathfrak{I}$ a collection of small ∞-categories, closely following [HA, 4.8.1].

Definition 3.2.2.1 ([HA, 4.8.1.2, 4.8.1.4 and 4.8.1.15])**.** Let $\mathfrak{I}$ be a collection of small ∞-categories. We define a monomorphism

$$\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes} \to \mathcal{C}\mathrm{at}_\infty^{\times}$$

corresponding as in Remark B.6.0.1 to a replete subcategory H of $\mathrm{Ho}(\mathcal{C}\mathrm{at}_\infty^{\times})$ that we describe next.

An object $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ of $(\mathcal{C}\mathrm{at}_\infty)^{\times}_{\langle n\rangle}$ with $\mathcal{C}_1, \ldots, \mathcal{C}_n$ ∞-categories is to be an object of H if and only if each $\mathcal{C}_i$ admits all $\mathfrak{I}$-indexed colimits. A morphism $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n \to \mathcal{C}'_1 \oplus \cdots \oplus \mathcal{C}'_m$ lying over $\varphi\colon \langle n\rangle \to \langle m\rangle$ is to be in H if and only if for each $1 \leq j \leq m$ the associated functor

$$\prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{C}'_j$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

Now let $\mathfrak{J}$ be the collection of all small ∞-categories. We define $\mathcal{P}\mathrm{r}^{\mathrm{L}^{\otimes}}$ to be the full subcategory of $\mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^{\otimes}$ spanned by those objects $\mathcal{C} \simeq \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ where each $\mathcal{C}_i$ is presentable. ♢

Remark 3.2.2.2. It is clear from the definitions that the functors

$$(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}))^{\otimes}_{\langle 1\rangle} \to (\mathcal{C}\mathrm{at}_\infty)^{\times}_{\langle 1\rangle} \qquad \text{and} \qquad (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\otimes}_{\langle 1\rangle} \to (\mathcal{C}\mathrm{at}_\infty)(\mathfrak{J})^{\times}_{\langle 1\rangle}$$

which are induced by the functors defined in Definition 3.2.2.1 can be identified with the functors

$$\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}) \to \mathcal{C}\mathrm{at}_\infty \qquad \text{and} \qquad \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{J})$$

from Definition 3.1.3.10. ♢

Proposition 3.2.2.3 ([HA, 4.8.1.4 and 4.8.1.15]). *Let $\mathfrak{J}$ be the collection of all small ∞-categories, let $\mathfrak{I}$ be a subcollection of $\mathfrak{J}$ and $\mathfrak{I}'$ a subcollection of $\mathfrak{I}$. Then the following statements hold.*

(0) The monomorphism $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$ from Definition 3.2.2.1 factors through the monomorphism $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}')^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$ from Definition 3.2.2.1. The lift obtained in this manner is also a monomorphism.

(1) The compositions

$$\mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times \to \mathsf{Fin}_*$$

and

$$\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty^\times \to \mathsf{Fin}_*$$

where the first functor is the monomorphism from Definition 3.2.2.1 and the second functor is the canonical morphism of ∞-operads, are cocartesian fibrations of ∞-operads.

(2) The functors

$$\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^\otimes \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I}')^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$$

from Definition 3.2.2.1 and (0) are lax symmetric monoidal with respect to the symmetric monoidal structures from (1).

(3) A morphism in $\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^\otimes$ or $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes}$ is inert if an only if its image in $\mathcal{C}\mathrm{at}_\infty^\times$ is inert.

(4) The functor

$$\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes$$

is symmetric monoidal with respect to the symmetric monoidal structure from (1).

(5) A morphism in $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes}$ is cocartesian with respect to the canonical morphism of ∞-operads $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathsf{Fin}_*$ if and only if its image in $\mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes$ is cocartesian with respect to the canonical morphism of ∞-operads $\mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes \to \mathsf{Fin}_*$. ♡

Proof. Proof of (0): Immediate from the definition together with Proposition B.4.3.1 and Proposition B.1.2.1.

Proof of (1) and (2) for the compositions to $\mathcal{C}\mathrm{at}_\infty^\times$: This is [HA, 4.8.1.4 and 4.8.1.15].

Proof of (4): This is [HA, 4.8.1.15].

Proof of(3) and (5): The functors $\mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$ and $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty^\times$ were already shown to be morphisms of ∞-operads, and $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes$ was already shown to be symmetric monoidal. As these functors are also monomorphisms[18] and hence conservative by Proposition B.4.1.2, we can apply Proposition E.1.2.1 to deduce the claims.

Proof of the rest of (2): Follows directly from (3). □

[18] That $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{J})^\otimes$ is a monomorphism follows from Proposition B.4.4.1 and that $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathcal{C}\mathrm{at}_\infty^\times$ is a monomorphism then follows from Proposition B.1.2.1.

3.2.2.2. The symmetric monoidal structure on $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)$

In this section we construct the symmetric monoidal structure on $\mathrm{Mon}_{\mathsf{Assoc}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)$. While defining $\mathrm{Mon}_{\mathsf{Assoc}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ is relatively straightforward, showing that this defines a symmetric monoidal structure (which is Proposition 3.2.2.10) will require a bit more work, requiring a comparison result between $\mathrm{Mon}_{\mathsf{Assoc}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ and $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\otimes}$ that will be shown in Proposition 3.2.2.8.

Definition 3.2.2.4 ([HA, 4.8.5.14])**.** Let $\mathfrak{I}$ be a collection of small ∞-categories and $\mathcal{O}$ an ∞-operad. We define a monomorphism[19]

$$\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$

corresponding as in Remark B.6.0.1 to a replete subcategory H of $\mathrm{Ho}(\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times})$ that we describe next.

An object $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ of $\mathrm{Ho}(\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times})$ is to be in H if and only if for each $1 \leq i \leq n$ the $\mathcal{O}$-monoidal ∞-category $\mathcal{C}_i$ is compatible with $\mathfrak{I}$-indexed colimits in the sense of [HA, 3.1.1.19 and 3.1.1.18]. A morphism in $\mathrm{Ho}(\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times})$ between objects of H is to be in H if and only if $\mathrm{Ho}((\mathrm{ev}_{\langle 1\rangle})^{\times})$ maps that morphism to a morphism in $\mathrm{Im}(\mathrm{Ho}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes}) \to \mathrm{Ho}(\mathcal{C}\mathrm{at}_\infty^{\times}))$[20].

Now let $\mathfrak{I}$ be the collection of all small ∞-categories. We define $\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ to be the full subcategory of $\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ spanned by those objects $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ for which for each $1 \leq i \leq n$ the $\mathcal{O}$-monoidal ∞-category $\mathcal{C}_i$ is presentable $\mathcal{O}$-monoidal in the sense of [HA, 3.4.4.1]. $\diamondsuit$

Remark 3.2.2.5. It is clear from the definitions that the functors

$$\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}_{\langle 1\rangle} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}_{\langle 1\rangle}$$

and

$$\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}_{\langle 1\rangle} \to \mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\times}_{\langle 1\rangle}$$

which are induced by the functors defined in Definition 3.2.2.4 can be identified with the functors

$$\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$

and

$$\mathrm{Mon}_{\mathcal{O}}^{\mathcal{P}\mathrm{r}}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)$$

from Definition 3.1.3.10. $\diamondsuit$

Remark 3.2.2.6. It follows directly from the definitions in Definition 3.2.2.4 together with Proposition B.4.3.1 that for $\mathfrak{I}$ a collection of small ∞-categories and $\mathfrak{I}'$ a subcollection of $\mathfrak{I}$ the monomorphism

$$\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$

[19] For products in $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ see Proposition F.2.0.1.

[20] This condition boils down to associated underlying functors of the form $\prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{C}'_j$ preserving $\mathfrak{I}$-indexed colimits separately in each variable.

factors through the monomorphism

$$\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}'}(\mathfrak{Cat}_\infty)^\otimes \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\times$$

and the lift is by Proposition B.1.2.1 again a monomorphism. ♢

For easier reference we introduce some notation that we are going to use in some statements and proof below.

Notation 3.2.2.7. The following notation will be used only when specifically invoked, but not elsewhere. In the notation below, $\mathfrak{I}$ will be a collection of small ∞-categories, $\mathfrak{I}'$ a subcollection of $\mathfrak{I}$, and $\mathcal{O}$ an ∞-operad.

- Some of the below notations will use a superscript or subscript $\mathfrak{I}$. In the case $\mathfrak{I} = \emptyset$ we will allow ourselves to drop this superscript or subscript.
- We denote by
$$p_{\mathcal{O}}\colon \mathcal{O}^\otimes \to \mathsf{Fin}_*$$
the canonical morphism of ∞-operads.
- We let α be the bifunctor defined as the following composition.
$$\mathsf{Fin}_* \times \mathcal{O}^\otimes \xrightarrow{\mathrm{id}_{\mathsf{Fin}_*} \times p_{\mathcal{O}}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{- \wedge -} \mathsf{Fin}_*$$
- We denote by
$$p_{\mathfrak{I}}\colon \mathfrak{Cat}_\infty(\mathfrak{I})^\otimes \to \mathsf{Fin}_*$$
$$p_{\mathcal{P}\mathrm{r}}\colon \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \to \mathsf{Fin}_*$$
$$p_{\mathrm{Alg},\mathfrak{I}}\colon \mathrm{Alg}_{\mathcal{O}}\big(\mathfrak{Cat}_\infty(\mathfrak{I})\big)^\otimes \to \mathsf{Fin}_*$$
$$p_{\mathrm{Alg},\mathcal{P}\mathrm{r}}\colon \mathrm{Alg}_{\mathcal{O}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}\right)^\otimes \to \mathsf{Fin}_*$$
the canonical morphism of ∞-operads, where for $p_{\mathrm{Alg},\mathfrak{I}}$ and $p_{\mathrm{Alg},\mathcal{P}\mathrm{r}}$ this is with respect to the induced symmetric monoidal structures as in Proposition E.4.2.3 with respect to the bifunctor α.
- We denote the lax symmetric monoidal functors from Proposition 3.2.2.3 as indicated below.
$$\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \xrightarrow{\left(\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}\right)^\otimes} \mathfrak{Cat}_\infty(\mathfrak{I})^\otimes \xrightarrow{\left(\Phi_{\mathfrak{I}'}^{\mathfrak{I}}\right)^\otimes} \mathfrak{Cat}_\infty(\mathfrak{I}')^\otimes \xrightarrow{\left(\Phi^{\mathfrak{I}'}\right)^\otimes} \mathfrak{Cat}_\infty^\times$$
We set $(\Phi^{\mathcal{P}\mathrm{r}})^\otimes := (\Phi^{\mathfrak{I}})^\otimes \circ (\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}})^\otimes$.

- We denote the monomorphisms from Definition 3.2.2.4 and Remark 3.2.2.6 as indicated below.

$$\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\left(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}\right)^{\otimes}} \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\left(\Psi^{\mathfrak{I}}_{\mathfrak{I}'}\right)^{\otimes}} \mathrm{Mon}^{\mathfrak{I}'}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\left(\Psi^{\mathfrak{I}'}\right)^{\otimes}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$

We set $(\Psi^{\mathcal{P}\mathrm{r}})^{\otimes} := (\Psi^{\mathfrak{I}})^{\otimes} \circ (\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}$.

- We denote by

$$p_{\mathrm{Mon}} \colon \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \to \mathsf{Fin}_*$$

the canonical morphism of ∞-operads, and define $p_{\mathrm{Mon},\mathfrak{I}}$ and $p_{\mathrm{Mon},\mathcal{P}\mathrm{r}}$ as the following compositions.

$$p_{\mathrm{Mon},\mathfrak{I}} \coloneqq p_{\mathrm{Mon}} \circ (\Psi^{\mathfrak{I}})^{\otimes}$$
$$p_{\mathrm{Mon},\mathcal{P}\mathrm{r}} \coloneqq p_{\mathrm{Mon}} \circ (\Psi^{\mathcal{P}\mathrm{r}})^{\otimes}$$

- The cartesian symmetric monoidal structure $\mathcal{C}\mathrm{at}_\infty^{\times}$ comes with a cartesian structure

$$\pi \colon \mathcal{C}\mathrm{at}_\infty^{\times} \to \mathcal{C}\mathrm{at}_\infty$$

that we will denote by π, see [HA, 2.4.1.5]. Similarly, we denote the cartesian structure

$$\pi_{\mathrm{Mon}} \colon \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$

of $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$ by π_{Mon}. $\diamondsuit$

Proposition 3.2.2.8 ([HA, 4.8.5.16 (1)]). *In this proposition we will make use of Notation 3.2.2.7.*

Let $\mathfrak{I}$ be a collection of small ∞-categories, $\mathfrak{I}'$ a subcollection of $\mathfrak{I}$, and $\mathcal{O}$ an ∞-operad. Then there is a commutative diagram as follows such that the horizontal functors are equivalences

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}\right)^{\otimes} \arrow[r, "\Theta^{\otimes}_{\mathcal{P}\mathrm{r}}", "\simeq"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}"'] & \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[d, "(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}))^{\otimes} \arrow[r, "\Theta^{\otimes}_{\mathfrak{I}}", "\simeq"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathfrak{I}}_{\mathfrak{I}'})^{\otimes}"'] & \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[d, "(\Psi^{\mathfrak{I}}_{\mathfrak{I}'})^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I}'))^{\otimes} \arrow[r, "\Theta^{\otimes}_{\mathfrak{I}'}", "\simeq"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathfrak{I}'})^{\otimes}"'] & \mathrm{Mon}^{\mathfrak{I}'}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[d, "(\Psi^{\mathfrak{I}'})^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[r, "\Theta^{\otimes}", "\simeq"'] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}
\end{tikzcd} \tag{3.11}$$

The functor $\Theta^{\otimes}$ can be chosen in such a way that for every object X in $\mathcal{O}$ there is a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[rr, "\Theta^{\otimes}", "\simeq"'] \arrow[dr, "\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}"] \arrow[ddr, bend right, "p_{\mathrm{Alg}}"'] & & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\times} \arrow[dl, "(\mathrm{ev}_X)^{\times}"'] \arrow[ddl, bend left, "p_{\mathrm{Mon}}"] \\
& \mathcal{C}\mathrm{at}_{\infty}^{\times} \arrow[d, "p"] & \\
& \mathsf{Fin}_* &
\end{tikzcd} \tag{3.12}$$

where the functors to $\mathcal{C}\mathrm{at}_{\infty}^{\times}$ are the symmetric monoidal forgetful functors[21].

Furthermore, $\Theta^{\otimes}$ can be chosen such that the underlying equivalence

$$\Theta\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$$

is the equivalence from [HA, 2.4.2.5], i. e. there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[r, "\Theta"] \arrow[d] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[dd] \\
\mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}^{\times}) \arrow[d] & \\
\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}^{\times}) \arrow[r, "\pi_*"'] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd} \tag{3.13}$$

where the vertical functors are the canonical projections or inclusions. ♡

Proof. We start by constructing $\Theta^{\otimes}$ together with diagram (3.12).

By Proposition F.3.0.2 there is a functor π_{Alg} making the following diagram commute

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[r, dashed, "\pi_{\mathrm{Alg}}"] \arrow[d, "\iota_{\mathrm{Alg}}"'] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[dd] \\
\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}^{\times}) \times_{\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \arrow[d, "\mathrm{pr}_1"'] & \\
\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}^{\times}) \arrow[r, "\pi_*"'] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd} \tag{3.14}$$

where ι_{Alg} is as in Proposition E.4.2.3 and the unlabeled vertical functor on the right is the inclusion. Furthermore, Proposition F.3.0.2 also shows that π_{Alg} is a cartesian

[21] See Proposition E.4.2.3 (5) for the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}^{\times}$ that is given by evaluation at X and Proposition F.2.0.1 for the forgetful functor $\mathrm{ev}_X\colon \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathcal{C}\mathrm{at}_{\infty}$ preserving products and hence inducing a functor $(\mathrm{ev}_X)^{\times}\colon \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\times} \to \mathcal{C}\mathrm{at}_{\infty}^{\times}$.

structure. Applying [HA, 2.4.1.7] we obtain a symmetric monoidal functor $\Theta^{\otimes}$ making the following diagram commute.

$$\begin{tikzcd}
 & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) & \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[ur, "\pi_{\mathrm{Alg}}"] \arrow[rr, "\Theta^{\otimes}"] \arrow[dr, "p_{\mathrm{Alg}}"'] & & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \arrow[ul, "\pi_{\mathrm{Mon}}"'] \arrow[dl, "p_{\mathrm{Mon}}"] \\
 & \mathsf{Fin}_* &
\end{tikzcd} \tag{3.15}$$

Of diagram (3.12) that we want to construct we have thus constructed $\Theta^{\otimes}$ as a functor over Fin_*. The two forgetful functors to $\mathcal{C}\mathrm{at}_\infty^{\times}$ are already given as functors over Fin_*, so it remains to construct a filler for the small triangle at the top, considered as a diagram over Fin_*.

So let X be an object of $\mathcal{O}$. As both the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}} \mathcal{C}\mathrm{at}_\infty^{\times}$$

as well as the composition

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\Theta^{\otimes}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \xrightarrow{(\mathrm{ev}_X)^{\times}} \mathcal{C}\mathrm{at}_\infty^{\times}$$

are symmetric monoidal, giving a homotopy between them as symmetric monoidal functors (and hence functors over Fin_*) is by [HA, 2.4.1.7] equivalent to giving a homotopy of weak cartesian structures between

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}} \mathcal{C}\mathrm{at}_\infty^{\times} \xrightarrow{\pi} \mathcal{C}\mathrm{at}_\infty$$

and the following composition.

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\Theta^{\otimes}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \xrightarrow{(\mathrm{ev}_X)^{\times}} \mathcal{C}\mathrm{at}_\infty^{\times} \xrightarrow{\pi} \mathcal{C}\mathrm{at}_\infty$$

Such a homotopy is encoded in the outer commutative diagram depicted below

$$\begin{tikzcd}
 & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \arrow[r, "(\mathrm{ev}_X)^{\times}"] \arrow[d, "\pi_{\mathrm{Mon}}"] & \mathcal{C}\mathrm{at}_\infty^{\times} \arrow[d, "\pi"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[ur, "\Theta^{\otimes}"] \arrow[r, "\pi_{\mathrm{Alg}}"'] \arrow[d, "\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}"'] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, "\mathrm{ev}_X"'] \arrow[d] & \mathcal{C}\mathrm{at}_\infty \arrow[d, equal] \\
\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty^{\times}) \arrow[r, "\pi_*"'] \arrow[dr, "\mathrm{ev}_X"'] & \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[r, "\mathrm{ev}_X"'] & \mathcal{C}\mathrm{at}_\infty \\
 & \mathcal{C}\mathrm{at}_\infty^{\times} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where the upper left commutative triangle is the one from (3.15), the upper right commutative square arises from the functoriality of the construction $(-)^{\times}$, the middle left commutative square is the one from (3.14), the middle lower commutative square is one by definition, and the bottom commutative square arises from naturality of ev_X.

We have now constructed $\Theta^{\otimes}$ as a functor over Fin_* as well as diagram (3.12) for every object X of $\mathcal{O}$. Let us now consider diagram (3.13) concerning the underlying functor Θ. The composition of the inclusion of $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\times}_{\langle 1\rangle}$ into $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\times}$ with π_{Mon} is by definition homotopic to the identity, so we obtain from the commutative diagram (3.15) a homotopy between Θ and the the composition

$$\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty) \to \mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes} \xrightarrow{\pi_{\mathrm{Alg}}} \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$$

The desired commutative diagram (3.13) can now be obtained by combining this with commutative diagram (3.14).

From this description of Θ it now follows from [HA, 2.4.2.5] that Θ is an equivalence. Using that $\Theta^{\otimes}$ is symmetric monoidal we can thus conclude from [HA, 2.1.3.8] that $\Theta^{\otimes}$ is an equivalence as well.

To construct diagram (3.11), we will show the following claims for each collection of small ∞-categories $\mathfrak{I}$.

(A) $(\Psi^{\mathfrak{I}})^{\otimes}$ is a monomorphism.

(B) $\mathrm{Alg}(\Phi^{\mathfrak{I}})^{\otimes}$ is a monomorphism.

(C) $\mathrm{Im}(\mathrm{Ho}(\Theta^{\otimes} \circ \mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathfrak{I}})^{\otimes}))$ is equal to $\mathrm{Im}(\mathrm{Ho}((\Psi^{\mathfrak{I}})^{\otimes}))$.

Let us assume claims (A), (B), and (C) for the moment and deduce the statements we have to prove.

Existence of an equivalences $\Theta^{\otimes}_{\mathfrak{I}}$ together with commutative squares of the form

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}\big(\mathfrak{Cat}_\infty(\mathfrak{I})\big)^{\otimes} \arrow[r, "\Theta^{\otimes}_{\mathfrak{I}}", "\simeq"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathfrak{I}})^{\otimes}"'] & \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes} \arrow[d, "(\Psi^{\mathfrak{I}})^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes} \arrow[r, "\Theta^{\otimes}", "\simeq"'] & \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\times}
\end{tikzcd}$$

then follows from Proposition B.4.3.1, see also Remark B.6.0.1. That there is a compatibility square between $\Theta^{\otimes}_{\mathfrak{I}'}$ and $\Theta^{\otimes}_{\mathfrak{I}}$ follows immediately from the uniqueness part of Proposition B.4.3.1 using that $(\Psi^{\mathfrak{I}'})^{\otimes}$ is a monomorphism.

Finally, we need to construct the dashed equivalence fitting into the square depicted at the top of the commutative diagram below, where $\mathfrak{I}$ is the collection of all small

∞-categories, and X is an object of $\mathcal{O}$.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}\right)^{\otimes} \arrow[rr, dashed, "\Theta^{\otimes}_{\mathcal{P}\mathrm{r}}", "\simeq"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}\left(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}}\right)^{\otimes}"'] & & \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[d, "\left(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}}\right)^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J}))^{\otimes} \arrow[rr, "\Theta^{\otimes}_{\mathfrak{J}}", "\simeq"'] \arrow[dr, "\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}} \circ \mathrm{Alg}_{\mathcal{O}}\left(\Phi^{\mathfrak{J}}\right)^{\otimes}"'] & & \mathrm{Mon}^{\mathfrak{J}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[dl, "(\mathrm{ev}_X)^{\times} \circ \left(\Psi^{\mathfrak{J}}\right)^{\otimes}"] \\
& \mathcal{C}\mathrm{at}^{\times}_{\infty} &
\end{tikzcd} \tag{3.16}$$

The functor $(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ is by definition the inclusion of the fully faithful subcategory of $\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J})^{\otimes}$ spanned by objects $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ such that the ∞-category $\Phi^{\mathfrak{J}}(\mathcal{C}_i)$ is presentable for each $1 \leq i \leq n$, see Definition 3.2.2.1. It follows from the definition of the induced functor $\mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ in Remark E.4.2.2 together with Proposition B.3.0.1, Proposition B.5.1.1, Remark B.5.1.2, and Proposition B.5.3.1, that $\mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ is again a fully faithful functor with essential image spanned by objects $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ such that the underlying ∞-category $(\mathrm{ev}_X \circ \mathrm{Alg}_{\mathcal{O}}(\Phi^{\mathfrak{J}}))(\mathcal{C}_i)$ of $\mathcal{C}_i$ is presentable for each $1 \leq i \leq n$ and object X of $\mathcal{O}$[22].

The functor $(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ is by definition (see Definition 3.2.2.4) the inclusion of the fully faithful subcategory described in the same way[23], so as $\Theta^{\otimes}_{\mathfrak{J}}$ is compatible with the forgetful functors to $\mathcal{C}\mathrm{at}^{\times}_{\infty}$ we can use Proposition B.4.3.1 to complete diagram (3.16).

We now turn towards proving (A), (B), and (C). We will simplify notation and write $\Phi \coloneqq \Phi^{\mathfrak{J}}$ and $\Psi \coloneqq \Psi^{\mathfrak{J}}$.

Proof of (A): That $\Psi^{\otimes}$ is a monomorphism holds by definition, see Definition 3.2.2.4.

Proof of (B): By Remark E.4.2.2, there is a commutative diagram as follows

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J}))^{\otimes} \arrow[r, "\iota'_{\mathrm{Alg}}"] \arrow[d, dashed, "\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}"'] & \mathrm{Fun}\left(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J})^{\otimes}\right) \times_{\mathrm{Fun}\left(\mathcal{O}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \arrow[d, "\left(\Phi^{\otimes}\right)_* \times_{\mathrm{id}} \mathrm{id}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[r, "\iota_{\mathrm{Alg}}"'] & \mathrm{Fun}\left(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}^{\times}_{\infty}\right) \times_{\mathrm{Fun}\left(\mathcal{O}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_*
\end{tikzcd}$$

where ι_{Alg} and ι'_{Alg} are as in Proposition E.4.2.3. $\Phi^{\otimes}$ is by definition (see Definition 3.2.2.1) a monomorphism, so $\left(\Phi^{\otimes}\right)_*$ is a monomorphism by Proposition B.5.1.1 and then it follows that $\left(\Phi^{\otimes}\right)_* \times_{\mathrm{id}} \mathrm{id}$ is a monomorphism by Proposition B.5.3.1. As ι_{Alg} and ι'_{Alg} are fully faithful by definition and hence monomorphisms by Proposition B.4.4.1, it follows from Proposition B.1.2.1 that $\mathrm{Alg}(\Phi)^{\otimes}$ is a monomorphism.

[22] We are using here that only functors preserving inert morphisms are in the essential image of $\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}$ – this implies that we only need to check the presentability condition for objects X of $\mathcal{O}$ rather than all of $\mathcal{O}^{\otimes}$.

[23] So spanned by objects $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$ such that the underlying ∞-category of $\mathcal{C}_i$ is presentable for each $1 \leq i \leq n$.

Proof of (C): To describe $\mathrm{Im}(\mathrm{Ho}(\Theta^{\otimes} \circ \mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$ we will go through the same steps of (B) and identify the replete image of the respective functor at each step. We start with $\Phi^{\otimes}$, for which $\mathrm{Im}(\mathrm{Ho}(\Phi^{\otimes}))$ is described in Definition 3.2.2.1.

Combining this with Proposition B.5.1.1 we can describe $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_*))$ as follows.

(ObjI) A functor $A\colon \mathcal{O}^{\otimes} \to \mathfrak{Cat}_{\infty}^{\times}$, considered as an object of $\mathrm{Ho}(\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}^{\times}))$, is in $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_*))$ if and only if the following hold.

(ObjI.1) For each object X of $\mathcal{O}^{\otimes}$, if $A(X) \simeq \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_k$ then for each $1 \le i \le k$ the ∞-category $\mathcal{C}_i$ admits all $\mathfrak{I}$-indexed colimits.

(ObjI.2) If β is a morphism in $\mathcal{O}^{\otimes}$, and

$$A(\beta)\colon \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_k \to \mathcal{C}'_1 \oplus \cdots \oplus \mathcal{C}'_l$$

lies over a morphism $\varphi\colon \langle k \rangle \to \langle l \rangle$ of Fin_*, then for each $1 \le j \le l$ the associated functor

$$\prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{C}'_j$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

(MorI) A natural transformation $f\colon A \to B$ of functors $\mathcal{O}^{\otimes} \to \mathfrak{Cat}_{\infty}^{\times}$, considered as a morphism of $\mathrm{Ho}(\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}^{\times}))$, is in $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_*))$ if and only if the following hold.

(MorI.1) A and B are in $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_*))$.

(MorI.2) For every object X of $\mathcal{O}^{\otimes}$ the morphism

$$f_X\colon \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_k \simeq A(X) \to B(X) \simeq \mathcal{C}'_1 \oplus \cdots \oplus \mathcal{C}'_l$$

lying over a morphism $\varphi\colon \langle k \rangle \to \langle l \rangle$ is such that for every $1 \le j \le l$ the associated functor

$$\prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{C}'_j$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

Describing $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_* \times_{\mathrm{id}} \mathrm{id}))$ needs little extra work, it follows from Proposition B.5.3.1 that an object or morphism of

$$\mathrm{Fun}\big(\mathcal{O}, \mathfrak{Cat}_{\infty}^{\times}\big) \times_{\mathrm{Fun}(\mathcal{O}, \mathsf{Fin}_*)} \mathsf{Fin}_*$$

is in $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_* \times_{\mathrm{id}} \mathrm{id}))$ if and only if its projection to the first factor is and object or morphism of $\mathrm{Im}(\mathrm{Ho}((\Phi^{\otimes})_*))$.

The functor ι'_{Alg} is defined as the inclusion of the full subcategory of objects whose projection to the first factor is a functor $\mathcal{O}^{\otimes} \to \mathfrak{Cat}_{\infty}(\mathfrak{I})^{\otimes}$ that preserves inert morphisms, and ι_{Alg} is defined analogously. As by Proposition 3.2.2.3 (3) a morphism in $\mathfrak{Cat}_{\infty}(\mathfrak{I})^{\otimes}$ is inert if and only if $\Phi^{\otimes}$ maps that morphism to an inert morphism in $\mathfrak{Cat}_{\infty}^{\times}$, we can conclude that an object or morphism of $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\otimes)$ is in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^\otimes))$ if and only if $\mathrm{Ho}(\iota_{\mathrm{Alg}})$ maps it into $\mathrm{Im}(\mathrm{Ho}((\Phi^\otimes)_* \times_{\mathrm{id}} \mathrm{id}))$. This leads to the following description of $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^\otimes))$.

We will notationally identify $\langle n\rangle \wedge \langle m\rangle$ with $(\langle n\rangle^\circ \times \langle m\rangle)_*$ and thus write non-basepoint elements of $\langle n\rangle \wedge \langle m\rangle$ as pairs (i,j) with $1 \le i \le n$ and $1 \le j \le m$.

(ObjII) An object A of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\otimes_{\langle n\rangle}$, considered as an object of $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\otimes)$, is in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^\otimes))$ if and only if the following hold.

(ObjII.1) For each $k \ge 0$ and object X in $\mathcal{O}^\otimes_{\langle k\rangle}$, if

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(X) \simeq \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)}$$

then for each $1 \le i_1 \le n$ and $1 \le i_2 \le k$ the ∞-category $\mathcal{C}_{(i_1,i_2)}$ admits all $\mathfrak{I}$-indexed colimits.

(ObjII.2) If $\varphi\colon \langle k\rangle \to \langle l\rangle$ is a morphism in Fin_* and $f\colon X \to Y$ a morphism in $\mathcal{O}^\otimes$ lying over φ, and

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(f)\colon \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)} \to \mathcal{C}'_{(1,1)} \oplus \cdots \oplus \mathcal{C}'_{(n,l)}$$

then for each $1 \le j_1 \le n$ and $1 \le j_2 \le l$ the associated functor

$$\prod_{\varphi(i)=j_2} \mathcal{C}_{(j_1,i)} \to \mathcal{C}'_{(j_1,j_2)}$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

(MorII) A morphism $f\colon A \to B$ of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\otimes$, lying over a morphism $\varphi\colon \langle n\rangle \to \langle m\rangle$ in Fin_* and considered as a morphism of $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\otimes)$, is in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^\otimes))$ if and only if the following hold.

(MorII.1) A and B are in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^\otimes))$.

(MorII.2) For every $k \ge 0$ and object X in $\mathcal{O}^\otimes_{\langle k\rangle}$ the morphism

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(f)_X\colon \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)} \to \mathcal{C}'_{(1,1)} \oplus \cdots \oplus \mathcal{C}'_{(m,k)}$$

is such that for every $1 \le j_1 \le m$ and $1 \le j_2 \le k$ the associated functor

$$\prod_{\varphi(i)=j_1} \mathcal{C}_{(i,j_2)} \to \mathcal{C}'_{(j_1,j_2)}$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

We will now replace these conditions with equivalent descriptions that are more amenable to describing what happens under the equivalence $\Theta^\otimes$.

Let $A \simeq A_1 \oplus \cdots \oplus A_n$ be an object of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^\otimes_{\langle n\rangle}$, let $k \ge 0$, let $X \simeq X_1 \oplus \cdots \oplus X_k$ be an object of $\mathcal{O}^\otimes_{\langle k\rangle}$, and let

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(X) \simeq \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)}$$

be the usual decomposition. Let $1 \leq i \leq n$ and $g_i \colon A \to A_i$ be an inert morphism lying over ρ^i. By Proposition E.4.2.3 (2) the morphism $(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(g_i)(X)$ can be identified with the inert morphism

$$\mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)} \to \mathcal{C}_{(i,1)} \oplus \cdots \oplus \mathcal{C}_{(i,k)}$$

in $\mathfrak{Cat}_\infty^\times$ over $\rho^i \wedge \mathrm{id}_{\langle k\rangle}$. Furthermore, as A_i lies in $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes}_{\langle 1\rangle} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$ (see Proposition E.4.2.3 (0)) and thus preserves inert morphisms, we also obtain an equivalence as follows.

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A_i)(X) \simeq \bigoplus_{1\leq j\leq k} (\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A_i)(X_j)$$

It follows that condition (ObjII.1) is equivalent to the following condition.

(ObjIII.1) For each $1 \leq i \leq n$ and object X of $\mathcal{O}$, the underlying ∞-category[24] $\mathrm{ev}_X(A_i)$ in $\mathfrak{Cat}_\infty$ is an ∞-category that admits all $\mathfrak{I}$-indexed colimits.

Similarly one obtains that if $f\colon X \to Y$ is a morphism in $\mathcal{O}^{\otimes}$ lying over $\varphi\colon \langle k\rangle \to \langle l\rangle$ then

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(f)\colon \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)} \to \mathcal{C}'_{(1,1)} \oplus \cdots \oplus \mathcal{C}'_{(n,l)}$$

can be identified with a sum $A_1(f) \oplus \cdots \oplus A_n(f)$ in $\mathfrak{Cat}_\infty^\times$, and for $1 \leq j_1 \leq n$ and $1 \leq j_2 \leq l$ the functor

$$\prod_{\varphi(i)=j_2} \mathcal{C}_{(j_1,i)} \to \mathcal{C}'_{(j_1,j_2)}$$

associated to $(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(f)$ can be identified with the analogous functor associated to

$$(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A_{j_1})(f)\colon \mathcal{C}_{(j_1,1)} \oplus \cdots \oplus \mathcal{C}_{(j_1,k)} \to \mathcal{C}'_{(j_1,1)} \oplus \cdots \oplus \mathcal{C}'_{(j_1,l)}$$

at index (j_1, j_2). It follows that condition (ObjII.2) is equivalent to the following condition.

(ObjIII.2) For each $1 \leq i \leq n$, the $\mathcal{O}$-monoidal ∞-category A_i (which we consider as an object of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes}_{\langle 1\rangle} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$) is such that for every morphism $f\colon X_1 \oplus \cdots \oplus X_k \to Y$ in $\mathcal{O}$ lying over $\varphi\colon \langle k\rangle \to \langle 1\rangle$ the associated functor

$$\prod_{1\leq j\leq k} \mathrm{ev}_{X_j} A_i \to \mathrm{ev}_Y A_i$$

is compatible with $\mathfrak{I}$-indexed colimits separately in each variable.

Reformulations (ObjIII.1) and (ObjIII.2) allow us to rephrase (ObjI) as follows, by using the definitions of Θ (given by postcomposing with π) and the monomorphism $\mathrm{Mon}_{\mathcal{O}}^{\mathfrak{I}}(\mathfrak{Cat}_\infty) \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$ from Definition 3.1.3.10, which we can identify with Ψ by Remark 3.2.2.5.

[24]Here we consider A_i as an object of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes}_{\langle 1\rangle} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$

(ObjIII) Let $A \simeq A_1 \oplus \cdots \oplus A_n$ be an object of $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes}_{\langle n\rangle}$, and consider A as an object of $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes})$. Then A is in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$ if and only if for each $1 \leq i \leq n$, the equivalence $\Theta\colon \mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty) \to \mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$ maps A_i to an object in $\mathrm{Im}(\mathrm{Ho}(\Psi))$.

Using that $\Theta^{\otimes}$ is lax monoidal and thus maps $A_1 \oplus \cdots \oplus A_n$ to $\Theta(A_1) \oplus \cdots \oplus \Theta(A_n)$, as well as the definition of $\Psi^{\otimes}$ in Definition 3.2.2.4, we finally obtain the following reformulation.

(ObjIV) Let A be an object $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes})$. Then A is in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$ if and only if $\mathrm{Ho}(\Theta^{\otimes})(A)$ is in $\mathrm{Im}(\mathrm{Ho}(\Psi^{\otimes}))$.

This shows (C) for objects. Let us now turn towards reformulating (MorII). Let $f\colon A \to B$ be a morphism in $\mathrm{Alg}_{\mathcal{O}}(\mathfrak{Cat}_\infty)^{\otimes}$, lying over a morphism $\varphi\colon \langle n\rangle \to \langle m\rangle$ in Fin_*, and let $X \simeq X_1 \oplus \cdots \oplus X_k$ be an object of $\mathcal{O}^{\otimes}_{\langle k\rangle}$. As $(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)$ and $(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(B)$ preserve inert morphisms, we can for $1 \leq j_2 \leq k$ identify the commutative diagram

$$\begin{tikzcd}
(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(X) \arrow[r, "(\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(f)_X"] \arrow[d, "(\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(A)(\rho^{j_2})"'] & (\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(B)(X) \arrow[d, "(\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(B)(\rho^{j_2})"] \\
(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(A)(X_{j_2}) \arrow[r, "(\mathrm{pr}_1\circ\iota_{\mathrm{Alg}})(f)_{X_{j_2}}"'] & (\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(B)(X_{j_2})
\end{tikzcd}$$

lying over

$$\begin{tikzcd}
\langle n\rangle \wedge \langle k\rangle \arrow[r, "\varphi\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\rho^{j_2}"'] & \langle m\rangle \wedge \langle k\rangle \arrow[d, "\mathrm{id}\wedge\rho^{j_2}"] \\
\langle n\rangle \wedge \langle 1\rangle \arrow[r, "\varphi\wedge\mathrm{id}"'] & \langle m\rangle \wedge \langle 1\rangle
\end{tikzcd}$$

with a diagram as indicated below

$$\begin{tikzcd}
\mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,k)} \arrow[r] \arrow[d] & \mathcal{C}'_{(1,1)} \oplus \cdots \oplus \mathcal{C}'_{(m,k)} \arrow[d] \\
\mathcal{C}_{(1,j_2)} \oplus \cdots \oplus \mathcal{C}_{(n,j_2)} \arrow[r] & \mathcal{C}'_{(1,j_2)} \oplus \cdots \oplus \mathcal{C}'_{(m,j_2)}
\end{tikzcd}$$

and the functor

$$\prod_{\varphi(i)=j_1} \mathcal{C}_{(i,j_2)} \to \mathcal{C}'_{(j_1,j_2)}$$

associated to the top horizontal morphism at index (j_1, j_2) with $1 \leq j_1 \leq m$ can be identified with the functor associated to the bottom horizontal morphism at the same index. This implies that (MorII.2) is equivalent to the following condition.

(MorIII.2) For every object X of $\mathcal{O}$, if f is such that

$$(\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}})(f) \colon \mathcal{C}_{(1,1)} \oplus \cdots \oplus \mathcal{C}_{(n,1)} \to \mathcal{C}'_{(1,1)} \oplus \cdots \oplus \mathcal{C}'_{(m,1)}$$

then for every $1 \leq j \leq m$ the associated functor

$$\prod_{\varphi(i)=j} \mathcal{C}_{(i,1)} \to \mathcal{C}'_{(j,1)}$$

preserves $\mathfrak{I}$-indexed colimits separately in each variable.

For X an object of $\mathcal{O}$, the composition $\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}$ is by commutativity of diagram (3.12) homotopic to the composition $(\mathrm{ev}_X)^{\times} \circ \Theta^{\otimes}$. Combining this with the definition of Ψ (see Definition 3.2.2.4) and the reformulation of (MorII.1) made possible by (ObjIV) we finally obtain the following.

(MorIV) Let $f \colon A \to B$ be a morphism in $\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes})$. Then f is a morphism in $\mathrm{Im}(\mathrm{Ho}(\mathrm{Alg}_{\mathcal{O}}(\Phi)^{\otimes}))$ if and only if $\mathrm{Ho}(\Theta^{\otimes})(f)$ is in $\mathrm{Im}(\mathrm{Ho}(\Psi^{\otimes}))$.

This shows (C) and thereby ends the proof. □

Remark 3.2.2.9. In this remark we will make use of Notation 3.2.2.7.

Let $\mathfrak{I}$ be a collection of small ∞-categories, $\mathcal{O}$ an ∞-operad, and X and object of the underlying category $\mathcal{O}$. Diagram (3.11) constructed in Equation (3.11) can be extended to a commutative diagram as follows

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}\right)^{\otimes} \arrow[rr, "\Theta^{\otimes}_{\mathcal{P}\mathrm{r}}", "\simeq"'] \arrow[dr, "E"'] \arrow[dd, "\mathrm{Alg}\left(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}\right)^{\otimes}"'] & & \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[dl, dashed] \arrow[dd, "\left(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}\right)^{\otimes}"] \\
& \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \arrow[dd, "\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}"] & \\
\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})\right)^{\otimes} \arrow[rr, "\Theta^{\otimes}_{\mathfrak{I}}", "\simeq"'] \arrow[dr, "E"'] \arrow[dd, "\mathrm{Alg}\left(\Phi^{\mathfrak{I}}\right)^{\otimes}"'] & & \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[dl, dashed] \arrow[dd, "\left(\Psi^{\mathfrak{I}}\right)^{\otimes}"] \\
& \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes} \arrow[dd, "\Psi^{\mathfrak{I}}"] & \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[rr, "\Theta^{\otimes}", "\simeq"'] \arrow[dr, "E"'] & & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times} \arrow[dl, "\left(\mathrm{ev}_{\langle 1 \rangle}\right)^{\times}"] \\
& \mathcal{C}\mathrm{at}_\infty^{\times} &
\end{tikzcd}$$

where we write E as an abbreviation for $\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}$.

We will refer to the functors

$$\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes}$$

and
$$\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}$$
as the forgetful functors and denote them by $(\mathrm{ev}_X)^{\otimes}$. ♢

Proposition 3.2.2.10 ([HA, 4.8.5.16 (1)])**.** *In this proposition we make use of Notation 3.2.2.7.*

Let $\mathfrak{I}$ be a collection of small ∞-categories, let $\mathfrak{I}'$ be a subcollection of $\mathfrak{I}$, and let $\mathfrak{J}$ be the collection of all small ∞-categories. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. Then the following statements hold.

(1) The functors $p_{\mathrm{Mon},\mathfrak{I}}$ and $p_{\mathrm{Mon},\mathcal{P}\mathrm{r}}$ are cocartesian fibrations of ∞-operads and thus exhibit $\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ and $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ as symmetric monoidal ∞-categories.

(2) The functors
$$\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}} \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{(\Psi^{\mathfrak{I}}_{\mathfrak{I}'})^{\otimes}} \mathrm{Mon}^{\mathfrak{I}'}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \xrightarrow{\left(\Psi^{\mathfrak{I}'}\right)^{\otimes}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$
are lax symmetric monoidal with respect to the symmetric monoidal structures from (1).

(3) A morphism in $\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ or $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ is inert if and only if its image under $\left(\Psi^{\mathfrak{I}}\right)^{\otimes}$ or $\left(\Psi^{\mathcal{P}\mathrm{r}}\right)^{\otimes}$ in $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$ is inert.

(4) The functor
$$\left(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}\right)^{\otimes} : \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$
is symmetric monoidal with respect to the symmetric monoidal structure from (1).

(5) A morphism f in $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$ is $p_{\mathrm{Mon},\mathcal{P}\mathrm{r}}$-cocartesian if and only if $\left(\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}\right)^{\otimes}(f)$ is $p_{\mathrm{Mon},\mathfrak{I}}$-cocartesian.

(6) Let X be an object in $\mathcal{O}$. The forgetful functors
$$(\mathrm{ev}_X)^{\otimes} : \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^{\otimes}$$
and
$$(\mathrm{ev}_X)^{\otimes} : \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \to \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}$$
from Remark 3.2.2.9 are symmetric monoidal. ♡

Proof. All of the statements will be shown by translating them to statements regarding $\mathrm{Alg}_{\mathcal{O}}\big(\mathcal{C}\mathrm{at}_\infty(\mathfrak{I})\big)^{\otimes}$ and $\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}\right)^{\otimes}$ using Proposition 3.2.2.8. The individual statements then all follow by combining parts of Proposition E.4.2.3 with parts of Proposition 3.2.2.3, as indicated in the table below.

Claim	Combine Proposition E.4.2.3	with Proposition 3.2.2.3
(1)	(3)	(1)
(2)	(7)	(2)
(3)	(2) and (9)	(3)
(4)	(8)	(4)
(5)	(4) and (9)	(5)
(6)	(5)	

□

3.2.2.3. The symmetric monoidal structure on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$

By Proposition 3.2.1.1 and Proposition C.2.0.3 the cocartesian fibration

$$q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$$

preserves products (see also Remark 3.2.1.4). By [HA, 2.4.1.8] we thus obtain an induced symmetric monoidal functor

$$q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\times}$$

between the respective cartesian symmetric monoidal structures.

In this section we upgrade $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ and $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$ to symmetric monoidal functors in a compatible way.

Definition 3.2.2.11 ([HA, 4.8.5.14])**.** Let $\mathfrak{I}$ be a collection of small ∞-categories. We define functors

$$q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}} \to \mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$$

and

$$q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \to \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes}$$

as pullbacks, as indicated in the following pullback diagrams

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}} \arrow[r, "(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}"] \arrow[d, "q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[d, "q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[r, "(\Psi^{\mathfrak{I}})^{\otimes}"'] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\times}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \arrow[r, "(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}})^{\otimes}"] \arrow[d, "q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[d, "q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[r, "(\Psi^{\mathcal{P}\mathrm{r}})^{\otimes}"'] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)^{\times}
\end{tikzcd}$$

where the lower horizontal functors are the ones defined in Notation 3.2.2.7. ♢

Remark 3.2.2.12. Passing to fibers over $\langle 1\rangle$ we obtain a pullback diagram

$$\begin{tikzcd} (\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}})^{\otimes}_{\langle 1\rangle} \arrow[r] \arrow[d, "(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}})^{\otimes}_{\langle 1\rangle}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times}_{\langle 1\rangle} \arrow[d, "(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})^{\times}_{\langle 1\rangle}"] \\ \mathrm{Mon}^{\mathfrak{J}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}_{\langle 1\rangle} \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\times}_{\langle 1\rangle} \end{tikzcd}$$

that can be identified using Remark 3.2.2.5 with the pullback diagram

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}} \arrow[r] \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\ \mathrm{Mon}^{\mathfrak{J}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty}) \end{tikzcd}$$

from Definition 3.1.3.11. A similar statement holds for $(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}})^{\otimes}_{\langle 1\rangle}$. ◊

Proposition 3.2.2.13 ([HA, 4.8.5.16 (1)]). *In this proposition we use notation from Notation 3.2.2.7.*

Let $\mathfrak{J}$ be a collection of small ∞-categories, $\mathfrak{J}'$ a subcollection of $\mathfrak{J}$, and $\mathfrak{J}$ the collection of all small ∞-categories.

(0) The functors $(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}$ and $(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}})^{\otimes}$ from Definition 3.2.2.11 are monomorphisms in $\mathcal{C}\mathrm{at}_{\infty}$, and $(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}$ factors as a composition of a monomorphisms $(\widetilde{\Psi}^{\mathfrak{J}}_{\mathfrak{J}'})^{\otimes}$ with $(\widetilde{\Psi}^{\mathfrak{J}'})^{\otimes}$. Similarly, $(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}})^{\otimes}$ factors as a composition of a monomorphism $(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ with $(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}$.

(1) The functors $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}}$ and $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$ as defined in Definition 3.2.2.11 are cocartesian fibrations of ∞-operads.

(2) The compositions $p_{\mathrm{Mon},\mathfrak{J}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}}$ and $p_{\mathrm{Mon},\mathcal{P}\mathrm{r}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$ exhibit $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{J}}$ and $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$ as symmetric monoidal ∞-categories.

(3) The morphisms of ∞-operads $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}}$ and $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$ are symmetric monoidal.

(4) Let f be a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{J}}$. Then f is $p_{\mathrm{Mon},\mathfrak{J}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}}$-cocartesian if and only if $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}}(f)$ is $p_{\mathrm{Mon},\mathfrak{J}}$-cocartesian and $(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}(f)$ is $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian. An analogous statement holds for morphisms in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$.

(5) The functors $(\widetilde{\Psi}^{\mathfrak{J}}_{\mathfrak{J}'})^{\otimes}$ and $(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}$ of Definition 3.2.2.11 are lax symmetric monoidal.

(6) Let f be a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{J}}$. Then f is inert if and only if $(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}(f)$ is inert. An analogous statement holds for morphisms in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$.

(7) The functor

$$(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes} \colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{J}}$$

of (0) is symmetric monoidal.

(8) Let f be a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$. Then f is $p_{\mathrm{Mon},\mathcal{P}\mathrm{r}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$-cocartesian if and only if $(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})^{\otimes}(f)$ is $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian. ♡

Proof. Proof of (0): That the functors factor as indicated follows from composability of pullback diagrams [HTT, 4.4.2.1] together with Remark 3.2.2.6. By Proposition B.5.2.1, pullbacks of monomorphisms are again monomorphisms, so that the functors in question are monomorphisms follows from Definition 3.2.2.4 and Remark 3.2.2.6.

Proof of (1): The functor $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ is a pullback of $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$, which is a cocartesian fibration of ∞-operads by Proposition 3.2.1.1 and Proposition C.2.0.6. As cocartesian fibrations of ∞-operads are stable under taking pullbacks along morphisms of ∞-operads[25] and $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\times}$ is a morphism of ∞-operads by Proposition 3.2.2.10 (2), we can conclude that $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ is also a cocartesian fibration of ∞-operads, and thus in particular a morphism of ∞-operads by [HA, 2.1.2.14].

Proof of (2): As the ∞-operad $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$ is in fact a symmetric monoidal ∞-category[26] by Proposition 3.2.2.10 (1), it follows[27] with (1) that $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$ is a symmetric monoidal ∞-category as well.

Proof of (3): Follows immediately from Proposition C.1.3.1.

Proof of (4): We do the case of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$, as the proof for $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$ is completely analogous. Let f be a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$. Because $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ maps $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian morphisms to $p_{\mathrm{Mon},\mathfrak{I}}$-cocartesian morphisms by (3), it follows from [HTT, 2.4.1.3 (3)] that f is $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian if and only if $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}(f)$ is $p_{\mathrm{Mon},\mathfrak{I}}$-cocartesian and f is $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian. The claim now follows from Proposition C.1.1.1.

Proof of (6): We again only discuss the case of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$, as the proof for $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}}$ is completely analogous. In light of (4) it suffices to show that if f is a morphism of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}}$ lying over an inert morphism in Fin_*, then $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}(f)$ is $p_{\mathrm{Mon}} \circ q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian if and only if $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}(f)$ and $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}(f)$ are inert.

Combining that $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ is a morphism of ∞-operads with [HTT, 2.4.1.3 (3)] we obtain that $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}(f)$ being $p_{\mathrm{Mon}} \circ q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian is equivalent to $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}(f)$ as well as

$$\left(q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \circ \left(\widetilde{\Psi}^{\mathfrak{I}}\right)^{\otimes}\right)(f) \simeq \left(\left(\Psi^{\mathfrak{I}}\right)^{\otimes} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}\right)(f)$$

being inert. The claim now follows by applying Proposition 3.2.2.10 (3).

Proof of (5): Immediate consequence of (6).

Proof of (8): Analogous to the proof of (6), using that $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ is even symmetric monoidal and Proposition 3.2.2.10 (5).

Proof of (7): Immediate consequence of (8). □

[25] This is a special case of the functoriality of cocartesian families of monoidal ∞-categories discussed in Remark 3.1.1.3 – in this case we consider [0]-families, which are just cocartesian fibrations of ∞-operads.

[26] I. e. the canonical morphism of ∞-operads $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \to \mathsf{Fin}_*$ is a cocartesian fibration.

[27] Cocartesian fibrations are closed under composition by [HTT, 2.4.2.3 (3)].

3.2.3. LMod as a symmetric monoidal functor

In Section 3.1 we constructed a natural transformation $\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod} \to \mathrm{pr}$ of functors $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_\infty$, see Definition 3.1.3.8. It was shown in Proposition 3.2.1.3 that $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ admits products and that LMod and pr preserve products. This makes $\mathrm{ev}_{\mathfrak{m}}$ into a morphism in $\mathrm{Fun}^\times(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}, \mathcal{C}\mathrm{at}_\infty)$, the full subcategory of $\mathrm{Fun}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}, \mathcal{C}\mathrm{at}_\infty)$ spanned by the product-preserving functors. [HA, 2.4.1.8] then implies that $\mathrm{ev}_{\mathfrak{m}}$ can be upgraded to a natural transformation $\mathrm{ev}_{\mathfrak{m}}{}^\times \colon \mathrm{LMod}^\times \to \mathrm{pr}^\times$ of symmetric monoidal functors $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\times \to \mathcal{C}\mathrm{at}_\infty^\times$.

We also investigated the behavior of $\mathrm{ev}_{\mathfrak{m}}$ with respect to algebras in *presentable* symmetric monoidal ∞-categories as input, and showed in Proposition 3.1.3.12 that $\mathrm{ev}_{\mathfrak{m}}$ lifts to a natural transformation of functors $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Finally, in Section 3.2.2 we constructed symmetric monoidal structures on $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and upgraded the inclusion functors to $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ and $\mathcal{C}\mathrm{at}_\infty$ to lax symmetric monoidal functors (see Proposition 3.2.2.3 and Proposition 3.2.2.13).

The situation is depicted in the non-dashed part of the following diagram. Squares that contain parallel arrows on opposing sides are to be interpreted as encoding two commutative diagrams, one considering only the arrows at the top, and one only considering the arrows at the bottom, as well as a compatible homotopy between the two natural transformations from the source corner to the target corner that one obtains by pre-composing and post-composing.

$$\begin{tikzcd}
 & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}^{\otimes} \arrow[dl] \arrow[rr, dashed, bend left=15] \arrow[rr, dashed, bend right=15] \arrow[rr, phantom, "\Downarrow\mathrm{ev}_{\mathfrak{m}}"] & & \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \arrow[dl] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[rr, bend left=15] \arrow[rr, bend right=15] \arrow[rr, phantom, "\Downarrow\mathrm{ev}_{\mathfrak{m}}^{\times}"] & & \mathcal{C}\mathrm{at}_\infty^{\times} & \\
 & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \arrow[uu] \arrow[dl] \arrow[rr, bend left=15] \arrow[rr, bend right=15] \arrow[rr, phantom, "\Downarrow\mathrm{ev}_{\mathfrak{m}}"] & & \mathcal{P}\mathrm{r}^{\mathrm{L}} \arrow[uu] \arrow[dl] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[uu] \arrow[rr, bend left=15] \arrow[rr, bend right=15] \arrow[rr, phantom, "\Downarrow\mathrm{ev}_{\mathfrak{m}}"] & & \mathcal{C}\mathrm{at}_\infty \arrow[uu] &
\end{tikzcd} \tag{3.17}$$

The vertical functors are all inclusions of the fiber over $\langle 1 \rangle$, the bottom square was constructed in Proposition 3.1.3.12, and the front square can be obtained from [HA, 2.4.1.8]. To be more precise about how the above cube is to be interpreted with regards to parallel arrows, we could also depict the cube (3.17) in the form shown below (as just a standard commuting cube in $\mathcal{C}\mathrm{at}_\infty$), using that natural transformations are equivalently encoded as functors out of a product with [1].

$$
\begin{tikzcd}
& {[1]} \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}^{\otimes} \arrow[rr, dashed, "\mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \arrow[dl] & & \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \arrow[dl] \\
{[1]} \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[rr, "\mathrm{ev}_{\mathfrak{m}}^{\times}"] & & \mathcal{C}\mathrm{at}_{\infty}^{\times} & \\
& {[1]} \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \arrow[uu] \arrow[rr, "\mathrm{ev}_{\mathfrak{m}}"] \arrow[dl] & & \mathcal{P}\mathrm{r}^{\mathrm{L}} \arrow[uu] \arrow[dl] \\
{[1]} \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[uu] \arrow[rr, "\mathrm{ev}_{\mathfrak{m}}"] & & \mathcal{C}\mathrm{at}_{\infty} \arrow[uu] &
\end{tikzcd}
\tag{3.18}
$$

The goal of this section is to complete the cube as indicated by the dashed arrows, and in such a way that $\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod} \to \mathrm{pr}$ in its incarnation as a natural transformation of functors $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is upgraded to a natural transformation of symmetric monoidal functors.

Proposition 3.2.3.1 ([HA, 4.8.5.16 (3) and (4)]). *Let $\mathfrak{J}$ be a collection of small ∞-categories that includes $\mathbf{\Delta}^{\mathrm{op}}$. Then the restriction to $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}^{\otimes}$ of the natural transformation $\mathrm{ev}_{\mathfrak{m}}^{\times}$ of symmetric monoidal functors $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \to \mathcal{C}\mathrm{at}_{\infty}^{\times}$ factors through $\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J})^{\otimes}$. Analogously, the restriction to $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}^{\otimes}$ factors through $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}$. The situation is depicted in the following commutative diagram.*

$$
\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}^{\otimes} \arrow[r, dashed, bend left, "\mathrm{LMod}^{\otimes}"] \arrow[r, dashed, bend right, "\mathrm{pr}^{\otimes}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \arrow[d, "(\widetilde{\Psi}_{\mathfrak{J}}^{\mathcal{P}\mathrm{r}})^{\otimes}"'] & \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \arrow[d, "(\Phi_{\mathfrak{J}}^{\mathcal{P}\mathrm{r}})^{\otimes}"] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{J}}^{\otimes} \arrow[r, dashed, bend left, "\mathrm{LMod}^{\otimes}"] \arrow[r, dashed, bend right, "\mathrm{pr}^{\otimes}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \arrow[d, "(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}"'] & \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J})^{\otimes} \arrow[d, "(\Phi^{\mathfrak{J}})^{\otimes}"] \\
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[r, bend left, "\mathrm{LMod}^{\times}"] \arrow[r, bend right, "\mathrm{pr}^{\times}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}^{\times}"] & \mathcal{C}\mathrm{at}_{\infty}^{\times}
\end{tikzcd}
\tag{3.19}
$$

Furthermore, the two natural transformations $\mathrm{ev}_{\mathfrak{m}}^{\otimes}$ that we obtain in this manner are natural transformations of symmetric monoidal functors, and the underlying diagram of *underlying ∞-categories of diagram (3.19) can be identified with diagram (3.10) from Proposition 3.1.3.12.* ♡

Proof. In this proof we will use Notation 3.2.2.7 as well as the notation from Definition 3.2.2.11 and Proposition 3.2.2.13.

Reformulation of the lifting problem: We first note that by Proposition 3.2.2.3 (0) and Definition 3.2.2.1 together with Proposition B.4.4.1 and Proposition B.1.2.1 the right vertical functors $\left(\Phi_{\mathfrak{I}}^{\mathcal{P}r}\right)^{\otimes}$ and $\left(\Phi^{\mathfrak{I}}\right)^{\otimes}$ in diagram (3.19) are monomorphisms. In this situation Proposition B.4.3.1 implies that the dashed lifts in the following diagram are essentially unique if they exist.

$$\begin{tikzcd}
{[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}r}^{\otimes}} \arrow[r, dashed, "\mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \arrow[d, "\mathrm{id} \times \left(\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}r}\right)^{\otimes}"'] & \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \arrow[d, "\left(\Phi_{\mathfrak{I}}^{\mathcal{P}r}\right)^{\otimes}"] \\
{[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}^{\otimes}} \arrow[r, dashed, "\mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \arrow[d, "\mathrm{id} \times \left(\widetilde{\Psi}^{\mathfrak{I}}\right)^{\otimes}"'] & \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})^{\otimes} \arrow[d, "\left(\Phi^{\mathfrak{I}}\right)^{\otimes}"] \\
{[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times}} \arrow[r, "\mathrm{ev}_{\mathfrak{m}}^{\times}"] & \mathcal{C}\mathrm{at}_{\infty}^{\times}
\end{tikzcd} \tag{$*$}$$

Furthermore, Proposition B.4.3.1 also implies that these lifts exists if and only if the following two inclusions of replete subcategories of $\mathrm{Ho}(\mathcal{C}\mathrm{at}_{\infty}^{\times})$ hold.

$$\begin{gathered}
\mathrm{Im}\left(\mathrm{Ho}\left(\mathrm{ev}_{\mathfrak{m}}^{\times} \circ \left(\mathrm{id} \times \left(\widetilde{\Psi}^{\mathfrak{I}}\right)^{\otimes}\right)\right)\right) \subseteq \mathrm{Im}\left(\mathrm{Ho}\left(\left(\Phi^{\mathfrak{I}}\right)^{\otimes}\right)\right) \\
\mathrm{Im}\left(\mathrm{Ho}\left(\mathrm{ev}_{\mathfrak{m}}^{\times} \circ \left(\mathrm{id} \times (\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}\right) \circ \left(\mathrm{id} \times (\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}r})^{\otimes}\right)\right)\right) \subseteq \mathrm{Im}\left(\mathrm{Ho}\left((\Phi^{\mathfrak{I}})^{\otimes} \circ (\Phi_{\mathfrak{I}}^{\mathcal{P}r})^{\otimes}\right)\right)
\end{gathered} \tag{A}$$

Verification of the inclusion of replete images for fibers over Fin_*: We start by checking those inclusions for objects and morphisms lying in a fiber over $\langle n \rangle$ for some $n \geq 0$. Because $(\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}r})^{\otimes}$, $(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}$, $(\Phi_{\mathfrak{I}}^{\mathcal{P}r})^{\otimes}$, and $(\Phi^{\mathfrak{I}})^{\otimes}$ are all morphisms of ∞-operads (see Proposition 3.2.2.3 (2) and Proposition 3.2.2.13 (5)), we can identify the diagram induced by ($*$) on fibers over $\langle n \rangle$ with the following diagram.

$$\begin{tikzcd}
{[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}r}^{\times n}} \arrow[r, dashed] \arrow[d, "\mathrm{id} \times \left(\widetilde{\Psi}_{\mathfrak{I}}^{\mathcal{P}r}\right)^{\otimes}"'] & \mathcal{P}\mathrm{r}^{\mathrm{L}\times n} \arrow[d, "\left(\Phi_{\mathfrak{I}}^{\mathcal{P}r}\right)^{\otimes}"] \\
{[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}^{\times n}} \arrow[r, dashed] \arrow[d, "\mathrm{id} \times \left(\widetilde{\Psi}^{\mathfrak{I}}\right)^{\otimes}"'] & \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})^{\times n} \arrow[d, "\left(\Phi^{\mathfrak{I}}\right)^{\otimes}"] \\
{[1] \times \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times n}} \arrow[r, "\mathrm{ev}_{\mathfrak{m}}^{\times n}"] & \mathcal{C}\mathrm{at}_{\infty}^{\times n}
\end{tikzcd} \tag{$**$}$$

By Remark 3.2.2.12 and Remark 3.2.2.2 this diagram can be identified with the n-fold product of the lifting problem solved in Proposition 3.1.3.12, so we deduce that the inclusions (A) hold for objects as well as for morphisms lying over an identity morphism in Fin_*.

Reduction of the presentable case to the other cases: Suppose for the moment that we have shown the first inclusion of (A) for all families of small ∞-categories. Given that we already know the second inclusion on objects, the second inclusion will follow if $(\Phi_{\mathfrak{J}}^{\mathcal{P}\mathrm{r}})^{\otimes}$ and $(\widetilde{\Psi}_{\mathfrak{J}}^{\mathcal{P}\mathrm{r}})^{\otimes}$ are fully faithful for $\mathfrak{J}$ the family of all small ∞-categories. That $\left(\Phi_{\mathfrak{J}}^{\mathcal{P}\mathrm{r}}\right)^{\otimes}$ is fully faithful is the case by Definition 3.2.2.1, and $\left(\widetilde{\Psi}_{\mathfrak{J}}^{\mathcal{P}\mathrm{r}}\right)^{\otimes}$ is fully faithful combining Proposition B.5.2.1 with Definition 3.2.2.11 and Definition 3.2.2.4.

Verification of the inclusion of replete images for morphisms: Let

$$\Gamma\colon A_1 \oplus \cdots \oplus A_n \to B_1 \oplus \cdots \oplus B_m$$

be a morphism in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times}$ lying over a morphism

$$G\colon \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n \to \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_m$$

in $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\times}$ lying over a morphism

$$\gamma\colon \langle n\rangle \to \langle m\rangle$$

in Fin_*. Note that by Remark 3.1.3.7 we can interpret A_i as an object of $\mathrm{Alg}(\mathcal{C}_i)$ and similarly for B_j. Assume that Γ lies in the replete image of $(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}$. By applying Proposition B.5.2.1, the definition of $(\widetilde{\Psi}^{\mathfrak{J}})^{\otimes}$ in Definition 3.2.2.11, as well as Definition 3.2.2.4 we can unpack this to see that this implies in particular that the underlying ∞-categories of $\mathcal{C}_1, \ldots, \mathcal{C}_n, \mathcal{D}_1, \ldots, \mathcal{D}_m$ admit $\mathfrak{J}$-indexed colimits, that the tensor product functors on $\mathcal{C}_1, \ldots, \mathcal{C}_n, \mathcal{D}_1, \ldots, \mathcal{D}_m$ are compatible with $\mathfrak{J}$-indexed colimits, and that for every $1 \leq j \leq m$ the functor

$$\prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{D}_j$$

associated to G preserves $\mathfrak{J}$-indexed colimits in each variable separately. Applying $\mathrm{ev}_{\mathfrak{m}}^{\times}$ to Γ we obtain a commutative diagram as follows in $\mathcal{C}\mathrm{at}_{\infty}^{\times}$ (see Remark 3.1.3.9).

$$\begin{tikzcd}
\mathrm{LMod}_{A_1}(\mathcal{C}_1) \oplus \cdots \oplus \mathrm{LMod}_{A_n}(\mathcal{C}_n) \arrow[r, "\mathrm{LMod}^{\times}(\Gamma)"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}^{\times}(A_1 \oplus \cdots \oplus A_n)"'] & \mathrm{LMod}_{B_1}(\mathcal{D}_1) \oplus \cdots \oplus \mathrm{LMod}_{B_m}(\mathcal{D}_m) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}^{\times}(B_1 \oplus \cdots \oplus B_m)"] \\
\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n \arrow[r, "\mathrm{pr}^{\times}(\Gamma)"'] & \mathcal{C}_1' \oplus \cdots \oplus \mathcal{D}_m
\end{tikzcd}$$

What we have to show is that this diagram is in the replete image of $(\Phi^{\mathfrak{J}})^{\otimes}$. What we have already shown when considering objects and morphisms in fibers over Fin_* already implies that the four objects as well as the vertical morphisms are in the replete image of $(\Phi^{\mathfrak{J}})^{\otimes}$, so it only remains to show this for the horizontal morphisms. By definition (see Definition 3.2.2.1) this means that we have to show that for every $1 \leq j \leq m$ the two horizontal functors in the following commutative diagram associated to the diagram above preserve $\mathfrak{I}$-indexed colimits separately in each variable (see Remark 3.1.3.9 for the identifications made here – in particular the functors called $\mathrm{ev}_{\mathfrak{m}}$ are the actual evaluation functors).

$$\begin{tikzcd} \prod_{\varphi(i)=j} \mathrm{LMod}_{A_i}(\mathcal{C}_i) \arrow[r] \arrow[d, "\prod_{\varphi(i)=j} \mathrm{ev}_{\mathfrak{m}}"'] & \mathrm{LMod}_{B_j}(\mathcal{D}_j) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\ \prod_{\varphi(i)=j} \mathcal{C}_i \arrow[r] & \mathcal{D}_j \end{tikzcd}$$

The bottom horizontal functor is the same one as the functor associated to G that we already mentioned preserving $\mathfrak{I}$-indexed colimits separately in each variable. We also already know that the left vertical functor is a product of functors that preserve $\mathfrak{I}$-indexed colimit, so it follows that the compositions from the top left to the bottom right preserve $\mathfrak{I}$-indexed colimits separately in each variable. As the tensor product in the monoidal ∞-category $\mathcal{D}_j$ is compatible with $\mathfrak{I}$-indexed colimits, we can now apply [HA, 4.2.3.5] to deduce that the top horizontal functor also preserves $\mathfrak{I}$-indexed colimits separately in each variable.

On showing that the induced functors are symmetric monoidal: We have now constructed a commutative diagram (3.19). We next need to prove that the induced functors $\mathrm{LMod}^{\otimes}$ and $\mathrm{pr}^{\otimes}$ are symmetric monoidal[28], i. e. that they preserve morphisms that are cocartesian with respect to the canonical morphism of ∞-operads to Fin_* (see [HA, 2.1.3.7]).

Proof that the induced functors are lax monoidal: As all solid arrows in diagram (3.19) are lax monoidal (so preserve inert morphisms)[29], and the right vertical morphisms of that diagram reflect inert morphisms by Proposition 3.2.2.3 (3), we can already conclude that the functors called $\mathrm{LMod}^{\otimes}$ and $\mathrm{pr}^{\otimes}$ preserve inert morphisms, i. e. are lax monoidal.

Reduction of what needs to be checked for symmetric monoidality: Let $\mathfrak{J}$ be the collection of all small ∞-categories. Note that in the commutative diagram

$$\begin{tikzcd}[column sep=huge] \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \arrow[r, bend left, "\mathrm{LMod}^{\otimes}"] \arrow[r, bend right, "\mathrm{pr}^{\otimes}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}^{\otimes}_{\mathfrak{m}}"] \arrow[d, "(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}"'] & \mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \arrow[d, "(\Phi^{\mathcal{P}\mathrm{r}}_{\mathfrak{J}})^{\otimes}"] \\ \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{J}} \arrow[r, bend left, "\mathrm{LMod}^{\otimes}"] \arrow[r, bend right, "\mathrm{pr}^{\otimes}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}^{\otimes}_{\mathfrak{m}}"] & \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{J})^{\otimes} \end{tikzcd} \tag{3.20}$$

[28]The ∞-category of symmetric monoidal functors from one symmetric monoidal ∞-category to another one is a full subcategory of the ∞-category of functors over Fin_* (see [HA, 2.1.3.7]), so there is no extra condition that we need to check for $\mathrm{ev}_{\mathfrak{m}}$.

[29]See Proposition 3.2.2.13 (5) for the left vertical functors and Proposition 3.2.2.3 (2) for the right vertical functors. The bottom horizontal functor is symmetric monoidal by construction.

the left vertical functor is symmetric monoidal by Proposition 3.2.2.13 (7) and the right vertical functor reflects cocartesian morphisms with respect to the canonical morphisms of ∞-operads to Fin_* by Proposition 3.2.2.3 (5). If we show that the two bottom horizontal morphisms of ∞-operads are symmetric monoidal it will thus follow that the same is true for the two top horizontal ones.

Taking into account Proposition E.1.1.1 it thus remains to show that the functors

$$\mathrm{LMod}^{\otimes}, \mathrm{pr}^{\otimes}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})^{\otimes}$$

map $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian lifts of μ and ϵ (see Proposition E.1.1.1 for the definitions) to $p_{\mathfrak{I}}$-cocartesian morphisms.

Cocartesian lifts of ϵ: Denote by $\emptyset$ the unique object in $(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}})^{\otimes}_{\langle 0\rangle}$, and let

$$\widetilde{E}'\colon \emptyset \to A$$

be a $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian lift of ϵ lying over a $p_{\mathrm{Mon},\mathfrak{I}}$-cocartesian morphism[30]

$$E'\colon \emptyset \to \mathcal{C}$$

in $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$.

That E' is $p_{\mathrm{Mon},\mathfrak{I}}$-cocartesian implies that the functor

$$E\colon \mathbb{1}_{\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})} \to \mathcal{C}$$

associated to E' is an equivalence, so that we can identify $\mathcal{C}$ with the unit[31] $\mathbb{1}_{\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})}$ in $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$.

By Proposition 3.2.2.13 (4) the morphism $(\Psi^{\mathfrak{I}})^{\otimes}(\widetilde{E}')$ is $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian. The commutative diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[r, "\pi_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \arrow[d, "q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\times} \arrow[r, "\pi_{\mathrm{Mon}}"'] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd}$$

where the horizontal functors are the cartesian structures is a pullback diagram by Proposition 3.2.1.1 and Proposition F.1.0.2. Applying Proposition C.1.1.1 we conclude that the functor

$$\mathbb{1}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \to A$$

associated to $(\Psi^{\mathfrak{I}})^{\otimes}(\widetilde{E}')$ (where $\mathbb{1}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ is the final object in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$, so the unit object in the cartesian symmetric monoidal structure) is a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian lift of the monoidal

[30] $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$ is symmetric monoidal by Proposition 3.2.2.13 (3).

[31] By Proposition 3.2.2.10 (6) the forgetful functor $(\mathrm{ev}_{\langle 1\rangle})^{\otimes}\colon \mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})^{\otimes}$ is symmetric monoidal, so the underlying ∞-category of the monoidal unit $\mathbb{1}_{\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})}$ of $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$ is given by the monoidal unit of $\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})$.

functor[32]

$$e\colon [0] \to \mathcal{C}$$

associated to $(\Phi^{\mathcal{I}})^{\otimes}(E')$. The final object $\mathbb{1}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ can then using Remark 3.1.3.7, Proposition 3.2.1.1, and Proposition C.2.0.3 be identified with the final object in

$$\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{[0]} \simeq \mathrm{Alg}([0])^{\mathrm{op}}$$

which is the unit object $\mathbb{1}_{[0]}$ in $[0]$[33]. That the morphism $\mathbb{1}_{[0]} \to A$ is $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian then implies using Remark 3.1.3.7 that A can be identified as an object of

$$\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{C}} \simeq \mathrm{Alg}(\mathcal{C})^{\mathrm{op}}$$

with $e(\mathbb{1}_{[0]}) \simeq \mathbb{1}_{\mathcal{C}}$.

Getting back to showing that $\mathrm{LMod}^{\otimes}$ and $\mathrm{pr}^{\otimes}$ map $\widetilde{E'}$ to a $p_{\mathfrak{I}}$-cocartesian morphism, we obtain the following commutative diagram in $\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})$ by applying $\mathrm{ev}_{\mathfrak{m}}^{\otimes}$ to $\widetilde{E'}$.

$$\begin{tikzcd}
\emptyset \arrow[r, "\mathrm{LMod}^{\otimes}(\widetilde{E'})"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"'] & \mathrm{LMod}_A(\mathcal{C}) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\
\emptyset \arrow[r, "\mathrm{pr}^{\otimes}(\widetilde{E'})"'] & \mathcal{C}
\end{tikzcd}$$

It suffices to show that the associated horizontal functors as depicted in the diagram below are equivalences.

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathrm{LMod}_A(\mathcal{C}) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\
\mathbb{1}_{\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})} \arrow[r, "E"'] & \mathcal{C}
\end{tikzcd}$$

That E is an equivalence was already noted, and the right vertical functor $\mathrm{ev}_{\mathfrak{m}}$ is an equivalence by [HA, 4.2.4.9], as A is the unit object in $\mathcal{C}$.

Cocartesian lifts of μ: Let $\mathcal{C}$ and $\mathcal{D}$ be two objects in $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$, let A be an algebra in $\mathcal{C}$, and let B be an algebra in $\mathcal{D}$. We can use an analysis completely analogous to the ϵ-case to describe a $p_{\mathrm{Mon},\mathfrak{I}} \circ q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}$-cocartesian lift $\widetilde{M'}\colon A \oplus B \to A \otimes_{\mathfrak{I}} B$. Let us just note that from the lax symmetric monoidal functor $\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$ we obtain a monoidal functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_{\mathfrak{I}} \mathcal{D}$, and the induced functor on algebras sends the pair (A, B) to an object $A \otimes_{\mathfrak{I}} B$ of $\mathrm{Alg}(\mathcal{C} \otimes_{\mathfrak{I}} \mathcal{D})$, and it is this algebra considered as an object in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ that is the target of $\widetilde{M'}$.

[32] The final object of $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$ (which is also the monoidal unit with respect to the cartesian symmetric monoidal structure) is by Proposition 3.2.2.10 (6) given by the essentially unique monoidal structure on the ∞-category that is final in $\mathcal{C}\mathrm{at}_{\infty}$, the discrete category $[0]$ that has a single object and only the identity morphism.

[33] In this case this is completely clear because there is only an essentially unique algebra in $[0]$, but we could also invoke [HA, 3.2.1.8].

$\mathrm{ev}_{\mathfrak{m}}^{\otimes}$ applied to $\widetilde{M}'$ yields a commutative diagram (after passing to the associated functors, as before)

$$\begin{tikzcd}
\mathrm{LMod}_A(\mathcal{C}) \times \mathrm{LMod}_B(\mathcal{D}) \arrow[r, "\simeq"] \arrow[d, "- \otimes_{\mathfrak{I}} -"'] & \mathrm{LMod}_{(A,B)}(\mathcal{C} \times \mathcal{D}) \arrow[d, "\mathrm{LMod}(- \otimes_{\mathfrak{I}} -)"] \\
\mathrm{LMod}_A(\mathcal{C}) \otimes_{\mathfrak{I}} \mathrm{LMod}_B(\mathcal{D}) \arrow[r] \arrow[d, "\mathrm{ev}_{\mathfrak{m}} \otimes_{\mathfrak{I}} \mathrm{ev}_{\mathfrak{m}}"'] & \mathrm{LMod}_{A \otimes_{\mathfrak{I}} B}(\mathcal{C} \otimes_{\mathfrak{I}} \mathcal{D}) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\
\mathcal{C} \otimes_{\mathfrak{I}} \mathcal{D} \arrow[r, "\mathrm{id}"'] & \mathcal{C} \otimes_{\mathfrak{I}} \mathcal{D}
\end{tikzcd}$$

and we have to show that the bottom and middle horizontal functors are equivalences. This can be done by applying [HA, 4.7.3.16], and the verification of the necessary hypotheses is carried out in [HA, Proof of 4.8.5.16 (4)]. While our settings are slightly different, for example our functor was constructed on an ∞-category where morphisms of algebras have the opposite variance compared to Lurie's ∞-category, these differences are not relevant in the proof, the most that would need to be changed for our setting is replacing RMod by LMod.

Note that this is the step that requires the assumption that $\mathbf{\Delta}^{\mathrm{op}}$ is contained in $\mathfrak{I}$.

Compatibility of the constructed diagram with diagram (3.10) *from Proposition 3.1.3.12*: Finally, it only remains to show that the underlying diagram of (3.19) on underlying ∞-categories can be identified with diagram (3.10) from Proposition 3.1.3.12. But this follows from $\Phi^{\mathfrak{I}}$ and $\Phi_{\mathfrak{I}}^{\mathcal{P}\mathrm{r}}$ being monomorphisms together with the uniqueness part of Proposition B.4.3.1. □

3.3. Bialgebras

Let C be a symmetric monoidal category. An (associative) *algebra* A in C consists of a multiplication $A \otimes A \to A$ and a unit $\mathbb{1}_{\mathsf{C}} \to A$ such that diagrams encoding associativity and unitality commute. The notion of (coassociative) *coalgebras* A in C is dual to this; instead of a multiplication we require a *comultiplication* $A \to A \otimes A$, and instead of a unit we require a *counit* $A \to \mathbb{1}_{\mathsf{C}}$, satisfying diagrams encoding coassociativity and counitality. Instead of defining coalgebras from scratch like this we can also define them in terms of algebras: A coalgebra in C is the same thing as an algebra in C^{op}.

We are often not only interested in individual algebras A in C, but the category of all (associative) algebras in C, which we denote by $\mathrm{Alg}_{\mathsf{Assoc}}(\mathsf{C})$. The data of a morphism of algebras $A \to B$ just consists of a morphism in C from the underlying object of A to the underlying object of B, but we require that this morphism is compatible with the respective multiplication and unit morphisms. If we want morphisms of coalgebras to similarly be given by morphisms of underlying objects that are compatible with comultiplication and counit, then we need to fix having passed to the opposite category by doing it a second time, leading to the definition of the category of (coassociative) coalgebras as

$$\mathrm{coAlg}_{\mathsf{Assoc}}(\mathsf{C}) := \mathrm{Alg}_{\mathsf{Assoc}}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$$

This is the perspective that is most suitable to extend the definition to the ∞-categorical setting.

Definition 3.3.0.1. Let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a morphisms of ∞-operads and $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ an $\mathcal{O}$-monoidal ∞-category. Then we set

$$\mathrm{coAlg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \coloneqq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$$

where $\mathcal{C}^{\mathrm{op}}$ carries the $\mathcal{O}$-monoidal structure described in [HA, 2.4.2.7][34]. ♢

Notation 3.3.0.2. We will use similar notational shortcuts for coAlg as for Alg. In particular, in the situation of Definition 3.3.0.1:

- If α is the identity, then we will shorten $\mathrm{coAlg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C})$ to $\mathrm{coAlg}\,/\mathcal{O}(\mathcal{C})$.
- If $\mathcal{O}^{\otimes} = \mathsf{Fin}_*$, then we write $\mathrm{coAlg}_{\mathcal{O}'}(\mathcal{C})$ instead of $\mathrm{coAlg}_{\mathcal{O}'/\mathsf{Comm}}(\mathcal{C})$.
- We write $\mathrm{coAlg}(\mathcal{C})$ for $\mathrm{coAlg}_{/\mathsf{Assoc}}(\mathcal{C})$ or $\mathrm{coAlg}_{\mathsf{Assoc}}(\mathcal{C})$.
- We write $\mathrm{coCAlg}(\mathcal{C})$ for $\mathrm{coAlg}_{\mathsf{Comm}}(\mathcal{C})$. ♢

The category $\mathrm{Alg}_{\mathsf{Assoc}}(\mathsf{C})$ inherits a symmetric monoidal structure from C, so that we can form the category

$$\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Assoc}}(\mathsf{C}) \coloneqq \mathrm{coAlg}_{\mathsf{Assoc}}\big(\mathrm{Alg}_{\mathsf{Assoc}}(\mathsf{C})\big)$$

of *bialgebras* in C. Unpacking the definition, a bialgebra in C consists of an object A in C together with a multiplication, unit, comultiplication, and counit, satisfying associativity, coassociativity, unitality, and counitality, and such that comultiplication and counit are morphisms of algebras. In this classical setting it is very easy to see that comultiplication and counit are morphisms of algebras if and only if multiplication and unit are morphisms of coalgebras, so that there is a canonical isomorphism

$$\mathrm{coAlg}_{\mathsf{Assoc}}\big(\mathrm{Alg}_{\mathsf{Assoc}}(\mathsf{C})\big) \cong \mathrm{Alg}_{\mathsf{Assoc}}\big(\mathrm{coAlg}_{\mathsf{Assoc}}(\mathsf{C})\big)$$

or ordinary categories, and we could have taken either side as a definition for the category of bialgebras $\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Assoc}}(\mathsf{C})$.

Unfortunately, the situation is not quite as easy in the setting of ∞-categories. For the case of commutative and cocommutative bialgebras in a symmetric monoidal ∞-category it is shown in [Lur18, 3.3.4] that the two possible definitions coincide. The case of either commutative or cocommutative bialgebras is handled in [Rak20, 2.1.2]. In all these cases, the crucial input to the proof is the fact that tensor products of commutative algebras happen to be coproducts in the ∞-category of commutative algebras [HA, 3.2.4.7], so the proof strategies do not generalize easily to bialgebras which are neither commutative nor cocommutative. Luckily we will not need to use that the two possible definitions are equivalent in this text. Instead, for us *bialgebra* will always mean *coalgebra in algebras*.

[34] So if the cocartesian fibration $p_{\mathcal{C}}$ is classified by a functor $F\colon \mathcal{O}^{\otimes} \to \mathfrak{Cat}_{\infty}$, then the cocartesian fibration $(\mathcal{C}^{\mathrm{op}})^{\otimes} \to \mathcal{O}^{\otimes}$ is classified by the composite $(-)^{\mathrm{op}} \circ F$.

Definition 3.3.0.3. Let $\alpha\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads, and $\mathcal{C}$ an $\mathcal{O}''$-monoidal ∞-category. Then we define

$$\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C}) \coloneqq \mathrm{coAlg}_{/\mathcal{O}}\Big(\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})\Big)$$

where $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})$ carries the $\mathcal{O}$-monoidal structure of Proposition E.4.2.3, and call $\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C})$ the ∞-category of $\mathcal{O}', \mathcal{O}$-bialgebras in $\mathcal{C}$. ♢

Warning 3.3.0.4. In the notation $\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C})$, the ∞-operad stated *first*, $\mathcal{O}'$, is employed in the *algebra* direction, and $\mathrm{Alg}_{\mathcal{O}'}$ is also what is applied first (i. e. innermost) to $\mathcal{C}$ in our definition. ♢

Remark 3.3.0.5. Let $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ and $p_{\mathcal{O}'}\colon \mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$ be ∞-operads and $\mathcal{C}$ a symmetric monoidal ∞-category.

There is a canonical bifunctor of ∞-operads

$$\alpha\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \xrightarrow{(-\wedge-)\circ\left(p_{\mathcal{O}}\times p_{\mathcal{O}'}\right)} \mathsf{Fin}_*$$

with respect to which we can form the ∞-category of $\mathcal{O}', \mathcal{O}$-bialgebras as in Definition 3.3.0.3.

Note that if we let β be the canonical bifunctor of ∞-operads

$$\beta\colon \mathsf{Fin}_* \times \mathcal{O}'^{\otimes} \xrightarrow{(-\wedge-)\circ\left(\mathrm{id}\times p_{\mathcal{O}'}\right)} \mathsf{Fin}_*$$

then α is the composition $\alpha = \beta \circ (p_{\mathcal{O}} \times \mathrm{id})$. Let $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})'^{\otimes}$ be the $\mathcal{O}$-monoidal category from Proposition E.4.2.3 with respect to α and let $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ be the symmetric monoidal ∞-category from Proposition E.4.2.3 with respect to β. It then follows from Remark E.4.2.4 that there is a pullback diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})'^{\otimes} \arrow[r] \arrow[d, "\mathrm{pr}_2\circ\iota_{\mathrm{Alg}}"'] & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \arrow[d, "\mathrm{pr}_2\circ\iota_{\mathrm{Alg}}"] \\
\mathcal{O}^{\otimes} \arrow[r, "p_{\mathcal{O}}"'] & \mathsf{Fin}_*
\end{tikzcd}$$

in $\mathfrak{Cat}_\infty$, and all morphisms in the square are morphisms of ∞-operads, while the vertical morphisms are even cocartesian fibrations of ∞-operads by Proposition E.4.2.3 (3).

Passing to fiberwise opposites, applying Remark E.2.0.4, and passing to opposites again we then obtain an induced equivalence

$$\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C}) = \mathrm{coAlg}_{/\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})'\big) \xrightarrow{\simeq} \mathrm{coAlg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})\big) \qquad ♢$$

3.3.1. Bialgebras in (co)cartesian symmetric monoidal ∞-categories

Let $\mathcal{C}$ be a cocartesian symmetric monoidal ∞-category[35]. Then if $\mathcal{O}$ is a reduced[36] ∞-operad, then [HA, 2.4.3.9] shows that the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ is an equivalence. In other words, every object of $\mathcal{C}$ carries an essentially unique $\mathcal{O}$-algebra structure. This implies analogous results for bialgebras of cocartesian or cartesian symmetric monoidal ∞-categories, as the next two propositions show.

The first of the two, Proposition 3.3.1.1 can be summarized as saying that every coalgebra in a cocartesian symmetric monoidal ∞-category can be upgraded to a bialgebra in an essentially unique way. The second, Proposition 3.3.1.2, instead says that any algebra in a cartesian symmetric monoidal ∞-category can be upgraded to a bialgebra in an essentially unique way.

Proposition 3.3.1.1. *Let $\mathcal{C}$ be a cocartesian symmetric monoidal ∞-category, let $\mathcal{O}$ be an ∞-operad, let $\mathcal{O}'$ be a reduced ∞-operad, and let $\mathfrak{o}$ be the essentially unique underlying object of $\mathcal{O}'$.*

Then the following composite functor is an equivalence

$$\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C}) \simeq \mathrm{coAlg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})\big) \xrightarrow{\mathrm{coAlg}_{\mathcal{O}}(\mathrm{ev}_{\mathfrak{o}})} \mathrm{coAlg}_{\mathcal{O}}(\mathcal{C})$$

where the first functor is the equivalence discussed in Remark 3.3.0.5 and the second functor is induced on coalgebras by the symmetric monoidal functor $\mathrm{ev}_{\mathfrak{o}}$ from Proposition E.4.2.3 (5). ♡

Proof. As the functor

$$\mathrm{ev}_{\mathfrak{o}}^{\otimes} \colon \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$$

is symmetric monoidal, with underlying functor an equivalence by [HA, 2.4.3.9] (as $\mathcal{C}$ is cocartesian symmetric monoidal), it follows from [HA, 2.1.3.8] that $\mathrm{ev}_{\mathfrak{o}}^{\otimes}$ is an equivalence of symmetric monoidal ∞-categories. It follows that the induced functor on $\mathcal{O}$-coalgebras is an equivalence. □

Proposition 3.3.1.2. *Let $\mathcal{C}$ be a cartesian symmetric monoidal ∞-category, let $\mathcal{O}$ be a reduced ∞-operad with essentially unique underlying object $\mathfrak{o}$, and let $\mathcal{O}'$ be an ∞-operad.*

Then the forgetful functor

$$\mathrm{BiAlg}_{\mathcal{O}',\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\mathrm{op}}\big)^{\mathrm{op}} \xrightarrow{\mathrm{ev}_{\mathfrak{o}}^{\mathrm{op}}} \big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\mathrm{op}}\big)^{\mathrm{op}} \simeq \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$$

is an equivalence, where the first equivalence is the one from Remark 3.3.0.5. ♡

Proof. By Proposition F.3.0.2, the symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ is cartesian, so the symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\mathrm{op}}$ is cocartesian, so that the statement follows from [HA, 2.4.3.9]. □

[35] See [HA, 2.4.0.1] for a definition and [HA, 2.4.3] for further discussion.
[36] See [HA, 2.3.4.1].

3.4. Modules over bialgebras

In Section 3.2 we upgraded LMod to a symmetric monoidal functor $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$. In this section we will try to better understand the functor induced on ∞-categories of $\mathcal{O}$-algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ when $\mathcal{O}$ is an ∞-operad. By Proposition 3.2.2.8 there is an equivalence $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \simeq \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, so that this functor can be interpreted as producing presentable monoidal ∞-categories out of $\mathcal{O}$-algebras in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ in a functorial way.

In Section 3.4.1 we will give a description of the domain of this functor. The result can be roughly summarized as follows: An $\mathcal{O}$-algebra in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}$ is given by a pair $(\mathcal{O}^{\otimes}, A)$ where $\mathcal{C}$ is an $\mathcal{O} \otimes \mathsf{Assoc}$-monoidal ∞-category and A is an $\mathsf{Assoc}, \mathcal{O}$-bialgebra in $\mathcal{C}$.

In Section 3.4.2 we will then discuss LMod as a functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}) \to \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$$

and describe the $\mathcal{O}$-monoidal structure on an $\mathsf{Assoc}, \mathcal{O}$-bialgebra in more concrete terms. We will thus see that this construction really implements the idea described in the introduction to Chapter 3.

3.4.1. Algebras in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$

The goal of this section is to give a description of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}})$. It will turn out that the presentability condition plays little role in the discussion, so to illustrate the results we will start by unpacking a bit what objects in $\mathrm{Mon}_{\mathsf{Fin}_*}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p})$ are. Specifically, let us try to understand the multiplication functor induced by the active morphism $\mu\colon \langle 2\rangle \to \langle 1\rangle$.

So let $\mathcal{C}$ be a monoidal ∞-category and let A be an Assoc-algebra in $\mathcal{C}$. By Remark 3.1.3.7 this specifies an object of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ lying over $\mathcal{C}$ that we denote by $(\mathcal{C}, A)$.

Suppose $(\mathcal{C}, A)$ is the underlying object of a commutative monoid in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. We want to describe the multiplication

$$(\mathcal{C}, A) \times (\mathcal{C}, A) \to (\mathcal{C}, A)$$

where the product is taken in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. Proposition 3.2.1.1 and Proposition C.2.0.3 that the product is given by $(\mathcal{C}\times\mathcal{C}, (A, A))$. So the multiplication map is given by a morphism

$$\big(\mathcal{C} \times \mathcal{C}, (A, A)\big) \to (\mathcal{C}, A)$$

in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. We can factor this morphism as indicated in the commutative triangle below

$$\begin{tikzcd}
 & \big(\mathcal{C}, F((A, A))\big) \arrow[d, "(\mathrm{id}_{\mathcal{C}}, f)"] \\
\big(\mathcal{C} \times \mathcal{C}, (A, A)\big) \arrow[ur, "\widetilde{F}"] \arrow[r] & (\mathcal{C}, A)
\end{tikzcd}$$

where $\widetilde{F}$ is a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphism lifting a monoidal functor $F^{\otimes}\colon (\mathcal{C}\times\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$, and f is a morphism of algebras $A \to F((A,A))$ (see also Remark 3.1.3.7). The monoidal functor $F^{\otimes}$ grants us a second tensor product functor on $\mathcal{C}$, which by the Eckmann-Hilton argument can be identified with the original one. Thus f can be identified with a morphism of algebras $\Delta\colon A \to A\otimes A$, and this provides the comultiplication of a bialgebra structure on A.

To approach such a description more rigorously, we use that the cocartesian fibration of ∞-operads $q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathcal{P}\mathrm{r}} \to \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$ (see Proposition 3.2.2.13 (1)) induces a cocartesian fibration

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}) \to \mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})\Big)$$

for every ∞-operad $\mathcal{O}$, see Definition 3.4.1.2 and Proposition 3.4.1.3 below.

We start this section by discussing in Construction 3.4.1.1 how we can identify the codomain of this cocartesian fibration $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty}))$ with the ∞-category of presentable $\mathcal{O}\otimes\mathsf{Assoc}$-monoidal ∞-categories $\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})$.

Most of the remainder of this section will then be occupied by determining the fiber of $\mathrm{Alg}_{\mathcal{O}}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}})$ over a presentable $\mathcal{O}\otimes\mathsf{Assoc}$-monoidal ∞-category $\mathcal{C}$, and in Proposition 3.4.1.15 we will show that the fiber over $\mathcal{C}$ can be identified with $\mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathcal{C})^{\mathrm{op}}$.

Construction 3.4.1.1. Let $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{O}''$ be ∞-operads, $\alpha\colon \mathcal{O}^{\otimes}\times\mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ a bifunctor of ∞-operads exhibiting $\mathcal{O}''$ as a tensor product of $\mathcal{O}$ and $\mathcal{O}'$, and $\mathfrak{I}$ a collection of small ∞-categories. Then there is a commutative diagram as follows, explained below. To save space we abbreviate expressions such as $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}\mathrm{at}_{\infty})$ by $\mathrm{Mon}_{\mathcal{O}'}$, i. e. we omit the $\mathcal{C}\mathrm{at}_{\infty}$ in parentheses.

$$\begin{tikzcd}
 & & \mathrm{Mon}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}'}\Big) \arrow[r] \arrow[d, dash, "\simeq"] & \mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}'}\Big) \arrow[r] \arrow[d, dash, "\simeq"] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Alg}_{\mathcal{O}'}\big(\mathcal{P}\mathrm{r}^{\mathrm{L}}\big)\Big) \arrow[r] \arrow[d, dash, "\simeq"] & \mathrm{Alg}_{\mathcal{O}}\Big(\mathrm{Alg}_{\mathcal{O}'}\big(\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})\big)\Big) \arrow[r] \arrow[d, dash, "\simeq"] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}_{\mathcal{O}''}\big(\mathcal{P}\mathrm{r}^{\mathrm{L}}\big) \arrow[r] \arrow[d, dash, "\simeq"] & \mathrm{Alg}_{\mathcal{O}''}\big(\mathcal{C}\mathrm{at}_{\infty}(\mathfrak{I})\big) \arrow[r] \arrow[d, dash, "\simeq"] & \mathrm{Alg}_{\mathcal{O}''} \arrow[d, dash, "\simeq"] \\
\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}''} \arrow[r] & \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}''} \arrow[r] & \mathrm{Mon}_{\mathcal{O}''}
\end{tikzcd}$$

The equivalence at the top right is the one from [HA, 2.4.2.5], i. e. is the one induced by $\pi_{\mathrm{Mon}*}$. The top two squares are induced on $\mathcal{O}$-algebras by the commutative diagram constructed in Proposition 3.2.2.8, which is a commutative diagram of ∞-operads by Proposition 3.2.2.10. The middle two squares are obtained from naturality of the equivalences constructed in Proposition E.5.0.2 and Proposition E.5.0.1 as discussed in

Remark F.3.0.4, applied to the morphisms of ∞-operads

$$\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes} \to \mathcal{C}\mathrm{at}_\infty(\mathfrak{I})^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$$

from Proposition 3.2.2.3 (2). Finally, the commutative diagram constructed in Proposition 3.2.2.8 induces a commutative diagram on underlying ∞-categories that yields the bottom two commutative squares. $\diamondsuit$

Definition 3.4.1.2. Let $\mathcal{O}$ be an ∞-operad and $\mathfrak{I}$ a collection of small ∞-categories. We define the following ∞-categories and morphisms of ∞-categories by applying $\mathrm{Alg}_\mathcal{O}$ to the morphisms of ∞-operads (see Proposition 3.2.2.13 (1)) $q^\otimes_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_\mathfrak{I}}$ and $q^\otimes_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}}$. The equivalences used are the ones from Construction 3.4.1.1.

$$\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\mathfrak{I}_\mathcal{O} \coloneqq \mathrm{Alg}_\mathcal{O}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_\mathfrak{I})$$
$$\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_\mathcal{O} \coloneqq \mathrm{Alg}_\mathcal{O}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}})$$
$$q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\mathfrak{I}_\mathcal{O}} \colon \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\mathfrak{I}_\mathcal{O} \xrightarrow{\mathrm{Alg}_\mathcal{O}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_\mathfrak{I}})} \mathrm{Alg}_\mathcal{O}\Big(\mathrm{Mon}^\mathfrak{I}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\Big) \simeq \mathrm{Mon}^\mathfrak{I}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$$
$$q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_\mathcal{O}} \colon \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_\mathcal{O} \xrightarrow{\mathrm{Alg}_\mathcal{O}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{P}\mathrm{r}}})} \mathrm{Alg}_\mathcal{O}\Big(\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\Big) \simeq \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$$

We will also write $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_\mathcal{O}}$ for $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\emptyset_\mathcal{O}}$ and $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_\mathcal{O}$ for $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\emptyset_\mathcal{O}$. $\diamondsuit$

Proposition 3.4.1.3. *In the situation of Definition 3.4.1.2, the functors $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\mathfrak{I}_\mathcal{O}}$ and $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_\mathcal{O}}$ are cocartesian fibrations.* $\heartsuit$

Proof. Combine Proposition 3.2.2.13 (1) with Proposition E.3.2.1. $\square$

We start the process of identifying the fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\mathfrak{I}_\mathcal{O}}$ and $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_\mathcal{O}}$ by reducing the problem to $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_\mathcal{O}}$.

Proposition 3.4.1.4. *We use Notation 3.2.2.7 in this proposition. Let $\mathfrak{I}$ be a collection of small ∞-categories and let $\mathcal{O}$ be an ∞-operad. Then there is a pullback diagram in $\mathcal{C}\mathrm{at}_\infty$ as follows.*

$$\begin{tikzcd}
\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\mathfrak{I}_\mathcal{O} \arrow[r, "\mathrm{Alg}_\mathcal{O}(\widetilde{\Psi}^\mathfrak{I})"] \arrow[d, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\mathfrak{I}_\mathcal{O}}"'] & \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_\mathcal{O} \arrow[d, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_\mathcal{O}}"] \\
\mathrm{Mon}^\mathfrak{I}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, "(\Psi^\mathfrak{I})"'] & \mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)
\end{tikzcd}$$

In particular, if $\mathcal{C}$ is an object in $\mathrm{Mon}^\mathfrak{I}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$, then we can identify the fiber $(\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\mathfrak{I}_\mathcal{O})_\mathcal{C}$ with $(\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_\mathcal{O})_{(\Psi^\mathfrak{I})^\otimes(\mathcal{C})}$, and if $F\colon \mathcal{C} \to \mathcal{D}$ is a morphism in the ∞-category $\mathrm{Mon}^\mathfrak{I}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ we can identify the induced functor on fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^\mathfrak{I}_\mathcal{O}}$ with the functor induced by $(\Psi^\mathfrak{I})^\otimes(F)$ on fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_\mathcal{O}}$.

Analogous statements hold for $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_\mathcal{O}}$. $\heartsuit$

Proof. We only prove the case of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}}}$, the case of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{P}\mathrm{r}}_{\mathcal{O}}}$ is completely analogous.

By Definition 3.2.2.11 we have a pullback diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\otimes}_{\mathfrak{I}} \arrow[r, "(\widetilde{\Psi}^{\mathfrak{I}})^{\otimes}"] \arrow[d, "q^{\otimes}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\times} \arrow[d, "q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes} \arrow[r, "(\Psi^{\mathfrak{I}})^{\otimes}"'] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})^{\times}
\end{tikzcd}$$

where $q^{\times}_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ is a cocartesian fibration of ∞-operads (see Proposition 3.2.2.13 (1)) and $(\Psi^{\mathfrak{I}})^{\otimes}$ is a morphism of ∞-operads (see Proposition 3.2.2.10 (2)). Combining Proposition E.1.3.1 and Proposition E.3.1.1 we conclude that the the top square in the following commutative diagram is a pullback square[37]

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(\widetilde{\Psi}^{\mathfrak{I}})"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathfrak{I}}})"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})"] \\
\mathrm{Alg}_{\mathcal{O}}\left(\mathrm{Mon}^{\mathfrak{I}}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})\right) \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(\Psi^{\mathfrak{I}})"] \arrow[d, "\simeq"'] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})) \arrow[d, "\simeq"] \\
\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[r, "\Psi^{\mathfrak{I}}"'] & \mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd}$$

where the lower commuting square is the one from Construction 3.4.1.1. This proves the claim, as the the left and right vertical compositions are by definition $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}}}$ and $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$. $\square$

Before starting to analyze the fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$, it will be helpful to describe the equivalences from Construction 3.4.1.1 more concretely as done in the following proposition.

Proposition 3.4.1.5. *Let $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{O}''$ be ∞-operads, and $\alpha\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ a bifunctor of ∞-operads exhibiting $\mathcal{O}''$ as a tensor product of $\mathcal{O}$ and $\mathcal{O}'$.*

[37] The two $\Psi^{\mathfrak{I}}$ in the diagram are different functors, the same notation only arises here because the operad does not occur in the notation.

Then there is a commutative diagram as follows

$$
\begin{tikzcd}
\mathrm{Mon}_{\mathcal{O}}\big(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}\mathrm{at}_\infty)\big) \arrow[r] \arrow[d, dash, "\simeq"'] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}_\infty\big)\Big) \\
\mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}\mathrm{at}_\infty)\big) \arrow[d, dash, "\simeq"'] & \\
\mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}\mathrm{at}_\infty)\big) \arrow[r] \arrow[d, dash, "\simeq"'] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}_\infty^{\times}\big)\Big) \arrow[uu, "(\pi_*)_*"'] \\
\mathrm{BiFunc}(\mathcal{O}, \mathcal{O}'; \mathcal{C}\mathrm{at}_\infty) \arrow[r] \arrow[d, dash, "\simeq"'] & \mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}_\infty^{\times}\big) \arrow[u, "\widehat{(-)}"'] \\
\mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[r] \arrow[d, dash, "\simeq"'] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{C}\mathrm{at}_\infty^{\times}\big) \arrow[u, "\alpha^*"'] \arrow[d, "\pi_*"] \\
\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[r] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{C}\mathrm{at}_\infty\big) \arrow[uuuuu, bend right, "\widehat{(-\circ\alpha)}"']
\end{tikzcd}
$$

where vertical functors on the left are the ones from Construction 3.4.1.1 (where we split up the equivalence in the middle in its two steps from Proposition E.5.0.2 and Proposition E.5.0.1) and the horizontal functors are the the compositions of the canonical inclusions and projections. ♡

Proof. The top square is obtained from the construction of the equivalence $\Theta^{\otimes}$ by combining commutative diagrams (3.15) and (3.14) occurring in the proof of Proposition 3.2.2.8. The two middle squares are from Proposition F.3.0.3. The bottom square is diagram (3.13) from Proposition 3.2.2.8. Finally, the commutative rectangle on the right is obtained from naturality of $\widehat{(-)}$. □

The cocartesian fibration $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$ is constructed in multiple steps from the universal cocartesian family of Assoc-monoidal ∞-categories, but ends up with $\mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ as a codomain. The next proposition relates the universal cocartesian family of Assoc-monoidal ∞-categories with the universal cocartesian family of $\mathsf{Assoc} \otimes \mathcal{O}$-monoidal ∞-categories.

Proposition 3.4.1.6. *Let $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{O}''$ be ∞-operads and $\alpha\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ a bifunctor of ∞-operads exhibiting $\mathcal{O}''$ as the tensor product of $\mathcal{O}$ and $\mathcal{O}'$. Then there is a commutative diagram as follows such that both squares are pullback diagrams, and where other parts of the diagram will be explained further below.*

$$
\begin{tikzcd}
\widetilde{\mathrm{Mon}}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[d, "p^{\mathcal{O}''}"'] & \widetilde{\mathrm{Mon}}_{\alpha}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[l] \arrow[r] \arrow[d, "p^{\alpha}"] & \widetilde{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)^{\otimes} \arrow[d, "p^{\mathcal{O}}"] \\
\mathcal{O}''^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[l] \arrow[r] & \mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)
\end{tikzcd}
\tag{3.21}
$$

The left and right vertical functors are the universal cocartesian families of monoidal ∞-categories defined in Definition 3.1.1.4, whereas the middle vertical functor is a functor we newly define here as the pullback of either the left or right square. The bottom left horizontal functor is $\alpha \times \mathrm{id}$, and the bottom right vertical functor is the the product of $\mathrm{id}_{\mathcal{O}^\otimes}$ with the following composition

$$\begin{aligned} &\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \to \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}'}\big(\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)\big) \\ &\quad \to \mathcal{O}'^{\otimes} \times \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)\big) \xrightarrow{\mathrm{ev}} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \end{aligned} \tag{3.22}$$

where the first functor uses the equivalence from Proposition 3.4.1.5 interpreting $\mathcal{O}''$ as the tensor product $\mathcal{O}'\otimes\mathcal{O}$ via $\alpha\circ\tau$, where τ is the symmetry equivalence $\mathcal{O}^{\otimes}\times\mathcal{O}'^{\otimes} \simeq \mathcal{O}'^{\otimes}\times\mathcal{O}^{\otimes}$, and the second functor is the product of the identity and the canonical inclusion. ♡

Proof. Both $p^{\mathcal{O}''}$ and $p^{\mathcal{O}}$ are by definition cocartesian fibrations, with $p^{\mathcal{O}}$ classified by[38] the composition

$$\mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \to \mathcal{O}^{\otimes} \times \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty\big) \xrightarrow{\mathrm{ev}} \mathcal{C}\mathrm{at}_\infty$$

where the first functor is the product of the identity functor and the canonical inclusion, and similarly for $p^{\mathcal{O}''}$. So by naturality of the Grothendieck construction[39] it suffices to show that the composition of the left bottom horizontal functor in diagram (3.21) with the functor the left vertical cocartesian fibration is classified by is homotopic to the composition of the right bottom horizontal functor with the functor the right vertical cocartesian fibration is classified by. For this consider the following three commutative diagrams, where we will denote the various canonical inclusions by ι.

$$\begin{tikzcd} \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, "\mathrm{id}\times\mathrm{id}\times\iota"] \arrow[d] & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}(\mathcal{O}''^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathrm{id}\times\mathrm{id}\times\alpha^*"] \\ \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}'}\big(\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)\big) \arrow[d, "\mathrm{id}\times\mathrm{id}\times\iota"'] & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathrm{id}\times\mathrm{id}\times\widehat{-\circ\tau}"] \\ \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)\big) \arrow[r, "\mathrm{id}\times\mathrm{id}\times\iota_*"] \arrow[d, "\mathrm{id}\times\mathrm{ev}"'] & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}\Big(\mathcal{O}'^{\otimes}, \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty\big)\Big) \arrow[d, "\mathrm{id}\times\mathrm{ev}"] \\ \mathcal{O}^{\otimes} \times \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, "\mathrm{id}\times\iota"] & \mathcal{O}^{\otimes} \times \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathrm{ev}"] \\ & \mathcal{C}\mathrm{at}_\infty \end{tikzcd} \tag{$*$}$$

In the above diagram, the top square arises from Proposition 3.4.1.5 and the bottom square uses naturality of evaluation. The next two commutative diagrams only use vari-

[38] See Definition 3.1.1.4.

[39] See [GHN17, A.32] and [Maz19b].

ous naturalities and functorialities.

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathfrak{Cat}_{\infty}\big) \arrow[d, "\mathrm{id}\times\mathrm{id}\times\widehat{-\circ\tau}"] \arrow[ddr, "\mathrm{ev}"] & \\
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}\Big(\mathcal{O}'^{\otimes}, \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}\big)\Big) \arrow[d, "\mathrm{id}\times\mathrm{ev}"] & \\
\mathcal{O}^{\otimes} \times \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathfrak{Cat}_{\infty}\big) \arrow[r, "\mathrm{ev}"] & \mathfrak{Cat}_{\infty}
\end{tikzcd} \tag{$**$}$$

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathfrak{Cat}_{\infty}) \arrow[r, "\mathrm{id}\times\mathrm{id}\times\iota"] \arrow[d, "\alpha\times\mathrm{id}"'] & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathfrak{Cat}_{\infty}\big) \arrow[d, "\mathrm{id}\times\mathrm{id}\times\alpha^{*}"] \arrow[ddl, "\alpha\times\mathrm{id}"'] \\
\mathcal{O}''^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathfrak{Cat}_{\infty}) \arrow[d, "\mathrm{id}\times\iota"'] & \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathfrak{Cat}_{\infty}\big) \arrow[d, "\mathrm{ev}"] \\
\mathcal{O}''^{\otimes} \times \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathfrak{Cat}_{\infty}\big) \arrow[r, "\mathrm{ev}"] & \mathfrak{Cat}_{\infty}
\end{tikzcd} \tag{$***$}$$

The composite of the lower left (right) horizontal functor in diagram (3.21) with the functor the left (right) vertical cocartesian fibration is classified by is precisely the composite via the bottom left corner from the top left to the bottom right corner in diagram ($***$) (in diagram ($*$)). Diagrams ($*$), ($**$), and ($***$) show that these two composites are homotopic, which proves the claim. □

We next go through the steps used to construct $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ from p^{Assoc} and show how we can identify $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ with a functor obtained from p^{α} as in Proposition 3.4.1.6. We will use the right pullback square in (3.21) to compare constructions obtained from p^{α} with the intermediate steps on the way to $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$, while using the left pullback square to be able to describe those constructions in a way helpful to ultimately describe fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$ as ∞-categories of bialgebras.

Definition 3.4.1.7. Let $\mathcal{O}'$ and $\mathcal{O}''$ be ∞-operads and $\alpha\colon \mathsf{Assoc}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ a bifunctor of ∞-operads that exhibits $\mathcal{O}''$ as the tensor product of Assoc and $\mathcal{O}$.

Using that the right square in (3.21) is a pullback diagram we can interpret p^{α} from Proposition 3.4.1.6 as a cocartesian $\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathfrak{Cat}_{\infty})$-family of Assoc-monoidal ∞-categories. Passing to Assoc-algebras we obtain by Proposition 3.1.2.10 a pullback, where we will denote the ∞-category on the top left and functor on the left as indicated, and the functor on the right is the one from Definition 3.1.3.3.

$$\begin{tikzcd}
\mathcal{A}^{\otimes} = \mathrm{Alg}_{/\mathsf{Assoc}}\Big(\widetilde{\mathrm{Mon}}_{\alpha}(\mathfrak{Cat}_{\infty})^{\otimes}\Big) \arrow[r] \arrow[d, "q_{\mathcal{A}}"'] & \mathcal{A}\mathrm{lg} \arrow[d, "q_{\mathcal{A}\mathrm{lg}}"] \\
\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathfrak{Cat}_{\infty}) \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathfrak{Cat}_{\infty})
\end{tikzcd}$$

◊

Remark 3.4.1.8. Let $\mathcal{C}$ be an ∞-category, $\mathcal{O}$ an ∞-operad, and $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{C}$ a cocartesian $\mathcal{C}$-family of $\mathcal{O}$-monoidal ∞-categories.

Note that the projection $\mathrm{pr}_2\colon \mathcal{O}^{\otimes} \times \mathcal{C} \to \mathcal{C}$ is a cocartesian fibration[40], and pr_2-cocartesian morphisms are those that are (equivalent to) an identity morphism in the first factor.

By [HTT, 2.4.2.3 (3)] and Proposition C.1.3.1 we obtain a morphism of cocartesian fibrations over $\mathcal{C}$ as follows.

$$\begin{tikzcd}
\mathcal{D}^{\otimes} \arrow[rr, "p"] \arrow[dr, "\mathrm{pr}_2 \circ p"'] & & \mathcal{O}^{\otimes} \times \mathcal{C} \arrow[dl, "\mathrm{pr}_2"] \\
& \mathcal{C} &
\end{tikzcd}$$

If $f\colon X \to Y$ is a morphism in $\mathcal{C}$, then we obtain an induced commutative square on fibers as follows.

$$\begin{tikzcd}
\mathcal{D}^{\otimes}_X \arrow[r, "f_!"] \arrow[d, "p_X"'] & \mathcal{D}^{\otimes}_Y \arrow[d, "p_Y"] \\
\mathcal{O}^{\otimes} \arrow[r, "\mathrm{id}"'] & \mathcal{O}^{\otimes}
\end{tikzcd}$$

By the description of pr_2-cocartesian morphisms given above the induced functor on fibers of pr_2 is the identity, and by assumption on p the two vertical functors are cocartesian fibrations of ∞-operads. We thus obtain a commuting triangle

$$\begin{tikzcd}
\mathcal{D}^{\otimes}_X \arrow[rr, "p_X"] \arrow[dr, "\mathrm{pr}_2 \circ p"'] & & \mathcal{D}^{\otimes}_Y \arrow[dl, "p_Y"] \\
& \mathcal{O}^{\otimes} &
\end{tikzcd}$$

that by Proposition 3.1.1.1 is an $\mathcal{O}$-monoidal functor. It is this $\mathcal{O}$-monoidal functor that we will refer to as the induced $\mathcal{O}$-monoidal functor on fibers over f. $\diamondsuit$

Proposition 3.4.1.9. *Assume we are in the situation of Definition 3.4.1.7, and let $\mathcal{C}$ be an $\mathcal{O}''$-monoidal ∞-category. Then the fiber of $q_{\mathcal{A}}$ over $\mathcal{C}$ (considered as an object of $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$) can be identified with the $\mathcal{O}'$-monoidal ∞-category of* Assoc*-algebras*[41] $\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes}$ *from Proposition E.4.2.3.*

Furthermore, if $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{O}''$-monoidal functor, then the induced $\mathcal{O}'$-monoidal functor on fibers of $q_{\mathcal{A}}$ fits into a commutative diagram as follows

$$\begin{tikzcd}
\mathcal{A}^{\otimes}_{\mathcal{C}} \arrow[r, "F_!"] \arrow[d, "\simeq"'] & \mathcal{A}^{\otimes}_{\mathcal{D}} \arrow[d, "\simeq"] \\
\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(F)^{\otimes}"'] & \mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{D})^{\otimes}
\end{tikzcd}$$

[40] This is for example easy to see by using that it is the pullback of the functor $\mathcal{O}^{\otimes} \to *$ along $\mathcal{C} \to *$.
[41] With respect to the bifunctor of ∞-operads $\alpha \circ \tau$.

where $\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(F)^{\otimes}$ *is the induced functor from* Proposition E.4.2.3 *and the vertical equivalences are the ones from the first claim of this proposition.* ♡

Proof. Consider the following commutative diagram, where the bottom square is the induced pullback square by applying $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, -)$ to the left pullback square in diagram (3.21) of Proposition 3.4.1.6, and the top pullback square is the one from Definition 3.1.2.1.

$$\begin{tikzcd}
\mathcal{A}^{\otimes} \arrow[d] \arrow[drr, "q_{\mathcal{A}}"] & & \\
\widetilde{\mathcal{A}}^{\otimes} \coloneqq \widetilde{\mathrm{Alg}}_{/\mathsf{Assoc}}\Big(\widetilde{\mathrm{Mon}}_{\alpha}(\mathcal{C}\mathrm{at}_{\infty})\Big) \arrow[rr, "\mathrm{pr}"] \arrow[d, "\mathrm{pr}_{\mathrm{Fun}}"'] & & \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[d, "\widehat{\mathrm{id}}_{\mathsf{Assoc}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}}"] \\
\mathrm{Fun}\Big(\mathsf{Assoc}^{\otimes}, \widetilde{\mathrm{Mon}}_{\alpha}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}\Big) \arrow[r, "p^{\alpha}_{*}"] \arrow[d] & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathsf{Assoc}^{\otimes} \times \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})\big) \arrow[rr, phantom, "(*)"] \arrow[d, "(\alpha \times \mathrm{id})_{*}"] & & {} \\
\mathrm{Fun}\Big(\mathsf{Assoc}^{\otimes}, \widetilde{\mathrm{Mon}}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}\Big) \arrow[r, "p^{\mathcal{O}''}_{*}"] & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{O}''^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})\big)
\end{tikzcd}$$

$\mathcal{A}^{\otimes}$ is by definition[42] the full subcategory of $\widetilde{\mathcal{A}}^{\otimes}$ spanned by those objects that are mapped by $\mathrm{pr}_{\mathrm{Fun}}$ to functors $\mathsf{Assoc}^{\otimes} \to \widetilde{\mathrm{Mon}}_{\alpha}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$ that send inert morphisms to p^{α}-cocartesian ones. By the description of p^{α}-cocartesian morphisms afforded by the left pullback square in diagram (3.21) of Proposition 3.4.1.6 in combination with Proposition C.1.1.1 we can thus identify $\mathcal{A}^{\otimes}$ with the full subcategory of $\widetilde{\mathcal{A}}^{\otimes}$ spanned by those objects that map to functors $\mathsf{Assoc}^{\otimes} \to \widetilde{\mathrm{Mon}}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$ which send inert morphisms to $p^{\mathcal{O}''}$-cocartesian ones. Similarly, we obtain from Proposition 3.1.2.17 that a morphism in $\mathcal{A}^{\otimes}$ is $q_{\mathcal{A}}$-cocartesian if and only if it maps to a natural transformation of functors $\mathsf{Assoc}^{\otimes} \to \widetilde{\mathrm{Mon}}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}$ that is pointwise $p^{\mathcal{O}''}$-cocartesian.

Now let $\mathcal{C}$ be an $\mathcal{O}''$-monoidal ∞-category. Then there is a commutative cube as follows[43].

$$\begin{tikzcd}
& \widetilde{\mathrm{Alg}}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[rr] \arrow[dl, dashed] \arrow[dd] & & \mathcal{O}'^{\otimes} \times \{\mathcal{C}\} \arrow[dl] \arrow[dd] \\
\widetilde{\mathcal{A}}^{\otimes} \arrow[rr, crossing over] \arrow[dd] & & \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''} & \\
& \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[rr] \arrow[dl] & & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{O}''^{\otimes} \times \{\mathcal{C}\}\big) \arrow[dl] \\
\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \widetilde{\mathrm{Mon}}_{\mathcal{O}''}\big) \arrow[rr] & & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{O}''^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}\big) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

The front square is the composite pullback diagram from (∗). The bottom square is the pullback square obtained by applying $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, -)$ to the pullback diagram of

[42]See Definition 3.1.2.3.

[43]We abbreviate $\widetilde{\mathrm{Mon}}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$ and $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$ as $\widetilde{\mathrm{Mon}}_{\mathcal{O}''}$ and $\mathrm{Mon}_{\mathcal{O}''}$.

the identification of $\mathcal{C}^{\otimes}$ as the fiber of $p^{\mathcal{O}''}$ over $\mathcal{C}$, see Remark 3.4.1.8. The back one is the pullback diagram from Proposition E.4.2.3. That there is a commutative square as indicated on the right, where the top functor is the product of the identity with the inclusion of $\{\mathcal{C}\}$, can be checked by unpacking the definitions and using naturality. We obtain the induced top and left square and filler for the cube (using that the front square is a pullback square), and it follows from [HTT, 4.4.2.1] that the top square is also a pullback diagram.

The description of $\mathcal{A}^{\otimes}$ as a full subcategory of $\widetilde{\mathcal{A}}^{\otimes}$ we gave above together with the definition of $\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes}$ as a full subcategory of $\widetilde{\mathrm{Alg}}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes}$ in Remark E.4.2.1 and an argument very similar to the one in the proof of Proposition 3.1.2.2 show that the dashed functor in the above diagram induces an equivalence

$$\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathcal{A}^{\otimes}$$

on full subcategories.

The description of the functor induced on fibers by a morphism $F\colon \mathcal{C} \to \mathcal{D}$ of $\mathcal{O}''$-monoidal ∞-categories follows from the description given above for $q_{\mathcal{A}}$-cocartesian morphisms together with the fact that the $\mathcal{O}''$-monoidal functor induced by F (considered as a morphism in $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$) on fibers of $p^{\mathcal{O}''}$ can by construction (see Definition 3.1.1.4) be identified with F. □

Proposition 3.4.1.10. *Assume we are in the situation of Definition 3.4.1.7. Then $q_{\mathcal{A}}$ is a cocartesian $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$-family of $\mathcal{O}'$-monoidal ∞-categories.* ♡

Proof. Follows immediately from the definition[44] together with Proposition 3.4.1.9 and Proposition E.4.2.3 (3). □

Definition 3.4.1.11. Assume we are in the situation of Definition 3.4.1.7. We let

$$q_{\mathcal{A}'}\colon \mathcal{A}'^{\otimes} \to \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$$

be the cocartesian fibration obtained by applying the functor

$$\begin{aligned}
&\mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}\big(\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})\big)\\
&\to \mathrm{Fun}\big(\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty}), \mathcal{C}\mathrm{at}_{\infty}\big)\\
&\xrightarrow{(-^{\mathrm{op}})_*} \mathrm{Fun}\big(\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty}), \mathcal{C}\mathrm{at}_{\infty}\big)\\
&\to \mathrm{co}\mathcal{C}\mathcal{F}\mathrm{ib}\big(\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})\big)
\end{aligned}$$

to $q_{\mathcal{A}}\colon \mathcal{A}^{\otimes} \to \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty})$. ◇

Proposition 3.4.1.12. *Assume we are in the situation of Definition 3.4.1.7. Then there is a pullback diagram as follows*

$$\begin{tikzcd}
\mathcal{A}'^{\otimes} \arrow[r] \arrow[d, "q_{\mathcal{A}'}"'] & \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \arrow[d, "q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}"] \\
\mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_{\infty}) \arrow[r] & \mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_{\infty})
\end{tikzcd}$$

[44] Definition 3.1.1.2 with variant Proposition 3.1.1.1 (2).

where the bottom functor is the composition (3.22). ♡

Proof. Follows immediately from Definition 3.4.1.11 and Definition 3.1.3.5 together with Proposition 3.4.1.6 and naturality of the Grothendieck construction. □

Proposition 3.4.1.13. *Assume we are in the situation of Definition 3.4.1.7. Then the following hold.*

(1) $q_{\mathcal{A}'}$ *from Definition 3.4.1.11 is again a cocartesian* $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)$*-family of* $\mathcal{O}'$*-monoidal* ∞*-categories.*

(2) Let $\mathcal{C}$ *be a* $\mathcal{O}''$*-monoidal* ∞*-category. Then the fiber of* $q_{\mathcal{A}'}$ *over* $\mathcal{C}$ *is, as an* $\mathcal{O}'$*-monoidal* ∞*-category, equivalent to* $(\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\mathrm{op}})^{\otimes}$*, the opposite* $\mathcal{O}'$*-monoidal* ∞*-category of* $\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\otimes}$*.*

(3) Let $F\colon \mathcal{C} \to \mathcal{D}$ *be a* $\mathcal{O}''$*-monoidal functor. Then there is a commutative square*

$$\begin{tikzcd}
\mathcal{A}'^{\otimes}_{\mathcal{C}} \arrow[rr, "F_!"] \arrow[d, "\simeq"'] & & \mathcal{A}'^{\otimes}_{\mathcal{D}} \arrow[d, "\simeq"] \\
(\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{C})^{\mathrm{op}})^{\otimes} \arrow[rr, "(\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(F)^{\mathrm{op}})^{\otimes}"'] & & (\mathrm{Alg}_{\mathsf{Assoc}/\mathcal{O}''}(\mathcal{D})^{\mathrm{op}})^{\otimes}
\end{tikzcd}$$

where the top functor is the one induced on fibers of $q_{\mathcal{A}'}$*, and the vertical functors are the equivalences from (2).* ♡

Proof. Follows directly from $q_{\mathcal{A}}$ being a cocartesian family of $\mathcal{O}'$-monoidal ∞-categories by Proposition 3.4.1.10 and the description of its fibers in Proposition 3.4.1.9. □

Proposition 3.4.1.14. *Let* $\mathcal{O}'$ *and* $\mathcal{O}''$ *be* ∞*-operads and* $\alpha\colon \mathsf{Assoc}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ *a bifunctor of* ∞*-operads that exhibits* $\mathcal{O}''$ *as the tensor product of* Assoc *and* $\mathcal{O}$*.*

Then there is a commutative triangle as follows such that the horizontal functor is an equivalence

$$\begin{tikzcd}
\mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}') \arrow[rr, "\simeq", no head] \arrow[dr, "\mathrm{pr}_{\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)}"'] & & \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}'} \arrow[dl, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}'}}"] \\
& \mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) &
\end{tikzcd}$$

where the functor on the left is as in Definition 3.1.2.3 and Definition 3.1.2.1, applied to the cocartesian family of $\mathcal{O}'$*-monoidal* ∞*-categories* $q_{\mathcal{A}'}$ *from Definition 3.4.1.11 and Proposition 3.4.1.13.* ♡

Proof. By naturality of the construction $-^{\times}$ and [HA, 2.4.2.5] there is a commutative diagram as follows

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}'}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})"'] & \mathrm{Mon}_{\mathcal{O}'}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[d, "\mathrm{Mon}_{\mathcal{O}'}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})"] \\
\mathrm{Alg}_{\mathcal{O}'}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)) \arrow[r, "\simeq"'] & \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty))
\end{tikzcd}$$

with the two horizontal functors equivalences. It follows from Definition 3.4.1.2 and Construction 3.4.1.1 that there is a commutative square

$$\begin{tikzcd}
\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}'} \arrow[r, dash, "\simeq"] \arrow[d, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}'}}"'] & \mathrm{Mon}_{\mathcal{O}'}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[d, "\mathrm{Mon}_{\mathcal{O}'}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})"] \\
\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, dash, "\simeq"'] & \mathrm{Mon}_{\mathcal{O}'}\big(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\big)
\end{tikzcd}$$

where the bottom horizontal functor is the equivalence from Construction 3.4.1.1.

Thus it suffices to show that there is a commutative square as follows

$$\begin{tikzcd}
\mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}') \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{pr}_{\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)}"'] & \mathrm{Mon}_{\mathcal{O}'}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[d, "\mathrm{Mon}_{\mathcal{O}'}(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}})"] \\
\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty) \arrow[r, dash, "\simeq"'] & \mathrm{Mon}_{\mathcal{O}'}\big(\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)\big)
\end{tikzcd}$$

where the bottom horizontal functor is the equivalence from Construction 3.4.1.1.

Now we consider the following diagram[45] that will be explained in detail below.

$$\begin{tikzcd}
 & \mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}') \arrow[rr, dashed] \arrow[d] & & \mathrm{Mon}_{\mathcal{O}'}(\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[d, dashed, "\varphi"] \arrow[dl, "\iota"'] \arrow[dd, bend left] \\
 & \widetilde{\mathrm{Alg}}_{/\mathcal{O}'}(\mathcal{A}') \arrow[rr, dashed, "\vartheta"] \arrow[dl, "\mathrm{pr}_{\mathrm{Fun}}"'] \arrow[dd] & & \mathcal{P} \arrow[dl, "\psi"] \arrow[dd] \\
\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{A}'^{\otimes}) \arrow[rr] \arrow[dd] & & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}) \arrow[dd] & \\
 & \mathrm{Mon}_{\mathcal{O}''} \arrow[rr] \arrow[dl] & & \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Mon}_{\mathsf{Assoc}}) \arrow[dl] \\
\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}'^{\otimes} \times \mathrm{Mon}_{\mathcal{O}''}) \arrow[rr] & & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathrm{Mon}_{\mathsf{Assoc}}) &
\end{tikzcd}$$

The front square is $\mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$ applied to the pullback square from Proposition 3.4.1.12. In particular, the front square is again a pullback square. The bottom square arises from naturality of $\widehat{-}$ and the fact that $\widehat{\mathrm{ev}} = \mathrm{id}$. The bottom back horizontal equivalence is the one from Construction 3.4.1.1 and Proposition 3.4.1.5. The left square is the pullback square defining $\widetilde{\mathrm{Alg}}_{/\mathcal{O}'}(\mathcal{A}')$, see Definition 3.1.2.1. We define the right square to be a pullback square.

As the left and right squares in the cube are pullback diagrams, we obtain an induced functor ϑ together with fillers for the top and back square and the cube.

The right big square arises from applying the natural transformation

$$\mathrm{Mon}_{\mathcal{O}'}(-) \to \mathrm{Fun}(\mathcal{O}'^{\otimes}, -)$$

to $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$. We obtain the induced functor φ and the two commutative triangles on the right. By definition, ι and the bottom functor from the back to the front on the right side

[45]We abbreviate $\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}\mathrm{at}_\infty)$ and $\mathrm{Mon}_{\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$ as $\mathrm{Mon}_{\mathcal{O}''}$ and $\mathrm{Mon}_{\mathsf{Assoc}}$.

are fully faithful. As the small square is a pullback square and taking pullbacks preserves fully faithful functors by Proposition B.5.2.1, ψ is fully faithful as well. By considering the top triangle on the right side we then deduce that φ is also fully faithful[46].

What we have to show is that there is a dashed top back horizontal functor making the back big rectangle commute and which is an equivalence. As the front, left, and right squares are pullback squares it follows from [HTT, 4.4.2.1] that the back lower square is a pullback square as well. As the lower back horizontal functor is an equivalence, it follows that ϑ is an equivalence too. It thus suffices to show that an object A of $\widetilde{\mathrm{Alg}}_{/\mathcal{O}'}(\mathcal{A}')$ is in the essential image of the functor from $\mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}')$ if and only if $\vartheta(A)$ is in the essential image of φ (see Proposition B.4.3.1).

We first consider the essential image of φ, which consists of precisely those objects that are mapped by ψ to an object that is in the essential image of ι i.e. is an $\mathcal{O}'$-monoid. By definition [HA, 2.4.2.1], a functor $F\colon \mathcal{O}'^{\otimes} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ is an $\mathcal{O}'$-monoid if and only if for every $n \geq 0$, objects X_i in $\mathcal{O}'$ for every $1 \leq i \leq n$, and inert morphisms $r_i\colon X_1 \oplus \cdots \oplus X_n \to X_i$ lying over ρ^i, the morphisms $F(r_i)$ exhibit $F(X_1 \oplus \cdots \oplus X_n)$ as the product of $(F(X_i))_{1\leq i\leq n}$. By the description of products in $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ from Proposition 3.2.1.1 and Proposition C.2.0.3 this is equivalent to the morphisms $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}(F(r_i))$ exhibiting $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}(F(X_1 \oplus \cdots \oplus X_n))$ as the product of $(q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}(F(X_i)))_{1\leq i\leq n}$ and $F(r_i)$ being $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian for every $1 \leq i \leq n$. Thus F is in the essential image of ι if and only if $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \circ F$ is an $\mathcal{O}'$-monoid and F maps inert morphisms to $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphisms. By Proposition B.5.2.1, a functor $F\colon \mathcal{O}'^{\otimes} \to \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ lies in the essential image of ψ if and only if $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}} \circ F$ is an $\mathcal{O}'$-monoid. It follows that an object A of $\mathcal{P}$ is in the essential image of φ if and only if $\psi(A)$ maps inert morphisms to $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphisms.

By definition[47], an object A of $\widetilde{\mathrm{Alg}}_{/\mathcal{O}'}(\mathcal{A}')$ is in the essential image of the inclusion from $\mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}')$ if and only if $\mathrm{pr}_{\mathrm{Fun}}(A)$ maps inert morphisms to $q_{\mathcal{A}'}$-cocartesian morphisms. By Proposition 3.4.1.12 and Proposition C.1.1.1 this is the case if and only if $\psi(\vartheta(A))$ maps inert morphisms to $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian morphisms. Thus an object A of $\widetilde{\mathrm{Alg}}_{/\mathcal{O}'}(\mathcal{A}')$ is in the essential image of the functor from $\mathrm{Alg}_{/\mathcal{O}'}(\mathcal{A}')$ if and only if $\vartheta(A)$ is in the essential image of φ, which finishes the proof. $\square$

With Proposition 3.4.1.14 we can now finally discuss the fibers of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}$.

Proposition 3.4.1.15. *Let $\mathfrak{I}$ be a collection of small ∞-categories, let $\mathcal{O}$ be an ∞-operad. Then the following hold.*

(1) Let $\mathcal{C}$ be an $\mathsf{Assoc} \otimes \mathcal{O}$-monoidal ∞-category that is compatible with $\mathfrak{I}$-indexed colimits, and that we also consider as an object of $\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. Then the fiber of $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}}}$ over $\mathcal{C}$ can be identified with $\mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathcal{C})^{\mathrm{op}}$.

[46] It follows immediately from Definition B.2.0.1 that functors being fully faithful satisfies the two-out-of-three-property.
[47] Definition 3.1.2.3

(2) *Let* $F\colon \mathcal{C} \to \mathcal{D}$ *be a morphism in* $\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$. *Then there is a commutative diagram*

$$\begin{tikzcd}
(\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}})_{\mathcal{C}} \arrow[rr, "F_!"] \arrow[d, "\simeq"'] & & (\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}})_{\mathcal{D}} \arrow[d, "\simeq"] \\
\mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathcal{C})^{\mathrm{op}} \arrow[rr, "\mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathcal{F})^{\mathrm{op}}"'] & & \mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathcal{D})^{\mathrm{op}}
\end{tikzcd}$$

where the top horizontal functor is the one induced by F *on fibers of* $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}}}$ *and the vertical equivalences are those from (1).*

Analogous statements holds for $q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}}$. ♡

Proof. By Proposition 3.4.1.4 and Proposition 3.4.1.14 we can consider fibers of

$$\mathrm{pr}_{\mathrm{Mon}_{\mathsf{Assoc}\otimes\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)}\colon \mathrm{Alg}_{/\mathcal{O}}(\mathcal{A}) \to \mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty)$$

instead. For this we can combine Proposition 3.4.1.13 with Remark 3.1.2.18 and then need only compare with the definition of BiAlg in Definition 3.3.0.3. □

3.4.2. LMod as a functor from $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$

In this short section we discuss LMod as a functor $\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}} \to \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$.

Definition 3.4.2.1. Let $\mathfrak{I}$ be a collection of small ∞-categories that includes $\mathbf{\Delta}^{\mathrm{op}}$ and $\mathcal{O}$ an ∞-operad.

Applying $\mathrm{Alg}_{\mathcal{O}}(-)$ to the natural transformation of symmetric monoidal functors denoted by $\mathrm{ev}^{\otimes}_{\mathfrak{m}}\colon \mathrm{LMod}^{\otimes} \to \mathrm{pr}^{\otimes}$ of Proposition 3.2.3.1 and postcomposing with the underlying equivalences of Proposition 3.2.2.8[48] we obtain natural transformations that we will again denote by $\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{LMod} \to \mathrm{pr}$, as depicted in the commutative diagram below

$$\begin{tikzcd}
\mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}"] & \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}} \arrow[l, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}}"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\widetilde{\Psi}^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}})"'] \arrow[r, bend left, "\mathrm{LMod}"] \arrow[r, bend right, "\mathrm{pr}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] & \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "\Psi^{\mathcal{P}\mathrm{r}}_{\mathfrak{I}}"] \\
\mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "\Psi^{\mathfrak{I}}"] & \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}} \arrow[l, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}^{\mathfrak{I}}_{\mathcal{O}}}"'] \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(\widetilde{\Psi}^{\mathfrak{I}})"'] \arrow[r, bend left, "\mathrm{LMod}"] \arrow[r, bend right, "\mathrm{pr}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] & \mathrm{Mon}^{\mathfrak{I}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "\Psi^{\mathfrak{I}}"] \\
\mathrm{Mon}_{\mathcal{O}\otimes\mathsf{Assoc}}(\mathcal{C}\mathrm{at}_\infty) & \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}} \arrow[l, "q_{\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}}}"'] \arrow[r, bend left, "\mathrm{LMod}"] \arrow[r, bend right, "\mathrm{pr}"'] \arrow[r, phantom, "\Downarrow \mathrm{ev}_{\mathfrak{m}}"] & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)
\end{tikzcd}$$

[48] So $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \simeq \mathrm{Mon}^{\mathcal{P}\mathrm{r}}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ etc.

where the functors Ψ and $\widetilde{\Psi}$ are as in Notation 3.2.2.7 and Definition 3.2.2.11, and the left part of the diagram is induced by the pullback squares of Definition 3.2.2.11, which are commutative squares of ∞-operads by Proposition 3.2.2.13. $\diamondsuit$

Remark 3.4.2.2. By Proposition E.4.2.3 (8) the functor induced on $\mathcal{O}$-algebras by a symmetric monoidal functor can again be upgraded to a symmetric monoidal functor with respect to the induced symmetric monoidal structures. It follows that the natural transformations $\mathrm{ev}_{\mathfrak{m}}$ defined in Definition 3.4.2.1 acquire the structure of natural transformations of symmetric monoidal functors between symmetric monoidal ∞-categories. $\diamondsuit$

Remark 3.4.2.3. Let $\mathcal{O}$ be an ∞-operad. Using [HA, 2.4.2.5] and the definition of the equivalence

$$\Theta\colon \operatorname{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \to \operatorname{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$$

as in diagram (3.13) of Proposition 3.2.2.8, we can identify the functor

$$\operatorname{LMod}\colon \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}} \to \operatorname{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty})$$

with the functor induced by the product-preserving functor $\operatorname{LMod}\colon \mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p} \to \mathcal{C}\mathrm{at}_{\infty}$ on $\mathcal{O}$-monoids.

Let $\mathcal{C}$ be a symmetric monoidal ∞-category, A an associative algebra in $\mathcal{C}$, and consider $(\mathcal{C}, A)$ as an object of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$. In the introduction to Section 3.4.1 we discussed how the multiplication morphism induced by the active morphism $\mu\colon \langle 2\rangle \to \langle 1\rangle$ looks like for a commutative monoid structure on $(\mathcal{C}, A)$. Concretely, the multiplication morphism factors as a composition

$$\big(\mathcal{C}\times\mathcal{C}, (A, A)\big) \xrightarrow{\widetilde{-\otimes-}} (\mathcal{C}, A\otimes A) \xrightarrow{(\mathrm{id},\Delta)} (\mathcal{C}, A)$$

where $\widetilde{-\otimes-}$ is a $q_{\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}}$-cocartesian lift of the tensor product functor $-\otimes-\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and (id,Δ) is a morphism in the fiber of $\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}$ over $\mathcal{C}$ – so in $\operatorname{Alg}(\mathcal{C})^{\mathrm{op}}$ – given by a morphism of algebras $\Delta\colon A\to A\otimes A$, encoding the comultiplication.

Let us now discuss the induced multiplication on $\operatorname{LMod}_A(\mathcal{C})$, using Remark 3.1.3.7. The multiplication functor can be identified with the composition

$$\begin{aligned}\operatorname{LMod}_A(\mathcal{C})\times\operatorname{LMod}_A(\mathcal{C}) &\xrightarrow{\simeq} \operatorname{LMod}_{(A,A)}(\mathcal{C}\times\mathcal{C})\\ &\xrightarrow{\operatorname{LMod}_{(A,A)}(\widetilde{-\otimes-})} \operatorname{LMod}_{A\otimes A}(\mathcal{C}) \xrightarrow{\operatorname{LMod}_{\Delta}(\mathcal{C})} \operatorname{LMod}_A(\mathcal{C})\end{aligned}$$

where the first functor arises from compatibility of LMod with products, the second is induced by $\widetilde{-\otimes-}$, and the last functor is given by restriction of the action along Δ.

Let now X and Y be two objects in $\operatorname{LMod}_A(\mathcal{C})$. Then $\operatorname{LMod}_{(A,A)}(\widetilde{-\otimes-})$ sends (X,Y) to the left $A\otimes A$-module in $\mathcal{C}$ whose underlying object in $\mathcal{C}$ is $X\otimes Y$ and where the action by $A\otimes A$ is the tensor-factor-wise one, i. e.

$$(A\otimes A)\otimes(X\otimes Y)\simeq(A\otimes X)\otimes(A\otimes Y)\to X\otimes Y \tag{3.23}$$

where the first morphism uses the symmetric monoidal structure on $\mathcal{C}$ and the second is the tensorwise action of A on X and Y, respectively. Finally, $\mathrm{LMod}_\Delta(\mathcal{C})$ restricts this action along Δ.

The unit morphism, as well as the case of ∞-operads other than the commutative one can be unpacked analogously, and hence the functor

$$\mathrm{LMod}\colon \mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}\mathcal{O}\mathrm{p}_{\mathcal{O}} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$$

really implements the construction sketched at the very beginning of Chapter 3. $\diamondsuit$

We end this section by considering the case of 1-categories, for which the constructions discussed so far reduce to the classical ones.

Remark 3.4.2.4. Let C be a 1-category. The data of a symmetric monoidal structure on C in the classical sense is equivalent to the the data of a symmetric monoidal structure on C considered as an ∞-category, so there is no ambiguity when talking about symmetric monoidal structures on C[49].

So assume now that C is a symmetric monoidal 1-category. By [HA, 4.1.1.2 and 2.1.3.3] the ∞-categories $\mathrm{Alg}(\mathsf{C})$ and $\mathrm{CAlg}(\mathsf{C})$ of associative and commutative algebras in C are 1-categories and can be identified with the usual classical 1-categories of associative and commutative algebras in C. Let $\mathcal{O}$ be either the ∞-operad Assoc or Comm. Then we can also conclude that the ∞-category $\mathrm{BiAlg}_{\mathsf{Assoc},\mathcal{O}}(\mathsf{C})$ can be identified with the classical 1-category of $\mathsf{Assoc}, \mathcal{O}$-bialgebras in C.

Similarly, if A is an associative algebra in C, then by [HA, 4.2.1.3] the ∞-category $\mathrm{LMod}_A(\mathsf{C})$ is a 1-category that can be identified with the usual classical 1-category of left modules over A. The discussion in Remark 3.4.2.3 furthermore implies that if A is an $\mathsf{Assoc}, \mathsf{Comm}$-bialgebra in C, then we can also identify the symmetric monoidal structure on $\mathrm{LMod}_A(\mathsf{C})$ with the classical one that was sketched in the introduction to Chapter 3. $\diamondsuit$

[49]The discussion in [HA, after 2.0.0.6 and condition (M2)] can be summarized as follows: The data of a symmetric monoidal structure on C in the classical sense (up to symmetric monoidal equivalence) is equivalent to the data of a cocartesian fibration of ∞-operads $p\colon \mathsf{C}^\otimes \to \mathsf{Fin}_*$ (up to symmetric monoidal equivalence) such that $\mathsf{C}^\otimes$ is a 1-category.

But if $p\colon \mathsf{C}^\otimes \to \mathsf{Fin}_*$ is any cocartesian fibration of ∞-operads with $\mathsf{C}^\otimes_{\langle 1\rangle} \simeq \mathsf{C}$, then $\mathsf{C}^\otimes$ is automatically a 1-category. Indeed, using that Fin_* is a 1-category it suffices to show that for every pair of objects X and Y of $\mathsf{C}^\otimes$ and morphism $f\colon p(X) \to p(Y)$ in Fin_* the fiber of the map $\mathrm{Map}_{\mathsf{C}^\otimes}(X,Y) \to \mathrm{Map}_{\mathsf{Fin}_*}(p(X), p(Y))$ over f is discrete. But by [HTT, 2.4.4.2], this fiber is equivalent to $\mathrm{Map}_{\mathsf{C}^\otimes_{p(Y)}}(f_! X, Y)$, which is discrete as $\mathsf{C}^\otimes_{p(Y)} \simeq \mathsf{C}^{\times n}$ is a 1-category (here n is such that $p(Y) \cong \langle n\rangle$).

Chapter 4.

Mixed complexes

Let A be an associative k-algebra. As will be discussed in Chapter 6, the Hochschild homology functor $\mathrm{HH}_{\mathbb{T}}$ produces out of A an object of $\mathcal{D}(k)$ with action by the circle group $\mathbb{T}$, so an object of $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$. It will be useful to have a strict model for $\mathrm{HH}_{\mathbb{T}}(A)$, by which we mean an object representing $\mathrm{HH}_{\mathbb{T}}(A)$ in a model category whose underlying ∞-category comes with an equivalence to $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$. This can indeed by done; there is a result of Hoyois [Hoy18], which we will discuss in more detail in Section 6.3.4.1, that provides us with a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{Alg}\big(\mathrm{LMod}_k(\mathsf{Ab})\big) \arrow[rr, "\text{Standard Hochschild complex}"] \arrow[d] & & \mathsf{Mixed} \arrow[d] \\
\mathrm{Alg}(\mathcal{D}(k)) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"'] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[r, dash, "\simeq"'] & \mathcal{M}\mathrm{ixed}
\end{tikzcd}$$

The standard Hochschild complex functor appearing in this diagram has as codomain the model category Mixed of *strict mixed complexes*, which are chain complexes of k-modules together with some extra structure that encodes the circle action. The ∞-category $\mathcal{M}\mathrm{ixed}$ of *mixed complexes* is (equivalent to) the underlying ∞-category of Mixed, and also equivalent to $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$, as we will see in Chapter 5.

In order to be able to make sense of this, this chapter will introduce and discuss Mixed and $\mathcal{M}\mathrm{ixed}$. We begin in Section 4.1 with reviewing chain complexes, primarily to fix notation. In Section 4.2 we will then discuss Mixed, including the closed symmetric monoidal structure that can be defined on it as well as the model structure. We then turn to the corresponding ∞-categories. We will collect the properties we need from $\mathcal{D}(k)$ in Section 4.3. Finally, we discuss the underlying ∞-categories of the model categories Mixed and $\mathrm{Alg}(\mathsf{Mixed})$ in Section 4.4.

4.1. Chain complexes

In this section we briefly review the 1-category of chain complexes of modules over the commutative ring k, to fix notation and sign conventions. We refer to books like [Wei94] for a thorough introduction to homological algebra. The book [Lod98], which we will use as our main reference for classical Hochschild homology, also reviews chain complexes in more detail than we do.

4.1.1. $\mathsf{Ch}(k)$ as a 1-category

To fix notation we briefly review the 1-category of chain complexes of k-modules.

Definition 4.1.1.1. We denote by $\mathsf{Ch}(k)$ the 1-category of chain complexes of k-modules. We use homological grading, so an object X of $\mathsf{Ch}(k)$ consists of k-modules X_n for every integer n together with boundary operators $\partial_n^X \colon X_n \to X_{n-1}$ (we will often omit the sub- and superscript when they are clear from context) satisfying $\partial \circ \partial = 0$.

If x is an element of X_n for some integer n, then we define $\deg_{\mathsf{Ch}}(x) := n$ and call n the *(chain) degree* of x.

If n is an integer, then we denote by $\mathsf{Ch}(k)_{\geq n} = \mathsf{Ch}(k)_{n\leq}$ the full subcategory of $\mathsf{Ch}(k)$ that is spanned by those objects X for which $X_m \cong 0$ if $m < n$. The full subcategories $\mathsf{Ch}(k)_{\leq n}$ and $\mathsf{Ch}(k)_{n_1\leq,\leq n_2}$ are defined analogously. ♢

Definition 4.1.1.2. Let X be an object of $\mathsf{Ch}(k)$ and n an integer. Then we denote by $X[n]$ the *n-fold shift* of X, which is also an object of $\mathsf{Ch}(k)$ that is defined as follows.

$$\big(X[n]\big)_m := X_{m-n} \qquad \partial_m^{X[n]} := (-1)^n \cdot \partial_{m-n}^X$$

We can extend the construction $X \mapsto X[n]$ to an endofunctor of $\mathsf{Ch}(k)$ by setting $(f[n])_m := f_{m-n}$ for morphisms f.

Note that some authors denote what we call $X[n]$ by $X[-n]$, see for example [Wei94, Translation 1.2.8]. The convention we use is chosen to be consistent with [HA, 1.1.2.7]. ♢

Definition 4.1.1.3. If X is a k-module, then we will often consider X as a chain complex of k-modules concentrated in degree 0 without comment. This is the chain complex X' defined as follows.

$$X'_n := \begin{cases} X & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

If we want to make clear we are considering X as a chain complex rather than a module we will use $X[0]$. ♢

4.1.2. $\mathsf{Ch}(k)$ as a closed symmetric monoidal 1-category

In this short section we recall the closed symmetric monoidal structure on $\mathsf{Ch}(k)$, in particular to fix signs.

Definition 4.1.2.1. We equip $\mathsf{Ch}(k)$ with the usual symmetric monoidal structure, described as follows. For X and Y two objects of $\mathsf{Ch}(k)$ and f and g two morphisms in $\mathsf{Ch}(k)$ their tensor product is given by the following formulas[1].

$$(X \otimes Y)_n := \bigoplus_{i+j=n} X_i \otimes Y_j$$

[1]When we write $X_i \otimes Y_j$ this refers to the tensor product in $\mathrm{LMod}_k(\mathsf{Ab})$, i. e. the relative tensor product over k.

$$\partial_n^{X\otimes Y}(x \otimes y) \coloneqq \partial^X(x) \otimes y + (-1)^{\deg_{\mathsf{Ch}}(x)} x \otimes \partial^Y(y)$$
$$(f \otimes g)(x \otimes y) \coloneqq f(x) \otimes g(y)$$

The monoidal unit is $k[0]$, and the symmetry isomorphism is given by the isomorphism $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ that sends $x \otimes y$ to $(-1)^{\deg_{\mathsf{Ch}}(x)\deg_{\mathsf{Ch}}(y)} y \otimes x$.

$\mathsf{Ch}(k)$ can be upgraded to a closed symmetric monoidal category, with internal homomorphism objects given by the following formulas.

$$\mathrm{HOM}_{\mathsf{Ch}(k)}(X,Y)_n = \prod_{i\in\mathbb{Z}} \mathrm{HOM}_{\mathrm{LMod}_k(\mathsf{Ab})}(X_i, Y_{i+n})$$
$$\left(\partial^{\mathrm{HOM}_{\mathsf{Ch}(k)}(X,Y)}(f)\right) = \partial^Y \circ f - (-1)^{\deg_{\mathsf{Ch}}(f)} f \circ \partial^X \qquad \diamondsuit$$

Remark 4.1.2.2. The tensor product is compatible with the shift functors defined in Definition 4.1.1.2; For every integer n there are isomorphisms natural in X and Y as follows

$$\big(X[n]\big) \otimes Y \xrightarrow{\cong} (X \otimes Y)[n] \xleftarrow{\cong} X \otimes \big(Y[n]\big) \tag{4.1}$$

where the first isomorphisms maps $x\otimes y$ to $x\otimes y$, but the second isomorphism introduces a sign by mapping $x \otimes y$ to $(-1)^{n \deg_{\mathsf{Ch}}(x)} x \otimes y$. That one of the two isomorphisms must introduce signs is related to the following compatibility: The first isomorphism in (4.1) is equal to the composition

$$\big(X[n]\big) \otimes Y \cong Y \otimes \big(X[n]\big) \cong (Y \otimes X)[n] \cong (X \otimes Y)[n]$$

where the first and third isomorphism is (induced by) the symmetry isomorphism τ and the middle isomorphism is the second one from (4.1).

The sign is easier to remember if one thinks of $Y[n]$ as $(-)[n]$ applied to Y. Then the shift construction is commuted past X, and hence introduces a sign if the degree of the element of x as well as n are both odd. $\diamondsuit$

4.1.3. $\mathsf{Ch}(k)$ as a model category

We recall the main properties of the projective model structure on $\mathsf{Ch}(k)$ for later use.

Fact 4.1.3.1. $\mathsf{Ch}(k)$ *can be given the* projective *model structure where the weak equivalences are the quasiisomorphisms and the fibrations are the levelwise surjective morphism, see [HA, 7.1.2.8] and [Hov99, 2.3.11]. This model structure is left proper and combinatorial [HA, 7.1.2.8]. Furthermore, with respect to the closed symmetric monoidal structure discussed in Section 4.1.2, this model structure is a symmetric monoidal model structure [HA, 7.1.2.11] with cofibrant unit*[2] *and satisfies the monoid axiom [HA, 7.1.4.3].* ♧

When we refer to the model structure on $\mathsf{Ch}(k)$, we will always mean the projective model structure from Fact 4.1.3.1 – while there are other model structures on $\mathsf{Ch}(k)$, the projective one is the only one we will use in this text.

[2]The definition of a (symmetric) monoidal model category in [HA, 4.1.7] differs slightly from the definition in [Hov99, 4.2.6]: Lurie requires that the unit object is cofibrant, while Hovey replaces this condition with a weaker condition. See Section 4.2.2.2 for a more detailed discussion.

4.1.4. Homotopies in $\mathsf{Ch}(k)$

In this section we record that the notion of homotopy between morphisms from a cofibrant to a fibrant chain complex coincides with the usual notion of chain homotopy.

Proposition 4.1.4.1 ([Hov99, Between 2.3.11 and 2.3.12])**.** *Let Y be a chain complex. Then the operator of degree -1 on the graded k-module $P \coloneqq Y \times Y \times Y[-1]$ defined as*

$$\partial\big((x,y,z)\big) \coloneqq \big(\partial x, \partial y, -\partial(z) + x - y\big)$$

upgrades P into a chain complex. Furthermore the assignments $x \mapsto (x,x,0)$ and $(x,y,z) \mapsto (x,y)$ define morphisms of chain complexes

$$Y \xrightarrow{i} P \xrightarrow{p} Y \times Y$$

which exhibit P as a path object for Y. ♡

Proof. The calculation

$$\begin{aligned} \partial\Big(\partial\big((x,y,z)\big)\Big) &= \partial\Big(\big(\partial x, \partial y, -\partial(z) + x - y\big)\Big) \\ &= \Big(\partial(\partial x), \partial(\partial y), -\partial\big(-\partial(z) + x - y\big) + \partial x - \partial y\Big) \\ &= (0,0,0) \end{aligned}$$

shows that P is a chain complex, and similarly simple calculations show that i and p are morphisms of chain complexes.

It is clear that p is levelwise surjective, so p is a fibration. It thus remains to show that i is a quasiisomorphism. For this consider $r\colon P \to Y$ defined by $(x,y,z) \mapsto x$. This is also a chain map, and $r \circ i = \mathrm{id}_Y$. It thus suffices to show that $i \circ r$ is chain homotopic to the identity. For this consider the chain homotopy h from P to P that is defined by $(x,y,z) \mapsto (0,z,0)$. Then we obtain

$$\begin{aligned} &\partial\Big(h\big((x,y,z)\big)\Big) + h\Big(\partial\big((x,y,z)\big)\Big) \\ &= \partial\big((0,z,0)\big) + h\Big(\big(\partial x, \partial y, -\partial(z) + x - y\big)\Big) \\ &= (0, \partial z, -z) + \big(0, -\partial(z) + x - y, 0\big) \\ &= (0, x - y, -z) \\ &= (x,x,0) - (x,y,z) \\ &= (i \circ r - \mathrm{id}_P)\big((x,y,z)\big) \end{aligned}$$

and thus h is a chain homotopy from $i \circ r$ to id_P. □

Proposition 4.1.4.2 ([Hov99, Between 2.3.11 and 2.3.12])**.** *Let X be a cofibrant chain complex, Y a fibrant chain complex, and f and g two morphisms $X \to Y$ in $\mathsf{Ch}(k)$. Then f and g are homotopic (in the sense of model categories) if and only if there exists a chain homotopy from f to g, i. e. there exists a morphism h of graded k-modules that increases degree by 1 from X to Y satisfying the following relation.*

$$\partial \circ h + h \circ \partial = f - g$$ ♡

Proof. By [Hov99, 1.2.6], as X is cofibrant and Y is fibrant, left and right homotopy define the same equivalence relations on morphisms from X to Y. Furthermore, to check for right homotopies, we can use any path object for Y. Thus f and g are homotopic if and only if there exists a morphism of chain complexes $H\colon X \to P$ such that $p \circ H = f \times g$, where P and p are as in Proposition 4.1.4.1. As a graded k-module, P is given by $Y \times Y \times Y[-1]$, so we can write H as $H = h_0 \times h_1 \times h$, where h_0, h_1, and h are morphisms of graded k-modules from X to Y, where h increases degree by 1. The condition $p \circ H$ amounts to $h_0 = f$ and $h_1 = g$. The remaining data of h is then only constrained by the requirement that H be a morphism of chain complexes. This amounts to the equation

$$\partial \circ H = H \circ \partial$$

needing to hold. The left hand side is given by

$$\partial \circ H = \partial \circ (f \times g \times h) = \big((\partial \circ f) \times (\partial \circ g) \times (-\partial \circ h + f - g)\big)$$

and the right hand side is given by

$$H \circ \partial = (f \times g \times h) \circ \partial = \big((f \circ \partial) \times (g \circ \partial) \times (h \circ \partial)\big)$$

so, as equality in the first two factors follows automatically from f and g being morphisms of chain complexes, this boils down to

$$-\partial \circ h + f - g = h \circ \partial$$

which is equivalent to the equation from the statement. □

4.1.5. Extension of scalars

While we will usually keep the commutative ring k fixed, it will sometimes be useful to consider functoriality in k. For this we record the following statement.

Fact 4.1.5.1 ([Hov99, Page 48 and before 4.2.17 on page 114])**.** *Let $\varphi\colon k \to k'$ be a morphism of commutative rings.*

Then extension and restriction of scalars along φ induces a Quillen adjunction as follows.

$$\begin{tikzcd} \mathsf{Ch}(k) \arrow[r, shift left, "k' \otimes_k -"] \arrow[r, phantom, "\perp"] & \mathsf{Ch}(k') \arrow[l, shift left, "\varphi^*"] \end{tikzcd}$$

Furthermore, $k' \otimes_k -$ preserves fibrations and can be upgraded to a symmetric monoidal functor, making the adjunction into a symmetric monoidal Quillen adjunction in the sense of [Hov99, 4.2.16]. The right adjoint φ^* then obtains the structure of a lax symmetric monoidal functor, but is in general not symmetric monoidal. ♧

4.2. Strict mixed complexes

In this section we discuss strict mixed complexes. Strict mixed complexes where introduced by Kassel in [Kas87], where they are called *mixed complexes*. We will use the additional adjective *strict* to distinguish between the model category of strict mixed complexes Mixed and its underlying ∞-category of mixed complexes $\mathcal{M}\mathrm{ixed}$. A strict mixed complex roughly consists of a chain complex X together with a homomorphism $\mathrm{d}_n \colon X_n \to X_{n+1}$ increasing degree by 1 for every integer n, and satisfying $\mathrm{d} \circ \mathrm{d} = 0$ and $\partial \mathrm{d} + \mathrm{d}\partial = 0$, see Remark 4.2.1.4. The main examples of strict mixed complexes arise in the setting of Hochschild homology: The standard Hochschild complex of an associative ring carries the natural structure of a mixed complex, as will be discussed in Section 6.3.1. This was already alluded to in the introduction of Chapter 4, and in that context the operator d is the extra structure that encodes the circle action.

In Section 4.2.1, we will start by discussing Mixed as a closed symmetric monoidal 1-category. We will then discuss model structures on Mixed as well as $\mathrm{Alg}(\mathsf{Mixed})$ in Section 4.2.2 and discuss their properties and how they relate to each other, for example along the various forgetful functors. Finally, in Section 4.2.3, we will discuss the notion of strongly homotopy linear morphisms of strict mixed complexes, which are a form of weak morphisms between strict mixed complexes that only commute with d up to coherent homotopy.

4.2.1. Mixed as a closed symmetric monoidal 1-category

In this section we define the 1-category of strict mixed complexes Mixed and discuss its closed symmetric monoidal structure as well as algebra objects in Mixed. As Mixed will be defined as the category of left modules over a cocommutative bialgebra D in $\mathsf{Ch}(k)$, we start in Section 4.2.1.1 by defining the D, which then allows us to define Mixed as a symmetric monoidal category in Section 4.2.1.2 by using the results from Section 3.4. We will unpack the symmetric monoidal structure in Section 4.2.1.4 and discuss algebras in Mixed in Section 4.2.1.5. The symmetric monoidal structure will then be upgraded to a closed symmetric monoidal structure in Section 4.2.1.6. Finally, when discussing examples in Chapter 10 it will be helpful to depict mixed complexes diagrammatically, so we introduce the conventions we will use for this in Section 4.2.1.3. Examples of such diagrams will also appear as Example 4.2.1.11 in Section 4.2.1.4.

4.2.1.1. The bialgebra D

Construction 4.2.1.1. Let D be the chain complex of k-modules $k \cdot \{1\} \oplus k \cdot \{\mathrm{d}\}$ with 1 of degree 0 and d of degree 1. In other words, D is the chain complex with zero differentials and a copy of k generated by 1 in degree 0, and a copy of k generated by an element we call d in degree 1.

Then D can be given a unique structure of a commutative algebra in $\mathsf{Ch}(k)$ such that the element 1 in degree 0 is the unit[3].

[3] 1 being the unit already pins down products $x \cdot y$ if one of x and y is in degree 0, and if x and y are

Furthermore, there is a unique way to extend this structure to a commutative and cocommutative bialgebra in $\mathsf{Ch}(k)$. Indeed, if $\epsilon\colon \mathrm{D} \to k$ is the counit of such a bialgebra structure, then $\epsilon(1) = 1$ is determined by the requirement that ϵ is a morphism of algebras, and $\epsilon(d) = 0$ is clear for degree reasons. If $\Delta\colon \mathrm{D} \to \mathrm{D}\otimes\mathrm{D}$ is the comultiplication of such a bialgebra structure, then again as Δ is an algebra morphism we must have $\Delta(1) = 1 \otimes 1$. We can write $\Delta(\mathrm{d})$ as $a \cdot (1 \otimes \mathrm{d}) + b \cdot (\mathrm{d} \otimes 1)$ for some elements a and b of k. But from counitality we can conclude that a and b must both be 1. Hence we must have $\Delta(\mathrm{d}) = \mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}$. That ϵ and Δ defined like this really define a commutative and cocommutative bialgebra can easily be checked.

While we will usually just write D, we will also denote this commutative and cocommutative bialgebra by D_k if we want to make the base ring explicit. It follows immediately from the construction that if $\varphi\colon k \to k'$ is a morphism of commutative rings, then the symmetric monoidal functor[4]

$$k' \otimes_k -\colon \mathsf{Ch}(k) \to \mathsf{Ch}(k')$$

maps D_k to $\mathrm{D}_{k'}$, as a commutative and cocommutative bialgebra. ♢

4.2.1.2. Definition of Mixed

We can now define the symmetric monoidal category of strict mixed complexes.

Definition 4.2.1.2. We denote by Mixed the symmetric monoidal category

$$\mathsf{Mixed} := \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)$$

and call Mixed the category of *strict mixed complexes*. The symmetric monoidal structure we consider here is the one from Definition 3.4.2.1, see also Remark 3.4.2.4.

We will sometimes have reason to use strict mixed complexes whose underlying chain complex is cofibrant with respect to the projective model structure (see Fact 4.1.3.1). We will thus use the notation

$$\mathsf{Mixed}_{\mathrm{cof}} := \mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)$$

for the full symmetric monoidal subcategory of Mixed spanned by those strict mixed complexes whose underlying chain complex is cofibrant.

If we want to make the base ring explicit we will also use the notation Mixed_k and $\mathsf{Mixed}_{k,\mathrm{cof}}$. ♢

Remark 4.2.1.3. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. The symmetric monoidal functor

$$k' \otimes_k -\colon \mathsf{Ch}(k) \to \mathsf{Ch}(k') \tag{4.2}$$

both in degree 1 then the product is 0 for degree reasons.

[4] See Fact 4.1.5.1.

from Fact 4.1.5.1 induces by Definition 3.4.2.1 and Remark 3.4.2.4 a symmetric monoidal functor as indicated at the top of the following commutative diagram.

$$\begin{tikzcd} \mathsf{Mixed}_k \arrow[r, "k'\otimes_k -"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"'] & \mathsf{Mixed}_{k'} \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\ \mathsf{Ch}(k) \arrow[r, "k'\otimes_k -"'] & \mathsf{Ch}(k') \end{tikzcd} \tag{4.3}$$

As (4.2) preserves cofibrant objects by Fact 4.1.5.1, the top horizontal functor restricts to a symmetric monoidal functor from $\mathsf{Mixed}_{k,\mathrm{cof}}$ to $\mathsf{Mixed}_{k',\mathrm{cof}}$.

Furthermore, as the forgetful functors $\mathrm{ev}_{\mathfrak{m}}$ detect colimits by [HA, 4.2.3.5 (2)] and the bottom horizontal functor in (4.3) preserves colimits by Fact 4.1.5.1, the top horizontal functor in (4.3) preserves colimits as well. ♢

Remark 4.2.1.4. Let us unpack what an object of Mixed is. A D-module consists of an underlying chain complex X together with a morphism $\mu\colon \mathrm{D}\otimes X \to X$ of chain complexes, the action of D on X, satisfying associativity and unitality.

Unpacking the definition of the tensor product in $\mathsf{Ch}(k)$ and the definition of D we see that the data of μ corresponds to the data of morphisms of abelian groups

$$\mu(1\otimes -)_n\colon X_n \to X_n \qquad \text{and} \qquad \mu(\mathrm{d}\otimes -)_n\colon X_n \to X_{n+1}$$

for every integer n. Those morphisms have to satisfy a condition corresponding to μ being a morphism of chain complexes.

Let us first note that unitality of the action is equivalent to $\mu(1\otimes -)_n$ being the identity for every n, so this piece of data is redundant. If x is an element of X_n for some n, let us write $\mathrm{d}(x)$ for $\mu(\mathrm{d}\otimes x)$. Then μ being a morphism of chain complexes is equivalent to $\partial\mathrm{d} + \mathrm{d}\partial = 0$. Finally, associativity of the action is equivalent to $\mathrm{d}\circ\mathrm{d} = 0$.

A morphism of D-modules $f\colon X \to Y$ can similarly be unpacked to be a morphism of underlying chain complexes (which we also denote by f) such that $f\circ \mathrm{d}^X = \mathrm{d}^Y \circ f$.

The upshot of the above discussion is that the category of strict mixed complexes is isomorphic to the category of chain complexes with an extra operator d that increases degree by 1, and that satisfies the two equations $\partial\mathrm{d} + \mathrm{d}\partial = 0$ and $\mathrm{d}^2 = 0$. In the rest of the text we will often switch back and forth between these two perspectives. ♢

As an example, we define a very basic family of strict mixed complexes.

Definition 4.2.1.5. Let n be an integer. Then we denote by D_n the strict mixed complex with underlying chain complex $\mathbb{Z}\cdot\{1\}\oplus\mathbb{Z}\cdot\{\delta\}[1]$ (so the same underlying chain complex as D itself), and with d defined by $\mathrm{d}(1) = n\cdot\delta$. ♢

Remark 4.2.1.6. As a D-module, D is isomorphic to D_1. Also note that D_n is isomorphic to D_{-n}. ♢

4.2.1.3. Diagrams depicting strict mixed complexes

Convention 4.2.1.7. It will sometimes be helpful to diagrammatically depict strict mixed complexes for which the underlying graded abelian group is free on some basis $(b_i)_{i \in I}$ for a set I. In that case we will use the following conventions.

- Basis elements are represented by vertices of the diagram.

- A non-squiggly black arrow from b_i to b_j is used to represent the b_j-coefficient of $\partial(b_i)$. More concretely, if we write $\partial(b_i)$ as a linear combination $\sum_{l \in I} a_l \cdot b_l$ of basis elements, with a_l elements of k, then the label of such a non-squiggly black arrow will be a_j. If $a_j = 0$, then we will omit the arrow.

- d is represented completely analogously with red squiggly arrows.

- If an arrow has no label without further comment, then the the missing label is to be interpreted as 1.

- Sometimes we will drop the signs of the labels, or the labels altogether. In these cases we will point this out in the text. ♢

Example 4.2.1.8. The strict mixed complex $\mathrm{D}_n \oplus \mathrm{D}_m[1]$ for n and m integers can be depicted as follows, where we use $1'$ and δ' for the basis elements of D_m.

$$\begin{tikzcd}
& \delta' \\
\delta & 1' \arrow[u, squiggly, "-m"'] \\
1 \arrow[u, squiggly, "n"'] &
\end{tikzcd}$$

The sign arises from the isomorphism $\mathrm{D} \otimes (\mathrm{D}_m[1]) \cong (\mathrm{D} \otimes \mathrm{D}_m)[1]$, see Remark 4.1.2.2. ♢

Example 4.2.1.9. Let n be an integer. The following is an example of an acyclic strict mixed complex.

$$\begin{tikzcd}
\delta' \arrow[dr] & \\
1' \arrow[u, squiggly, "-n"] \arrow[dr] & \delta \\
& 1 \arrow[u, squiggly, "n"']
\end{tikzcd}$$

♢

4.2.1.4. The symmetric monoidal structure on Mixed

Remark 4.2.1.10. Let us unpack the symmetric monoidal structure on Mixed. By Definition 3.4.2.1 the forgetful functor $\mathsf{Mixed} \to \mathsf{Ch}(k)$ is symmetric monoidal, so if X and Y are two strict mixed complexes, then the underlying chain complex of $X \otimes Y$ must be the tensor product of underlying chain complexes, and it remains to figure out how d acts. Using Remark 3.4.2.4, this action arises from the composition

$$\mathrm{D} \otimes X \otimes Y \xrightarrow{\Delta \otimes \mathrm{id}_X \otimes \mathrm{id}_Y} \mathrm{D} \otimes \mathrm{D} \otimes X \otimes Y \xrightarrow{\mathrm{id}_{\mathrm{D}} \otimes \tau_{\mathrm{D},X} \otimes \mathrm{id}_Y} \mathrm{D} \otimes X \otimes \mathrm{D} \otimes Y \xrightarrow{\mu^X \otimes \mu^Y} X \otimes Y$$

where Δ is the comultiplication of D as defined in Construction 4.2.1.1, τ is the symmetry isomorphism reviewed in Definition 4.1.2.1, and μ^X and μ^Y are the action morphisms on X and Y, respectively.

By unpacking the definitions we obtain the following.

$$\begin{aligned}
\mathrm{d}^{X\otimes Y}(x \otimes y) &= \left(\mu^X \otimes \mu^Y\right) \circ \left(\mathrm{id}_{\mathrm{D}} \otimes \tau_{\mathrm{D},X} \otimes \mathrm{id}_Y\right) \circ (\Delta \otimes \mathrm{id}_X \otimes \mathrm{id}_Y)(\mathrm{d} \otimes x \otimes y) \\
&= \left(\mu^X \otimes \mu^Y\right) \circ \left(\mathrm{id}_{\mathrm{D}} \otimes \tau_{\mathrm{D},X} \otimes \mathrm{id}_Y\right) \circ (\mathrm{d} \otimes 1 \otimes x \otimes y + 1 \otimes \mathrm{d} \otimes x \otimes y) \\
&= \left(\mu^X \otimes \mu^Y\right) \circ \left(\mathrm{d} \otimes x \otimes 1 \otimes y + (-1)^{\deg_{\mathsf{Ch}}(x)} 1 \otimes x \otimes \mathrm{d} \otimes y\right) \\
&= \mathrm{d}^X(x) \otimes y + (-1)^{\deg_{\mathsf{Ch}}(x)} x \otimes \mathrm{d}^Y(y)
\end{aligned}$$

The monoidal unit of Mixed is the unique strict mixed complex with underlying chain complex $k[0]$. ♢

Example 4.2.1.11. As an example, let us discuss the tensor product $\mathrm{D}_n \otimes \mathrm{D}_m$ for n and m positive integers.

The strict mixed complex $\mathrm{D}_n \otimes \mathrm{D}_m$ can be depicted as follows.

$$\begin{tikzcd}
& \delta \otimes \delta & \\
\delta \otimes 1 \arrow[ur, squiggly, "-m"] & & 1 \otimes \delta \arrow[ul, squiggly, "n"'] \\
& 1 \otimes 1 \arrow[ul, squiggly, "n"] \arrow[ur, squiggly, "m"'] &
\end{tikzcd}$$

Let i, j be integers such that $\gcd(n, m) = in + jm$. Then another basis for the free abelian group generated by $\delta \otimes 1$ and $1 \otimes \delta$ is given by the two elements

$$\frac{n}{\gcd(n,m)} \cdot \delta \otimes 1 + \frac{m}{\gcd(n,m)} \cdot 1 \otimes \delta \qquad \text{and} \qquad j \cdot \delta \otimes 1 - i \cdot 1 \otimes \delta.$$

Thus we can also depict $\mathrm{D}_n \otimes \mathrm{D}_m$ as follows.

$$\begin{tikzcd}
 & \delta \otimes \delta & \\
\frac{n}{\gcd(n,m)} \cdot \delta \otimes 1 + \frac{m}{\gcd(n,m)} \cdot 1 \otimes \delta & & j \cdot \delta \otimes 1 - i \cdot 1 \otimes \delta \arrow[ul, squiggly, "-\gcd(n,m)"'] \\
 & 1 \otimes 1 \arrow[ul, squiggly, "\gcd(n,m)"] &
\end{tikzcd}$$

Thus $\mathrm{D}_n \otimes \mathrm{D}_m$ is isomorphic in Mixed to $\mathrm{D}_{\gcd(n,m)} \oplus \mathrm{D}_{\gcd(n,m)}[1]$. ♢

4.2.1.5. Algebras in Mixed

As we will later also consider algebras in Mixed, we unpack the definition in the following remark.

Remark 4.2.1.12. As the forgetful functor from Mixed to $\mathsf{Ch}(k)$ is symmetric monoidal, every algebra in strict mixed complexes has an underlying differential graded algebra (i. e. an algebra in $\mathsf{Ch}(k)$). An algebra in Mixed then consists of a differential graded algebra together with a strict mixed complex structure on the underlying chain complex A, such that the unit morphism $k \to A$ and the multiplication morphism $A \otimes A \to A$ are morphisms of strict mixed complexes.

Making use of Remark 4.2.1.10 we can rephrase this as the requirement that $\mathrm{d}(1) = 0$ and that the Leibniz rule

$$\mathrm{d}(x \cdot y) = \mathrm{d}(x) \cdot y + (-1)^{\deg_{\mathsf{Ch}}(x)} x \cdot \mathrm{d}(y)$$

is satisfied for every element x and y of A.

Note that the Leibniz rule for $x = y = 1$ implies $\mathrm{d}(1) = 2\,\mathrm{d}(1)$ and hence $\mathrm{d}(1) = 0$, so if the Leibniz rule holds, then this condition is redundant.

Commutative algebras in Mixed have the analogous description, they consist of a commutative differential graded algebra together with a strict mixed complex structure on the underlying chain complex satisfying the Leibniz rule. ♢

4.2.1.6. The closed symmetric monoidal structure on Mixed

Construction 4.2.1.13. Let X and Y be two strict mixed complexes. We can define an operator d increasing degree by one on $\mathrm{HOM}_{\mathsf{Ch}(k)}(X, Y)$ by letting d act on f by the following formula.

$$\mathrm{d}(f) = \mathrm{d}^Y \circ f - (-1)^{\deg_{\mathsf{Ch}}(f)} f \circ \mathrm{d}^X$$

By unwrapping the definitions it is straightforward to check that this definition satisfies $\mathrm{d} \circ \mathrm{d} = 0$ and $\mathrm{d} \circ \partial + \partial \circ \mathrm{d} = 0$ and thus defines a strict mixed complex, which we will denote by $\mathrm{HOM}_{\mathsf{Mixed}}(X, Y)$. ♢

Proposition 4.2.1.14. *Let*

$$\begin{aligned}\varphi\colon\ \mathrm{Mor}_{\mathsf{Ch}(k)}(-_1\otimes -_2, -_3) &\xrightarrow{\cong} \mathrm{Mor}_{\mathsf{Ch}(k)}\big(-_1, \mathrm{HOM}_{\mathsf{Ch}(k)}(-_2, -_3)\big)\\ f &\mapsto \Big(x\mapsto \big(y\mapsto f(x\otimes y)\big)\Big)\end{aligned}$$

be the natural isomorphism that is part of the closed symmetric monoidal structure on $\mathsf{Ch}(k)$*. Then* φ *restricts to a natural isomorphism as follows.*

$$\mathrm{Mor}_{\mathsf{Mixed}}(-_1\otimes -_2, -_3) \xrightarrow{\cong} \mathrm{Mor}_{\mathsf{Mixed}}\big(-_1, \mathrm{HOM}_{\mathsf{Mixed}}(-_2, -_3)\big)$$

In particular, this makes Mixed *into a* closed *symmetric monoidal category.* ♡

Proof. Let X, Y, and Z be strict mixed complexes and $f : X\otimes Y\to Z$ a morphism of chain complexes. The statement then follows from the following chain of equivalences.

$$\begin{aligned}
&f \text{ is a morphism of strict mixed complexes}\\
\iff &\forall x\in X\colon \forall y\in Y\colon\\
&\mathrm{d}^Z\big(f(x\otimes y)\big) = f\Big(\mathrm{d}^X(x)\otimes y\Big) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x\otimes \mathrm{d}^Y(y))\\
\iff &\forall x\in X\colon \forall y\in Y\colon\\
&\mathrm{d}^Z(\varphi(f)(x)(y)) = \varphi(f)\Big(\mathrm{d}^X(x)\Big)(y) + (-1)^{\deg_{\mathsf{Ch}}(x)}\varphi(f)(x)\Big(\mathrm{d}^Y(y)\Big)\\
\iff &\forall x\in X\colon \forall y\in Y\colon\\
&\mathrm{d}^Z\big(\varphi(f)(x)(y)\big) - (-1)^{\deg_{\mathsf{Ch}}(x)}\varphi(f)(x)\Big(\mathrm{d}^Y(y)\Big) = \varphi(f)\Big(\mathrm{d}^X(x)\Big)(y)\\
\iff &\forall x\in X\colon\ \mathrm{d}\big(\varphi(f)(x)\big) = \varphi(f)\Big(\mathrm{d}^X(x)\Big)\\
\iff &\varphi(f) \text{ is a morphism of strict mixed complexes}
\end{aligned}$$

□

4.2.2. Mixed and Alg(Mixed) as model categories

In this section we construct model structures on Mixed and $\mathrm{Alg}(\mathsf{Mixed})$ and discuss various properties that they have. We will start in Section 4.2.2.1 by reviewing a general result by Schwede and Shipley concerning when one can lift a model structure from a closed symmetric monoidal category with compatible model structure to a model structure on categories of algebras or modules over an algebra. We then apply this in Section 4.2.2.2 to $\mathsf{Ch}(k)$ in order to obtain a model structure on $\mathsf{Mixed} = \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$. We will also show that this model structure is again suitably compatible with the closed symmetric monoidal structure on Mixed, so that we can further lift the model structure from Mixed to $\mathrm{Alg}(\mathsf{Mixed})$, which we do in Section 4.2.2.3. As discussed in Section 4.2.1.5, an algebra in Mixed consists of a chain complex that has both an algebra structure as well as a strict mixed complex structure, satisfying that the Leibniz rule. We thus obtain two forgetful functors on $\mathrm{Alg}(\mathsf{Mixed})$: One forgetting the strict mixed complex structure and mapping to $\mathrm{Alg}(\mathsf{Ch}(k))$, and one forgetting the algebra structure and mapping to Mixed.

Together with the forgetful functors from $\mathrm{Alg}(\mathsf{Ch}(k))$ and Mixed to $\mathsf{Ch}(k)$ they fit into a commutative diagram, and the main result of Section 4.2.2.3 is Proposition 4.2.2.12, in which various properties of those forgetful functors are shown. Finally, it will in practice be helpful to have a concrete description of homotopies in the model categories Mixed as well as $\mathrm{Alg}(\mathsf{Ch}(k))$ and $\mathrm{Alg}(\mathsf{Mixed})$, so we discuss them in Sections 4.2.2.4, 4.2.2.5 and 4.2.2.6.

4.2.2.1. Model categories of algebras and modules

In order to construct model structures on $\mathsf{Mixed} = \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$ and $\mathrm{Alg}(\mathsf{Mixed})$ we will make use of a general theorem by Schwede and Shipley that allows one to lift model structures to categories of modules and algebras. We recall their result as Theorem 4.2.2.1 below.

Theorem 4.2.2.1 ([SS00, Theorem 4.1])**.** *Let C be a combinatorial model category with a closed symmetric monoidal structure such that the tensor product functor is a Quillen bifunctor (i. e. the pushout product axiom is satisfied) and satisfying the monoid axiom (see [SS00, 3.3]).*

Then there is a combinatorial model structure on $\mathrm{Alg}(\mathsf{C})$ such that the following statements hold.

(1) The adjunction

$$\mathrm{Free}^{\mathrm{Alg}}\colon \mathsf{C} \rightleftarrows \mathrm{Alg}(\mathsf{C}) : \mathrm{ev}_{\mathfrak{a}}$$

where $\mathrm{Free}^{\mathrm{Alg}}$ is the free algebra functor and $\mathrm{ev}_{\mathfrak{a}}$ is the forgetful functor, is a Quillen adjunction.

(2) $\mathrm{Alg}(\mathsf{C})$ is cofibrantly generated with the set of generating (acyclic) cofibrations given by application of $\mathrm{Free}^{\mathrm{Alg}}$ to the set of generating (acyclic) cofibrations of C.

(3) $\mathrm{ev}_{\mathfrak{a}}$ preserves and reflects weak equivalences and fibrations.

(4) If the unit of C is cofibrant, then $\mathrm{ev}_{\mathfrak{a}}$ preserves cofibrant objects and cofibrations between cofibrant objects.

Let A be an algebra in C. Then there is a combinatorial model structure on $\mathrm{LMod}_A(\mathsf{C})$ such that the following statements hold.

(5) The adjunction

$$\mathrm{Free}^{\mathrm{LMod}_A}\colon \mathsf{C} \rightleftarrows \mathrm{LMod}_A(\mathsf{C}) : \mathrm{ev}_{\mathfrak{m}}$$

where $\mathrm{Free}^{\mathrm{LMod}_A}$ is the functor sending an object X to the free A-module $A \otimes X$ and $\mathrm{ev}_{\mathfrak{m}}$ is the forgetful functor, is a Quillen adjunction.

(6) $\mathrm{LMod}_A(\mathsf{C})$ is cofibrantly generated with set of generating (acyclic) cofibrations given by application of $\mathrm{Free}^{\mathrm{LMod}_A}$ to the set of generating (acyclic) cofibrations of C.

(7) $\mathrm{ev}_{\mathfrak{m}}$ preserves and reflects weak equivalences and fibrations.

(8) *If the underlying object of A is cofibrant in C, then $\mathrm{ev}_{\mathfrak{m}}$ preserves cofibrations.* ♡

Proof. Construction of the model structures: By definition (see [HTT, A.2.6.1]), a combinatorial model category has presentable underlying category, so in particular every object is small (see [HTT, A.1.1.2]). Furthermore, combinatorial model categories are by definition also cofibrantly generated, so all the conditions to applying [SS00, 4.1] are satisfied. We thus obtain the existence of cofibrantly generated model structures on $\mathrm{Alg}(\mathsf{C})$ and $\mathrm{LMod}_A(\mathsf{C})$. Let us now turn to the various properties of these model structures that we claimed.

Proof of claims (1), (2), (3), (5), (6), and (7): See the proof of [SS00, 4.1] as well as [SS00, 2.3 and the description right before 2.3].

Proof of (4): Part of the statement of [SS00, 4.1 (3)].

Proof that the model structures are combinatorial: It remains to show that $\mathrm{Alg}(\mathsf{C})$ and $\mathrm{LMod}_A(\mathsf{C})$ are presentable. We refer to [HTT, A.1.1.2] for a definition of presentable categories. That the two categories are cocomplete is already part of them being model categories, and as the forgetful functors to C are faithful it is also clear that the morphisms sets are small. It thus suffices to show that the two categories are accessible[5]; condition [HTT, A.1.1.2 (2)] then follows directly from definition and [HTT, A.1.1.2 (3)] follows from [AR94, 2.2 (3) and 1.16]. See also [HTT, 5.5.1.1 and 5.5.0.1].

But both $\mathrm{Alg}(\mathsf{C})$ and $\mathrm{LMod}_A(\mathsf{C})$ are categories of algebras over an accessible monad on C[6], so they are again accessible by [AR94, 2.78].

Proof of claim (8): $\mathrm{ev}_{\mathfrak{m}}$ preserves colimits[7], so to show that $\mathrm{ev}_{\mathfrak{m}}$ preserves cofibrations it suffices to show that $\mathrm{ev}_{\mathfrak{m}}$ preserves generating cofibrations. So let $i\colon X \to Y$ be a cofibration in C. We claim that $\mathrm{ev}_{\mathfrak{m}}(\mathrm{Free}^{\mathrm{LMod}_A}(i)) = \mathrm{id}_A \otimes i$ is again a cofibration. But this follows from $- \otimes -$ being a Quillen bifunctor[8]. □

4.2.2.2. The model structure on Mixed

The general result Theorem 4.2.2.1 allows us to define a combinatorial model structure on Mixed that is lifted from the projective model structure on $\mathsf{Ch}(k)$ – all prerequisites to apply Theorem 4.2.2.1 are covered by Fact 4.1.3.1.

Definition 4.2.2.2. We equip $\mathsf{Mixed} = \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))$ with the combinatorial model structure from Theorem 4.2.2.1 that is lifted from the projective model structure on $\mathsf{Ch}(k)$. ♢

[5] See [AR94, 2.2 (1)] for a definition. An object is called κ-presentable (presentable) in [AR94, 1.13] precisely if it is called κ-compact (small) in [HTT, A.1.1.1]. Thus (keeping in mind we already know that the categories in question are cocomplete), [AR94, 2.2 (1)] asks for existence of a regular cardinal κ and a small set of κ-compact objects such that every object can be obtained as a κ-filtered colimit of objects from that set.

[6] The proof of [SS00, 4.1] uses this fact, so see there for more details.

[7] Because we assume that the symmetric monoidal structure on C is closed, the tensor product preserves colimits separately in each variable, so we can apply [HA, 4.2.3.5].

[8] See [Hov99, 4.2.1] for a definition. We apply the property to the cofibrations $0 \to A$ and i, and use that the morphism $(0 \to A) \,\square\, i$ can be identified with $\mathrm{id}_A \otimes i$.

Proposition 4.2.2.3. *Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the extension of scalars functor*

$$k' \otimes_k - \colon \mathsf{Mixed}_k \to \mathsf{Mixed}_{k'}$$

from Remark 4.2.1.3 preserves cofibrations as well as weak equivalences between objects with cofibrant underlying chain complex. ♡

Proof. We first show that the functor preserves cofibrations. As it preserves colimits by Remark 4.2.1.3, it suffices to show that the functor preserves generating cofibrations. But this follows immediately from compatibility with the free module functors by Proposition E.7.4.1 in combination with

$$k' \otimes_k - \colon \mathsf{Ch}(k) \to \mathsf{Ch}(k') \qquad (*)$$

preserving cofibrations by Fact 4.1.5.1.

That the functor preserves weak equivalences between objects with cofibrant underlying chain complex follows directly from the forgetful functors ev_{m} detecting weak equivalences, the diagram (4.3) in Remark 4.2.1.3 commuting, and $(*)$ preserving weak equivalences between cofibrant objects by Fact 4.1.5.1. □

Proposition 4.2.2.4. *The underlying chain complex of* D *is cofibrant.* ♡

Proof. Follows from [Hov99, 2.3.6]. □

So we have now obtained a model structure on Mixed. We also already discussed a closed symmetric monoidal structure on Mixed, see Proposition 4.2.1.14. We would like to show that these to structures are in fact compatible and make Mixed into a symmetric monoidal model structure. However, there are slightly different definitions of what properties a monoidal model structure needs to satisfy, and not all are true in this case. What all definitions require is that the tensor product is a Quillen bifunctor. As explained in [SS00, 3.2] and [Hov99, below 4.2.6], this does not quite suffice to obtain an induced monoidal structure on the homotopy category, a condition on the unit object is also necessary. This is because the derived tensor product is formed by tensoring cofibrant replacements of the two objects one wants to tensor. If the unit object is not cofibrant, there is no guarantee that the derived tensor product with the unit object is weakly equivalent to the original object. One condition to guarantee that this is nevertheless the case is given in [Hov99, 4.2.6] as part of Hovey's definition of monoidal model structures. This condition is always satisfied when the unit is in fact cofibrant, and Lurie requires this more restrictive condition for monoidal model categories [HA, Start of 4.1.7].

The unit object in Mixed is $\mathbb{Z}$ (see Remark 4.2.1.10), which is unfortunately not cofibrant (see Proposition 4.2.2.5 directly below), so we can not directly apply some of the result concerning monoidal model categories proven in [HA], like the result on rectification of algebras [HA, 4.1.8.4]. However, Hovey's condition is satisfied, and we will be able to work around the obstacles to deducing the analogous result to [HA, 4.1.8] in Proposition 4.4.2.3 in Section 4.4.2.

Proposition 4.2.2.5. *The unit object $\mathbb{Z}$ of Mixed (see Remark 4.2.1.10) is not cofibrant with respect to the model structure from Definition 4.2.2.2.* ♡

Proof. Consider the counit $\epsilon\colon \mathrm{D} \to \mathbb{Z}$. This is a morphism of mixed complexes, and also a fibration in Mixed as it is levelwise surjective and $\mathrm{ev}_{\mathfrak{m}}$ detects fibrations by Theorem 4.2.2.1 (7). If $\mathbb{Z}$ were cofibrant in Mixed, then there would have to exist a section of ϵ as strict mixed complexes. However, the unique section in $\mathsf{Ch}(k)$ is not a morphism of strict mixed complexes, as $\mathrm{d}(1) = \mathrm{d} \neq 0$ in D. □

Proposition 4.2.2.6. *The model structure on Mixed from Definition 4.2.2.2 is a symmetric monoidal model structure (in the sense of [Hov99, 4.2.6]) with respect to the closed symmetric monoidal structure from Definition 4.2.1.2 and Proposition 4.2.1.14.* ♡

Proof. Proof that $- \otimes -$ is a Quillen bifunctor: Let $f\colon W \to X$ be a cofibration and $p\colon Y \to Z$ a fibration in Mixed. By [Hov99, 4.2.2] if suffices to show that the induced morphism

$$\mathrm{HOM}_{\mathsf{Mixed}}(X, Y) \to \mathrm{HOM}_{\mathsf{Mixed}}(X, Z) \times_{\mathrm{HOM}_{\mathsf{Mixed}}(W,Z)} \mathrm{HOM}_{\mathsf{Mixed}}(W, Y)$$

is a fibration in Mixed, and acyclic if f or p is acyclic. But this follows immediately from $\mathsf{Ch}(k)$ having the corresponding property by Fact 4.1.3.1 and [Hov99, 4.2.2], in combination with $\mathrm{ev}_{\mathfrak{m}}$ preserving and detecting fibrations and weak equivalences by Theorem 4.2.2.1 (7), preserving cofibrations by Theorem 4.2.2.1 (8) and Proposition 4.2.2.4, and mapping $\mathrm{HOM}_{\mathsf{Mixed}}$ to $\mathrm{HOM}_{\mathsf{Ch}(k)}$ by Construction 4.2.1.13.

Proof of [Hov99, 4.2.6 (2)]: We have to show that if $0 \to \mathbb{Z}^{\mathrm{cof}} \xrightarrow{f} \mathbb{Z}$ is a factorization in Mixed of $0 \to \mathbb{Z}$ into a cofibration followed by an acyclic fibration, then tensoring f with the identity of any cofibrant object on either side yields a weak equivalence. By Proposition 4.2.2.4 and Theorem 4.2.2.1 (7) and (8), the forgetful functor $\mathrm{ev}_{\mathfrak{m}}\colon \mathsf{Mixed} \to \mathsf{Ch}(k)$ preserves weak equivalences as well as cofibrations, and also detects weak equivalences. Furthermore, $\mathrm{ev}_{\mathfrak{m}}$ is also symmetric monoidal.

Hence it suffices to show that for a cofibrant chain complex X it holds that

$$\mathrm{ev}_{\mathfrak{m}}(\mathbb{Z}^{\mathrm{cof}}) \otimes X \xrightarrow{\mathrm{ev}_{\mathfrak{m}}(f) \otimes \mathrm{id}_X} \mathrm{ev}_{\mathfrak{m}}(\mathbb{Z}) \otimes X$$

is a weak equivalence in $\mathsf{Ch}(k)$. But note that while $\mathbb{Z}$ is not cofibrant an an object in Mixed, it *is* cofibrant as a chain complex. Hence $\mathrm{ev}_{\mathfrak{m}}(\mathbb{Z}^{\mathrm{cof}}) \xrightarrow{\mathrm{ev}_{\mathfrak{m}}(f)} \mathrm{ev}_{\mathfrak{m}}(\mathbb{Z}) = \mathbb{Z}$ is a weak equivalence between cofibrant objects. As $\mathsf{Ch}(k)$ is a symmetric monoidal model category, $- \otimes X$ preserves acyclic cofibrations, and hence sends weak equivalences between cofibrant objects to weak equivalences (see [Hov99, 1.1.12]), so the claim follows. □

We next show that Mixed satisfies the monoid axiom. Definitions of the monoid axiom can be found in [SS00, 3.3] and [HA, 4.1.8.1], however these two definitions are stated in a slightly different way, so we briefly discuss them first in the next remark.

Remark 4.2.2.7. Let C be a combinatorial model category that is equipped with a symmetric monoidal structure.

Let U be the subclass of morphisms of C that are of the form $\mathrm{id}_X \otimes i$, with X an object in C and i an acyclic cofibration. Let $\overline{U}$ be the weakly saturated class of morphisms generated by U[9]. Let $\widetilde{U}$ be the subclass of morphisms of C that can be obtained as a transfinite composition of pushouts of morphisms in U. Finally, let $\widetilde{U}'$ be the subclass of morphisms of C that are retracts of morphisms in $\widetilde{U}$.

Then [SS00, 3.3] asks that all morphisms in $\widetilde{U}$ are weak equivalences, and [HA, 4.1.8.1] asks that all morphisms in $\overline{U}$ are weak equivalences.

From the definitions it is clear that $\widetilde{U}'$ is contained in $\overline{U}$. On the other hand, [HTT, A.1.2.8] implies that $\overline{U}$ is contained in $\widetilde{U}'$. As weak equivalences are closed under retracts, $\widetilde{U}$ is contained in the class of weak equivalences if and only if $\widetilde{U}' = \overline{U}$ is, so definitions [SS00, 3.3] and [HA, 4.1.8.1] are equivalent. $\diamondsuit$

Proposition 4.2.2.8. *The symmetric monoidal model category*[10] Mixed *satisfies the monoid axiom.* $\heartsuit$

Proof. In this proof we use the following notation. If S is a class of morphisms in some monoidal category C, then we denote by $\mathsf{C} \otimes S$ the class of all morphisms of the form $\mathrm{id}_X \otimes s$ where X is an object of C and s is an element of S. We denote by $\overline{S}$ the weakly saturated class of morphisms generated by S in the sense of [HTT, A.1.2.2].

Denote by W the class of weak equivalences of $\mathsf{Ch}(k)$, and by I a set of generating acyclic cofibrations of $\mathsf{Ch}(k)$. We also define $\mathrm{Free}^{\mathsf{Mixed}}$ to be $\mathrm{Free}^{\mathrm{LMod}_{\mathrm{D}}}$, the left adjoint to the forgetful functor $\mathrm{ev}_{\mathfrak{m}} \colon \mathsf{Mixed} \to \mathsf{Ch}(k)$.

What we have to show is that $\overline{\mathsf{Mixed} \otimes \{\text{acyclic cofibrations in } \mathsf{Mixed}\}}$ is contained in the class of weak equivalences of Mixed, which by Theorem 4.2.2.1 (8) is equivalent to showing that $\mathrm{ev}_{\mathfrak{m}}(\overline{\mathsf{Mixed} \otimes \{\text{acyclic cofibrations in } \mathsf{Mixed}\}})$ is contained in W.

This will follow from the following easy claims.

(1) $\overline{\mathsf{Mixed} \otimes \{\text{acyclic cofibrations in } \mathsf{Mixed}\}} = \overline{\mathsf{Mixed} \otimes \mathrm{Free}^{\mathsf{Mixed}}(I)}$

(2) $\mathrm{ev}_{\mathfrak{m}}\left(\overline{\mathsf{Mixed} \otimes \mathrm{Free}^{\mathsf{Mixed}}(I)}\right) \subseteq \overline{\mathrm{ev}_{\mathfrak{m}}\left(\mathsf{Mixed} \otimes \mathrm{Free}^{\mathsf{Mixed}}(I)\right)}$

(3) $\mathrm{ev}_{\mathfrak{m}}\left(\mathsf{Mixed} \otimes \mathrm{Free}^{\mathsf{Mixed}}(I)\right) \subseteq \mathsf{Ch}(k) \otimes \{\text{acyclic cofibrations in } \mathsf{Ch}(k)\}$

(4) $\overline{\mathsf{Ch}(k) \otimes \{\text{acyclic cofibrations in } \mathsf{Ch}(k)\}} \subseteq W$

Proof of claim (1): The class of acyclic cofibrations in Mixed is by Theorem 4.2.2.1 (6) equal to $\overline{\mathrm{Free}^{\mathsf{Mixed}}(I)}$. As the tensor product functor on Mixed preserves colimits in each variable the claim follows.

Proof of claim (2): Follows from $\mathrm{ev}_{\mathfrak{m}}$ preserving colimits.

[9] See [HTT, A.1.2.2] for a definition. This is smallest subclass of morphisms of C containing U that is closed under taking pushouts along morphisms of C, transfinite compositions, and retracts.

[10] In the sense of [Hov99, 4.2.6].

Proof of claim (3): Let i be a generating acyclic cofibration of $\mathsf{Ch}(k)$ and X a strict mixed complex. Then we have

$$\mathrm{ev}_{\mathfrak{m}}\left(\mathrm{id}_X \otimes \mathrm{Free}^{\mathsf{Mixed}}(i)\right) \cong \mathrm{id}_{X\otimes \mathrm{D}} \otimes i$$

where we use that $\mathrm{ev}_{\mathfrak{m}}$ is symmetric monoidal, so the claim follows.

Proof of claim (4): True as $\mathsf{Ch}(k)$ satisfies the monoid axiom, see Fact 4.1.3.1. □

4.2.2.3. The model structure on $\mathrm{Alg}(\mathsf{Mixed})$

We can now put together the various results regarding the model structure on Mixed and apply Theorem 4.2.2.1 in order to obtain a combinatorial model structure on $\mathrm{Alg}(\mathsf{Mixed})$.

Proposition 4.2.2.9. *There is a combinatorial model structure on* $\mathrm{Alg}(\mathsf{Mixed})$ *as well as* $\mathrm{Alg}(\mathsf{Ch}(k))$ *with the properties listed in Theorem 4.2.2.1.* ♡

Proof. By Definition 4.2.2.2 the model structure on Mixed is combinatorial, by Proposition 4.2.1.14 there is a closed symmetric monoidal structure on Mixed, by Proposition 4.2.2.6 the model structure satisfies the pushout product axiom, and by Proposition 4.2.2.8 the monoid axiom is satisfied. $\mathsf{Ch}(k)$ has all these properties as well by Fact 4.1.3.1. We can thus apply Theorem 4.2.2.1. □

We end this section by discussing the various forgetful functors, and show some properties that they have that will be useful later.

Notation 4.2.2.10. There is a commutative diagram of forgetful functors as follows.

$$\begin{tikzcd}
 & \mathrm{Alg}(\mathsf{Mixed}) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}"'] \arrow[dr, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})"] & \\
\mathsf{Mixed} \arrow[dr, "\mathrm{ev}_{\mathfrak{m}}"'] & & \mathrm{Alg}(\mathsf{Ch}(k)) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}"] \\
 & \mathsf{Ch}(k) &
\end{tikzcd} \tag{4.4}$$

To be able to distinguish the two forgetful functors from categories of algebras to their underlying categories, we give the forgetful functor $\mathrm{Alg}(\mathsf{Mixed}) \to \mathsf{Mixed}$ an extra superscript Mixed.

The functors $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$, $\mathrm{ev}_{\mathfrak{m}}$, and $\mathrm{ev}_{\mathfrak{a}}$ all have left adjoints according to Theorem 4.2.2.1. We denote

- the left adjoint to $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ by $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$.
- the left adjoint to $\mathrm{ev}_{\mathfrak{m}}$ by $\mathrm{Free}^{\mathsf{Mixed}}$.
- the left adjoint to $\mathrm{ev}_{\mathfrak{a}}$ by $\mathrm{Free}^{\mathrm{Alg}}$. ◇

Proposition 4.2.2.11. *The commutative square*

$$\begin{tikzcd} \mathsf{Alg}(\mathsf{Mixed}) \arrow[r, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}"] \arrow[d, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})"'] & \mathsf{Mixed} \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\ \mathrm{Alg}(\mathsf{Ch}(k)) \arrow[r, "\mathrm{ev}_{\mathfrak{a}}"'] & \mathsf{Ch}(k) \end{tikzcd}$$

from Notation 4.2.2.10 is left adjointable[11]*, i. e. the push-pull transformation*

$$\mathrm{Free}^{\mathrm{Alg}} \circ \mathrm{ev}_{\mathfrak{m}} \to \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}}) \circ \mathrm{Free}_{\mathsf{Mixed}}^{\mathsf{Alg}(\mathsf{Mixed})}$$

is a natural isomorphism. ♡

Proof. As the symmetric monoidal structures on Mixed and $\mathsf{Ch}(k)$ are compatible with colimits[12], and $\mathrm{ev}_{\mathfrak{m}}$ is symmetric monoidal and preserves colimits[13], this is a special case of Proposition E.7.2.2 (2). □

We can now collect some properties of the various forgetful functors.

Proposition 4.2.2.12. *The following table summarizes what kind of morphisms or constructions the various forgetful functors from Notation 4.2.2.10 preserve (marked with a P) or detect (marked with a D).*

Functor	*isos*[14]	*w. e.*[15]	*fib*[16]	*cofib*[17]	*cofib'*[18]	*lim*	*sifted colim*	*colim*
$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$	*D*	*D*	*D*			*D*	*D*	
$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$	*D*	*D*	*D*	*P*	*P*	*D*	*D*	*D*
$\mathrm{ev}_{\mathfrak{m}}$	*D*	*D*	*D*	*P*	*P*	*D*	*D*	*D*
$\mathrm{ev}_{\mathfrak{a}}$	*D*	*D*	*D*		*P*	*D*	*D*	

All properties that make use of a model structure are to be understood with respect to the model structures from Fact 4.1.3.1, Definition 4.2.2.2, and Proposition 4.2.2.9. ♡

Proof. Weak equivalences and fibrations: That $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$, $\mathrm{ev}_{\mathfrak{m}}$, and $\mathrm{ev}_{\mathfrak{a}}$ detect weak equivalences and fibrations is Theorem 4.2.2.1 (3) and (7). From commutativity of the diagram (4.4) we obtain the same for $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$.

[11] See [HTT, 7.3.1.1] for a definition.
[12] As both symmetric monoidal categories are closed symmetric monoidal, see Definition 4.1.2.1 and Proposition 4.2.1.14.
[13] See for example [HA, 4.2.3.5].
[14] Isomorphisms.
[15] Weak equivalences.
[16] Fibrations.
[17] Cofibrations.
[18] Cofibrant objects and cofibrations between cofibrant objects.

Limits and sifted colimits: That limits and colimits in module categories[19] are calculated on underlying objects is a standard categorical fact, see for example [HA, 4.2.3.3 and 4.2.3.5]. Similarly, it is standard that limits and sifted colimits[20] of algebras are calculated on underlying objects, see for example [HA, 3.2.2.5] and [HA, 3.2.3.1]. Again, as the three other functors detect limits and sifted colimits, this also follows for $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$.

Isomorphisms: That $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$, $\mathrm{ev}_{\mathfrak{m}}$, and $\mathrm{ev}_{\mathfrak{a}}$ are conservative, i. e. detect isomorphisms, is standard, and then it again follows that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ is conservative as well. However, we could also deduce this from all four functors detecting sifted colimits, as detecting isomorphisms is equivalent to detecting $[0]$-colimits.

Colimits: That $\mathrm{ev}_{\mathfrak{m}}$ detects colimits was already mentioned above. As $\mathrm{ev}_{\mathfrak{m}}$ is also symmetric monoidal, it then follows from Proposition E.7.3.1 that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ preserves colimits as well. As $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ is conservative, this implies that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ even detects colimits.

Cofibrations and cofibrations between cofibrant objects: It follows from Theorem 4.2.2.1 (8) in combination with D being cofibrant in $\mathsf{Ch}(k)$ by Proposition 4.2.2.4 that $\mathrm{ev}_{\mathfrak{m}}$ preserves cofibrations. It follows from Theorem 4.2.2.1 (4) in combination with the monoidal unit of $\mathsf{Ch}(k)$ being cofibrant by Fact 4.1.3.1 that $\mathrm{ev}_{\mathfrak{a}}$ preserves cofibrant objects and cofibrations between cofibrant objects.

It remains to show that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ preserves cofibrations. As we already showed that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ preserves colimits, it suffices to show that $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ maps generating cofibrations to cofibrations. Generating cofibrations of $\mathrm{Alg}(\mathsf{Mixed})$ are by Theorem 4.2.2.1 (2) and (6) morphisms of the form $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}(\mathrm{Free}^{\mathsf{Mixed}}(i))$ with i a (generating) cofibration in $\mathsf{Ch}(k)$. By Proposition 4.2.2.11 there is a natural isomorphism as follows.

$$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}}) \circ \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})} \circ \mathrm{Free}^{\mathsf{Mixed}} \cong \mathrm{Free}^{\mathrm{Alg}} \circ \mathrm{ev}_{\mathfrak{m}} \circ \mathrm{Free}^{\mathsf{Mixed}}$$

As $\mathrm{Free}^{\mathrm{Alg}}$ and $\mathrm{Free}^{\mathsf{Mixed}}$ preserve cofibrations as left Quillen functors[21] and $\mathrm{ev}_{\mathfrak{m}}$ was already shown to preserve cofibrations, the claim follows. □

Proposition 4.2.2.13. *Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the extension of scalars functor*

$$k' \otimes_k -\colon\ \mathrm{Alg}(\mathsf{Mixed}_k) \to \mathrm{Alg}(\mathsf{Mixed}_{k'})$$

that is induced on algebras by the symmetric monoidal functor $k' \otimes_k -\colon \mathsf{Mixed}_k \to \mathsf{Mixed}_{k'}$ from Remark 4.2.1.3 preserves colimits and cofibrations. ♡

Proof. The extension of scalars functor

$$k' \otimes_k -\colon \mathsf{Mixed}_k \to \mathsf{Mixed}_{k'}$$

[19] This is true for categories of modules in a monoidal category whose tensor product functor preserves colimits in each variable separately, which is the case for $\mathsf{Ch}(k)$, as it is a closed symmetric monoidal category.

[20] This again requires the assumption that the tensor product preserves sifted colimits in each variable separately, which is the case for both $\mathsf{Ch}(k)$ and Mixed.

[21] See Theorem 4.2.2.1 (1) and (5).

is by Remark 4.2.1.3 symmetric monoidal and preserves colimits. As the tensor product functors of Mixed_k and $\mathsf{Mixed}_{k'}$ also preserve colimits in each variable separately by Proposition 4.2.2.6 we can apply Proposition E.7.3.1 to conclude that the induced functor

$$k' \otimes_k -\colon \operatorname{Alg}(\mathsf{Mixed}_k) \to \operatorname{Alg}(\mathsf{Mixed}_{k'})$$

preserves colimits.

To show that this functor also preserves cofibrations it now suffices to show that it maps generating cofibrations to cofibrations. So let $i\colon X \to Y$ be a cofibration in Mixed_k. We have to show that

$$k' \otimes_k \operatorname{Free}_{\mathsf{Mixed}_k}^{\operatorname{Alg}(\mathsf{Mixed}_k)}(i)$$

is a cofibration in $\operatorname{Alg}(\mathsf{Mixed}_{k'})$. But by Proposition E.7.2.2 we can identify this morphism with

$$\operatorname{Free}_{\mathsf{Mixed}_{k'}}^{\operatorname{Alg}(\mathsf{Mixed}_{k'})}(k' \otimes_k i)$$

which is a cofibration as $\operatorname{Free}_{\mathsf{Mixed}_{k'}}^{\operatorname{Alg}(\mathsf{Mixed}_{k'})}$ is a left Quillen functor by Theorem 4.2.2.1 (5) and

$$k' \otimes_k -\colon \mathsf{Mixed}_k \to \mathsf{Mixed}_{k'}$$

preserves cofibrations by Proposition 4.2.2.3. □

4.2.2.4. Homotopies in Mixed

In this section we describe homotopies in Mixed, continuing from and proceeding analogously to Section 4.1.4.

Proposition 4.2.2.14. *Let Y be a strict mixed complex. Then defining an operator d that increases degree by one on P from Proposition 4.1.4.1 as*

$$\mathrm{d}\big((x, y, z)\big) \coloneqq (\mathrm{d}\,x, \mathrm{d}\,y, -\mathrm{d}\,z)$$

upgrades P to a strict mixed complex. Furthermore, the morphisms i and p that were defined in Proposition 4.1.4.1 are compatible with this strict mixed structure, exhibiting P as a path object for Y in Mixed. ♡

Proof. It is clear that d as defined in the statement is k-linear and increases degree by 1. Let (x, y, z) be an element in P. Then the short calculation

$$\mathrm{d}\Big(\mathrm{d}\big((x, y, z)\big)\Big) = \mathrm{d}\big((\mathrm{d}\,x, \mathrm{d}\,y, -\mathrm{d}\,z)\big) = \big(\mathrm{d}(\mathrm{d}\,x), \mathrm{d}(\mathrm{d}\,y), \mathrm{d}(\mathrm{d}\,z)\big) = (0, 0, 0)$$

shows that d squares to zero, and the following calculation shows that $\mathrm{d} \circ \partial + \partial \circ \mathrm{d} = 0$, so that P indeed becomes a strict mixed complex.

$$\begin{aligned}&(\mathrm{d} \circ \partial + \partial \circ \mathrm{d})\big((x, y, z)\big)\\ &= \mathrm{d}\Big(\big(\partial x, \partial y, -\partial(z) + x - y\big)\Big) + \partial\big((\mathrm{d}\,x, \mathrm{d}\,y, -\mathrm{d}\,z)\big)\end{aligned}$$

$$\begin{aligned}
&= \Big(\mathrm{d}\big(\partial(x)\big), \mathrm{d}\big(\partial(y)\big), -\mathrm{d}\big(-\partial(z) + x - y\big)\Big) \\
&\quad + \Big(\partial\big(\mathrm{d}(x)\big), \partial\big(\mathrm{d}(y)\big), -\partial\big(-\mathrm{d}(z)\big) + \mathrm{d}\, x - \mathrm{d}\, y\Big) \\
&= \Big(\mathrm{d}\big(\partial(x)\big) + \partial\big(\mathrm{d}(x)\big), \mathrm{d}\big(\partial(y)\big) + \partial\big(\mathrm{d}(y)\big), \\
&\qquad \mathrm{d}\big(\partial(z)\big) - \mathrm{d}(x) + \mathrm{d}(y) + \partial\big(\mathrm{d}(z)\big) + \mathrm{d}(x) - \mathrm{d}(y)\Big) \\
&= (0, 0, 0)
\end{aligned}$$

It is clear that i and p are compatible with d, making them into morphisms in Mixed. As the forgetful functor $\mathrm{ev}_{\mathfrak{m}}\colon \mathsf{Mixed} \to \mathsf{Ch}(k)$ detects weak equivalences and fibrations by Proposition 4.2.2.12, it now follows from Proposition 4.1.4.1 that i and p exhibit P as a path object for Y. □

Proposition 4.2.2.15. *Let X be a cofibrant and Y a fibrant object in Mixed, with respect to the model structure of Definition 4.2.2.2, and f and g two morphisms $X \to Y$ in Mixed. Then f and g are homotopic if and only if there exists a* chain homotopy of strict mixed complexes *h from f to g, by which we mean a chain homotopy h from f to g in the sense of Proposition 4.1.4.2 satisfying additionally*[22]

$$h\big(\mathrm{d}(x)\big) = -\,\mathrm{d}\big(h(x)\big) \tag{4.5}$$

for all elements x of X. ♡

Proof. Note that by [Hov99, 1.2.6], as X is cofibrant and Y is fibrant, the left and right homotopy relations coincide, and the right homotopy relation can be tested using any path object for Y. For this we use the path object P from Proposition 4.2.2.14.

Arguing completely analogously to the proof of Proposition 4.1.4.2, we see that f and g are homotopic as morphisms of strict mixed complexes if and only if there exists a morphism of strict mixed complexes $H = f \times g \times h\colon X \to P$. That H is a morphism of chain complexes amounts, just like in Proposition 4.1.4.2, to

$$\partial \circ h + h \circ \partial = f - g$$

but this time H needs to additionally commute with d, so for x an element of X the following equality must hold.

$$\Big(f\big(\mathrm{d}(x)\big), g\big(\mathrm{d}(x)\big), h\big(\mathrm{d}(x)\big)\Big) = \mathrm{d}\Big(\big(f(x), g(x), h(x)\big)\Big) \tag{$*$}$$

The right hand side is given by

$$\mathrm{d}\Big(\big(f(x), g(x), h(x)\big)\Big) = \Big(\mathrm{d}\big(f(x)\big), \mathrm{d}\big(g(x)\big), -\,\mathrm{d}\big(h(x)\big)\Big)$$

so as f and g are morphisms of strict mixed complexes we can conclude that equality $(*)$ is equivalent to the following equation.

$$h\big(\mathrm{d}(x)\big) = -\,\mathrm{d}\big(h(x)\big)$$ □

[22]To remember the sign, note that both d and h have odd degree, so commuting them should be expected to introduce a sign.

4.2.2.5. Homotopies in $\mathrm{Alg}(\mathsf{Ch}(k))$

In this section we describe homotopies in $\mathrm{Alg}(\mathsf{Ch}(k))$. The statements of the first two propositions, concerning an appropriate path object and a concrete description of the resulting homotopies, are completely analogous to the propositions in Sections 4.1.4 and 4.2.2.4. However, this section has an additional helpful result that reduces the amount of data that needs to be specified and the amount of properties that need to be checked to construct homotopies out of differential graded algebras whose underlying $\mathbb{Z}$-graded k-algebra is free.

Proposition 4.2.2.16. *Let Y be a differential graded k-algebra. Then defining a multiplication on the chain complex P that was defined in Proposition 4.1.4.1 as*

$$(x, y, z) \cdot (x', y', z') := \Big(xx', yy', zy' + (-1)^{\deg_{\mathsf{Ch}}(x)} xz'\Big)$$

upgrades P to a differential graded k-algebra with unit $(1, 1, 0)$. Furthermore, the morphisms i and p that were defined in Proposition 4.1.4.1 are compatible with this multiplicative structure, exhibiting P as a path object for Y in $\mathrm{Alg}(\mathsf{Ch}(k))$. ♡

Proof. It is clear that $(1, 1, 0)$ is a unit for the multiplication that was defined in the statement, and that multiplication is k-linear in both factors. For associativity we carry out the following calculations.

$$\begin{aligned}
&\Big((x, y, z) \cdot (x', y', z')\Big) \cdot (x'', y'', z'') \\
&= \Big(xx', yy', zy' + (-1)^{\deg_{\mathsf{Ch}}(x)} xz'\Big) \cdot (x'', y'', z'') \\
&= \Big(xx'x'', yy'y'', zy'y'' + (-1)^{\deg_{\mathsf{Ch}}(x)} xz'y'' + (-1)^{\deg_{\mathsf{Ch}}(x) + \deg_{\mathsf{Ch}}(x')} xx'z''\Big)
\end{aligned}$$

$$\begin{aligned}
&(x, y, z) \cdot \Big((x', y', z') \cdot (x'', y'', z'')\Big) \\
&= (x, y, z) \cdot \Big(x'x'', y'y'', z'y'' + (-1)^{\deg_{\mathsf{Ch}}(x')} x'z''\Big) \\
&= \Big(xx'x'', yy'y'', zy'y'' + (-1)^{\deg_{\mathsf{Ch}}(x)} xz'y'' + (-1)^{\deg_{\mathsf{Ch}}(x) + \deg_{\mathsf{Ch}}(x')} xx'z''\Big)
\end{aligned}$$

The next calculations show that the Leibniz rule is also satisfied, making P into a differential graded algebra.

$$\begin{aligned}
&\partial\Big((x, y, z) \cdot (x', y', z')\Big) \\
&= \partial\Big(\Big(xx', yy', zy' + (-1)^{\deg_{\mathsf{Ch}}(x)} xz'\Big)\Big) \\
&= \Big(\partial(xx'), \partial(yy'), -\partial\Big(zy' + (-1)^{\deg_{\mathsf{Ch}}(x)} xz'\Big) + xx' - yy'\Big)
\end{aligned}$$

$$= \Big(\partial(x)x' + (-1)^{\deg_{\mathsf{Ch}}(x)} x\partial\big(x'\big), \partial(y)y' + (-1)^{\deg_{\mathsf{Ch}}(x)} y\partial\big(y'\big),$$
$$-\partial(z)y' - (-1)^{\deg_{\mathsf{Ch}}(x)+1} z\partial(y') - (-1)^{\deg_{\mathsf{Ch}}(x)}\partial(x)z' - x\partial(z') + xx' - yy'\Big)$$

$$\begin{aligned}
&\partial\big((x,y,z)\big)\cdot\big(x',y',z'\big) + (-1)^{\deg_{\mathsf{Ch}}(x)}(x,y,z)\cdot\partial\Big(\big(x',y',z'\big)\Big)\\
&= \big(\partial(x),\partial(y),-\partial(z)+x-y\big)\cdot\big(x',y',z'\big)\\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(x)}(x,y,z)\cdot\Big(\partial\big(x'\big),\partial\big(y'\big),-\partial\big(z'\big)+x'-y'\Big)\\
&= \Big(\partial(x)x',\partial(y)y',-\partial(z)y' + xy' - yy' - (-1)^{\deg_{\mathsf{Ch}}(x)}\partial(x)z'\Big)\\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(x)}\Big(x\partial\big(x'\big), y\partial\big(y'\big),\\
&\quad z\partial(y') - (-1)^{\deg_{\mathsf{Ch}}(x)}x\partial(z') + (-1)^{\deg_{\mathsf{Ch}}(x)}xx' - (-1)^{\deg_{\mathsf{Ch}}(x)}xy'\Big)\\
&= \Big(\partial(x)x' + (-1)^{\deg_{\mathsf{Ch}}(x)}x\partial\big(x'\big), \partial(y)y' + (-1)^{\deg_{\mathsf{Ch}}(x)}y\partial\big(y'\big),\\
&\quad -\partial(z)y' + xy' - yy' - (-1)^{\deg_{\mathsf{Ch}}(x)}\partial(x)z' + (-1)^{\deg_{\mathsf{Ch}}(x)}z\partial(y') - x\partial(z') + xx' - xy'\Big)\\
&= \Big(\partial(x)x' + (-1)^{\deg_{\mathsf{Ch}}(x)}x\partial\big(x'\big), \partial(y)y' + (-1)^{\deg_{\mathsf{Ch}}(x)}y\partial\big(y'\big),\\
&\quad -\partial(z)y' + (-1)^{\deg_{\mathsf{Ch}}(x)}z\partial(y') - (-1)^{\deg_{\mathsf{Ch}}(x)}\partial(x)z' - x\partial(z') + xx' - yy'\Big)
\end{aligned}$$

It is immediate from the formula for multiplication on P that the the morphisms of chain complexes $i\colon Y \to P$ and $p\colon P \to Y \times Y$ from Proposition 4.1.4.1 become morphisms of differential graded algebras. As weak equivalences and fibrations in $\mathrm{Alg}(\mathsf{Ch}(k))$ are detected by the forgetful functor to $\mathsf{Ch}(k)$ by Proposition 4.2.2.12, it now follows from Proposition 4.1.4.1 that i and p exhibit P as a path object for Y. We remark that a more conceptual approach to constructing this path object is described in [SS00, Section *Chain complexes* on pages 503 and 504], though there are some differences in signs. □

Proposition 4.2.2.17. *Let X be a cofibrant and Y a fibrant object in* $\mathrm{Alg}(\mathsf{Ch}(k))$*, with respect to the model structure of Proposition 4.2.2.9, and f and g two morphisms $X \to Y$ in* $\mathrm{Alg}(\mathsf{Ch}(k))$*. Then f and g are homotopic if and only if there exists a* chain homotopy of differential graded k-algebras *h from f to g, by which we mean a chain homotopy h from f to g in the sense of Proposition 4.1.4.2 satisfying additionally*

$$h\big(x\cdot x'\big) = h(x)g\big(x'\big) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)h\big(x'\big) \tag{4.6}$$

for all elements x and x' of X. ♡

Proof. Note that by [Hov99, 1.2.6], as X is cofibrant and Y is fibrant, the left and right homotopy relations coincide, and the right homotopy relation can be tested using any path object for Y. For this we use the path object P from Proposition 4.2.2.16.

Arguing completely analogously to the proof of Proposition 4.1.4.2, we see that f and g are homotopic as morphisms of differential graded algebras if and only if there exists a morphism of differential graded algebras $H = f \times g \times h\colon X \to P$. That H is a morphism of chain complexes amounts, just like in Proposition 4.1.4.2, to

$$\partial \circ h + h \circ \partial = f - g$$

but this time H needs to additionally preserve the unit, which is equivalent to $h(1) = 0$, and the multiplication, so for x and x' elements of X the following equality must hold.

$$\big(f(x \cdot x'), g(x \cdot x'), h(x \cdot x')\big) = \big(f(x), g(x), h(x)\big) \cdot \big(f(x'), g(x'), h(x')\big) \tag{$*$}$$

The right hand side is given by

$$\begin{aligned} &\big(f(x), g(x), h(x)\big) \cdot \big(f(x'), g(x'), h(x')\big) \\ = &\Big(f(x) \cdot f(x'), g(x) \cdot g(x'), h(x)g(x') + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)h(x')\Big) \end{aligned}$$

so as f and g are multiplicative we conclude that equality ($*$) is equivalent to the following equation.

$$h(x \cdot x') = h(x)g(x') + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)h(x')$$

Finally, note that this equation holding for $x = x' = 1$ implies that $h(1) = 2h(1)$ and hence $h(1) = 0$. $\square$

The following proposition will sometimes be helpful in defining homotopies of differential graded k-algebras.

Proposition 4.2.2.18. *Let X and Y be objects in $\mathrm{Alg}(\mathsf{Ch}(k))$, and assume that the underlying $\mathbb{Z}$-graded k-algebra of X is free on a $\mathbb{Z}$-graded subset Z of X.*

Let f and g be morphisms of differential graded algebras from X to Y and h a map from Z to Y that increases degree by 1. Then there is a unique extension of h to a morphism of $\mathbb{Z}$-graded k-modules of degree 1 from X to Y such that

$$h\big(x \cdot x'\big) = h(x)g\big(x'\big) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)h\big(x'\big) \tag{4.7}$$

holds for all elements x and x' of X. That unique extension is given by defining h on the basis given by words in Z by

$$h(z_1 \cdots z_l) \coloneqq \sum_{1 \le i \le l} (-1)^{\sum_{1 \le j \le i-1} \deg_{\mathsf{Ch}}(z_j)} \cdot f(z_1 \cdots z_{i-1}) \cdot h(z_i) \cdot g(z_{i+1} \cdots z_l) \tag{4.8}$$

for $l \ge 0$ and $z_1, \ldots, z_l \in Z$, and then extending k-linearly.

Furthermore, such an extension h satisfies

$$\partial \circ h + h \circ \partial = f - g \tag{4.9}$$

if and only if this holds on elements of Z. ♡

Proof. We first show uniqueness of the extension. As h must be k-linear, it suffices to show that the h is already uniquely given on words in Z. This we do by induction on the word length. By the Leibniz rule (4.7), h must map 1 to 0 (use $x = x' = 1$), so h is uniquely determined on words in Z of length 0. It is also uniquely determined on elements of Z themselves, as we prescribe the value on those elements. The induction step then follows directly from (4.7).

Now define h as in (4.8). It is clear from the definition that this definition extends the prescribed valued on Z. To verify that (4.7) holds we first note that both sides of the equation are k-linear in both x and x', so that it suffices to check this on a k-basis of X. So let $w = z_1 \cdots z_l$ and $w' = z'_1 \cdots z'_{l'}$ be words in Z. Then the following calculation shows that (4.7) is satisfied.

$$\begin{aligned}
&h\big(w \cdot w'\big)\\
&= \sum_{1\le i\le l} (-1)^{\sum_{1\le j\le i-1} \deg_{\mathsf{Ch}}\left(z_j\right)} \cdot f(z_1 \cdots z_{i-1}) \cdot h(z_i) \cdot g\big(z_{i+1} \cdots z_l \cdot w'\big)\\
&\quad + \sum_{1\le i\le l'} (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (-1)^{\sum_{1\le j\le i-1} \deg_{\mathsf{Ch}}\left(z'_j\right)} \cdot f\big(w \cdot z'_1 \cdots z'_{i-1}\big) \cdot h\big(z'_i\big) \cdot g\big(z'_{i+1} \cdots z'_{l'}\big)\\
&= \left(\sum_{1\le i\le l} (-1)^{\sum_{1\le j\le i-1} \deg_{\mathsf{Ch}}\left(z_j\right)} \cdot f(z_1 \cdots z_{i-1}) \cdot h(z_i) \cdot g(z_{i+1} \cdots z_l)\right) \cdot g\big(w'\big)\\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot\\
&\qquad \left(\sum_{1\le i\le l'} (-1)^{\sum_{1\le j\le i-1} \deg_{\mathsf{Ch}}\left(z'_j\right)} \cdot f\left(z'_1 \cdots z'_{i-1}\right) \cdot h\big(z'_i\big) \cdot g\big(z'_{i+1} \cdots z'_{l'}\big)\right)\\
&= h(w) \cdot g\big(w'\big) + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot h\big(w'\big)
\end{aligned}$$

It remains to show the assertion concerning (4.9). That if equality holds in general, then it also holds on Z is clear. So assume that (4.9) holds on Z. As both sides of the equation are k-linear it again suffices to show (4.9) on the k-basis given by words in Z. We show this by induction on the word length. For the element 1 (i. e. the unique word of length 0) we obtain $h(1) = 0$ and $\partial(1) = 0$ from the respective Leibniz rules, and the right hand side of (4.9) is zero as well as $f(1) = 1 = g(1)$. On words of length 1, i. e. elements of Z, the equation (4.9) holds by assumption. So now let w be an element of X on which (4.9) holds, and z an element of Z. Then the following calculation shows that (4.9) also holds for $w \cdot z$, thereby finishing the proof.

$$\partial\big(h(w \cdot z)\big) + h\big(\partial(w \cdot z)\big)$$

We first apply the Leibniz rule twice, for both h and ∂.

$$\begin{aligned}
&= \partial\Big(h(w) \cdot g(z) + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot h(z)\Big)\\
&\quad + h\Big(\partial(w) \cdot z + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot w \cdot \partial(z)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \partial\big(h(w)\big) \cdot g(z) + (-1)^{\deg_{\mathsf{Ch}}(w)+1} h(w) \cdot \partial\big(g(z)\big) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot \partial\big(f(w)\big) \cdot h(z) + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot \partial\big(h(z)\big) \\
&\quad + h\big(\partial(w)\big) \cdot g(z) + (-1)^{\deg_{\mathsf{Ch}}(w)-1} \cdot f\big(\partial(w)\big) \cdot h(z) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot h(w) \cdot g\big(\partial(z)\big) + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot h\big(\partial(z)\big)
\end{aligned}
$$

Next we reorder the summands.

$$
\begin{aligned}
&= \partial\big(h(w)\big) \cdot g(z) + h\big(\partial(w)\big) \cdot g(z) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)+1} h(w) \cdot \partial\big(g(z)\big) + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot h(w) \cdot g\big(\partial(z)\big) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot \partial\big(f(w)\big) \cdot h(z) + (-1)^{\deg_{\mathsf{Ch}}(w)-1} \cdot f\big(\partial(w)\big) \cdot h(z) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot \partial\big(h(z)\big) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot f(w) \cdot h\partial\big((z)\big) \\
&= \Big(\partial\big(h(w)\big) + h\big(\partial(w)\big)\Big) \cdot g(z) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} h(w) \cdot \Big(-\partial\big(g(z)\big) + g\big(\partial(z)\big)\Big) \\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(w)} \cdot \Big(\partial\big(f(w)\big) - f\big(\partial(w)\big)\Big) \\
&\quad + f(w) \cdot \Big(\partial\big(h(z)\big) + h\partial\big((z)\big)\Big)
\end{aligned}
$$

Now we can apply the induction hypothesis, and that f and g are morphisms of chain complexes.

$$
\begin{aligned}
&= \big(f(w) - g(w)\big) \cdot g(z) + f(w) \cdot \big(f(z) - g(z)\big) \\
&= f(w) \cdot g(z) - g(w) \cdot g(z) + f(w) \cdot f(z) - f(w) \cdot g(z) \\
&= f(w \cdot z) - g(w \cdot z)
\end{aligned}
$$

□

4.2.2.6. Homotopies in $\mathrm{Alg}(\mathsf{Mixed})$

Now we turn to homotopies of algebras in strict mixed complexes. This results in this section are analogous to those in the preceding Section 4.2.2.5, and obtained by combining those results with those from Section 4.2.2.4.

Proposition 4.2.2.19. *Let Y be an object in $\mathrm{Alg}(\mathsf{Mixed})$. Then the strict mixed structure defined in Proposition 4.2.2.14 on the chain complex P from Proposition 4.1.4.1 satisfies the Leibniz rule with respect to the multiplication from Proposition 4.2.2.16, upgrading P to an object in $\mathrm{Alg}(\mathsf{Mixed})$. Furthermore, the morphisms i and p exhibit P as a path object for Y in $\mathrm{Alg}(\mathsf{Mixed})$.* ♡

Proof. Let (x, y, z) and (x', y', z') be two elements of P. Then the following calculation shows that d satisfies the Leibniz rule.

$$
\begin{aligned}
&\mathrm{d}\Big((x, y, z) \cdot (x', y', z')\Big) \\
&= \mathrm{d}\Big(\Big(x \cdot x', y \cdot y', z \cdot y' + (-1)^{\deg_{\mathsf{Ch}}(x)} \cdot x \cdot z'\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\mathrm{d}\big(x\cdot x'\big), \mathrm{d}\big(y\cdot y'\big), -\mathrm{d}\Big(z\cdot y' + (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot z'\Big)\Big)\\
&= \Big(\mathrm{d}(x)\cdot x' + (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot \mathrm{d}\big(x'\big), \mathrm{d}(y)\cdot y' + (-1)^{\deg_{\mathsf{Ch}}(y)}\cdot y\cdot \mathrm{d}\big(y'\big),\\
&\qquad -\mathrm{d}(z)\cdot y' - (-1)^{\deg_{\mathsf{Ch}}(z)}\cdot z\cdot \mathrm{d}\big(y'\big)\\
&\qquad\qquad - (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot \mathrm{d}(x)\cdot z' - (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot \mathrm{d}\big(z'\big)\Big)\\
&= \Big(\mathrm{d}(x)\cdot x', \mathrm{d}(y)\cdot y', -\mathrm{d}(z)\cdot y' - (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot \mathrm{d}(x)\cdot z'\Big)\\
&\quad + \Big((-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot \mathrm{d}\big(x'\big), (-1)^{\deg_{\mathsf{Ch}}(y)}\cdot y\cdot \mathrm{d}\big(y'\big),\\
&\quad - (-1)^{\deg_{\mathsf{Ch}}(z)}\cdot z\cdot \mathrm{d}\big(y'\big) - (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot \mathrm{d}\big(z'\big)\Big)\\
&= \Big(\mathrm{d}(x)\cdot x', \mathrm{d}(y)\cdot y', \big(-\mathrm{d}(z)\big)\cdot y' + (-1)^{\deg_{\mathsf{Ch}}(\mathrm{d}\,x)}\cdot \mathrm{d}(x)\cdot z'\Big)\\
&\quad + \Big((-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot \mathrm{d}\big(x'\big), (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot y\cdot \mathrm{d}\big(y'\big),\\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot z\cdot \mathrm{d}\big(y'\big) + (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot x\cdot \Big(-\mathrm{d}\big(z'\big)\Big)\Big)\\
&= \big(\mathrm{d}(x), \mathrm{d}(y), -\mathrm{d}(z)\big)\cdot \big(x', y', z'\big) + (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot (x,y,z)\cdot \Big(\mathrm{d}\big(x'\big), \mathrm{d}\big(y'\big), -\mathrm{d}\big(z'\big)\Big)\\
&= \mathrm{d}\big((x,y,z)\big)\cdot \big(x', y', z'\big) + (-1)^{\deg_{\mathsf{Ch}}(x)}\cdot (x,y,z)\cdot \mathrm{d}\Big(\big(x', y', z'\big)\Big)
\end{aligned}
$$

This upgrades P to an object in $\mathrm{Alg}(\mathsf{Mixed})$. As i and p are compatible with both the strict mixed structure by Proposition 4.2.2.14 and the multiplicative structure by Proposition 4.2.2.16 we can conclude that i and p also lift to morphisms in $\mathrm{Alg}(\mathsf{Mixed})$. As weak equivalences and fibrations in $\mathrm{Alg}(\mathsf{Mixed})$ are detected by the forgetful functor to $\mathsf{Ch}(k)$ by Proposition 4.2.2.12, it now follows from Proposition 4.1.4.1 that i and p exhibit P as a path object for Y. $\square$

Proposition 4.2.2.20. *Let X be a cofibrant and Y a fibrant object in* $\mathrm{Alg}(\mathsf{Mixed})$*, with respect to the model structure of Proposition 4.2.2.9, and f and g two morphisms $X \to Y$ in* $\mathrm{Alg}(\mathsf{Mixed})$*. Then f and g are homotopic if and only if there exists a* chain homotopy of algebras of strict mixed complexes *h from f to g, by which we mean a chain homotopy h from f to g in the sense of Proposition 4.1.4.2 that is simultaneously a chain homotopy of differential graded algebras from f to g in the sense of Proposition 4.2.2.17 and a chain homotopy of strict mixed complexes from f to g in the sense of Proposition 4.2.2.15.* ♡

Proof. Note that by [Hov99, 1.2.6], as X is cofibrant and Y is fibrant, the left and right homotopy relations coincide, and the right homotopy relation can be tested using any path object for Y. For this we use the path object P from Proposition 4.2.2.19.

Arguing completely analogously to the proof of Proposition 4.1.4.2, we see that f and g are homotopic as morphisms of algebras in strict mixed complexes if and only if there exists a morphism of algebras in strict mixed complexes $H = f \times g \times h\colon X \to P$. While an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ is more than a chain complex that is equipped with both a strict mixed and an algebra structure, as d needs to additionally satisfy the

Leibniz rule, morphisms of algebras in strict mixed complexes are just morphisms of chain complexes that are compatible with both multiplication and the strict mixed structure. Thus the claim now follows directly by combining the proofs of Propositions 4.2.2.15 and 4.2.2.17. □

The following proposition is an analogue of Proposition 4.2.2.18 and will sometimes be helpful when trying to define a chain homotopy of algebras in strict mixed complexes.

Proposition 4.2.2.21. *Let X and Y be objects in $\mathrm{Alg}(\mathsf{Mixed})$, and let Z be a $\mathbb{Z}$-graded subset of X. Assume that Z is disjoint from $\mathrm{d}\,Z$ and that the underlying $\mathbb{Z}$-graded k-algebra of X is free on $Z \cup \mathrm{d}\,Z$.*

Let f and g be morphisms of algebras of strict mixed complexes from X to Y, and h a map from Z to Y that increases degree by 1. Then there is a unique extension of h to a morphism of $\mathbb{Z}$-graded k-modules of degree 1 from X to Y such that

$$h\big(x \cdot x'\big) = h(x)g\big(x'\big) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)h\big(x'\big) \tag{4.10}$$

and

$$h\big(\mathrm{d}(x)\big) = -\,\mathrm{d}\big(h(x)\big) \tag{4.11}$$

holds for all elements x and x' of X. That unique extension is given by first extending h to $Z \cup \mathrm{d}\,Z$ via

$$h(\mathrm{d}\,z) \coloneqq -\,\mathrm{d}\big(h(z)\big) \tag{4.12}$$

for z an element of Z, and then defining h on the basis given by words in Z and $\mathrm{d}\,Z$ by

$$h(z_1 \cdots z_l) \coloneqq \sum_{1 \le i \le l} (-1)^{\sum_{1 \le j \le i-1} \deg_{\mathsf{Ch}}(z_j)} \cdot f(z_1 \cdots z_{i-1}) \cdot h(z_i) \cdot g(z_{i+1} \cdots z_l) \tag{4.13}$$

for $l \ge 0$ and $z_1, \ldots, z_l \in Z \cup \mathrm{d}\,Z$, and then extending k-linearly.

Furthermore, such an extension h satisfies

$$\partial \circ h + h \circ \partial = f - g \tag{4.14}$$

if and only if this holds on elements of Z. ♡

Proof. We first show uniqueness of the extension. By (4.11) the extension to $Z \cup \mathrm{d}\,Z$ as in (4.12) is uniquely determined, and then uniqueness of the extension from $Z \cup \mathrm{d}\,Z$ to X follows from Proposition 4.2.2.18.

Now define h as in (4.12) and (4.13). Then h is extended from $Z \cup \mathrm{d}\,Z$ as in Proposition 4.2.2.18, so Proposition 4.2.2.18 show that (4.10) holds. To show that (4.11) holds, we start by noting that (4.11) holds on elements of $Z \cup \mathrm{d}\,Z$. For elements of Z this is by construction, and for $\mathrm{d}\,Z$ this is shown by the following small calculation, where $z \in Z$.

$$h\big(\mathrm{d}(\mathrm{d}\,z)\big) = h(0) = 0 = \mathrm{d}\Big(\mathrm{d}\big(h(z)\big)\Big) = -\,\mathrm{d}\Big(h\big(\mathrm{d}(z)\big)\Big)$$

As both sides of (4.11) are k-linear, it suffices to show (4.11) on the k-basis given by words in $Z \cup \mathrm{d}\,Z$. By what we just argued (4.11) holds on words of length 1, and as

$\mathrm{d}(1) = 0$ and $h(1) = 0$ by the respective Leibniz rules we also have that (4.11) holds for words of length 0. We now show that (4.11) holds for words of length greater than 1 by induction. So let z and z' be elements of Z such that (4.11) holds on them. Then we have to show that (4.11) also holds for $z \cdot z'$, which we do with the following calculation, using the Leibniz rule for d as well as the Leibniz rule for h (i. e. (4.10)), which we already showed.

$$\begin{aligned}
&h\Big(\mathrm{d}\big(z \cdot z'\big)\Big)\\
&= h\Big(\mathrm{d}\big(z\big) \cdot z' + (-1)^{\deg_{\mathsf{Ch}}(z)} z \cdot \mathrm{d}\big(z'\big)\Big)\\
&= h\big(\mathrm{d}(z)\big) \cdot g\big(z'\big) + (-1)^{\deg_{\mathsf{Ch}}\left(\mathrm{d}(z)\right)} \cdot f\big(\mathrm{d}(z)\big) \cdot h\big(z'\big)\\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot h(z) \cdot g\Big(\mathrm{d}\big(z'\big)\Big) + (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot f(z) \cdot h\Big(\mathrm{d}\big(z'\big)\Big)\\
&= -\,\mathrm{d}\big(h(z)\big) \cdot g\big(z'\big) - (-1)^{\deg_{\mathsf{Ch}}(z)}\,\mathrm{d}\big(f(z)\big) \cdot h\big(z'\big)\\
&\quad - (-1)^{\deg_{\mathsf{Ch}}\left(h(z)\right)} h(z) \cdot \mathrm{d}\Big(g\big(z'\big)\Big) - (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot (-1)^{\deg_{\mathsf{Ch}}(z)} f(z) \cdot \mathrm{d}\Big(h\big(z'\big)\Big)\\
&= -\,\mathrm{d}\big(h(z)\big) \cdot g\big(z'\big) - (-1)^{\deg_{\mathsf{Ch}}\left(h(z)\right)} h(z) \cdot \mathrm{d}\Big(g\big(z'\big)\Big)\\
&\quad - (-1)^{\deg_{\mathsf{Ch}}(z)}\,\mathrm{d}\big(f(z)\big) \cdot h\big(z'\big) - (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot (-1)^{\deg_{\mathsf{Ch}}(z)} f(z) \cdot \mathrm{d}\Big(h\big(z'\big)\Big)\\
&= -\,\mathrm{d}\Big(h(z) \cdot g\big(z'\big)\Big) - (-1)^{\deg_{\mathsf{Ch}}(z)} \cdot \mathrm{d}\Big(f(z) \cdot h\big(z'\big)\Big)\\
&= -\,\mathrm{d}\Big(h(z) \cdot g\big(z'\big) + (-1)^{\deg_{\mathsf{Ch}}(z)} f(z) \cdot h\big(z'\big)\Big)\\
&= -\,\mathrm{d}\Big(h\big(z \cdot z'\big)\Big)
\end{aligned}$$

It remains to show the assertion concerning (4.14). So assume that (4.14) holds on elements of Z. Then we first show that (4.14) also holds on elements of $\mathrm{d}\,Z$. Indeed, the following calculation verifies (4.14) for $\mathrm{d}\,z$ if z is an element of Z, where we use the compatibility of all the involved morphisms and operators with d.

$$\begin{aligned}
\partial\big(h(\mathrm{d}\,z)\big) + h\big(\partial(\mathrm{d}\,z)\big) &= -\partial\Big(\mathrm{d}\big(h(z)\big)\Big) - h\Big(\mathrm{d}\big(\partial(z)\big)\Big) = \mathrm{d}\Big(\partial\big(h(z)\big)\Big) + \mathrm{d}\Big(h\big(\partial(z)\big)\Big)\\
&= \mathrm{d}\big((\partial \circ h + h \circ \partial)(z)\big) = \mathrm{d}\big(f(z) - g(z)\big)\\
&= f\big(\mathrm{d}(z)\big) - g\big(\mathrm{d}(z)\big)
\end{aligned}$$

Now that we know that (4.14) is satisfied on all of $Z \cup \mathrm{d}\,Z$ it immediately follows from Proposition 4.2.2.18 that (4.14) already holds on all of Z. □

4.2.3. Strongly homotopy linear morphisms of strict mixed complexes

Let X and Y be strict mixed complexes and $f\colon X \to Y$ a morphism of the underlying chain complexes. We might then want to lift f to a morphism of strict mixed complexes, which is possible if and only if f commutes with the differential d, or equivalently if $f \circ d - d \circ f$ is zero. In practice it may however happen that f only commutes with d up to homotopy rather than strictly. In this case $f \circ d - d \circ f$ is nullhomotopic, but not zero, and we could record this by letting $f^{(1)}$ be a nullhomotopy[23] of $f \circ d - d \circ f$. We can now ask whether this additional data $f^{(1)}$ commutes with d. Again, this may only be the case up to a homotopy $f^{(2)}$. If we keep going in this manner we arrive at the notion of a *strongly homotopy linear morphism* of strict mixed complexes. We will give a full definition in Section 4.2.3.1.

To relate the notion of strongly homotopy linear morphisms with the homotopy theory of strict mixed complexes as developed in Section 4.2.2, we are then going to show in Section 4.2.3.2 that a strongly homotopy linear morphism $f\colon X \to Y$ corresponds to a (strict) morphism $f^{\mathrm{strict}}\colon X \to Y^{\mathrm{shl}}$ of strict mixed complexes, where Y^{shl} is a thickened version of Y coming with a quasiisomorphism of strict mixed complexes $Y \to Y^{\mathrm{shl}}$. We can thus interpret the strongly homotopy linear morphism f as encoding a zigzag as depicted below.

$$\begin{tikzcd} X \arrow[r, "f^{\mathrm{strict}}"'] \arrow[rr, bend left, dashed, "f"] & Y^{\mathrm{shl}} & Y \arrow[l, "\simeq"'] \end{tikzcd}$$

4.2.3.1. Definition of strongly homotopy linear morphisms

Below we record the definition of strongly homotopy linear morphisms that was sketched in the introduction to Section 4.2.3.

Definition 4.2.3.1 ([Kas87, 2.2] and [Lod98, 2.5.14])**.** Let X and Y be strict mixed complexes. A strongly homotopy linear morphism from X to Y consists of morphisms of graded k-modules $f^{(i)}\colon X \to Y$ of degree $2i$ for all $i \geq 0$, satisfying

$$\partial \circ f^{(i)} - f^{(i)} \circ \partial = f^{(i-1)} \circ \mathrm{d} - \mathrm{d} \circ f^{(i-1)} \tag{4.15}$$

where we set $f^{(-1)} = 0$. Note that the condition for $i = 0$ implies that $\partial \circ f^{(0)} = f^{(0)} \circ \partial$, so that $f^{(0)}$ is a morphism of chain complexes. ♢

Remark 4.2.3.2. We can compose strongly homotopy linear morphisms with (strict) morphisms of strict mixed complexes. To be more concrete, let X and Y be strict mixed complexes, $g^{(\bullet)}\colon X \to Y$ a strongly homotopy linear morphism, and $f\colon X' \to X$ and $h\colon Y \to Y'$ morphisms of strict mixed complexes. Then we make the following definition.

$$(hgf)^{(i)} := h \circ g^{(i)} \circ f \qquad \text{for } i \geq 0$$

This defines a strongly homotopy linear morphism hgf from X' to Y', whose underlying morphism of chain complexes is the composition of underlying morphisms of chain complexes. That hgf really is a strongly homotopy linear morphism can be easily checked

[23] As $f \circ d - d \circ f$ is a morphism of odd degree, this would take the form $\partial f^{(1)} - f^{(1)}\partial = fd - df$, compare with Definition 4.1.2.1.

using that f and h commute with both ∂ and d, as seen below.

$$\begin{aligned}\partial(hgf)^{(i)} - (hgf)^{(i)}\partial &= \partial hg^{(i)}f - hg^{(i)}f\partial \\ &= h\Big(\partial g^{(i)} - g^{(i)}\partial\Big)f \\ &= h\Big(g^{(i-1)}\mathrm{d} - \mathrm{d}g^{(i-1)}\Big)f \\ &= hg^{(i-1)}f\mathrm{d} - \mathrm{d}hg^{(i-1)}f \\ &= (hgf)^{(i-1)}\mathrm{d} - \mathrm{d}(hgf)^{(i-1)}\end{aligned}$$

◊

4.2.3.2. Strongly homotopy linear morphisms as zigzags

We begin this section with the construction of the strict mixed complex Y^{shl} that was mentioned in the introduction to Section 4.2.3, before explaining how to reinterpret a strongly homotopy linear morphism $f\colon X \to Y$ as a morphism of strict mixed complexes $f^{\mathrm{strict}}\colon X \to Y^{\mathrm{shl}}$.

Definition 4.2.3.3. Let Y be a strict mixed complex. Then define Y^{shl} to be the $\mathbb{Z}$-graded k-module

$$Y^{\mathrm{shl}}_n := \prod_{m\geq 0} Y[-m]$$

so that $Y^{\mathrm{shl}}_n = \prod_{m\geq n} Y_m$ for any integer n. We furthermore define operators ∂ and d of degrees -1 and 1 on Y^{shl} as follows, where $(y_n, y_{n+1}, \ldots)$ is an element of Y^{shl}_n and e. g. $\partial(y_n, y_{n+1}, \ldots)_m$ refers to the Y_m-component of Y^{shl}_{n-1}.

$$\partial(y_n, y_{n+1}, \ldots)_{n-1+i} := \begin{cases}\partial(y_n) & \text{if } i = 0\\ -\partial(y_{n+i}) & \text{if } i > 0 \text{ is odd}\\ \partial(y_{n+i}) - y_{n-1+i} & \text{if } i > 0 \text{ is even}\end{cases}$$

$$\mathrm{d}(y_n, y_{n+1}, \ldots)_{n+1+i} := \begin{cases}-\mathrm{d}(y_{n+i}) & \text{if } i \geq 0 \text{ is odd}\\ \mathrm{d}(y_{n+i}) + y_{n+1+i} & \text{if } i \geq 0 \text{ is even}\end{cases}$$

The special case for $i = 0$ in the formula for ∂ can be avoided by declaring y_{n-1} to be 0.

Finally, we let $\iota^{\mathrm{shl}}_Y\colon Y \to Y^{\mathrm{shl}}$ be the morphism of $\mathbb{Z}$-graded k-modules that is given by $\iota^{\mathrm{shl}}_Y(y) := (y, 0, 0, \ldots)$ for every element y of Y. ◊

Remark 4.2.3.4. The following diagram[24] depicts how one can think of Y^{shl}. The picture only shows part of Y^{shl}, which continues towards the right, top, and bottom, but

[24]This diagram uses some of the pictorial elements from Convention 4.2.1.7, but is only meant to help with intuition rather than as a precise depiction of an isomorphism class of strict mixed complexes. For example Y^{shl} is the product of the rows, whereas interpreting the picture while following Convention 4.2.1.7 too closely would suggest taking the sum.

not towards the left.

$$\begin{tikzcd}
Y_{n+1} \arrow[d, "\partial"'] & & Y_{n+2} \arrow[d, "-\partial"] \arrow[dr, "-\mathrm{id}"] & Y_{n+3} \arrow[d, "\partial"'] & & Y_{n+4} \arrow[d, "-\partial"] \arrow[dr, "-\mathrm{id}"] & Y_{n+5} \arrow[d, "\partial"'] \\
Y_{n} \arrow[u, squiggly, red, "\mathrm{d}"'] \arrow[d, "\partial"'] & & Y_{n+1} \arrow[u, squiggly, red, "-\mathrm{d}"] \arrow[ull, squiggly, red, "\mathrm{id}"'] \arrow[d, "-\partial"] \arrow[dr, "-\mathrm{id}"] & Y_{n+2} \arrow[u, squiggly, red, "\mathrm{d}"'] \arrow[d, "\partial"'] & & Y_{n+3} \arrow[u, squiggly, red, "-\mathrm{d}"] \arrow[ull, squiggly, red, "\mathrm{id}"'] \arrow[d, "-\partial"] \arrow[dr, "-\mathrm{id}"] & Y_{n+4} \arrow[u, squiggly, red, "\mathrm{d}"'] \arrow[d, "\partial"'] \\
Y_{n-1} \arrow[u, squiggly, red, "\mathrm{d}"'] & & Y_{n} \arrow[u, squiggly, red, "-\mathrm{d}"] \arrow[ull, squiggly, red, "\mathrm{id}"'] & Y_{n+1} \arrow[u, squiggly, red, "\mathrm{d}"'] & & Y_{n+2} \arrow[u, squiggly, red, "-\mathrm{d}"] \arrow[ull, squiggly, red, "\mathrm{id}"'] & Y_{n+3} \arrow[u, squiggly, red, "\mathrm{d}"']
\end{tikzcd}$$

$\diamondsuit$

Proposition 4.2.3.5. *Let Y be a strict mixed complex and Y^{shl} as in Definition 4.2.3.3. Then ∂ and d as defined in Definition 4.2.3.3 define a strict mixed complex structure on Y^{shl} which makes $\iota_Y^{\mathrm{shl}}\colon Y \to Y^{\mathrm{shl}}$ into a quasiisomorphism of strict mixed complexes.* $\heartsuit$

Proof. We begin by showing that ∂ and d upgrade Y^{shl} to a strict mixed complex. It is easiest to convince oneself of this by considering the diagram in Remark 4.2.3.4, but we also provide a proof by unpacking the formulas. So let $(y_n, y_{n+1}, \ldots)$ be an element of Y_n^{shl}. Then we obtain the following calculations, first for odd i and then for even i[25], showing that ∂ squares to zero.

$$\begin{aligned}
&\partial\Big(\partial\big((y_n, y_{n+1}, \ldots)\big)\Big)_{n-2+i} && \text{(assuming i is odd)}\\
&= -\partial\Big(\partial\big((y_n, y_{n+1}, \ldots)\big)_{n-1+i}\Big)\\
&= -\partial\big(-\partial(y_{n+i})\big)\\
&= \partial\big(\partial(y_{n+i})\big)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
&\partial\Big(\partial\big((y_n, y_{n+1}, \ldots)\big)\Big)_{n-2+i} && \text{(assuming i is even)}\\
&= \partial\Big(\partial\big((y_n, y_{n+1}, \ldots)\big)_{n-1+i}\Big) - \partial\big((y_n, y_{n+1}, \ldots)\big)_{n-2+i}\\
&= \partial\big(\partial(y_{n+i}) - y_{n-1+i}\big) + \partial(y_{n-1+i})\\
&= 0 - \partial(y_{n-1+i}) + \partial(y_{n-1+i})\\
&= 0
\end{aligned}$$

The proof that d squares to 0 is completely analogous. Similarly, the following calculation shows $\partial\mathrm{d} + \mathrm{d}\partial = 0$.

$$\Big((\partial\mathrm{d} + \mathrm{d}\partial)\big((y_n, y_{n+1}, \ldots)\big)\Big)_{n+i} \qquad \text{(assuming i is odd)}$$

[25] In the case of $i = 0$ we set $y_{n-1} = 0$ so that we can use the same formulas as for even $i > 0$.

$$
\begin{aligned}
&= -\partial\Big(\mathrm{d}\big((y_n, y_{n+1}, \ldots)\big)_{n+1+i}\Big) - \mathrm{d}\Big(\partial\big((y_n, y_{n+1}, \ldots)\big)_{n-1+i}\Big)\\
&= \partial\big(\mathrm{d}(y_{n+i})\big) + \mathrm{d}\big(\partial(y_{n+i})\big)\\
&= (\partial\mathrm{d} + \mathrm{d}\partial)(y_{n+i})\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&\Big((\partial\mathrm{d} + \mathrm{d}\partial)\big((y_n, y_{n+1}, \ldots)\big)\Big)_{n+i} \qquad\qquad \text{(assuming } i \text{ is even)}\\
&= \partial\Big(\mathrm{d}\big((y_n, y_{n+1}, \ldots)\big)_{n+1+i}\Big) - \mathrm{d}\big((y_n, y_{n+1}, \ldots)\big)_{n+i}\\
&\quad + \mathrm{d}\Big(\partial\big((y_n, y_{n+1}, \ldots)\big)_{n-1+i}\Big) + \partial\big((y_n, y_{n+1}, \ldots)\big)_{n+i}\\
&= \partial\big(\mathrm{d}(y_{n+i}) + y_{n+1+i}\big) + \mathrm{d}(y_{n-1+i})\\
&\quad + \mathrm{d}\big(\partial(y_{n+i}) - y_{n-1+i}\big) - \partial(y_{n+1+i})\\
&= (\partial\mathrm{d} + \mathrm{d}\partial)(y_{n+i}) + \partial(y_{n+1+i}) + \mathrm{d}(y_{n-1+i})\\
&\quad - \mathrm{d}(y_{n-1+i}) - \partial(y_{n+1+i})\\
&= 0
\end{aligned}
$$

It remains to show that $\iota_Y^{\mathrm{shl}}\colon Y \to Y^{\mathrm{shl}}$ is a morphism of strict mixed complexes as well as a quasiisomorphism. That ι_Y^{shl} is compatible with the boundary operator and differential is clear from the formulas. It thus remains to show that it is a quasiisomorphism. For this, let $Y^{\mathrm{shl},i}$ for $i \geq 1$ be the sub-$\mathbb{Z}$-graded k-module of Y^{shl} given by the factor $Y[-(2i-1)] \times Y[-2i]$. If we let $Y^{\mathrm{shl},0}$ be the first factor of Y^{shl}, i. e. $Y^{\mathrm{shl},0} = Y$, then we obtain a product decomposition

$$
Y^{\mathrm{shl}} \cong \prod_{i\geq 0} Y^{\mathrm{shl},i}
$$

as $\mathbb{Z}$-graded k-modules. It is immediate from the formulas for the boundary operator that each $Y^{\mathrm{shl},i}$ is closed under ∂, making this also product decomposition considered as chain complexes. As ι_Y^{shl} is the inclusion of the first factor it thus remains to show that for each $i \geq 1$ the chain complex $Y^{\mathrm{shl},i}$ is acyclic. To do so, we define a contracting homotopy as follows.

$$
\begin{aligned}
h\colon Y_n^{\mathrm{shl},i} = Y_{n+2i-1} \oplus Y_{n+2i} &\to Y_{n+1}^{\mathrm{shl},i} = Y_{n+2i} \oplus Y_{n+2i+1}\\
(y_{n+2i-1}, y_{n+2i}) &\mapsto (-y_{n+2i}, 0)
\end{aligned}
$$

The following calculations shows that h is a contracting homotopy of $Y^{\mathrm{shl},i}$, where (y_{n+2i-1}, y_{n+2i}) is an element of $Y_n^{\mathrm{shl},i}$.

$$
\begin{aligned}
&(\partial h + h\partial)\big((y_{n+2i-1}, y_{n+2i})\big)\\
&= \partial\big((-y_{n+2i}, 0)\big) + h\Big(\big(-\partial(y_{n+2i-1}), \partial(y_{n+2i}) - y_{n+2i-1}\big)\Big)\\
&= \big(-\partial(-y_{n+2i}), 0 - (-y_{n+2i})\big) + \big(-\partial(y_{n+2i}) + y_{n+2i-1}, 0\big)\\
&= \big(\partial(y_{n+2i}) - \partial(y_{n+2i}) + y_{n+2i-1}, y_{n+2i}\big)
\end{aligned}
$$

$$= (y_{n+2i-1}, y_{n+2i})$$

The following diagram depicts the situation for $i = 1$ diagrammatically as in Remark 4.2.3.4, with the contracting homotopy h indicated with the dashed blue arrow.

$$\begin{tikzcd}
Y_{n+2} \arrow[d, "-\partial"] \arrow[dr, "-\mathrm{id}"'] & Y_{n+3} \arrow[d, "\partial"'] \\
Y_{n+1} \arrow[d, "-\partial"] \arrow[dr, "-\mathrm{id}"'] & Y_{n+2} \arrow[ul, dashed, blue, "-\mathrm{id}"'] \arrow[d, "\partial"'] \\
Y_n & Y_{n+1} \arrow[ul, dashed, blue, "-\mathrm{id}"']
\end{tikzcd}$$

□

The proof of Proposition 4.2.3.5 shows that ι_Y^{shl} has a retraction given by the projection to the first factor, but only as chain complexes. While the projection to the first factor is not compatible with the differential, it can however be upgraded to a strongly homotopy linear morphism, as we will explain next.

Proposition 4.2.3.6. *Let Y be a strict mixed complex. Define $(p_Y^{\mathrm{shl}})^{(i)}$ for each $i \geq 0$ to be the morphisms of $\mathbb{Z}$-graded k-modules from Y^{shl} to Y of degree $2i$ that is the projection to the $2i$-th factor, i. e. is defined as follows.*

$$(p_Y^{\mathrm{shl}})_n^{(i)} : Y_n^{\mathrm{shl}} \to Y_n, \qquad (y_n, y_{n+1}, y_{n+2}, \ldots) \mapsto y_{n+2i}$$

Then this makes p_Y^{shl} into a strongly homotopy linear morphism from Y^{shl} to Y. Furthermore, the underlying morphism of chain complexes of p_Y^{shl} is a quasiisomorphism. ♡

Proof. That $(p_Y^{\mathrm{shl}})^{(0)}$ is a morphism of chain complexes is clear. As $(p_Y^{\mathrm{shl}})^{(0)}$ is a left inverse of ι_Y^{shl}, it also follows immediately from ι_Y^{shl} being a quasiisomorphism by Proposition 4.2.3.5 that $(p_Y^{\mathrm{shl}})^{(0)}$ is a quasiisomorphism as well.

It remains to show that the compatibility relations required of $(p_Y^{\mathrm{shl}})^{(i)}$ for $i \geq 0$ in order to make p_Y^{shl} into a strongly homotopy linear morphism are satisfied. So let $i \geq 1$ be an integer and $(y_n, y_{n+1}, \ldots)$ an element of Y_n^{shl}. Then the following calculations show the claim.

$$\begin{aligned}
&\Big(\partial \circ (p_Y^{\mathrm{shl}})^{(i)} - (p_Y^{\mathrm{shl}})^{(i)} \circ \partial\Big)\big((y_n, y_{n+1}, y_{n+2}, \ldots)\big) \\
&= \partial(y_{n+2i}) - \partial\big((y_n, y_{n+1}, y_{n+2}, \ldots)\big)_{n-1+2i} \\
&= \partial(y_{n+2i}) - \big(\partial(y_{n+2i}) - y_{n-1+2i}\big) \\
&= y_{n-1+2i}
\end{aligned}$$

$$\Big((p_Y^{\mathrm{shl}})^{(i-1)} \circ \mathrm{d} - \mathrm{d} \circ (p_Y^{\mathrm{shl}})^{(i-1)}\Big)\big((y_n, y_{n+1}, y_{n+2}, \ldots)\big)$$

$$
\begin{aligned}
&= \mathrm{d}\big((y_n, y_{n+1}, y_{n+2}, \ldots)\big)_{n+1+2i-2} - \mathrm{d}(y_{n+2i-2}) \\
&= \big(\mathrm{d}(y_{n+2i-2}) + y_{n+1+2i-2}\big) - \mathrm{d}(y_{n+2i-2}) \\
&= y_{n+1+2i-2} = y_{n-1+2i}
\end{aligned}
\qquad \square
$$

The relevance of Y^{shl} and p_Y^{shl} stems from the fact that p_Y^{shl} is the *universal* strongly homotopy linear morphism to Y; we show next that any other strongly homotopy morphism with codomain Y factors uniquely as the composition of a (strict) morphism of strict mixed complexes to Y^{shl} with p_Y^{shl}.

Proposition 4.2.3.7. *Let X and Y be strict mixed complexes and $f\colon Y \to Y$ a strongly homotopy linear morphism. Then there is a unique morphism of strict mixed complexes $g\colon X \to Y^{\mathrm{shl}}$ such that $f = p_Y^{\mathrm{shl}} \circ g$*[26]*.* ♡

Proof. We first show existence. Define a morphism of $\mathbb{Z}$-graded k-modules g as

$$
\begin{aligned}
g\colon X \to Y^{\mathrm{shl}} &= \prod_{m \geq 0} Y[-m] \\
g(x)_{n+2i} &= f^{(i)}(x) \\
g(x)_{n+2i+1} &= \Big(f^{(i)}\mathrm{d} - \mathrm{d}f^{(i)}\Big)(x) = \Big(\partial f^{(i+1)} - f^{(i+1)}\partial\Big)(x)
\end{aligned}
$$

for $i \geq 0$ and x elements of X_n, and where $g(x)_{n+m}$ refers to the component in Y_{n+m}. As $\big(p_Y^{\mathrm{shl}}\big)^{(i)}$ is projection to the $2i$-th factor, it is clear that f is the composition $p_Y^{\mathrm{shl}} \circ g$, so it only remains to show that g is a morphism of strict mixed complexes. This is proven by the following calculations, where $i \geq 0$ and x is an element of X_n.

$$
\begin{aligned}
&(\partial g - g\partial)(x)_{n-1+2i} \\
&= \partial\big(g(x)\big)_{n-1+2i} - f^{(i)}\big(\partial(x)\big) \\
&= \partial\big(g(x)_{n+2i}\big) - g(x)_{n-1+2i} - f^{(i)}\big(\partial(x)\big) \\
&= \partial\Big(f^{(i)}(x)\Big) - \Big(\partial f^{(i)} - f^{(i)}\partial\Big)(x) - f^{(i)}\big(\partial(x)\big) \\
&= 0
\end{aligned}
$$

This shows what is needed for g to be a morphism of chain complexes for only the even components, now we check the odd components.

$$
\begin{aligned}
&(\partial g - g\partial)(x)_{n+2i} \\
&= -\partial\big(g(x)_{n+2i+1}\big) - \Big(\partial f^{(i+1)} - f^{(i+1)}\partial\Big)\big(\partial(x)\big) \\
&= -\partial\Big(\Big(\partial f^{(i+1)} - f^{(i+1)}\partial\Big)(x)\Big) - \Big(\partial f^{(i+1)} - f^{(i+1)}\partial\Big)\big(\partial(x)\big) \\
&= \Big(-\partial\partial f^{(i+1)} + \partial f^{(i+1)}\partial - \partial f^{(i+1)}\partial + f^{(i+1)}\partial\partial\Big)(x)
\end{aligned}
$$

[26] See Remark 4.2.3.2 for the composition of a strongly homotopy linear morphism with a morphism of strict mixed complexes.

$$= 0$$

Next we verify that g commutes with d, beginning with the even components.

$$\begin{aligned}
&(\mathrm{d}g - g\mathrm{d})(x)_{n+1+2i} \\
&= \mathrm{d}\big(g(x)_{n+2i}\big) + g(x)_{n+1+2i} - f^{(i)}\big(\mathrm{d}(x)\big) \\
&= \mathrm{d}\Big(f^{(i)}(x)\Big) + \Big(f^{(i)}\mathrm{d} - \mathrm{d}f^{(i)}\Big)(x) - f^{(i)}\big(\mathrm{d}(x)\big) \\
&= 0
\end{aligned}$$

Finally, we check compatibility with d on odd components.

$$\begin{aligned}
&(\mathrm{d}g - g\mathrm{d})(x)_{n+2+2i} \\
&= -\,\mathrm{d}\big(g(x)_{n+1+2i}\big) - \Big(f^{(i)}\mathrm{d} - \mathrm{d}f^{(i)}\Big)\big(\mathrm{d}(x)\big) \\
&= -\,\mathrm{d}\bigg(\Big(f^{(i)}\mathrm{d} - \mathrm{d}f^{(i)}\Big)(x)\bigg) - \Big(f^{(i)}\mathrm{d} - \mathrm{d}f^{(i)}\Big)\big(\mathrm{d}(x)\big) \\
&= \Big(-\mathrm{d}f^{(i)}\mathrm{d} + \mathrm{d}\mathrm{d}f^{(i)} - f^{(i)}\mathrm{d}\mathrm{d} + \mathrm{d}f^{(i)}\mathrm{d}\Big)(x) \\
&= 0
\end{aligned}$$

This shows existence. It remains to show that such a lift g is already uniquely determined by f. So let $g\colon X \to Y^{\mathrm{shl}}$ be any morphism of strict mixed complexes such that $f = p_Y^{\mathrm{shl}} \circ g$. We can immediately read off that the even components must be given by

$$g(x)_{n+2i} = f^{(i)}(x) \qquad \text{for } n \in \mathbb{Z}, i \geq 0 \text{ and } x \in X_n.$$

So now let x be an element of X_n and $i \geq 0$. Then the following calculation, using that g is a morphism of chain complexes, shows that $g(x)_{n+2i+1}$ is also already determined by f.

$$\begin{aligned}
&g(x)_{n+2i+1} \\
&= \partial\Big(g(x)_{n+2i+2}\Big) - \bigg(\partial\Big(g(x)_{n+2i+2}\Big) - g(x)_{n+2i+1}\bigg) \\
&= \partial\Big(g(x)_{n+2i+2}\Big) - \partial\big(g(x)\big)_{n+2i+1} \\
&= \partial\Big(g(x)_{n+2i+2}\Big) - g\big(\partial(x)\big)_{n-1+2i+2} \\
&= \partial\Big(f^{(i+1)}(x)\Big) - f^{(i+1)}\big(\partial(x)\big)
\end{aligned}$$

□

Definition 4.2.3.8. Let X and Y be strict mixed complexes and $f\colon X \to Y$ a strongly homotopy linear morphism. Then we denote by f^{strict} the unique morphism of strict mixed complexes $X \to Y^{\mathrm{shl}}$ lifting f as in Proposition 4.2.3.7. The assignment $f \mapsto f^{\mathrm{strict}}$ defines a bijection from the set of strongly homotopy linear morphisms $X \to Y$ to the set of morphisms of strict mixed complexes $X \to Y^{\mathrm{shl}}$. ♢

4.3. The derived category of k

The derived category of k is an ∞-category $\mathcal{D}(k)$ that can be constructed by inverting the quasiisomorphisms in the category $\mathsf{Ch}(k)$ of chain complexes of (ordinary) k-modules. In this section we discuss $\mathcal{D}(k)$ and record the main properties that we will need later – most of them are proven in various places in [HA].

We begin in Section 4.3.1 by proving some useful statements concerning semiadditive ∞-categories, which we will need in Section 4.3.2, where we will collect the main properties of $\mathcal{D}(k)$. We finish this section with Section 4.3.4, where we state some properties of the truncation functors on $\mathcal{D}(k)$ that we will need in Chapter 5.

4.3.1. Semiadditive ∞-categories

In this section we prove some small helpful results regarding semiadditive ∞-categories that we will need in Section 4.3.2.

Proposition 4.3.1.1. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category such that the underlying ∞-category $\mathcal{C}$ is semiadditive ∞-category*[27]*. Then $\mathcal{C}^{\otimes}$ is cartesian if and only if it is cocartesian.* ♡

Proof. The property of symmetric monoidal structures being (co)cartesian is defined in [HA, 2.4.0.1]. The symmetric monoidal structure $\mathcal{C}^{\otimes}$ is cartesian if the unit object $\mathbb{1}_{\mathcal{C}}$ is final and if for every pair of objects X and Y of $\mathcal{C}$ the morphisms

$$X \simeq X \otimes \mathbb{1}_{\mathcal{C}} \leftarrow X \otimes Y \rightarrow \mathbb{1}_{\mathcal{C}} \otimes Y \simeq Y$$

induced by the essentially unique morphisms $X \to \mathbb{1}_{\mathcal{C}}$ and $Y \to \mathbb{1}_{\mathcal{C}}$ exhibit $X \otimes Y$ as a product of X and Y.

Analogously, for $\mathcal{C}^{\otimes}$ being cocartesian the unit object must be initial, and the analogously defined morphisms

$$X \simeq X \otimes \mathbb{1}_{\mathcal{C}} \rightarrow X \otimes Y \leftarrow \mathbb{1}_{\mathcal{C}} \otimes Y \simeq Y$$

must exhibit $X \otimes Y$ as a coproduct of X and Y.

As $\mathcal{C}$ is assumed to be semiadditive, every initial object is automatically final as well, and every final object is automatically initial, which shows equivalence of the first part of the respective definitions. For the second part, let X and Y be two objects of $\mathcal{C}$. Note that the compositions

$$X \otimes \mathbb{1}_{\mathcal{C}} \rightarrow X \otimes Y \rightarrow X \otimes \mathbb{1}_{\mathcal{C}}$$

[27] By this we mean that $\mathcal{C}$ admits finite products and finite coproducts and has the following two properties. Firstly, the (essentially unique) morphism from an initial object to a final object must be an equivalence (i. e. $\mathcal{C}$ has *zero objects*). Secondly, for any two objects X and Y of $\mathcal{C}$ the morphism

$$X \amalg Y \xrightarrow{\left[\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right]} X \times Y$$

must be an equivalence (i. e. $\mathcal{C}$ has *biproducts*).

and

$$\mathbb{1}_{\mathcal{C}} \otimes Y \to X \otimes Y \to \mathbb{1}_{\mathcal{C}} \otimes Y$$

are, by functoriality of the tensor product, homotopic to the identity. Functoriality also implies that the following square commutes

$$\begin{tikzcd} X \otimes \mathbb{1}_{\mathcal{C}} \arrow[r] \arrow[d] & X \otimes Y \arrow[d] \\ \mathbb{1}_{\mathcal{C}} \otimes \mathbb{1}_{\mathcal{C}} \arrow[r] & \mathbb{1}_{\mathcal{C}} \otimes Y \end{tikzcd}$$

which shows that the composition

$$X \otimes \mathbb{1}_{\mathcal{C}} \to X \otimes Y \to \mathbb{1}_{\mathcal{C}} \otimes Y$$

and analogously

$$\mathbb{1}_{\mathcal{C}} \otimes Y \to X \otimes Y \to X \otimes \mathbb{1}_{\mathcal{C}}$$

are zero morphisms. We can conclude that the following triangle commutes.

$$\begin{tikzcd} X \amalg Y \arrow[rr, "\begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix}"] \arrow[dr] & & X \times Y \\ & X \otimes Y \arrow[ur] & \end{tikzcd}$$

The second condition for $\mathcal{C}^{\otimes}$ being (co)cartesian is that the morphism on the right (left) is an equivalence for every X and Y. As the horizontal morphism is an equivalence by virtue of $\mathcal{C}$ being semiadditive, it follows that those two conditions are equivalent. $\square$

Proposition 4.3.1.2. *Let $\mathcal{C}$ be a semiadditive ∞-category, let $\mathcal{D}$ be an ∞-category admitting finite products, and let F_1 and F_2 be two functors*

$$F_1, F_2 \colon \mathcal{C} \to \mathrm{CMon}(\mathcal{D})$$

such that F_1 preserves products.

Denote the forgetful functor $\mathrm{CMon}(\mathcal{D}) \to \mathcal{D}$ *by V and assume that $V \circ F_1$ is naturally equivalent to $V \circ F_2$. Then there is also a natural equivalence between F_1 and F_2.* $\heartsuit$

Proof. As $\mathcal{D}$ has finite products we can upgrade $\mathcal{D}$ to a symmetric monoidal ∞-category with respect to the cartesian symmetric monoidal structure $\mathcal{D}^{\times}$ (see [HA, 2.4.1.5]). Applying [HA, 2.4.1.5 (5) and 2.4.2.5] we obtain an equivalence of ∞-categories

$$\mathrm{CMon}(\mathcal{D}) \simeq \mathrm{CAlg}(\mathcal{D})$$

which is compatible with the respective forgetful functors to $\mathcal{D}$. Denote the composite of F_i with this equivalence by F'_i. It suffices to show that F'_1 is naturally equivalent to F'_2.

Note that as V detects products by Proposition F.2.0.1 the equivalence $V \circ F_1 \simeq V \circ F_2$ and F_1 preserving products implies that F_2 preserves products as well. Hence both F_1' and F_2' preserve products too, so they induce symmetric monoidal functors as follows (see [HA, 2.4.1.8]).

$$F_i'^{\times} : \mathcal{C}^{\times} \to \mathrm{CAlg}(\mathcal{D})^{\times}$$

We obtain the following commutative diagram for $i = 1$ and $i = 2$

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{C}) \arrow[r, "\mathrm{CAlg}(F_i')"] \arrow[d, "U_{\mathcal{C}}"'] & \mathrm{CAlg}\big(\mathrm{CAlg}(\mathcal{D})\big) \arrow[d, "U_{\mathrm{CAlg}(\mathcal{D})}"] \\
\mathcal{C} \arrow[r, "F_i'"'] & \mathrm{CAlg}(\mathcal{D})
\end{tikzcd}$$

where the vertical functors are the forgetful functors forgetting the "outer" algebra structure. By Proposition 4.3.1.1, the cartesian symmetric monoidal structure $\mathcal{C}^{\times}$ is also cocartesian, so it follows from [HA, 2.4.3.9] that $U_{\mathcal{C}}$ is an equivalence. It thus suffices to show that $U_{\mathrm{CAlg}(\mathcal{D})} \circ \mathrm{CAlg}(F_1') \simeq U_{\mathrm{CAlg}(\mathcal{D})} \circ \mathrm{CAlg}(F_2')$.

The symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{D})$ used in forming $\mathrm{CAlg}\big(\mathrm{CAlg}(\mathcal{D})\big)$ in the above diagram is the cartesian one $\mathrm{CAlg}(\mathcal{D})^{\times}$. There is also a symmetric monoidal structure induced by $\mathcal{D}^{\times}$ on $\mathrm{CAlg}(\mathcal{D})$, which we denote by $\mathrm{CAlg}(\mathcal{D})^{\otimes}$, see Proposition E.4.2.3 and Proposition E.6.0.1. By Proposition F.3.0.2 in combination with [HA, 2.4.1.7] and [HA, 2.4.2.5], there is a symmetric monoidal equivalence $\mathrm{CAlg}(\mathcal{D})^{\otimes} \simeq \mathrm{CAlg}(\mathcal{D})^{\times}$ whose underlying functor of ∞-categories is the identity. We can thus replace $\mathrm{CAlg}(\mathcal{D})^{\times}$ implicitly used in $\mathrm{CAlg}\big(\mathrm{CAlg}(\mathcal{D})\big)$ with $\mathrm{CAlg}(\mathcal{D})^{\otimes}$.

By Proposition E.6.0.1 there is then a natural equivalence between $U_{\mathrm{CAlg}(\mathcal{D})}$ and $\mathrm{CAlg}(U_{\mathcal{D}})$, where $U_{\mathcal{D}} \colon \mathrm{CAlg}(\mathcal{D}) \to \mathcal{D}$ is the forgetful functor. We obtain

$$U_{\mathrm{CAlg}(\mathcal{D})} \circ \mathrm{CAlg}(F_i') \simeq \mathrm{CAlg}(U_{\mathcal{D}}) \circ \mathrm{CAlg}(F_i') \simeq \mathrm{CAlg}\big(U_{\mathcal{D}} \circ F_i'\big) \simeq \mathrm{CAlg}(V \circ F_i)$$

so as $V \circ F_1 \simeq V \circ F_2$ by assumption we conclude

$$U_{\mathrm{CAlg}(\mathcal{D})} \circ \mathrm{CAlg}(F_2') \simeq U_{\mathrm{CAlg}(\mathcal{D})} \circ \mathrm{CAlg}(F_2')$$

which is what we needed to show. □

4.3.2. Properties of $\mathcal{D}(k)$

Proposition 4.3.2.1. *The following hold.*

(1) *$\mathcal{D}(k)$ is[28] the presentable symmetric monoidal ∞-category underlying the combinatorial and symmetric monoidal model category $\mathsf{Ch}(k)$ carrying the projective model structure from Fact 4.1.3.1.*

[28] We will take this as the definition for $\mathcal{D}(k)$, but will also also point out in the proof below why other possible definitions used in [HA] are equivalent.

We will denote the symmetric monoidal functor $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ *by* γ*. We will also sometimes denote the composition of* γ *with the cofibrant replacement functor* $\mathsf{Ch}(k) \to \mathsf{Ch}(k)^{\mathrm{cof}}$ *by* γ *again*[29]*.*

(2) $\mathcal{D}(k)$ *is stable.*

(3) $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ *preserves coproducts.*

(4) *There are natural equivalences for every integer* n *as follows*[30]*.*

$$\gamma(-)[n] \simeq \gamma\big(-[n]\big)$$

From now on we will write k *for* $\gamma(k)$*.*

(5) *There is a natural isomorphism of functors* $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ab}$ *as follows.*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}\big(k[n], \gamma(-)\big) \cong \mathrm{H}_n(-)$$

(6) *Let* $\mathsf{Ch}(k)'_{\geq 0}$ *and* $\mathsf{Ch}(k)'_{\leq 0}$ *be the full subcategories of* $\mathsf{Ch}(k)$ *spanned by the chain complexes whose homology is concentrated in non-negative and non-positive, respectively, degree. Let* $\mathcal{D}(k)_{\geq 0}$ *be the essential image of the restriction of* γ *to* $\big(\mathsf{Ch}(k)\big)'_{\geq 0}$*, and analogously for* $\mathcal{D}(k)_{\leq 0}$*. Then the pair* $\big(\mathcal{D}(k)_{\geq 0}, \mathcal{D}(k)_{\leq 0}\big)$ *determines a t-structure on* $\mathcal{D}(k)$*.*

Furthermore, $\mathcal{D}(k)_{\geq 0}$ *is also the essential image of* $\mathsf{Ch}(k)_{\geq 0}$ *from Definition 4.1.1.1 and* $\mathcal{D}(k)_{\leq 0}$ *is the essential image of* $\mathsf{Ch}(k)_{\leq 0}$*.*

(7) *There is a symmetric monoidal equivalence preserving the respective t-structures between* $\mathcal{D}(k)$ *and the* ∞*-category of* k*-modules in spectra* $\mathrm{LMod}_k(\mathcal{S}\mathrm{p})$ *(where the tensor product is the tensor product over* k*, see [HA, 4.5], and the t-structure is defined in [HA, 7.1.1.10 and 7.1.1.13]).*

(8) *The t-structure on* $\mathcal{D}(k)$ *is compatible with the symmetric monoidal structure in the sense of [HA, 2.2.1.3].*

(9) *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{LMod}_k(\mathsf{Ab}) \arrow[r, "(-)[0]"] \arrow[d, dashed] & \mathsf{Ch}(k) \arrow[d, "\gamma"] \\
\mathcal{D}(k)^{\heartsuit} \arrow[r] & \mathcal{D}(k)
\end{tikzcd}$$

of ∞*-categories, where* $\mathcal{D}(k)^{\heartsuit} = \mathcal{D}(k)_{\geq 0} \cap \mathcal{D}(k)_{\leq 0}$ *is the heart of* $\mathcal{D}(k)$*, see [HA, 1.2.1.11], and the lower horizontal functor the inclusion.*

Furthermore, the dashed functor is an equivalence. We can thus identify the heart of $\mathcal{D}(k)$ *with* $\mathrm{LMod}_k(\mathsf{Ab})$*.* ♡

[29] Note that the restriction of this functor to $\mathsf{Ch}(k)^{\mathrm{cof}}$ is homotopic to the original functor γ.

[30] See Definition 4.1.1.2 for a definition of the shift in $\mathsf{Ch}(k)$ and [HA, 1.1.2.7] for a definition of the shift in the stable ∞-category $\mathcal{D}(k)$.

Proof. Proof of Claim (1): The projective model structure on chain complexes with the required properties was discussed in Fact 4.1.3.1. For the construction of $\mathcal{D}(k)$ as the symmetric monoidal ∞-category underlying $\mathsf{Ch}(k)$ see [HA, 7.1.2.12]. The proof that $\mathcal{D}(k)$ is *presentable* symmetric monoidal can be found in the proof of [HA, 7.1.2.13].

Finally, let us note that different ways of constructing $\mathcal{D}(k)$ are used in [HA]. They are however all equivalent by [HA, 7.1.2.9] and [HA, 1.3.5.15][31], so there is no problem in using results concerning $\mathcal{D}(k)$ from different places in [HA].

Proof of Claim (2): This is [HA, 1.3.5.9].

Proof of Claim (3): By (1) and [HA, 1.3.4.25 and 1.3.4.24] this follows from the fact that coproducts of cofibrant chain complexes are already homotopy coproducts[32].

Proof of Claim (4): We start by proving that $\gamma(k[n]) \cong \gamma(k)[n]$. First note that as k is projective as a k-module, the chain complexes $k[n]$ are cofibrant in the projective model structure by [Hov99, 2.3.6]. Now consider the following pushout diagram of cofibrant objects in $\mathsf{Ch}(k)$

$$\begin{tikzcd} k[n] \arrow[r] \arrow[d] & D^{n+1}(k) \arrow[d] \\ 0 \arrow[r] & k[n+1] \end{tikzcd}$$

where $D^{n+1}(k)$ is the chain complex with $D^{n+1}(k)_m = k$ if $m = n$ or $m = n+1$ and $D^{n+1}(k)_m = 0$ otherwise, and with differential from degree $n+1$ to degree n the identity, and where the morphisms $k[n] \to D^{n+1}(k)$ and $D^{n+1}(k) \to k[n+1]$ are the obvious inclusion and projection. The morphism $k[n] \to D^{n+1}(k)$ is a cofibration[33], so it follows from [HTT, A.2.4.4, variant (i)] that this diagram is a homotopy pushout diagram in $\mathsf{Ch}(k)$. Applying [HA, 1.3.4.24] and using that $D^{n+1}(k)$ is acyclic we can conclude that for every integer n there is a pushout diagram in $\mathcal{D}(k)$ of the following form.

$$\begin{tikzcd} \gamma\big(k[n]\big) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \gamma\big(k[n+1]\big) \end{tikzcd}$$

Using that $\gamma(k)[0] = \gamma(k) = \gamma(k[0])$ it now follows that $\gamma(k[n]) \cong \gamma(k)[n]$ by inducting up and down[34] from 0.

The general statement now follows by combining that by Remark 4.1.2.2 there is a natural isomorphism

$$-[n] \cong (k \otimes -)[n] \cong k[n] \otimes -$$

[31] The construction of $\mathcal{D}(\mathcal{A})$ considered in [HA, 1.3.5] applies to the case $\mathcal{A} = \mathrm{LMod}_k(\mathsf{Ab})$ (the category of ordinary k-modules), as $\mathrm{LMod}_k(\mathsf{Ab})$ is a Grothendieck abelian category in the sense of [HA, 1.3.5.1].

[32] Unpacking the projective model structure (see [HTT, A.2.8.2]) on $\mathrm{Fun}(\mathsf{J}, \mathsf{Ch}(k))$ for a discrete category J one can easily see that such a functor is cofibrant if and only if it is pointwise cofibrant.

[33] It is even one of the generating cofibrations discussed in [Hov99, 2.3.3 and 2.3.11].

[34] The downwards induction uses that $\mathcal{D}(k)$ is stable.

of endofunctors of $\mathsf{Ch}(k)^{\mathrm{cof}}$ and that as the tensor product functor of $\mathcal{D}(k)$ preserves colimits in each variable separately, there is also such a natural equivalence of endofunctors of $\mathcal{D}(k)$, with the fact that γ is symmetric monoidal.

Proof of Claim (5): We start by showing that the compositions of the two functors with the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ are naturally equivalent. Applying Proposition A.1.0.1 to $\mathsf{Ch}(k)$, we obtain a natural isomorphism as follows.

$$\mathrm{Mor}_{\mathrm{Ho}\,\mathcal{D}(k)}\big(\gamma(-),\gamma(-)\big) \cong \mathrm{Mor}_{\mathrm{Ho}\,\mathsf{Ch}(k)}(-,-)$$

A standard calculation using left homotopies (see [Hov99, 1.2.4 in combination with 1.2.6 and 1.2.10]) shows that[35]

$$\mathrm{Mor}_{\mathrm{Ho}\,\mathsf{Ch}(k)}\big(k[n],-\big) \cong \mathrm{H}_n(-) \tag{4.16}$$

so that we have obtained a natural equivalence between the respective compositions with the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$. This forgetful functor factors as the composition of the forgetful functors $\mathsf{Ab} \to \mathrm{CMon}(\mathsf{Set})$ and $\mathrm{CMon}(\mathsf{Set}) \to \mathsf{Set}$. As $\mathsf{Ab} \to \mathrm{CMon}(\mathsf{Set})$ is the inclusion of a full subcategory, it suffices to show that the two functors in question are naturally equivalent as functors to $\mathrm{CMon}(\mathsf{Set})$.

For this we apply Proposition 4.3.1.2. The category $\mathsf{Ch}(k)^{\mathrm{cof}}$ is semiadditive (coproducts of cofibrant objects are again cofibrant by [Hov99, 1.1.11]) and Set admits finite products, so it remains to show that $\mathrm{H}_n(-)$ as a functor $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathrm{CMon}(\mathsf{Set})$ preserve products. The forgetful functor from commutative monoids to sets detects products (see Proposition F.2.0.1), so it suffices to show that $\mathrm{H}_n(-)$ preserves products as a functor into Set. But this is clear, as direct sums in $\mathsf{Ch}(k)$ are formed levelwise, and direct sums are both limits as well as colimits, so are compatible with forming kernels and cokernels.

Proof of Claim (6): The first part is [HA, 1.3.5.16 and 1.3.5.21]. The second part follows immediately from the observation that every chain complex with homology concentrated in nonnegative or nonpositive degrees is quasiisomorphic to a chain complex itself concentrated in those degrees, by truncating.

Proof of Claim (7): By [HA, 7.1.2.13] there is an equivalence

$$\theta\colon\ \mathcal{D}(k) \to \mathrm{LMod}_k(\mathcal{S}\mathrm{p})$$

of symmetric monoidal ∞-categories. It remains to show that θ is compatible with the respective t-structures.

As a monoidal equivalence, θ preserves monoidal units, so $\theta(k) \simeq k$, which implies that there is a sequence of natural isomorphism for $n \geq 2$ of functors $\mathcal{D}(k) \to \mathsf{Set}$ as follows.

$$\mathrm{H}_n(-)$$

[35] The main point is that a cylinder object for $k[n]$ is given by $k[n] \amalg k[n] \xrightarrow{i_0 \amalg i_1} C \xrightarrow{s} k[n]$ where on underlying graded abelian groups $i_0 \amalg i_1$ is the inclusion into $k[n] \oplus k[n] \oplus k[n+1]$, and where ∂^C_{n+1} sends 1 to $(1,0)-(0,1)$. One also needs to use that every object of $\mathsf{Ch}(k)$ is fibrant, and then the rest is unpacking the definition.

Using Claim (5).

$$\cong \mathrm{Mor}_{\mathrm{Ho}\,\mathcal{D}(k)}\big(k[n], -\big)$$

Applying $\mathrm{Ho}\,\theta$.

$$\cong \mathrm{Mor}_{\mathrm{Ho}\,\mathrm{LMod}_k(\mathcal{S}\mathrm{p})}\big(k[n], \theta(-)\big)$$

Using that the functor $\mathrm{Free}\colon \mathcal{S}\mathrm{p} \to \mathrm{LMod}_k(\mathcal{S}\mathrm{p})$ is left adjoint to the forgetful functor. See [HA, 4.2.4.8] and [HTT, 5.2.2.9].

$$\cong \mathrm{Mor}_{\mathrm{Ho}\,\mathcal{S}\mathrm{p}}\big(\mathbb{S}[n], \theta(-)\big) \cong \pi_0\Big(\mathrm{Map}_{\mathcal{S}\mathrm{p}}\big(\mathbb{S}[n], \theta(-)\big)\Big)$$

Using that $n \geq 0$.

$$\cong \pi_n\Big(\mathrm{Map}_{\mathcal{S}\mathrm{p}}\big(\mathbb{S}, \theta(-)\big)\Big)$$

Using the adjunction $\Sigma^\infty \dashv \Omega^\infty_*$.

$$\cong \pi_n\Big(\mathrm{Map}_{\mathcal{S}_*}\big(S^0, \Omega^\infty_*\theta(-)\big)\Big) \cong \pi_n\big(\Omega^\infty_*\theta(-)\big)$$

Using [HA, 1.4.3.8].

$$\cong \pi_n\big(\theta(-)\big)$$

By using that H_* and π_* are both compatible with shifts[36], we can conclude[37] that $\mathrm{H}_n(-) \cong \pi_n(\theta(-))$ for every integer n, which implies that θ is compatible with the respective t-structures on $\mathcal{D}(k)$ and $\mathrm{LMod}_k(\mathcal{S}\mathrm{p})$ as follows directly from their respective definitions.

Proof of Claim (8): The t-structure on $\mathrm{LMod}_k(\mathcal{S}\mathrm{p})$ is compatible with the symmetric monoidal structure by [HA, 7.1.3.10], so this also holds for $\mathcal{D}(k)$ by Claim (7).

Proof of Claim (9): Every chain complex concentrated in degree 0 has obviously vanishing homology outside of degree 0, so $\gamma \circ (-)[0]$ factors through the full subcategory $\mathcal{D}(k)^\heartsuit$ of $\mathcal{D}(k)$.

The induced functor is essentially surjective by the second part of (6). If two morphisms f and g in $\mathrm{LMod}_k(\mathsf{Ab})$ map to homotopic morphisms, then they induce the same morphisms on $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}(k[0], -)$, so by (5) $\mathrm{H}_0(f[0]) = \mathrm{H}_0(g[0])$, and hence $f = g$. Thus $\mathrm{Ho}(\mathrm{LMod}_k(\mathsf{Ab})) \to \mathrm{Ho}(\mathcal{D}(k)^\heartsuit)$ is faithful. Finally, let X and Y be k-modules and $f\colon \gamma(X[0]) \to \gamma(Y[0])$ a morphism in $\mathrm{Ho}(\mathcal{D}(k))$. There is a zigzag of quasiisomorphisms

$$X[0] \cong (\tau_{\leq 0} \circ \tau_{\geq 0})(X^{\mathrm{cof}}) \leftarrow \tau_{\geq 0}(X[0]^{\mathrm{cof}}) \to X[0]^{\mathrm{cof}}$$

in $\mathsf{Ch}(k)$. As $Y[0]$ is fibrant we can by Proposition A.1.0.1 and [Hov99, 1.2.10 (iii)] find a morphism $\overline{f}\colon X^{\mathrm{cof}} \to Y[0]$ representing f, i. e. the composite

$$\gamma(X[0]) \simeq \gamma\Big((\tau_{\leq 0} \circ \tau_{\geq 0})(X^{\mathrm{cof}})\Big) \xleftarrow{\simeq} \gamma\Big(\tau_{\geq 0}(X[0]^{\mathrm{cof}})\Big) \xrightarrow{\simeq} \gamma\Big(X[0]^{\mathrm{cof}}\Big) \xrightarrow{\gamma(\overline{f})} \gamma\big(Y[0]\big)$$

[36]For π_* this is by definition, see [HA, 1.2.1.11], for H_* this follows from Claim (5) and (4)

[37]A priori this is only a natural bijection – which is also all we need, as an abelian group is isomorphic to 0 if and only if its underlying set consists of a single element – but one can also apply Proposition 4.3.1.2 to deduce that this bijection in fact preserves the group structure.

where the first two morphisms are obtained by applying γ to the above zigzag, is homotopic to a representative of f. But it is easy to see that $\overline{f}$ can be strictly lifted to a morphism from $X[0]$, as $Y[0]$ is concentrated in degree 0. This shows that the functor $\mathrm{Ho}(\mathrm{LMod}_k(\mathsf{Ab})) \to \mathrm{Ho}(\mathcal{D}(k)^{\heartsuit})$ is full.

As the ∞-category $\mathcal{D}(k)^{\heartsuit}$ is a 1-category by [HA, 1.2.1.12], this shows that the functor $\mathrm{LMod}_k(\mathsf{Ab}) \to \mathcal{D}(k)^{\heartsuit}$ is an equivalence. □

Remark 4.3.2.2. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the symmetric monoidal functor

$$k' \otimes_k - \colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ch}(k')^{\mathrm{cof}}$$

from Fact 4.1.5.1 preserves weak equivalences and so induces by [HA, 4.1.7.4] a commutative diagram of symmetric monoidal functors as follows.

$$\begin{tikzcd} \mathsf{Ch}(k)^{\mathrm{cof}} \arrow[r, "k'\otimes_k -"] \arrow[d, "\gamma"'] & \mathsf{Ch}(k')^{\mathrm{cof}} \arrow[d, "\gamma"] \\ \mathcal{D}(k) \arrow[r, "k'\otimes_k -"'] & \mathcal{D}(k') \end{tikzcd}$$

Furthermore, it follows from Fact 4.1.5.1 using [HA, 1.3.4.27] that the functor

$$k' \otimes_k - \colon\ \mathcal{D}(k) \to \mathcal{D}(k') \tag{4.17}$$

is left adjoint to the functor

$$\varphi^* \colon\ \mathcal{D}(k') \to \mathcal{D}(k)$$

that is induced by the composition

$$\mathsf{Ch}(k')^{\mathrm{cof}} \xrightarrow{\varphi^*} \mathsf{Ch}(k) \xrightarrow{(-)^{\mathrm{cof}}} \mathsf{Ch}(k)^{\mathrm{cof}}$$

where the second functor is the cofibrant replacement functor. In particular, functor (4.17) preserves small colimits.

As $k' \otimes_k -$ is a symmetric monoidal functor, we can use [HA, 7.3.2.7] to upgrade the adjunction $k' \otimes_k - \dashv \varphi^*$ to an adjunction[38]

$$\begin{tikzcd} \mathcal{D}(k)^{\otimes} \arrow[rr, shift left, "(k'\otimes_k -)^{\otimes}"] \arrow[rr, phantom, "\perp"] \arrow[dr] & & \mathcal{D}(k')^{\otimes} \arrow[ll, shift left, "(\varphi^*)^{\otimes}"] \arrow[dl] \\ & \mathsf{Fin}_* & \end{tikzcd}$$

relative to Fin_* in the sense of [HA, 7.3.2.3], and such that $(\varphi^*)^{\otimes}$ is lax symmetric monoidal. ♢

[38] The functors to Fin_* are to be the canonical cocartesian fibrations of ∞-operads.

4.3.3. Homology

Homology is a very important invariant of chain complexes, and for $\mathcal{D}(k)$ as well. In this section we will discuss how the different definitions are compatible, as well as some properties that we will need.

Definition 4.3.3.1 ([HA, 1.2.1.11])**.** Let n be an integer. We define a functor

$$\mathrm{H}_n \colon \mathcal{D}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$$

to be the composition

$$\mathcal{D}(k) \xrightarrow{(-)[-n]} \mathcal{D}(k) \xrightarrow{\tau_{\geq 0} \circ \tau_{\leq 0}} \mathcal{D}(k)^{\heartsuit} \simeq \mathrm{LMod}_k(\mathsf{Ab})$$

where the equivalence is the one from Proposition 4.3.2.1 (9). ♢

Proposition 4.3.3.2. *Let n be an integer. Then there is a commutative diagram*

$$\begin{tikzcd} \mathsf{Ch}(k) \arrow[dr, "\mathrm{H}_n"] \arrow[dd, "\gamma"'] & \\ & \mathrm{LMod}_k(\mathsf{Ab}) \\ \mathcal{D}(k) \arrow[ur, "\mathrm{H}_n"'] & \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$. ♡

Proof. We need to show that $\mathrm{H}_n \circ \gamma$ and H_n are naturally isomorphic.

Denote by φ the equivalence $\mathrm{LMod}_k(\mathsf{Ab}) \to \mathcal{D}(k)^{\heartsuit}$ from Proposition 4.3.2.1 (9) and assume we have already shown the claim for $n = 0$. Then we can deduce the claim for general n using Proposition 4.3.2.1 (4), as we obtain equivalences of functors $\mathsf{Ch}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$ as follows.

$$\begin{aligned} &\mathrm{H}_n \circ \gamma \\ &= \varphi^{-1} \circ \tau_{\geq 0} \circ \tau_{\leq 0} \circ (-)[-n] \circ \gamma \\ &\cong \varphi^{-1} \circ \tau_{\geq 0} \circ \tau_{\leq 0} \circ \gamma \circ (-)[-n] \\ &\cong \mathrm{H}_0 \circ \gamma \circ (-)[-n] \\ &\cong \mathrm{H}_0 \circ (-)[-n] \\ &\cong \mathrm{H}_n \end{aligned}$$

We now turn to the case $n = 0$. Consider the natural transformations of endofunctors of $\mathsf{Ch}(k)$

$$\mathrm{id}_{\mathsf{Ch}(k)} \to \tau_{\leq 0} \leftarrow \tau_{\geq 0} \circ \tau_{\leq 0} \tag{4.18}$$

where $\tau_{\leq 0}$ and $\tau_{\geq 0}$ refer to the truncation functors for chain complexes. The endofunctor $\tau_{\geq 0} \circ \tau_{\leq 0}$ factors over the inclusion of chain complexes that are concentrated in degree 0, so it suffices to show the following.

(1) The precompositions of $\mathrm{H}_0\colon \mathsf{Ch}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$ with the two natural transformations in (4.18) are natural isomorphisms.

(2) The precompositions of $\mathrm{H}_0 \circ \gamma\colon \mathsf{Ch}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$ with the two natural transformations in (4.18) are natural isomorphisms.

(3) The precompositions of H_0 and $\mathrm{H}_0 \circ \gamma$ with the inclusion of chain complexes concentrated in degree 0 are naturally isomorphic.

Proof of (1): Clear.

Proof of (2): We only consider the first natural transformation, the other case is similar. We need to show that the natural transformation

$$\tau_{\geq 0} \circ \tau_{\leq 0} \circ \gamma \circ \mathrm{id}_{\mathsf{Ch}(k)} \to \tau_{\geq 0} \circ \tau_{\leq 0} \circ \gamma \circ \tau_{\leq 0}$$

is a natural equivalence. Let X be a chain complex, and let f be the natural morphism $X \to \tau_{\leq 0} X$. Then f is an isomorphism in homology in non-positive degrees, while $\tau_{\leq 0} X$ has homology concentrated in non-positive degrees, so the homotopy fiber $\mathrm{hofib}(f)$ has homology concentrated in positive degrees. We obtain a pullback diagram

$$\begin{tikzcd}
\gamma\big(\mathrm{hofib}(f)\big) \arrow[r] \arrow[d] & \gamma(X) \arrow[d] \\
0 \arrow[r] & \gamma\big(\tau_{\leq 0} X\big)
\end{tikzcd}$$

in $\mathcal{D}(k)$, with $\gamma\big(\mathrm{hofib}(f)\big)$ lying in $\mathcal{D}(k)_{\geq 1}$. Applying $\tau_{\leq 0}\colon \mathcal{D}(k) \to \mathcal{D}(k)$ we obtain a pullback diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \tau_{\leq 0}\big(\gamma(X)\big) \arrow[d] \\
0 \arrow[r] & \tau_{\leq 0}\Big(\gamma\big(\tau_{\leq 0} X\big)\Big)
\end{tikzcd}$$

in $\mathcal{D}(k)$, which, as $\mathcal{D}(k)$ is stable, is also a pushout diagram, from which it follows that

$$\tau_{\leq 0}\big(\gamma(X)\big) \to \tau_{\leq 0}\Big(\gamma\big(\tau_{\leq 0} X\big)\Big)$$

is an equivalence. The claim follows.

Proof of (3): What we need to show is that $\mathrm{H}_0 \circ (-)[0]$ and $\mathrm{H}_0 \circ \gamma \circ (-)[0]$ are naturally isomorphic as functors from $\mathrm{LMod}_k(\mathsf{Ab})$ to $\mathrm{LMod}_k(\mathsf{Ab})$.

$\mathrm{H}_0 \circ (-)[0]$ is naturally isomorphic to the identity functor right from the definition. For $\mathrm{H}_0 \circ \gamma \circ (-)[0]$ we can apply Proposition 4.3.2.1 (9) to obtain equivalences as follows.

$$\begin{aligned}
&\mathrm{H}_0 \circ \gamma \circ (-)[0] \\
\simeq\ &\big(\varphi^{-1} \circ \tau_{\geq 0} \circ \tau_{\leq 0}\big) \circ \Big(\big(\mathcal{D}(k)^{\heartsuit} \to \mathcal{D}(k)\big) \circ \varphi\Big)
\end{aligned}$$

$$\begin{aligned}&\simeq \varphi^{-1} \circ \mathrm{id}_{\mathcal{D}(k)^{\heartsuit}} \circ \varphi \\ &\simeq \varphi^{-1} \circ \varphi \\ &\simeq \mathrm{id}_{\mathrm{LMod}_k(\mathsf{Ab})}\end{aligned}$$ □

Proposition 4.3.3.3. *Let n be an integer. Then there is a commutative diagram*

$$\begin{tikzcd} & \mathrm{LMod}_k(\mathsf{Ab}) \arrow[dd, "\mathrm{ev}_{\mathfrak{m}}"] \\ \mathcal{D}(k) \arrow[ur, "\mathrm{H}_n"] \arrow[dr, "\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}(k[n],-)"'] & \\ & \mathsf{Ab} \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$. ♡

Proof. By [HA, 1.3.4.1] it suffices to show that there is a homotopy

$$\mathrm{ev}_{\mathfrak{m}} \circ \mathrm{H}_n \circ \gamma \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}\big(k[n], \gamma(-)\big)$$

of functors $\mathcal{D}(k) \to \mathsf{Ab}$. The former functor is by Proposition 4.3.3.2 homotopic to the composition

$$\mathsf{Ch}(k) \xrightarrow{\mathrm{H}_n} \mathrm{LMod}_k(\mathsf{Ab}) \xrightarrow{\mathrm{ev}_{\mathfrak{m}}} \mathsf{Ab} \tag{$*$}$$

and the latter functor is by Proposition 4.3.2.1 (5) homotopic to the functor

$$\mathsf{Ch}(k) \xrightarrow{\mathrm{H}_n} \mathsf{Ab}$$

which is by definition the same as the composition ($*$). □

Notation 4.3.3.4. Let n be an integer. In light of Proposition 4.3.3.3 we will also denote the functor

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}\big(k[n], -\big)\colon\ \mathcal{D}(k) \to \mathsf{Ab}$$

by H_n. However, if it is not clear from context that we mean this functor, then usage of the notation H_n should be understood to refer to the functor with image in $\mathrm{LMod}_k(\mathsf{Ab})$. ♢

Proposition 4.3.3.5. *Let n be an integer. The functor*

$$\mathrm{H}_n\colon\ \mathcal{D}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$$

preserves products and coproducts. ♡

Proof. As the forgetful functor $\mathrm{ev}_{\mathfrak{m}}\colon\ \mathrm{LMod}_k(\mathsf{Ab}) \to \mathsf{Ab}$ detects limits and colimits, it suffices to show that the functor

$$\mathrm{H}_n\colon\ \mathcal{D}(k) \to \mathsf{Ab}$$

preserves products and coproducts.

We start by showing that it preserves products. As the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ preserves products, it suffices to show that the functor $\mathcal{D}(k) \to \mathsf{Set}$

$$\mathrm{Mor}_{\mathrm{Ho}(\mathcal{D}(k))}\big(k[n], -\big) \cong \pi_0\Big(\mathrm{Map}_{\mathcal{D}(k)}\big(k[n], -\big)\Big)\colon\ \mathcal{D}(k) \to \mathsf{Set}$$

preserves products, but this is clear as both $\mathrm{Map}_{\mathcal{D}(k)}\big(k[n], -\big)$ and π_0 preserve products.

For coproducts we use the commutative diagram constructed in Proposition 4.3.2.1 (5) that is depicted below.

$$\begin{tikzcd}
\mathsf{Ch}(k)^{\mathrm{cof}} \arrow[dr, "\mathrm{H}_n"] \arrow[dd, "\gamma"'] & \\
& \mathsf{Ab} \\
\mathcal{D}(k) \arrow[ur, "\mathrm{H}_n"'] &
\end{tikzcd}$$

As every object of $\mathcal{D}(k)$ is represented by a cofibrant chain complex (by definition) and γ preserves coproducts[39] it suffices to show that the functor H_n on chain complexes preserves coproducts, which is a classical exercise in homological algebra[40]. $\square$

Remark 4.3.3.6. The functor

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D}(k))}(k, -)\colon\ \mathrm{Ho}\big(\mathcal{D}(k)\big) \to \mathsf{Ab}$$

is by [Nee01, 1.1.10] homological in the sense of [Nee01, 1.1.7]. As the forgetful functor from $\mathrm{LMod}_k(\mathsf{Ab})$ to Ab detects exact sequences, it follows from Proposition 4.3.3.3 that the functor

$$\mathrm{H}_0\colon\ \mathrm{Ho}\big(\mathcal{D}(k)\big) \to \mathrm{LMod}_k(\mathsf{Ab})$$

is an homological functor as well.

Any cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{D}(k)$ thus induces a long exact sequence

$$\cdots \xrightarrow{\mathrm{H}_0(-h[-1])} \mathrm{H}_0(X) \xrightarrow{\mathrm{H}_0(f)} \mathrm{H}_0(Y) \xrightarrow{\mathrm{H}_0(g)} \mathrm{H}_0(Z) \xrightarrow{\mathrm{H}_0(h)} \mathrm{H}_0(X[1]) \xrightarrow{\mathrm{H}_0(-f[1])} \cdots$$

in $\mathrm{LMod}_k(\mathsf{Ab})$ that we can identify with a long exact sequence

$$\cdots \to \mathrm{H}_1(Z) \to \mathrm{H}_0(X) \xrightarrow{\mathrm{H}_0(f)} \mathrm{H}_0(Y) \xrightarrow{\mathrm{H}_0(g)} \mathrm{H}_0(Z) \xrightarrow{\mathrm{H}_0(h)} \mathrm{H}_{-1}(X) \to \cdots \qquad \diamondsuit$$

[39] Coproducts of cofibrant objects are homotopy coproducts, then use [HA, 1.3.4.25 and 1.3.4.24].

[40] See for example [Rot08, Exercise 6.9]. One way to show this is as follows. One first considers finite coproducts, which are biproducts, so one can for example use additivity. Arbitrary coproducts can be written as filtered colimits of their finite subcoproducts (this is true also for ∞-categories by [HTT, Special case of the proof of 4.2.3.11] but can of course also be shown in a more elementary way for our application), so it then suffices to show that filtered colimits in $\mathrm{LMod}_k(\mathsf{Ab})$ are exact, which is done in [Wei94, Theorem 2.6.15].

Proposition 4.3.3.7. *Let X be an object of $\mathcal{D}(k)$ so that $\mathrm{H}_n(X)$ is a free k-module with basis*[41] $\{b_i\colon k[n] \to X\}_{i\in I_n}$ *for every integer n.*

Then the morphism

$$\coprod_{n\in\mathbb{Z}, i\in I_n} k[n] \xrightarrow{\coprod_{n\in\mathbb{Z},i\in I_n} b_i} X$$

is an equivalence in $\mathcal{D}(k)$. ♡

Proof. Represent X by a chain complex. Unpacking and using the natural equivalence from Proposition 4.3.2.1 (5) and Proposition 4.3.3.2 we obtain that the morphism in question is represented by a quasiisomorphism of chain complexes and is thus an equivalence. □

Proposition 4.3.3.8. *Let n be an integer, $\mathcal{I}$ a small ∞-category, and $F\colon \mathcal{I} \to \mathcal{D}(k)$ a functor.*

Assume that $F(I)$ lies in $\mathcal{D}(k)_{\geq n}$ for every object I of $\mathcal{I}$. Then the canonical morphism

$$\operatorname*{colim}_{\mathcal{I}} \mathrm{H}_n\big(F(\bullet)\big) \to \mathrm{H}_n\left(\operatorname*{colim}_{\mathcal{I}} F\right)$$

is an isomorphism.

Analogously, if $F(I)$ lies in $\mathcal{D}(k)_{\leq n}$ for every object I of $\mathcal{I}$, then the canonical morphism

$$\mathrm{H}_n\left(\lim_{\mathcal{I}} F\right) \to \lim_{\mathcal{I}} \mathrm{H}_n\big(F(\bullet)\big)$$

is an isomorphism. ♡

Proof. It suffices to consider the case $n = 0$. By [HA, 1.2.1.6], the colimit of F is again in $\mathcal{D}(k)_{\geq 0}$ in the first case and in $\mathcal{D}(k)_{\leq 0}$ in the second case, and thus forms the colimit in that full subcategory by [HTT, 1.2.13.7]. The statement now follows from the fact that $\tau_{\leq 0}\colon \mathcal{D}(k)_{\geq 0} \to \mathcal{D}(k)^{\heartsuit}$ is left adjoint and thus preserves colimits and $\tau_{\geq 0}\colon \mathcal{D}(k)_{\leq 0} \to \mathcal{D}(k)^{\heartsuit}$ is a right adjoint and thus preserves limits. □

4.3.4. Properties of the truncation functors

Let n be an integer. The categories $\mathcal{D}(k)_{\geq n}$ and $\mathcal{D}(k)_{\leq n}$ defined as in [HA, 1.2.1.4] with respect to the t-structure discussed in Proposition 4.3.2.1 are the full subcategories of objects X with $\mathrm{H}_m(X) \cong 0$ for $m < n$ and $m > n$, respectively. By [HA, 1.2.1.6 and 1.2.1.7] we obtain adjunctions

$$\begin{tikzcd} \mathcal{D}(k) \arrow[r, shift left, "\tau_{\leq n}"] \arrow[r, phantom, "\perp"] & \mathcal{D}(k)_{\leq n} \arrow[l, shift left, "\iota_{\leq n}"] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{D}(k)_{\geq n} \arrow[r, shift left, "\iota_{\geq n}"] \arrow[r, phantom, "\perp"] & \mathcal{D}(k) \arrow[l, shift left, "\tau_{\geq n}"] \end{tikzcd}$$

[41] Such a morphism b_i represents an element in $\mathrm{H}_n(X)$ via Proposition 4.3.3.3.

with $\iota_{\leq n}$ and $\iota_{\geq n}$ the inclusions of the respective full subcategories.

We will sometimes omit $\iota_{\leq n}$ and $\iota_{\geq n}$ from the notation and consider $\tau_{\leq n}$ and $\tau_{\geq n}$ as endofunctors of $\mathcal{D}(k)$.

As the t-structure on $\mathcal{D}(k)$ is compatible with the symmetric monoidal structure, we get more, as the following proposition records.

Proposition 4.3.4.1. *The following list of statements hold.*

(1) $\mathcal{D}(k)_{\geq 0}$ inherits a symmetric monoidal structure from $\mathcal{D}(k)$.

(2) The adjunction $\iota_{\geq 0} \dashv \tau_{\geq 0}$ can be upgraded to an adjunction $\iota_{\geq 0}^{\otimes} \dashv \tau_{\geq 0}^{\otimes}$ of lax monoidal functors relative to Fin_* (in the sense of [HA, 7.3.2.3]).

(3) The lax monoidal functor $\iota_{\geq 0}^{\otimes}$ is symmetric monoidal.

(4) For $n \geq 0$, the full subcategory $\left(\mathcal{D}(k)_{\geq 0}\right)_{\leq n}$ inherits a symmetric monoidal structure from $\mathcal{D}(k)_{\geq 0}$.

(5) The adjunction $\tau_{\leq n} \dashv \iota_{\geq 0, \leq n}$, where $\iota_{\geq 0,\leq n} : \left(\mathcal{D}(k)_{\geq 0}\right)_{\leq n} \to \mathcal{D}(k)_{\geq 0}$ is the inclusion, can be upgraded to an adjunction $\tau_{\leq n}^{\otimes} \dashv \iota_{\geq 0,\leq n}^{\otimes}$ of lax monoidal functors relative to $N(\mathsf{Fin}_*)$.

(6) The lax monoidal functor $\tau_{\leq n}^{\otimes} \colon \mathcal{D}(k)_{\geq 0}^{\otimes} \to \left(\mathcal{D}(k)_{\geq 0}\right)_{\leq n}^{\otimes}$ is symmetric monoidal.

Let $\mathcal{O}^{\otimes}$ be an ∞-operad. Then the following statements hold as well.

(7) The adjunction $\iota_{\geq 0}^{\otimes} \dashv \tau_{\geq 0}^{\otimes}$ induces an adjunction

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}\big(\mathcal{D}(k)_{\geq 0}\big) \arrow[r, shift left, "\mathrm{Alg}_{\mathcal{O}}(\iota_{\geq 0})"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathcal{O}}\big(\mathcal{D}(k)\big) \arrow[l, shift left, "\mathrm{Alg}_{\mathcal{O}}(\tau_{\geq 0})"] \end{tikzcd}$$

and $\mathrm{Alg}_{\mathcal{O}}\big(\iota_{\geq 0}\big)$ is fully faithful with essential image spanned by those $\mathcal{O}$-algebras A in $\mathcal{D}(k)$ such that for every object X of $\mathcal{O}$, the underlying object $\mathrm{ev}_X(A)$ of A lies in $\mathcal{D}(k)_{\geq 0}$.

(8) The adjunction $\tau_{\leq n}^{\otimes} \dashv \iota_{\geq 0,\leq n}^{\otimes}$ induces an adjunction

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}\big(\mathcal{D}(k)_{\geq 0}\big) \arrow[r, shift left, "\mathrm{Alg}_{\mathcal{O}}(\tau_{\leq n})"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathcal{O}}\Big(\big(\mathcal{D}(k)_{\geq 0}\big)_{\leq n}\Big) \arrow[l, shift left, "\mathrm{Alg}_{\mathcal{O}}(\iota_{\geq 0,\leq n})"] \end{tikzcd}$$

and $\mathrm{Alg}_{\mathcal{O}}\big(\iota_{\geq 0,\leq n}\big)$ is fully faithful with essential image spanned by those $\mathcal{O}$-algebras A in $\mathcal{D}(k)_{\geq 0}$ such that for every object X of $\mathcal{O}$, the underlying object $\mathrm{ev}_X(A)$ of A lies in $\left(\mathcal{D}(k)_{\geq 0}\right)_{\leq n}$. ♡

Proof. By Proposition 4.3.2.1, the t-structure on $\mathcal{D}(k)$ is compatible with with the symmetric monoidal structure in the sense of [HA, 2.2.1.3], so the statements (1), (2), and (3) follow from [HA, 2.2.1.1], and the statements (4), (5), and (6) follow from [HA, 2.2.1.10 and 2.2.1.9].

That we obtain induced adjunctions on algebras as in (7) and (8) now follows from Proposition E.3.3.1, see also [HA, 2.2.1.5]. Finally, that the functors induced on algebra categories by the inclusions are again fully faithful as well as the descriptions of the essential images follow from Proposition E.3.5.1. □

We also record the following for later use.

Proposition 4.3.4.2 ([HA, 1.2.1.6])**.** *Let n be an integer.*

Then $\mathcal{D}(k)_{\leq n}$ is closed under small limits and coproducts. In particular, $\mathcal{D}(k)_{\leq n}$ admits all small limits and finite biproducts and $\iota_{\leq n}$ preserves them.

Analogously, $\mathcal{D}(k)_{\geq n}$ is closed under small colimits and finite products. In particular, $\mathcal{D}(k)_{\leq n}$ admits all small colimits and finite biproducts and $\iota_{\geq n}$ preserves them. ♡

Proof. The closure properties for limits and colimits are [HA, 1.2.1.6] and closure under finite biproducts follows from the definition using that $\mathrm{H}_m(-)$ commutes with finite biproducts.

The rest of the claims now follow from the closure claims by [HTT, 1.2.13.7] □

4.4. The ∞-category of mixed complexes

In Notation 4.2.2.10 we constructed a commutative diagram of forgetful functors as follows.

$$\begin{tikzcd}
 & \mathrm{Alg}(\mathsf{Mixed}) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}"'] \arrow[dr, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})"] & \\
\mathsf{Mixed} \arrow[dr, "\mathrm{ev}_{\mathfrak{m}}"'] & & \mathrm{Alg}(\mathsf{Ch}(k)) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}"] \\
 & \mathsf{Ch}(k) &
\end{tikzcd} \tag{4.19}$$

All four functors preserve weak equivalences by Proposition 4.2.2.12 so we obtain a commutative diagram on underlying ∞-categories. For this, let us use the following notation.

Notation 4.4.0.1. Denote by W_{Ch}, W_{Alg}, W_{Mixed} and $W_{\mathrm{Alg}(\mathsf{Mixed})}$ the classes of weak equivalences in $\mathsf{Ch}(k)$, $\mathrm{Alg}(\mathsf{Ch}(k))$, Mixed, and $\mathrm{Alg}(\mathsf{Mixed})$, respectively, where we use the weak equivalences from the model structures defined in Fact 4.1.3.1, Definition 4.2.2.2, and Proposition 4.2.2.9.

In contexts in which we only consider a full subcategory of those model categories, we will use the same notation for the class of weak equivalences between objects in that subcategory. ♢

Diagram (4.19) now induces a commutative diagram of ∞-categories as follows.

$$\begin{tikzcd} & \mathrm{Alg}(\mathsf{Mixed})[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}{}'"'] \arrow[dr, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})'"] & \\ \mathsf{Mixed}[W^{-1}_{\mathsf{Mixed}}] \arrow[dr, "\mathrm{ev}'_{\mathfrak{m}}"'] & & \mathrm{Alg}(\mathsf{Ch}(k))[W^{-1}_{\mathrm{Alg}}] \arrow[dl, "\mathrm{ev}'_{\mathfrak{a}}"] \\ & \mathsf{Ch}(k)[W^{-1}_{\mathsf{Ch}}] & \end{tikzcd} \tag{4.20}$$

$\mathsf{Ch}(k)[W^{-1}_{\mathsf{Ch}}]$ can be identified with the derived category, $\mathcal{D}(k)$[42]. The canonical symmetric monoidal functor $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ induces a functor on commutative and cocommutative bialgebras, so we can apply it to the cofibrant commutative and cocommutative bialgebra D (see Construction 4.2.1.1 and Proposition 4.2.2.4) to obtain a commutative and cocommutative bialgebra $\gamma(\mathrm{D})$ in $\mathcal{D}(k)$.

Notation 4.4.0.2. We will denote the object $\gamma(\mathrm{D})$ of $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Comm}}(\mathcal{D}(k))$ by D (or D_k if we want to make k explicit).

By the results of Section 3.4 we obtain an induced symmetric monoidal structure on $\mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)$. We will denote this symmetric monoidal ∞-category by $\mathcal{M}\mathrm{ixed}$, or, if we want to make the base ring k explicit, by $\mathcal{M}\mathrm{ixed}_k$. ♢

We can construct from the symmetric monoidal ∞-category $\mathcal{D}(k)$ and cocommutative bialgebra D in $\mathcal{D}(k)$ the following commutative diagram that is analogous to (4.19).

$$\begin{tikzcd} & \mathrm{Alg}(\mathcal{M}\mathrm{ixed}) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}"'] \arrow[dr, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})"] & \\ \mathcal{M}\mathrm{ixed} \arrow[dr, "\mathrm{ev}_{\mathfrak{m}}"'] & & \mathrm{Alg}\big(\mathcal{D}(k)\big) \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}"] \\ & \mathcal{D}(k) & \end{tikzcd} \tag{4.21}$$

The goal of this section is to show that diagram (4.21) can be identified with diagram (4.20).

For algebras, there is a relevant result: For a monoidal model category $\mathcal{A}$ with certain properties, [HA, 4.1.8.4] shows that there is an equivalence

$$\mathrm{Alg}(\mathcal{A})^{\mathrm{cof}}[W'^{-1}] \xrightarrow{\simeq} \mathrm{Alg}\Big(\mathcal{A}^{\mathrm{cof}}[W^{-1}]\Big)$$

where W and W' are the respective classes of weak equivalences. The reason only the full subcategory of cofibrant objects is considered is that we want the tensor product to

[42] By Proposition 4.3.2.1 (1) $\mathcal{D}(k) \simeq \mathsf{Ch}(k)^{\mathrm{cof}}[W^{-1}]$, but the inclusion of $\mathsf{Ch}(k)^{\mathrm{cof}}$ into $\mathsf{Ch}(k)$ and the cofibrant replacement functor induce mutually inverse equivalences after inverting weak equivalences, see [HA, 1.3.4.16] and Proposition A.3.2.1.

be automatically derived. The pushout product axiom ensures that the tensor product of two cofibrant objects is again cofibrant, so the tensor product restricts to the full subcategory of cofibrant objects. A monoidal category also needs a unit object, so in order to ensure that the subcategory is again a monoidal category, Lurie requires that the unit object in $\mathcal{A}$ is cofibrant. Unfortunately, this does not hold for the monoidal model category $\mathsf{Mixed} = \mathrm{LMod}_D(\mathsf{Ch}(k))$ that we considered above[43], so we can not directly apply Lurie's result. However, we proved that Mixed satisfies the monoid axiom (Proposition 4.2.2.8), which ensures that even though the unit object is not cofibrant, tensoring with it nevertheless results in the correct derived tensor product. Another (related) viewpoint would be to note that the tensor product in $\mathsf{Mixed} = \mathrm{LMod}_D(\mathsf{Ch}(k))$ is calculated on the underlying chain complexes, and in $\mathsf{Ch}(k)$ the unit object *is* cofibrant. This will open the possibility of nevertheless proving a result similar to [HA, 4.1.8.4] for our situation.

We will start in Section 4.4.1 by constructing a comparison natural transformation from diagram (4.20) to diagram (4.21), and then show that the comparison functors are equivalences in Section 4.4.2. Finally, in the very short Section 4.4.3 we show that $\mathcal{M}\mathrm{ixed}$ is a stable ∞-category, and in the also short section Section 4.4.4 we discuss how strongly homotopy linear morphisms of strict mixed complexes induce morphisms in $\mathcal{M}\mathrm{ixed}$.

4.4.1. Construction of comparison functors

In this section we will construct a comparison natural transformation from diagram (4.20) to diagram (4.21).

Construction 4.4.1.1. By Fact 4.1.3.1, the subcategory $\mathsf{Ch}(k)^{\mathrm{cof}}$ inherits a symmetric monoidal structure from $\mathsf{Ch}(k)$. As the underlying chain complex of D is cofibrant by Proposition 4.2.2.4, we can view D as an object of $\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Comm}}(\mathsf{Ch}(k)^{\mathrm{cof}})$. By Proposition 3.4.1.15 we can thus consider the pair $(\mathsf{Ch}(k)^{\mathrm{cof}}, \mathrm{D})$ as an object of $\mathcal{B}i\mathcal{A}lg\mathcal{O}p_{\mathsf{Comm}}$.

The symmetric monoidal functor $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ is a morphism in the ∞-category $\mathrm{Mon}_{\mathsf{Comm}}(\mathcal{C}\mathrm{at}_\infty)$. Denote by $\overline{\gamma}$ a $q_{\mathcal{B}i\mathcal{A}lg\mathcal{O}p_{\mathsf{Comm}}}$-cocartesian lift of γ with source $(\mathsf{Ch}(k)^{\mathrm{cof}}, \mathrm{D})$. By Proposition 3.4.1.15 we can identify the codomain of the morphism $\overline{\gamma}$ with the bialgebra $\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Comm}}(\gamma)(\mathrm{D})$, which we also denote by D.

Applying the natural transformation $\mathrm{ev}_{\mathfrak{m}}\colon\ \mathrm{LMod} \to \mathrm{pr}$ of functors from $\mathcal{B}i\mathcal{A}lg\mathcal{O}p_{\mathsf{Comm}}$ to $\mathrm{Mon}_{\mathsf{Comm}}(\mathcal{C}\mathrm{at}_\infty)$ from Definition 3.4.2.1 we obtain a commutative diagram of symmetric monoidal ∞-categories as follows.

$$\begin{tikzcd}
\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)^{\otimes} \arrow[r, "\mathrm{LMod}_{\mathrm{D}}(\gamma)^{\otimes}"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}^{\otimes}"'] & \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)^{\otimes} \arrow[d, "\mathrm{ev}_{\mathfrak{m}}^{\otimes}"] \\
\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)^{\otimes} \arrow[r, "\gamma^{\otimes}"'] & \mathcal{D}(k)^{\otimes}
\end{tikzcd}$$

[43]See the discussion in Section 4.2.2.2.

Applying the natural transformation

$$\mathrm{ev}_{\mathfrak{a}} \colon \mathrm{Alg}(-) \to - \times_{\mathsf{Fin}_*} \{\langle 1 \rangle\}$$

we obtain the following commutative cube.

$$\begin{tikzcd}
& \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\Big) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)\Big) \arrow[dl] \arrow[dd] \\
\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr] \arrow[dd] & & \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big) \arrow[dd] & \\
& \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr] \arrow[dl] & & \mathrm{Alg}\big(\mathcal{D}(k)\big) \arrow[dl] \\
\mathsf{Ch}(k)^{\mathrm{cof}} \arrow[rr] & & \mathcal{D}(k) &
\end{tikzcd}$$

where the horizontal functors are all induced by γ, and the left and right squares are made up of the various forgetful functors. ♢

Notation 4.4.1.2. We will also denote by γ_{Mixed} the functor

$$\mathsf{Mixed}_{\mathrm{cof}} = \mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \xrightarrow{\mathrm{LMod}_{\mathrm{D}}(\gamma)} \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big) = \mathcal{M}\mathrm{ixed}$$

induced by γ. ♢

Remark 4.4.1.3. Let $\varphi \colon k \to k'$ be a morphism of commutative rings.
Then the symmetric monoidal and weak-equivalence preserving functor

$$k' \otimes_k - \colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ch}(k')^{\mathrm{cof}}$$

from Fact 4.1.5.1 maps by Construction 4.2.1.1 D_k to $\mathrm{D}_{k'}$ and thus induces a transformation from the cube constructed in Construction 4.4.1.1 with respect to k to the same cube with respect to k' (i. e. a four-dimensional hypercube). In particular, there is an induced commutative diagram of symmetric monoidal functors as follows.

$$\begin{tikzcd}
\mathsf{Mixed}_{k,\mathrm{cof}} \arrow[r, "k' \otimes_k -"] \arrow[d, "\gamma_{\mathsf{Mixed}}"'] & \mathsf{Mixed}_{k',\mathrm{cof}} \arrow[d, "\gamma_{\mathsf{Mixed}}"] \\
\mathcal{M}\mathrm{ixed}_k \arrow[r, "k' \otimes_k -"'] & \mathcal{M}\mathrm{ixed}_{k'}
\end{tikzcd}$$

See also Remark 4.2.1.3, Proposition 4.2.2.3, and Remark 4.3.2.2. ♢

Proposition 4.4.1.4. *The functors*

$$\gamma \colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$$
$$\gamma_{\mathsf{Mixed}} \colon \mathsf{Mixed}_{\mathrm{cof}} \to \mathcal{M}\mathrm{ixed}$$

$$\begin{aligned} \mathrm{Alg}(\gamma)\colon\ \mathrm{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) &\to \mathrm{Alg}\big(\mathcal{D}(k)\big) \\ \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\colon\ \mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}}) &\to \mathrm{Alg}(\mathcal{M}\mathrm{ixed}) \end{aligned}$$

all map the respective weak equivalences to equivalences.

In particular, the commutative cube constructed in Construction 4.4.1.1 induces a commutative cube as follows.

$$\begin{tikzcd}
 & \mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}(\mathcal{M}\mathrm{ixed}) \arrow[dl] \arrow[dd] \\
\mathsf{Mixed}_{\mathrm{cof}}[W^{-1}_{\mathsf{Mixed}}] \arrow[rr, crossing over] \arrow[dd] & & \mathcal{M}\mathrm{ixed} & \\
 & \mathrm{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)[W^{-1}_{\mathrm{Alg}}] \arrow[rr] \arrow[dl] & & \mathrm{Alg}\big(\mathcal{D}(k)\big) \arrow[dl] \\
\mathsf{Ch}(k)^{\mathrm{cof}}[W^{-1}_{\mathsf{Ch}}] \arrow[rr] & & \mathcal{D}(k) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the horizontal functors are all induced by γ and the functors on the left and right sides are (induced by) the various forgetful functors. ♡

Proof. The following discussion refers to the cube constructed in Construction 4.4.1.1. Note that by Proposition 4.2.2.12 all the functors on the left side preserve weak equivalences, so that we obtain a commutative square as claimed after inverting the respective classes of weak equivalences. It remains to show that the horizontal functors map weak equivalences to equivalences.

The two functors $\mathrm{ev}_{\mathfrak{a}}$ on the right detect equivalences by [HA, 3.2.2.6], and by [HA, 4.2.3.3] the left vertical functor $\mathrm{ev}_{\mathfrak{m}}$ on the right side also detects equivalences. It follows that equivalences on the right side are detected in $\mathcal{D}(k)$, so it suffices to show that the compositions from the four categories on the left side to $\mathcal{D}(k)$ map weak equivalences to equivalences. But as all functors (or compositions) to $\mathsf{Ch}(k)^{\mathrm{cof}}$ preserve weak equivalences as already mentioned, it actually suffices to show that $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ maps weak equivalences to equivalences. But this is true by definition, see Proposition 4.3.2.1 (1). □

The commutative cube from Proposition 4.4.1.4 is pretty close to being a comparison natural transformation from diagram (4.20) to diagram (4.21). However, the left side is not quite given as (4.20) as we are only considering cofibrant underlying chain complexes. The next proposition shows that this does not make a difference.

Construction 4.4.1.5. We obtain a commutative cube completely analogous to the one constructed in Construction 4.4.1.1 from the symmetric monoidal inclusion functor $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ch}(k)$. Using Proposition 4.2.2.12 we obtain the following induced commutative cube

$$\begin{tikzcd}
& \mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}(\mathsf{Mixed})[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[dl] \arrow[dd] \\
\mathsf{Mixed}_{\mathrm{cof}}[W^{-1}_{\mathsf{Mixed}}] \arrow[rr] \arrow[dd] & & \mathsf{Mixed}[W^{-1}_{\mathsf{Mixed}}] \arrow[dd] & \\
& \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W^{-1}_{\mathrm{Alg}}] \arrow[rr] \arrow[dl] & & \mathrm{Alg}\big(\mathsf{Ch}(k)\big)[W^{-1}_{\mathrm{Alg}}] \arrow[dl] \\
\mathsf{Ch}(k)^{\mathrm{cof}}[W^{-1}_{\mathsf{Ch}}] \arrow[rr] & & \mathsf{Ch}(k)[W^{-1}_{\mathsf{Ch}}] &
\end{tikzcd}$$

where the horizontal functors are induced by the inclusion $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ch}(k)$ and the functors on the left and right are the various forgetful functors. ♢

Construction 4.4.1.6. By Proposition 4.2.2.12 the cofibrant objects in

$$\mathrm{Alg}(\mathsf{Mixed}), \qquad \mathsf{Mixed}, \qquad \mathrm{Alg}\big(\mathsf{Ch}(k)\big), \quad \text{and} \quad \mathsf{Ch}(k)$$

all have cofibrant underlying chain complex[44]. We thus obtain a commutative cube as follows

$$\begin{tikzcd}
& \mathrm{Alg}(\mathsf{Mixed})^{\mathrm{cof}}[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[dl] \arrow[dd] \\
\mathsf{Mixed}^{\mathrm{cof}}[W^{-1}_{\mathsf{Mixed}}] \arrow[rr] \arrow[dd] & & \mathsf{Mixed}_{\mathrm{cof}}[W^{-1}_{\mathsf{Mixed}}] \arrow[dd] & \\
& \mathrm{Alg}\big(\mathsf{Ch}(k)\big)^{\mathrm{cof}}[W^{-1}_{\mathrm{Alg}}] \arrow[rr] \arrow[dl] & & \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W^{-1}_{\mathrm{Alg}}] \arrow[dl] \\
\mathsf{Ch}(k)^{\mathrm{cof}}[W^{-1}_{\mathsf{Ch}}] \arrow[rr] & & \mathsf{Ch}(k)^{\mathrm{cof}}[W^{-1}_{\mathsf{Ch}}] &
\end{tikzcd}$$

where the horizontal functors are induced by the inclusions and the functors on the left and right are the various forgetful functors. ♢

Proposition 4.4.1.7. *The horizontal functors in the commutative cubes of Construction 4.4.1.5 and Construction 4.4.1.6 are equivalences.* ♡

Proof. The proof is very similar for the eight functors, so we only discuss the functor

$$\mathsf{Mixed}_{\mathrm{cof}}[W^{-1}_{\mathsf{Mixed}}] \to \mathsf{Mixed}[W^{-1}_{\mathsf{Mixed}}]$$

as the example case.

As already mentioned in Construction 4.4.1.6, by Proposition 4.2.2.12 the forgetful functor $\mathrm{ev}_{\mathfrak{m}}$ from Mixed to $\mathsf{Ch}(k)$ preserves cofibrant objects, so the cofibrant replacement functor of Mixed lands in $\mathsf{Mixed}_{\mathrm{cof}}$. Let

$$\iota\colon \mathsf{Mixed}_{\mathrm{cof}} \to \mathsf{Mixed}$$

[44] While $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ was not shown in Proposition 4.2.2.12 to preserve cofibrant objects, this is not a problem, as both $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ and $\mathrm{ev}_{\mathfrak{a}}$ preserve cofibrant objects by Proposition 4.2.2.12, so their composition does so too.

be the inclusion functor and

$$-^{\mathrm{cof}}\colon \mathsf{Mixed} \to \mathsf{Mixed}_{\mathrm{cof}}$$

the cofibrant replacement functor. The compositions $\iota \circ -^{\mathrm{cof}}$ and $-^{\mathrm{cof}} \circ \iota$ come with natural transformations to the identity functors that are pointwise weak equivalences. As both ι and $-^{\mathrm{cof}}$ preserve weak equivalences, we obtain induced functors after inverting weak equivalences, and by Proposition A.3.2.1 the natural transformations just mentioned become natural equivalences. Thus the functor induced by ι,

$$\mathsf{Mixed}_{\mathrm{cof}}[W_{\mathsf{Mixed}}^{-1}] \to \mathsf{Mixed}[W_{\mathsf{Mixed}}^{-1}]$$

is an equivalence. □

Definition 4.4.1.8. By composing the cube from Proposition 4.4.1.4 with the inverse of the cube from Construction 4.4.1.5 (where the horizontal functors are equivalences by Proposition 4.4.1.7), we obtain the following commutative cube.

$$\begin{tikzcd}
 & \mathrm{Alg}(\mathsf{Mixed})[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}(\mathcal{M}\mathrm{ixed}) \arrow[dl] \arrow[dd] \\
\mathsf{Mixed}[W_{\mathsf{Mixed}}^{-1}] \arrow[rr, crossing over] \arrow[dd] & & \mathcal{M}\mathrm{ixed} & \\
 & \mathrm{Alg}(\mathsf{Ch}(k))[W_{\mathrm{Alg}}^{-1}] \arrow[rr] \arrow[dl] & & \mathrm{Alg}(\mathcal{D}(k)) \arrow[dl] \\
\mathsf{Ch}(k)[W_{\mathsf{Ch}}^{-1}] \arrow[rr] & & \mathcal{D}(k) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

The horizontal functors are induced by the composition of the respective cofibrant replacement functors and γ, and the other functors are (induced by) the various forgetful functors. ♢

4.4.2. The comparison functors are equivalences

In this section we show that the horizontal functors in the cube of Definition 4.4.1.8 are all equivalences.

Proposition 4.4.2.1 ([HA, 4.1.8.4]). *The functor*

$$\mathrm{Alg}(\mathsf{Ch}(k))[W_{\mathrm{Alg}}^{-1}] \to \mathrm{Alg}(\mathcal{D}(k))$$

from Definition 4.4.1.8 is an equivalence. ♡

Proof. By Proposition 4.4.1.7 it suffices to show that the related functor

$$\mathrm{Alg}(\mathsf{Ch}(k))^{\mathrm{cof}}[W_{\mathrm{Alg}}^{-1}] \to \mathrm{Alg}(\mathcal{D}(k))$$

induced by γ is an equivalence.

By Fact 4.1.3.1 $\mathsf{Ch}(k)$ is a combinatorial symmetric monoidal model category with cofibrant unit object, satisfies the monoid axiom, is left proper, and the class of cofibrations is generated by cofibrations between cofibrant objects[45]. The statement thus follows from [HA, 4.1.8.4, variant (B)]. □

Proposition 4.4.2.2 ([HA, 4.3.3.17])**.** *The functor*

$$\mathsf{Mixed}[W_{\mathsf{Mixed}}^{-1}] \to \mathcal{M}\mathrm{ixed}$$

from Definition 4.4.1.8 is an equivalence. ♡

Proof. The proof is very similar to the proof of Proposition 4.4.2.1. Again it suffices by Proposition 4.4.1.7 to show that the functor

$$\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)^{\mathrm{cof}}[W_{\mathsf{Mixed}}^{-1}] \to \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)$$

is an equivalence.

By Fact 4.1.3.1 $\mathsf{Ch}(k)$ is a combinatorial monoidal model category with cofibrant unit object, and by Proposition 4.2.2.4 D is cofibrant. The statement thus follows from [HA, 4.3.3.17]. □

We now come to the last functor from Definition 4.4.1.8 that we still need to prove is an equivalence. As mentioned in the introduction to Section 4.4, we will not be able to merely cite an appropriate result from [HA], as the unit of Mixed is not cofibrant. We explain in more detail in Remark 4.4.2.4 below how the condition of the unit being cofibrant is used in the proof of [HA, 4.1.8.4].

Proposition 4.4.2.3. *The functor*

$$\mathrm{Alg}(\mathsf{Mixed})[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}] \to \mathrm{Alg}(\mathcal{M}\mathrm{ixed})$$

from Definition 4.4.1.8 is an equivalence. ♡

Proof. This proof will follow the proof of [HA, 4.1.8.4] closely. As in Proposition 4.4.2.1 and Proposition 4.4.2.2 it suffices by Proposition 4.4.1.7 to show that the functor

$$\mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)\Big)^{\mathrm{cof}}[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}] \to \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)\Big)$$

which we will call $\gamma_{\mathrm{Alg}(\mathsf{Mixed})}$ in this proof, is an equivalence.

By Proposition 4.4.1.4 and Construction 4.4.1.6 there is a commutative square

$$\begin{tikzcd}
\mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)\Big)^{\mathrm{cof}}[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}] \arrow[r, "\gamma_{\mathrm{Alg}(\mathsf{Mixed})}"] \arrow[d, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}"'] & \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)\Big) \arrow[d, "\mathrm{ev}_{\mathfrak{a}}"] \\
\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)[W_{\mathsf{Mixed}}^{-1}] \arrow[r, "\gamma_{\mathsf{Mixed}}"'] & \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)
\end{tikzcd}$$

[45] For this last bit see the description of the generating cofibrations in [Hov99, 2.3.11 and 2.3.3] in combination with the description of cofibrant objects in [Hov99, 2.3.6].

where the horizontal functors are induced by γ, and $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}$ is induced by $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$. Proposition 4.4.2.2 shows that γ_{Mixed} is an equivalence.

Like the proof of [HA, 4.1.8.4], we will apply [HA, 4.7.3.16] to show that $\gamma_{\mathrm{Alg}(\mathsf{Mixed})}$ is an equivalence. For this it suffices to verify the following.

(1) $\mathrm{ev}_{\mathfrak{a}}$ has a left adjoint, which we will call $\mathrm{Free}_{\mathcal{M}\mathrm{ixed}}^{\mathrm{Alg}(\mathcal{M}\mathrm{ixed})}$.

(2) $\mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)\Big)$ admits geometric realizations of simplicial objects.

(3) $\mathrm{ev}_{\mathfrak{a}}$ preserves geometric realizations of simplicial objects.

(4) $\mathrm{ev}_{\mathfrak{a}}$ is conservative.

(1') $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}$ has a left adjoint, which we will call $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})'}$.

(2') $\mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)\Big)^{\mathrm{cof}}[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}]$ admits geometric realizations of simplicial objects.

(3') $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}$ preserves geometric realizations of simplicial objects.

(4') $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}$ is conservative.

(5) The push-pull natural transformation[46]

$$\mathrm{Free}_{\mathcal{M}\mathrm{ixed}}^{\mathrm{Alg}(\mathcal{M}\mathrm{ixed})} \circ \gamma_{\mathsf{Mixed}} \to \gamma_{\mathrm{Alg}(\mathsf{Mixed})} \circ \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})'}$$

is a natural equivalence.

Proof of claim (2) and (3): By Proposition 4.3.2.1 (1) $\mathcal{D}(k)$ is presentable symmetric monoidal ∞-category, so by the discussions leading to Definition 3.4.2.1, $\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$ is also a presentable symmetric monoidal ∞-category. The claims now follow from [HA, 3.2.3.1] and Proposition E.2.0.2.

Proof of claim (1): Follows from Proposition E.7.2.1, again using that $\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$ is presentable symmetric monoidal.

Proof of claim (4): This is [HA, 3.2.2.6].

Proof of claim (2'): By Proposition 4.2.2.9 the model structure on $\mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big))$ is combinatorial, so it follows from [HA, 1.3.4.22] that $\mathrm{Alg}(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big))^{\mathrm{cof}}[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}]$ is presentable and hence in particular admits geometric realizations of simplicial objects.

Proof of claim (3'): This is the part of the proof where we need to do something differently than the proof of [HA, 4.1.8.4], as this is the point where the unit being cofibrant is used – see Remark 4.4.2.4 below for more details.

[46]See [HTT, Beginning of 7.3.1].

Consider the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Alg}\big(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))\big)^{\mathrm{cof}}[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[dl, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}\prime}"'] \arrow[dr, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})'"] & \\
\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))[W^{-1}_{\mathsf{Mixed}}] \arrow[dr, "\mathrm{ev}'_{\mathfrak{m}}"'] & & \mathrm{Alg}(\mathsf{Ch}(k))[W^{-1}_{\mathrm{Alg}}] \arrow[dl, "\mathrm{ev}'_{\mathfrak{a}}"] \\
 & \mathsf{Ch}(k)[W^{-1}_{\mathsf{Ch}}] &
\end{tikzcd}$$

that already appeared above as diagram (4.20)[47]. As the diagram commutes, it suffices to show the following three claims.

(a) The functor $\mathrm{ev}'_{\mathfrak{m}}$ in the above diagram detects geometric realizations of simplicial objects[48].

(b) The functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})'$ in the above diagram preserves geometric realizations of simplicial objects.

(c) The functor $\mathrm{ev}'_{\mathfrak{a}}$ in the above diagram preserves geometric realizations of simplicial objects.

Proof of claim (a): By Definition 4.4.1.8, Proposition 4.4.2.2, and Proposition 4.3.2.1 (1), we can identify the functor $\mathrm{ev}'_{\mathfrak{m}}$ in question with the functor

$$\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big) \to \mathcal{D}(k)$$

which, as $\mathcal{D}(k)$ is presentable symmetric monoidal by Proposition 4.3.2.1 (1), detects small colimits by [HA, 4.2.3.5 (2)].

Proof of claim (b): By [HA, 1.3.4.24 and 1.3.4.25], it suffices to show that the functor

$$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}}) \colon \mathrm{Alg}\big(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))\big) \to \mathrm{Alg}(\mathsf{Ch}(k))$$

preserves homotopy colimits. Homotopy colimits can be calculated by taking the colimit of a cofibrant replacement of the diagram with respect to the projective model structure on diagram categories, see [HTT, A.2.8]. As $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ preserves ordinary colimits and weak equivalences by Proposition 4.2.2.12 it hence suffices to show that

$$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})_* \colon \mathrm{Fun}\Big(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Alg}\big(\mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}(k))\big)\Big) \to \mathrm{Fun}\big(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Alg}(\mathsf{Ch}(k))\big)$$

preserves generating cofibrations. But this follows from their description [HTT, A.2.8.5] and the fact that $\mathrm{Alg}(V)$ preserves colimits and cofibrations by Proposition 4.2.2.12.

[47] With the tiny difference that we added a $-^{\mathrm{cof}}$ at the top, but by Proposition 4.4.1.7 this doesn't matter anyway.

[48] In other words it detects $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits.

Proof of claim (c): By Definition 4.4.1.8, Proposition 4.4.2.1, and Proposition 4.3.2.1 (1), we can identify the functor $\mathrm{ev}'_{\mathfrak{a}}$ in question with the functor

$$\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{Alg}\big(\mathcal{D}(k)\big) \to \mathcal{D}(k)$$

which, as $\mathcal{D}(k)$ is presentable symmetric monoidal by Proposition 4.3.2.1 (1), preserves sifted colimits by [HA, 3.2.3.1].

Proof of claim (4'): It suffices to show that the induced functor on homotopy categories is conservative, i. e. reflects isomorphisms. By Proposition A.1.0.1 we can identify that functor with the functor induced by

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\colon \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)\Big) \to \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)$$

on homotopy categories of the model categories, i. e.

$$\mathrm{Ho}_{W_{\mathrm{Alg}(\mathsf{Mixed})}}\Bigg(\mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)\Big)\Bigg) \to \mathrm{Ho}_{W_{\mathsf{Mixed}}}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)\Big)$$

which is conservative by the classical constructions for homotopy categories[49], as $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ detects weak equivalences by Proposition 4.2.2.12.

Proof of claims (1') and (5): We consider the symmetric monoidal functor

$$\mathrm{LMod}_{\mathrm{D}}(\gamma)^{\otimes}\colon \mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)^{\otimes} \to \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)^{\otimes}$$

from Construction 4.4.1.1. We want to show that the underlying functor preserves coproducts and that both $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$ and $\mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)$ admit coproducts and have tensor product functors that preserve coproducts in each variable separately.

That $\mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$ is a presentable symmetric monoidal ∞-category was already mentioned above.

As the forgetful functor $\mathrm{ev}_{\mathfrak{m}}$ from $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)$ to $\mathsf{Ch}(k)$ preserves colimits by Proposition 4.2.2.12, we can conclude that the subcategory $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$ is closed under coproducts[50] and hence admits coproducts, which are calculated in $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)$ (see [HTT, 1.2.13.7]). As $\mathrm{ev}_{\mathfrak{m}}$ detects colimits and is symmetric monoidal, and the tensor product in $\mathsf{Ch}(k)$ is compatible with colimits[51] we can conclude that the tensor product of $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$ preserves coproducts in each variable separately.

Finally, we show that the functor $\mathrm{LMod}_{\mathrm{D}}(\gamma)$ preserves coproducts. To see this, note that as argued in the proof of claim (a), the functor

$$\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big) \to \mathcal{D}(k)$$

detects small colimits, and as by the discussion above the forgetful functor

$$\mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathsf{Ch}(k)^{\mathrm{cof}}$$

[49] See [Hov99, 1.2].

[50] Cofibrant objects in a model category are closed under coproducts, which can be checked using the lifting property that defines cofibrations, see [Hov99, 1.1.10].

[51] As the symmetric monoidal structure is closed by Definition 4.1.2.1.

preserves coproducts, it suffices to show that the functor

$$\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$$

preserves coproducts, which is true by Proposition 4.3.2.1 (3).

We have now verified that $\mathrm{LMod}_{\mathrm{D}}(\gamma)^{\otimes}$ satisfies the assumptions of variant (2) of Proposition E.7.2.2. We thus obtain a left adjoint

$$\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}\colon \mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)\Big)$$

to the forgetful functor $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$, which can be identified with a restriction of the functor of the same name defined in Notation 4.2.2.10. More crucially, Proposition E.7.2.2 shows that the push-pull transformation

$$\mathrm{Free}_{\mathcal{M}\mathrm{ixed}}^{\mathrm{Alg}(\mathcal{M}\mathrm{ixed})} \circ \mathrm{LMod}_{\mathrm{D}}(\gamma) \to \mathrm{Alg}\big(\mathrm{LMod}_{\mathrm{D}}(\gamma)\big) \circ \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$$

is an equivalence.

The functor

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\colon \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)\Big) \to \mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)$$

preserves weak equivalences by Proposition 4.2.2.12. We next show that the functor

$$\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}\colon \mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)\Big)$$

also preserves weak equivalences. As the functor

$$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\colon \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)\Big) \to \mathrm{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)$$

detects weak equivalences by Proposition 4.2.2.12, it suffices to check that the composition preserves weak equivalences. This composition is by Proposition 4.2.2.11 naturally isomorphic to the composition of

$$\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathsf{Ch}(k)^{\mathrm{cof}}$$

with $\mathrm{Free}^{\mathrm{Alg}}$. But by Proposition 4.2.2.12, $\mathrm{ev}_{\mathfrak{m}}$ preserves weak equivalences, and $\mathrm{Free}^{\mathrm{Alg}}$ preserves weak equivalences between cofibrant objects as a left Quillen functor.

As $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ and $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$ preserve weak equivalences, they induce functors on the ∞-categories obtained by inverting weak equivalences. Additionally, unit and counit of the adjunction induce unit and counit of an adjunction as follows[52]

$$\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W_{\mathsf{Mixed}}^{-1}] \underset{\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}\prime}}{\overset{\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})\prime}}{\rightleftarrows}} \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\Big)[W_{\mathrm{Alg}(\mathsf{Mixed})}^{-1}]$$

[52]See the universal property of inverting morphisms in ∞-categories in [HA, 1.3.4.1].

where we think of adjunctions in terms of units and counits as in Proposition D.2.1.1.

In the non-dashed commutative square

$$\begin{tikzcd}
\mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)^{\mathrm{cof}}\Big)[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \arrow[r, "\gamma_{\mathrm{Alg}(\mathsf{Mixed})}"] \arrow[d, shift left, "\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}"] & \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)\Big) \arrow[d, shift left, "\mathrm{ev}_{\mathfrak{a}}"] \\
\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W^{-1}_{\mathsf{Mixed}}] \arrow[r, "\gamma_{\mathsf{Mixed}}"'] \arrow[u, dashed, shift left, "\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})'}"] & \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big) \arrow[u, shift left, "\mathrm{Free}_{\mathcal{M}\mathrm{ixed}}^{\mathrm{Alg}(\mathcal{M}\mathrm{ixed})}"]
\end{tikzcd} \qquad (*)$$

from Proposition 4.4.1.4, there is thus an induced left adjoint of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}$ as indicated. Furthermore, as unit and counit of the adjunction on the left are induced by the unit and counit of the adjunction $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})} \dashv \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$, we can identify the push-pull transformation associated to the square with the natural transformation induced by the push-pull transformation

$$\mathrm{Free}_{\mathcal{M}\mathrm{ixed}}^{\mathrm{Alg}(\mathcal{M}\mathrm{ixed})} \circ \mathrm{LMod}_{\mathrm{D}}(\gamma) \to \mathrm{Alg}\big(\mathrm{LMod}_{\mathrm{D}}(\gamma)\big) \circ \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$$

by passing from $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$ to $\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)[W^{-1}_{\mathsf{Mixed}}]$. As the latter is a natural equivalence, it follow that the push-pull transformation associated to diagram $(*)$ is also a natural equivalence.

Finally, the functor

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}\colon\ \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)^{\mathrm{cof}}\Big)[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \to \mathrm{LMod}_{\mathrm{D}}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)[W^{-1}_{\mathsf{Mixed}}]$$

discussed so far is by Proposition 4.4.1.7 equivalent to the functor

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}'}\colon\ \mathrm{Alg}\Big(\mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)\Big)^{\mathrm{cof}}[W^{-1}_{\mathrm{Alg}(\mathsf{Mixed})}] \to \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}(k)\big)[W^{-1}_{\mathsf{Mixed}}]$$

so this proves claims (1') and (5). □

Remark 4.4.2.4. While the statement [HA, 4.1.8.4] is formulated in such a way as to require the unit object to be cofibrant, thereby preventing us from using the result directly to show Proposition 4.4.2.3, let us remark on where this is used in the proof.

The main step in proving [HA, 4.1.8.4] is the lemma [HA, 4.1.8.13], which shows that if C is a monoidal model category satisfying certain assumptions and J is a small sifted category, then the forgetful functor $\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{Alg}(\mathsf{C}) \to \mathsf{C}$ preserves J-indexed homotopy colimits.

The proof proceeds by showing that every projectively cofibrant object A of the functor category $\mathrm{Fun}(\mathsf{J}, \mathrm{Alg}(\mathsf{C}))$ is a retract of a certain transfinite composition with favorable properties[53]. What needs to be shown is that $(\mathrm{ev}_{\mathfrak{a}})_*(A)$ is *good*, an ad hoc property used in the proof, which is shown by transfinite induction.

The induction start needs that $(\mathrm{ev}_{\mathfrak{a}})_*(\mathrm{const}_{\mathbb{1}_{\mathsf{C}}})$ is good. The argument in [HA, (3) on page 500] shows that every constant functor whose value is a cofibrant object in C is

[53]See [HA, End of page 500 and start of page 501].

good, so if one assumes that the unit $\mathbb{1}_{\mathsf{C}}$ is cofibrant in C, then this proves the induction start. Combining [HA, (3) on page 500] with the definition of good objects [HA, Middle of page 499] one sees that a constant functor $\mathsf{J} \to \mathsf{C}$ is actually good if and only if the constant value is cofibrant in C.

So if $\mathsf{C} = \mathsf{Mixed}$, where the unit is not cofibrant by Proposition 4.2.2.5, then the induction start fails, so $\mathrm{ev}_{\mathfrak{a}}$ preserving homotopy colimits needs to be proven in a different way than [HA, 4.1.8.13]. ♢

4.4.3. $\mathcal{M}\mathrm{ixed}$ is stable

In this section we show that $\mathcal{M}\mathrm{ixed}$ is a stable ∞-category.

Proposition 4.4.3.1. *The ∞-category $\mathcal{M}\mathrm{ixed}$ is stable*[54]*.* ♡

Proof. The statement follows by combining that $\mathcal{D}(k)$ is stable by Proposition 4.3.2.1 (2) with $\mathcal{M}\mathrm{ixed}$ admitting all small limits and colimits by [HA, 4.2.3.3 (1) and 4.2.3.5 (1)] and $\mathrm{ev}_{\mathfrak{m}}\colon \mathcal{M}\mathrm{ixed} \to \mathcal{D}(k)$ detecting small colimits and limits as well as equivalences by [HA, 4.2.3.3 (2) and 4.2.3.5 (2)]. □

4.4.4. Strongly homotopy linear morphisms

In Section 4.2.3 we introduced the notion of strongly homotopy linear morphisms between strict mixed complexes. In this short section we discuss how they induce morphisms in the ∞-category of mixed complexes.

Construction 4.4.4.1. Let X and Y be strict mixed complexes with cofibrant underlying chain complexes, and $f\colon X \to Y$ a strongly homotopy linear morphism. Recall from Proposition 4.2.3.7 and Definition 4.2.3.8 that f lifts to a morphism $f^{\mathrm{strict}}\colon X \to Y^{\mathrm{shl}}$ of strict mixed complexes, and from Proposition 4.2.3.5 that Y^{shl} comes with a quasiisomorphism of strict mixed complexes $\iota_Y^{\mathrm{shl}}\colon Y \to Y^{\mathrm{shl}}$.

We can't directly apply γ_{Mixed} to f^{strict}, as Y^{shl} might not have cofibrant underlying chain complex[55]. However we obtain a commutative diagram

$$\begin{tikzcd} X^{\mathrm{cof}} \arrow[r, "(f^{\mathrm{strict}})^{\mathrm{cof}}"] \arrow[d] & (Y^{\mathrm{shl}})^{\mathrm{cof}} \arrow[d] & Y^{\mathrm{cof}} \arrow[l, "(\iota_Y^{\mathrm{shl}})^{\mathrm{cof}}"'] \arrow[d] \\ X \arrow[r, "f^{\mathrm{strict}}"'] & Y^{\mathrm{shl}} & Y \arrow[l, "\iota_Y^{\mathrm{shl}}"] \end{tikzcd} \tag{$*$}$$

in Mixed, where the vertical morphisms are the cofibrant replacements in Mixed, and by Proposition 4.2.2.12 all strict mixed complexes except possibly Y^{shl} in this diagram have

[54] See [HA, 1.1.1.9] for a definition.

[55] This problem is related to the fact that Y^{shl} involves an infinite product (rather than an infinite coproduct, which would not be a problem).

cofibrant underlying chain complex. We can thus apply γ_{strict} to the part of the diagram not involving Y^{shl}.

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\left(X^{\mathrm{cof}}\right) \arrow[r, "\gamma_{\mathsf{Mixed}}\left(\left(f^{\mathrm{strict}}\right)^{\mathrm{cof}}\right)"] \arrow[d, "\simeq"'] & \gamma_{\mathsf{Mixed}}\left(\left(Y^{\mathrm{shl}}\right)^{\mathrm{cof}}\right) & \gamma_{\mathsf{Mixed}}\left(Y^{\mathrm{cof}}\right) \arrow[l, "\gamma_{\mathsf{Mixed}}\left(\left(\iota_Y^{\mathrm{shl}}\right)^{\mathrm{cof}}\right)"', "\simeq"] \arrow[d, "\simeq"] \\
\gamma_{\mathsf{Mixed}}(X) \arrow[rr, dashed, "\gamma_{\mathsf{Mixed}}(f)"'] & & \gamma_{\mathsf{Mixed}}(Y)
\end{tikzcd} \tag{4.22}$$

As the vertical morphisms in diagram (∗) as well as $\left(\iota_Y^{\mathrm{shl}}\right)^{\mathrm{cof}}$ are quasiisomorphisms, the corresponding morphisms in diagram (4.22) are equivalences. We can thus form the composition from X to Y, yielding a morphism in $\mathcal{M}\mathrm{ixed}$ that we will denote by $\gamma_{\mathsf{Mixed}}(f)$ and call the *morphism in* $\mathcal{M}\mathrm{ixed}$ *induced by* f. ♢

Remark 4.4.4.2. Let X and Y be strict mixed complexes with cofibrant underlying chain complex, and let $f\colon X \to Y$ be a strongly homotopy linear quasiisomorphism[56]. Then the induced morphism $\gamma_{\mathsf{Mixed}}(f)\colon \gamma_{\mathsf{Mixed}}(X) \to \gamma_{\mathsf{Mixed}}(Y)$ is an equivalence. Indeed, considering diagram (4.22) in Construction 4.4.4.1, it is enough to show that f^{strict} is a quasiisomorphism. As the underlying morphism of chain complexes of f is by definition the composition of f^{strict} with the underlying morphism of chain complexes of p_Y^{shl}, which is a quasiisomorphism by Proposition 4.2.3.6, this follows from the underlying morphism of chain complexes of f being a quasiisomorphism. ♢

[56]By this we mean a strongly homotopy linear morphism whose underlying morphism of chain complexes is a quasiisomorphism.

Chapter 5.

Mixed complexes and circle actions

In Section 6.2.1 we will see that Hochschild homology carries a natural action by the circle group $\mathbb{T}$, i. e. Hochschild homology forms a functor

$$\mathrm{HH}_{\mathbb{T}}\colon\ \mathrm{Alg}\big(\mathcal{D}(k)\big) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} = \mathrm{Fun}(\mathrm{B}\,\mathbb{T}, k)$$

where $\mathrm{B}\,\mathbb{T}$ can be thought of as the ∞-groupoid with one object $*$ and $\mathrm{Aut}_{\mathrm{B}\,\mathbb{T}}(*) \simeq \mathbb{T}$, where $\mathbb{T}$ can be defined as $\left\{\, z \in \mathbb{C} \,\middle|\, |z| = 1 \,\right\}$. We will define $\mathbb{T}$ properly in Section 5.2.1 and $\mathrm{B}\,\mathbb{T}$ in Section 5.3.

For calculations it will be helpful to have model categories available that represent the involved ∞-categories. We have seen in Section 4.3.2 that $\mathcal{D}(k)$ is the underlying ∞-category of $\mathsf{Ch}(k)$ with the projective model structure. By [HA, 4.1.8.4], the model structure on $\mathrm{Alg}(\mathsf{Ch}(k))$ discussed in Theorem 4.2.2.1 has $\mathrm{Alg}\big(\mathcal{D}(k)\big)$ as underlying ∞-category. This takes care of the domain of $\mathrm{HH}_{\mathbb{T}}$. How about the codomain?

If $\mathrm{B}\,\mathbb{T}$ were a 1-category, then we could apply [HA, 1.3.4.25], which would then imply that $\mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{D}(k))$ is the underlying ∞-category of the injective or projective model structure on $\mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathsf{Ch}(k))$. This is however not the case – $\mathrm{B}\,\mathbb{T}$ is a 2-category, but not a 1-category. We must thus proceed differently.

In Section 5.2 we will define a cocommutative bialgebra $k \boxtimes \mathbb{T}$ in $\mathcal{D}(k)$, and in Section 5.3 we will show that there is a symmetric monoidal equivalence

$$\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{k\boxtimes\mathbb{T}}\big(\mathcal{D}(k)\big)$$

where $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ carries the pointwise symmetric monoidal structure and $\mathrm{LMod}_{k\boxtimes\mathbb{T}}\big(\mathcal{D}(k)\big)$ the one from Definition 3.4.2.1.

By [HA, 4.3.3.17] the model category $\mathrm{LMod}_A(\mathsf{Ch}(k))$, with model structure as in Theorem 4.2.2.1, has $\mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k))$ as its underlying ∞-category if A is a differential graded algebra with cofibrant underlying complex and such that $\gamma(A) \simeq k \boxtimes \mathbb{T}$ as associative algebras.

We will show in Section 5.1 that the differential graded algebra D defined in Construction 4.2.1.1 represents $k \boxtimes \mathbb{T}$ as an associative algebra. In fact, we show more – D even represents $k \boxtimes \mathbb{T}$ as an associative and coassociative *bialgebra*. There is thus a monoidal (though not symmetric monoidal!) equivalence as follows.

$$\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{k\boxtimes\mathbb{T}}\big(\mathcal{D}(k)\big) \simeq \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big) = \mathcal{M}\mathrm{ixed}$$

Let us end by briefly going over the contents of the individual sections. We will start in Section 5.1 by showing a formality statement for commutative and coassociative bialgebras in $\mathcal{D}(k)$ with homology isomorphic to the homology of D and $k \boxtimes \mathbb{T}$. We will actually define $\mathbb{T}$ and $k \boxtimes \mathbb{T}$ in Section 5.2, and then use the result of Section 5.1 to conclude in Section 5.2.4 that $\mathrm{D} \simeq k \boxtimes \mathbb{T}$ as bialgebras. We show that there are symmetric monoidal equivalences of the form $\mathrm{Fun}(\mathrm{B}\,G, \mathcal{C}) \simeq \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}\boxtimes G}(\mathcal{C})}$ for presentable ∞-categories $\mathcal{C}$ and grouplike associative monoids G in $\mathcal{S}$ in Section 5.3. Finally, we put everything together to obtain the monoidal equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ in Section 5.4.

5.1. Formality of certain $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebras

In this section we show that any two commutative and coassociative bialgebras in $\mathcal{D}(k)$ with homology concentrated in degrees 0 and 1, where it is k, are equivalent as commutative and coassociative bialgebras.

Let us summarize the strategy used to prove this, which was suggested by Achim Krause. Let R be a commutative and coassociative bialgebra with homology as described. Then it suffices to construct another such commutative and coassociative bialgebra independently of R and construct an equivalence between that commutative bialgebra and R.

How could we go about to construct a morphism of commutative bialgebras? Or more generally, of algebras or coalgebras over some ∞-operad? There is one class of algebras where is easy to define morphisms out of, the *free* algebras, using that the free algebra functor is left adjoint to the forgetful functor. Analogously, it is easy to define morphisms of coalgebras into *cofree* algebras. While these concepts are in principle dual to each other, (by passing to opposite ∞-categories), it is in practice easier to work with free algebras than with cofree coalgebras. This is because the theory of free algebras works particularly well when the tensor products are compatible with colimits, see [HA, 3.1.3.5], which is usually the case in the kind of examples that we are interested in. Analogously, we would want the tensor products to be compatible with limits in order to obtain a good theory of cofree coalgebras, but this is usually *not* the case in examples of interest.

The discussion so far points us towards trying to find some kind of free resolution of the commutative and coassociative bialgebra R. Unfortunately, free *commutative* algebras are not quite as easy to describe as free associative algebras[1], as imposing commutativity requires taking certain (homotopy) orbits of actions by the symmetric groups Σ_n. Commutative algebras being more difficult to deal with in some respects is also reflected in the following fact. Let C be a reasonably nice symmetric monoidal model category that one finds in nature. Then it is often the case that $\mathrm{Alg}(\mathsf{C})$ inherits a nice model structure from C such that its underlying ∞-category is the ∞-category of algebras in the underlying ∞-category of C. However it is unreasonable to expect the analogous statement to

[1] [HA, 3.1.3.13] offers a description of free commutative and free associative algebras. We discuss the special case of associative algebras in Proposition E.7.2.1, and will unpack the statement for commutative algebras in the proof of Proposition 5.1.5.3.

hold for *commutative* algebras, which has to do with Σ_n orbits of the action of Σ_n on $X^{\otimes n}$ not necessarily being homotopy orbits[2].

So we would prefer to work with free associative algebras. To do so, we dualize the problem: R is dualizable in the symmetric monoidal ∞-category $\mathcal{D}(k)$, and the functor mapping a dualizable object to its dual,

$$(-)^\vee\colon (\mathcal{D}(k)_{\mathrm{fd}})^{\mathrm{op}} \to \mathcal{D}(k)_{\mathrm{fd}}$$

is symmetric monoidal equivalence and thus induces an equivalence

$$\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}\big(\mathcal{D}(k)_{\mathrm{fd}}\big) = \mathrm{coAlg}\Big(\mathrm{CAlg}\big(\mathcal{D}(k)_{\mathrm{fd}}\big)\Big) \simeq \mathrm{Alg}\Big(\mathrm{coCAlg}\big(\mathcal{D}(k)_{\mathrm{fd}}\big)\Big)^{\mathrm{op}}$$

so that it actually suffices to show that $R^\vee$ is formal.

To do so we will define a diagram

$$\begin{tikzcd}
\underline{B}_2 \arrow[r] \arrow[d] & k \arrow[d] & \underline{B}_4 \arrow[r] \arrow[d] & k \arrow[d] & \\
A_1 \arrow[r] & A_2 \arrow[r] & A_3 \arrow[r] & A_4 \arrow[r] & \ldots \\
& \underline{B}_3 \arrow[r] \arrow[u] & k \arrow[u] & &
\end{tikzcd} \tag{5.1}$$

in $\mathrm{Alg}\Big(\mathrm{coCAlg}\big(\mathcal{D}(k)\big)\Big)$ such that each square is a pushout square and the colimit of $A_1 \to A_2 \to \ldots$ has homology isomorphic to $\mathrm{H}_*(R^\vee)$. Furthermore, every $\underline{B}_n$ as well as A_1 will be free as an associative algebra on the underlying pointed object in $\mathrm{coCAlg}(\mathcal{D}(k))$.

[2]The relevant compatibility result for associative algebras is [HA, 4.1.8.4], and for commutative algebras [HA, 4.5.4.7]. The assumptions necessary for the result on associative algebras are mild enough to usually hold in examples one is interested in. The assumptions made for commutative algebras however include that every cofibration must be a power cofibration (see [HA, 4.5.4.2]). This is a strong condition that one can not expect to hold in general for otherwise nice examples found in nature. For example $\mathsf{Ch}(k)$ with the projective model structure (see Fact 4.1.3.1) does not in general have this property. The chain complex $k[0]$ is cofibrant, so we would need $k[0]$ to be power cofibrant. Let $n > 1$ and let X be the chain complex concentrated in degrees 0 and 1 with $X_0 = X_1 = k^{\oplus n}$, with $\partial_1^X = \mathrm{id}$, and with Σ_n acting by permutation, and let Y be the chain complex concentrated in degrees 0 and 1 with $Y_0 = Y_1 = k$, with $\partial_1^Y = \mathrm{id}$, and with Σ_n acting trivially. There is an Σ_n-equivariant chain morphism $f\colon X \to Y$ that maps a tuple $(a_1, \ldots, a_n)$ to $\sum_{1\leq i\leq n} a_i$. This morphism is an acyclic fibration in the projective model structure on $\mathsf{Ch}(k)^{\mathrm{B}\Sigma_n}$. Let $\varphi\colon k[0] \cong k[0]^{\otimes n} \to Y$ be the inclusion (i. e. the identity in level 0). If $k[0]$ were power cofibrant, then it would need to be possible to lift φ in a Σ_n-equivariant manner to a chain morphism $\overline{\varphi}\colon k[0] \to X$. Suppose $\overline{\varphi}$ is such a lift. Let $\overline{\varphi}(1) = (a_1, \ldots, a_n)$. That $\overline{\varphi}$ is Σ_n equivariant implies that $a \coloneqq a_1 = \cdots = a_n$. We must then have

$$1 = \varphi(1) = f(\overline{\varphi})(1) = f\big((a, \ldots, a)\big) = n \cdot a$$

in k, so n must be invertible in k. But there are many interesting commutative rings that do not contain $\mathbb{Q}$.

It will then be possible to define a morphism $A_1 \to R^\vee$ that is surjective on homology, so that it suffices to show that this morphism can be lifted inductively to each A_n. As k is a zero object in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$ (this will be shown in Remark 5.1.2.9), this amounts to showing that the composites $\underline{B}_n \to A_{n-1} \to R^\vee$ are nullhomotopic in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$. Using freeness, dualizing again, and calculations that exploit the homology of $R^\vee$, it will actually be possible to show that in fact *any* morphism $\underline{B}_n \to R^\vee$ in $\mathrm{Alg}\Big(\mathrm{coCAlg}\big(\mathcal{D}(k)\big)\Big)$ is nullhomotopic.

We now briefly summarize the content of the individual subsections. We start in Section 5.1.1 by discussing dualizable objects in symmetric monoidal ∞-categories and the symmetric monoidal duality functor. In Section 5.1.2 we will then construct diagram (5.1). In order to show that any two morphism $\underline{B}_n \to R^\vee$ are homotopic as discussed above, we will need a formality statement for certain associative algebras, which we show in Section 5.1.3, and of commutative algebras like R as commutative algebras in $\mathcal{D}(k)$, which we will show in Section 5.1.5. As the case of commutative algebras involves arguing about orbits of actions of Σ_n, there is also a short Section 5.1.4 discussing the relationship of orbits of group actions in $\mathcal{D}(k)$ with group homology. The result regarding mapping spaces that we discussed above will then be shown in Section 5.1.6, and everything will be put together in Section 5.1.7 to show formality of R as a commutative and coassociative bialgebra in $\mathcal{D}(k)$.

The subsections do not all depend on all the previous ones. The following diagram shows the dependencies.

$$\begin{tikzcd}
\text{Section 5.1.1} \arrow[dddrr] & \text{Section 5.1.2} \arrow[dddr] & \text{Section 5.1.4} \arrow[d] & \text{Section 5.1.3} \arrow[dl] \arrow[dd] \arrow[dddl] \\
& & \text{Section 5.1.5} \arrow[dr] \arrow[dd] & \\
& & & \text{Section 5.1.6} \arrow[dl] \\
& & \text{Section 5.1.7} &
\end{tikzcd}$$

5.1.1. Duality

In this section we discuss the notion of dualizable objects in symmetric monoidal ∞-categories, and we mostly follow [HA, 4.6.1], [HA, 5.2.1 and 5.2.2], and [Lur18, 3.2]. We start by recalling the definition of dualizable objects.

Definition 5.1.1.1 ([HA, 4.6.1.7, see also 4.6.1.12])**.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let C be an object of $\mathcal{C}$. The object C is called *dualizable* if there exists an object B of $\mathcal{C}$ and morphisms $c\colon \mathbb{1}_\mathcal{C} \to C \otimes B$ and $e\colon B \otimes C \to \mathbb{1}$ such that the composites

$$C \simeq \mathbb{1}_\mathcal{C} \otimes C \xrightarrow{c \otimes \mathrm{id}_C} C \otimes B \otimes C \xrightarrow{\mathrm{id}_C \otimes e} C \otimes \mathbb{1}_\mathcal{C} \simeq C$$

and

$$B \simeq B \otimes \mathbb{1}_{\mathcal{C}} \xrightarrow{\mathrm{id}_B \otimes c} B \otimes C \otimes B \xrightarrow{e \otimes \mathrm{id}_B} \mathbb{1}_{\mathcal{C}} \otimes B \simeq B$$

are homotopic to the identity.

In this case, we call B the *dual* of C, and write $B = C^\vee$; by [HA, 4.6.1.6 and 4.6.1.10] $C^\vee$ as well as c and e are essentially uniquely determined by C. We will also call C together with B, c, e, and homotopies as above a *duality datum*.

We let $\mathcal{C}_{\mathrm{fd}}$ be the full subcategory of $\mathcal{C}$ spanned by the dualizable objects. ♢

Remark 5.1.1.2. It follows easily from the definition that if C and C' are dualizable objects in a symmetric monoidal ∞-category $\mathcal{C}$, with c and e as in Definition 5.1.1.1 exhibiting $C^\vee$ as the dual of C and similarly c' and e' exhibiting $C'^\vee$ as a dual of C', then the compositions

$$\mathbb{1}_{\mathcal{C}} \simeq \mathbb{1}_{\mathcal{C}} \otimes \mathbb{1}_{\mathcal{C}} \xrightarrow{c \otimes c'} C \otimes C^\vee \otimes C' \otimes C'^\vee \xrightarrow{\mathrm{id}_C \otimes \tau \otimes \mathrm{id}_{C'^\vee}} (C \otimes C') \otimes (C^\vee \otimes C'^\vee)$$

and

$$(C^\vee \otimes C'^\vee) \otimes (C \otimes C') \xrightarrow{\mathrm{id}_{C^\vee} \otimes \tau \otimes \mathrm{id}_{C'}} C^\vee \otimes C \otimes C'^\vee \otimes C' \xrightarrow{e \otimes e'} \mathbb{1}_{\mathcal{C}} \otimes \mathbb{1}_{\mathcal{C}} \simeq \mathbb{1}_{\mathcal{C}}$$

exhibit $C^\vee \otimes C'^\vee$ as a dual of $C \otimes C'$, where τ is the symmetry equivalence and $\mathbb{1}_{\mathcal{C}} \simeq \mathbb{1}_{\mathcal{C}} \otimes \mathbb{1}_{\mathcal{C}}$ is the unitality equivalence. In particular the tensor product of two dualizable objects is again dualizable. Furthermore, $\mathbb{1}_{\mathcal{C}}$ is dualizable with dual $\mathbb{1}_{\mathcal{C}}$, so it follows from [HA, 2.2.1.2] that $\mathcal{C}_{\mathrm{fd}}$ inherits a symmetric monoidal structure from $\mathcal{C}$ such that the inclusion can be upgraded to a symmetric monoidal functor. ♢

It is easy to see from the definition that if C is dualizable with dual $C^\vee$, then $C^\vee$ is again dualizable with dual $C^{\vee\vee} \simeq C$. It is also clear from the definition that a symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ maps $\mathcal{C}_{\mathrm{fd}}$ to $\mathcal{D}_{\mathrm{fd}}$ and so restricts to a symmetric monoidal functor $F\colon \mathcal{C}_{\mathrm{fd}} \to \mathcal{D}_{\mathrm{fd}}$. In fact, the following is true.

Fact 5.1.1.3 ([Lur18, 3.2.4])**.** *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the assignment $C \mapsto C^\vee$ sending an object of $\mathcal{C}$ to a dual can be upgraded to an equivalence of symmetric monoidal ∞-categories*

$$(-)^\vee\colon (\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} \to \mathcal{C}_{\mathrm{fd}}$$

with inverse $((-)^\vee)^{\mathrm{op}}$.

Furthermore, this equivalence is compatible with symmetric monoidal functors in the following sense. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. Then there is a commutative diagram of symmetric monoidal functors as follows.

$$\begin{tikzcd}
(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} \arrow[r, "(-)^\vee"] \arrow[d, "F^{\mathrm{op}}"'] & \mathcal{C}_{\mathrm{fd}} \arrow[d, "F"] \\
(\mathcal{D}_{\mathrm{fd}})^{\mathrm{op}} \arrow[r, "(-)^\vee"'] & \mathcal{D}_{\mathrm{fd}}
\end{tikzcd}$$

♧

Remark 5.1.1.4. While the part of the statement of Fact 5.1.1.3 about compatibility with symmetric monoidal functors is not stated explicitly in [Lur18, 3.2.4][3], this becomes clear by going though every step of the proof. In this remark we provide some pointers to the relevant parts of the proof of [Lur18, 3.2.4] as well as the relevant material in [HA, 5.2.1 and 5.2.2] that is relevant for checking this.

First, as F maps dualizable objects to dualizable objects, it suffices to consider the case in which every object in $\mathcal{C}$ and $\mathcal{D}$ is dualizable.

Then the construction of the pairing of ∞-categories $\lambda = \mathrm{pr}_1\colon (\mathcal{C}\times\mathcal{C})\times_{\mathcal{C}}\mathcal{C}_{/\mathbb{1}_{\mathcal{C}}} \to \mathcal{C}\times\mathcal{C}$, as well as its upgrade to a pairing of symmetric monoidal ∞-categories, is compatible with F. Furthermore, the description of left and right universal objects in $(\mathcal{C}\times\mathcal{C})\times_{\mathcal{C}}\mathcal{C}_{/\mathbb{1}_{\mathcal{C}}}$ from the proof of [Lur18, 3.2.4] together with the fact that F preserves duality data implies that the morphism of pairings of ∞-categories induced by F is left and right representable (see [HA, 5.2.1.16]). The symmetric monoidal functor $(-)^\vee$ for $\mathcal{C}$ is constructed in [HA, 5.2.2.25] as a lax symmetric monoidal functor – it is the left duality functor $\mathfrak{D}_\lambda^\otimes$ that uses that λ is left representable. It is shown in [Lur18, 3.2.4] that this functor is actually symmetric monoidal, but as symmetric monoidal functors form a *full* subcategory of lax symmetric monoidal functors [HA, 2.1.3.7] it suffices to consider these functors as lax symmetric monoidal functors when discussing compatibility with F.

So one only needs to check that the construction of the lax symmetric monoidal left duality functors of left representable pairings of symmetric monoidal ∞-categories are compatible with left representable morphisms of left representable pairings of symmetric monoidal ∞-categories. The lax symmetric monoidal functor $\mathfrak{D}_\lambda^\otimes$ for $\mathcal{C}$ is constructed as the composition of the inverse of a symmetric monoidal equivalence $\varphi_{\mathcal{C}}\colon (\mathcal{C}_\lambda^0)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ with the lax monoidal inclusion $\iota_{\mathcal{C}}\colon (\mathcal{C}_\lambda^0)^{\mathrm{op}} \to (\mathcal{C}_\lambda)^{\mathrm{op}}$ and a symmetric monoidal functor $\psi_{\mathcal{C}}\colon (\mathcal{C}_\lambda)^{\mathrm{op}} \to \mathcal{C}$, so it suffices to check that each of those is suitably compatible with F.

The inclusion $\iota_{\mathcal{C}}$ is defined in [HA, 5.2.1.28], and can be upgraded to a lax symmetric monoidal functor by the discussion in [HA, 5.2.2.25] together with [HA, 2.2.1.9]. That it is compatible with F follows from the definition together with $\iota_{\mathcal{D}}^\otimes$ being fully faithful and [HA, 5.2.1.17].

The symmetric monoidal equivalence $\varphi_{\mathcal{C}}$ is the composition of $\iota_{\mathcal{C}}$ with the functor constructed in [HA, 5.2.1.29]. It is clear from definition that this latter functor is compatible with F.

Finally, $\psi_{\mathcal{C}}$ arises from the counit of an adjunction as discussed in [HA, 5.2.2.24] and is thus compatible with F. ♢

Remark 5.1.1.5. Let us give some hints regarding the opposite of the dualization functor being its inverse. Let us – as in Remark 5.1.1.4 – reduce to the case where every object of $\mathcal{C}$ is dualizable. The duality functor discussed so far, in particular in Remark 5.1.1.4 starts with the pairing of ∞-categories $\lambda = \mathrm{pr}_1\colon \mathcal{M} = (\mathcal{C}\times\mathcal{C})\times_{\mathcal{C}}\mathcal{C}_{/\mathbb{1}_{\mathcal{C}}} \to \mathcal{C}\times\mathcal{C}$ that is both left and right representable. This pairing can be upgraded to a pairing of symmetric monoidal ∞-categories $\lambda^\otimes$, and then left representability of λ is used to construct a lax

[3] Functoriality *is* however used and alluded to with [Lur18, 3.2.6].

symmetric monoidal morphism of pairings of symmetric monoidal ∞-categories

$$\begin{tikzcd}
\mathrm{TwArr}(\mathcal{C})^{\otimes} \arrow[r] \arrow[d] & \mathcal{M}^{\otimes} \arrow[d, "\lambda^{\otimes}"] \\
\mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} (\mathcal{C}^{\mathrm{op}})^{\otimes} \arrow[r, "\mathrm{id}_{\mathcal{C}^{\otimes}} \times \mathfrak{D}_{\lambda}^{\otimes}"'] & \mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathcal{C}^{\otimes}
\end{tikzcd}$$

where the bottom functor is on the second factor precisely the lax symmetric monoidal left duality functor that we are interested in and called $(-)^{\vee}$. See [HA, 5.2.2.25] and also Remark 5.1.1.4.

Now the important point is that the underlying morphism of pairings of ∞-categories is right representable. If we assume this for the moment, then we can use functoriality of *right* duality functors (which can be shown completely analogously to the case of left duality functors sketched in Remark 5.1.1.4) to obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[r, "\mathrm{id}"] \arrow[d, "(\mathfrak{D}_{\lambda}^{\mathrm{op}})^{\otimes}"'] & \mathcal{C}^{\otimes} \arrow[d, "\mathrm{id}"] \\
(\mathcal{C}^{\mathrm{op}})^{\otimes} \arrow[r, "\mathfrak{D}_{\lambda}'^{\otimes}"'] & \mathcal{C}^{\otimes}
\end{tikzcd}$$

where the top horizontal functor is the right duality functor of $\mathrm{TwArr}(\mathcal{C})^{\otimes}$, which can be identified with the identity. This shows that the opposite of the right duality functor of λ (which is symmetric monoidal by analogous considerations as the left duality functor) is an inverse to the left duality functor, as symmetric monoidal functors.

Next, there is a commutative diagram as follows.

$$\begin{tikzcd}
\big((\mathcal{C} \times \mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/\mathbb{1}_{\mathcal{C}}}\big)^{\otimes} \arrow[r, "\tau'"] \arrow[dd, "\mathrm{pr}_1^{\otimes}"'] & \big((\mathcal{C} \times \mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/\mathbb{1}_{\mathcal{C}}}\big)^{\otimes} \arrow[d, "\mathrm{pr}_1^{\otimes}"] \\
 & \mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathcal{C}^{\otimes} \arrow[d, "\tau^{\otimes}"] \\
\mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathcal{C}^{\otimes} \arrow[r, "\mathrm{id}"'] & \mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathcal{C}^{\otimes}
\end{tikzcd}$$

where τ' maps a tuple $(C, D, C \otimes D \to \mathbb{1})$ to $(D, C, D \otimes C \simeq C \otimes D \to \mathbb{1})$, where we use the symmetry equivalence of $\mathcal{C}$, and τ swaps the two factors. As $\mathrm{pr}_1^{\otimes}$ was a pairing of symmetric monoidal ∞-categories with left representable underlying pairing, one can see that the composition on the right is a pairing of symmetric monoidal ∞-categories with right representable underlying pairing, and the right duality functor can be identified with the left duality functor of $\mathrm{pr}_1^{\otimes}$. Furthermore, it follows from the description of left and right universal objects in [Lur18, 3.2.4] that the morphism of pairings encoded in the diagram is right representable. By functoriality of right duality functors we thus obtain a commutative diagram of symmetric monoidal functors

$$\begin{tikzcd} (\mathcal{C}^{\mathrm{op}})^{\otimes} \arrow[r, "\mathfrak{D}'^{\otimes}_{\lambda}"] \arrow[d, "\mathrm{id}"'] & \mathcal{C}^{\otimes} \arrow[d, "\mathrm{id}"] \\ (\mathcal{C}^{\mathrm{op}})^{\otimes} \arrow[r, "\mathfrak{D}^{\otimes}_{\lambda}"'] & \mathcal{C}^{\otimes} \end{tikzcd}$$

that shows that $\mathfrak{D}^{\otimes}_{\lambda} \simeq \mathfrak{D}'^{\otimes}_{\lambda}$ as lax symmetric monoidal (and hence also as symmetric monoidal) functors. As we previously obtained an equivalence $(\mathfrak{D}'^{\mathrm{op}}_{\lambda})^{\otimes} \simeq (\mathfrak{D}^{-1}_{\lambda})^{\otimes}$, this shows that $(\mathfrak{D}^{\mathrm{op}}_{\lambda})^{\otimes} \simeq (\mathfrak{D}^{-1}_{\lambda})^{\otimes}$.

Finally, let us say a few words on why, given a perfect[4] pairing of ∞-categories[5] $\lambda\colon \mathcal{M} \to \mathcal{C} \times \mathcal{D}$, the morphism of pairings

$$\begin{tikzcd} \mathrm{TwArr}(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\lambda"] \\ \mathcal{C} \times (\mathcal{C}^{\mathrm{op}}) \arrow[r, "\mathrm{id}_{\mathcal{C}} \times \mathfrak{D}_{\lambda}"'] & \mathcal{C} \times \mathcal{D} \end{tikzcd}$$

constructed in [HA, 5.2.2.24 and 5.2.2.25] is right representable, which means that the top horizontal functor needs to preserve right universal objects, see [HA, 5.2.1.13 and 5.2.1.8]. To start, we first see that by unwrapping the definition[6] we have to show that the composition of two morphisms of pairings of ∞-categories as depicted in the follow diagram preserves right universal objects.

$$\begin{tikzcd} \mathrm{TwArr}^0_{\lambda}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{TwArr}_{\lambda}(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\lambda"] \\ \mathcal{C} \times \mathcal{C}^{0\mathrm{op}}_{\lambda} \arrow[r] & \mathcal{C} \times \mathcal{C}^{\mathrm{op}}_{\lambda} \arrow[r] & \mathcal{C} \times \mathcal{D} \end{tikzcd}$$

Unpacking the definition using [HA, 5.2.1.24] and in particular [HA, 5.2.1.28] we see that we can describe objects of $\mathcal{C}^{\mathrm{op}}_{\lambda}$ as tuples (C_r, D, ϕ), with C_r an object of $\mathcal{C}^{\mathrm{op}}$, D an object of $\mathcal{D}$, and ϕ a morphism $D \to \mathfrak{D}_{\lambda}(C_r)$ in $\mathcal{D}$. The fiber in $\mathrm{TwArr}_{\lambda}(\mathcal{C})$ of a pair $(C_l, (C_r, D, \phi))$ in $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}_{\lambda}$ can be identified with $\mathrm{Map}_{\mathcal{C}}(C_l, C_r)$. An object in $\mathrm{TwArr}_{\lambda}(\mathcal{C})$ that is given by a morphism $f\colon C_l \to C_r$ as just described is then mapped to the object in $\mathcal{M}$ described as follows. As λ is left representable, there is a left universal object M_r over C_r in $\mathcal{M}$, lying over $(C_r, \mathfrak{D}_{\lambda}(C_r))$. A λ-cartesian lift of the morphism (f, ϕ) is then a morphism $M_l \to M_r$ in $\mathcal{M}$ where M_l lies over (C_l, D). f is mapped to this object M_l.

By definition (see [HA, 5.2.1.28]) $\mathcal{C}^{0\mathrm{op}}_{\lambda}$ is the full subcategory of $\mathcal{C}^{\mathrm{op}}_{\lambda}$ spanned by those tuples where ϕ is an equivalence, and the left square in the above commutative diagram is a pullback. One can then see that an object in $\mathrm{TwArr}^0_{\lambda}(\mathcal{C})$ is right universal precisely

[4] See [HA, 5.2.1.20 and 5.2.1.22].
[5] Of which the λ we discussed so far is an example by the proof of [Lur18, 3.2.4].
[6] See [HA, 5.2.2.24 and 5.2.2.25] and also Remark 5.1.1.4.

if the associated morphism $f: C_l \to C_r$ as before is an equivalence. This then implies that (f, ϕ) will be an equivalence, so the λ-cartesian lift $M_l \to M_r$ is also an equivalence, and hence M_l is left universal, as M_r is so by assumption. But as λ is perfect, this means that M_l is also right universal, see [HA, 5.2.1.22]. ♢

We make a bit more explicit how $(-)^\vee$ applies to morphisms in the following remark.

Remark 5.1.1.6. Let $f: C \to D$ be a morphism of dualizable objects in a symmetric monoidal ∞-category $\mathcal{C}$. Then the functor $(-)^\vee$ from Fact 5.1.1.3 sends f to a morphism $f^\vee: D^\vee \to C^\vee$. Unpacking the definitions[7], one can see that this morphism fits into a commutative diagram as follows

$$\begin{tikzcd}
D^\vee \arrow[rr, "f^\vee"] \arrow[d, "\simeq"'] & & C^\vee \arrow[d, "\simeq"] \\
D^\vee \otimes \mathbb{1}_{\mathcal{C}} \arrow[d, "\mathrm{id}\otimes c"'] & & \mathbb{1}_{\mathcal{C}} \otimes C^\vee \\
D^\vee \otimes C \otimes C^\vee \arrow[rr, "\mathrm{id}\otimes f\otimes\mathrm{id}"'] & & D^\vee \otimes D \otimes C^\vee \arrow[u, "e\otimes\mathrm{id}"']
\end{tikzcd}$$

where the top two vertical equivalences are the unitality equivalences of $\mathcal{C}$, the morphism c takes part in a duality datum for C, and e takes part in a duality datum for D. ♢

Applying Fact 5.1.1.3 to the symmetric monoidal functor $\gamma: \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ (see Proposition 4.3.2.1) we obtain the following.

Corollary 5.1.1.7. *There is a commutative diagram of symmetric monoidal functors as follows*

$$\begin{tikzcd}
\left(\mathsf{Ch}(k)^{\mathrm{cof}}_{\mathrm{fd}}\right)^{\mathrm{op}} \arrow[r, "(-)^\vee"] \arrow[d, "\gamma^{\mathrm{op}}"'] & \mathsf{Ch}(k)^{\mathrm{cof}}_{\mathrm{fd}} \arrow[d, "\gamma"] \\
\left(\mathcal{D}(k)_{\mathrm{fd}}\right)^{\mathrm{op}} \arrow[r, "(-)^\vee"'] & \mathcal{D}(k)_{\mathrm{fd}}
\end{tikzcd}$$

and both horizontal functors are equivalences. ♡

Example 5.1.1.8. Consider the commutative and cocommutative bialgebra D in $\mathsf{Ch}(k)$ from Construction 4.2.1.1. Its underlying chain complex is $k \cdot \{1\} \oplus k \cdot \{\mathrm{d}\}$ with 1 in degree 0 and d in degree 1. This chain complex is dualizable with dual[8] $k \cdot \{1\} \oplus k \cdot \{\mathrm{d}^\vee\}$ with 1 in degree 0 and $\mathrm{d}^\vee$ in degree -1.

By Fact 5.1.1.3 the commutative and cocommutative bialgebra structure on D induces again a commutative and cocommutative bialgebra structure on $\mathrm{D}^\vee$, with unit the basis element we called 1 in degree 0 (see Remark 5.1.1.6). The rest of the bialgebra structure is then already uniquely determined just as in Construction 4.2.1.1, with in particular $\Delta(\mathrm{d}^\vee) = 1 \otimes \mathrm{d}^\vee + \mathrm{d}^\vee \otimes 1$. ♢

[7] See in particular [Lur18, 3.2.4] and [HA, 5.2.1.9].
[8] A duality datum is given by defining e by $e(1 \otimes 1) = 1 = e(\mathrm{d}^\vee \otimes \mathrm{d})$ and c by $c(1) = \mathrm{d} \otimes \mathrm{d}^\vee + 1 \otimes 1$.

As $(-)^\vee$ is a symmetric monoidal equivalence, it induces an equivalence that converts algebras into coalgebras and vice versa, as we note next.

Remark 5.1.1.9. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $\mathcal{O}$ and $\mathcal{O}'$ two ∞-operads. Note that the symmetric monoidal duality functor

$$(-)^\vee \colon (\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} \to \mathcal{C}_{\mathrm{fd}}$$

from Fact 5.1.1.3 induces a symmetric monoidal equivalence

$$\begin{aligned} \mathrm{BiAlg}_{\mathcal{O},\mathcal{O}'}(\mathcal{C}_{\mathrm{fd}}) &\simeq \mathrm{Alg}_{\mathcal{O}'}\big(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}}\big)^{\mathrm{op}} \\ &\xrightarrow{(-)^\vee} \mathrm{Alg}_{\mathcal{O}'}\Big(\mathrm{Alg}_{\mathcal{O}}\big(\mathcal{C}_{\mathrm{fd}}^{\mathrm{op}}\big)^{\mathrm{op}}\Big)^{\mathrm{op}} \simeq \mathrm{Alg}_{\mathcal{O}'}\big(\mathrm{coAlg}_{\mathcal{O}}(\mathcal{C}_{\mathrm{fd}})\big)^{\mathrm{op}} \end{aligned} \qquad \diamondsuit$$

5.1.2. Construction of a resolution

The goal of this section is to construct diagram (5.1) that was discussed in the introduction to Section 5.1, and we refer to there for motivation. We will construct such a diagram in $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$ first and then show that its image in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$ has the required properties. While we are still discussing algebras and coalgebras in the symmetric monoidal 1-category $\mathsf{Ch}(k)$, one should keep in mind that, as explained in Section 3.3, there is a canonical isomorphism

$$\mathrm{Alg}\Big(\mathrm{coCAlg}\big(\mathsf{Ch}(k)\big)\Big) \cong \mathrm{coCAlg}\Big(\mathrm{Alg}\big(\mathsf{Ch}(k)\big)\Big) = \mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Comm}}\big(\mathsf{Ch}(k)\big)$$

so we will be justified in identifying these categories and talking about objects as cocommutative bialgebras.

Let us now briefly go over the content of the subsections. In order to make it easier to talk about certain differential graded algebras that have free underlying $\mathbb{Z}$-graded k-algebras, we start in Section 5.1.2.1 by introducing some convenient notation. We will then begin the actual construction of diagram (5.1) in Section 5.1.2.2 by constructing a sequence of cocommutative bialgebras

$$A_0 \to A_1 \to A_2 \to \dots$$

in $\mathsf{Ch}(k)$. Section 5.1.2.3 will then be devoted to calculating the homology of $\mathrm{colim}_n A_n$. In Section 5.1.2.4 we will construct pushout diagrams

$$\begin{tikzcd} \underline{B}_n \arrow[r] \arrow[d] & B_n \arrow[d] \\ A_{n-1} \arrow[r] & A_n \end{tikzcd}$$

of cocommutative bialgebras in $\mathsf{Ch}(k)$. The cocommutative bialgebra B_n itself is not isomorphic to k, but maps to a cocommutative bialgebra in $\mathcal{D}(k)$ that is equivalent to k, as we will see in Section 5.1.2.5. We then combine the previous results in Section 5.1.2.6 to describe the induced diagram (5.1) in $\mathrm{Alg}\Big(\mathrm{coCAlg}\big(\mathcal{D}(k)\big)\Big)$ and show that it has the required properties. Finally, in Section 5.1.2.7 we describe A_1 and $\underline{B}_n$ as free associative algebras on underlying pointed cocommutative coalgebras.

5.1.2.1. Notation for freely generated differential graded algebras

In this short section we introduce some notation for differential graded algebras whose underlying $\mathbb{Z}$-graded k-algebra is free associative.

Notation 5.1.2.1. Let X be a set and let[9] $\deg_{\mathsf{Ch}}(x)$ be an integer for every element x of X. Then we can form a $\mathbb{Z}$-graded k-module with basis X as follows.

$$k \cdot X \coloneqq \bigoplus_{x \in X} k[\deg_{\mathsf{Ch}}(x)]$$

We will denote the free associative $\mathbb{Z}$-graded k-algebra generated by $k \cdot X$ by

$$\mathrm{Free}^{\mathsf{Assoc}}(X)$$

and if $X = \{ x_1, x_2, \ldots \}$ then we will often write

$$\mathrm{Free}^{\mathsf{Assoc}}(x_1, x_2, \ldots) = \mathrm{Free}^{\mathsf{Assoc}}(X)$$

instead. A basis of $\mathrm{Free}^{\mathsf{Assoc}}(X)$ is given by elements of the form $x_{i_1} \cdots x_{i_n}$ for $n \geq 0$[10] with x_{i_j} elements of X for $1 \leq j \leq n$.

We can make $\mathrm{Free}^{\mathsf{Assoc}}(X)$ into an associative differential graded algebra by furnishing it with the zero boundary operator. But we will sometimes want to define associative differential graded algebras that have a free underlying $\mathbb{Z}$-graded k-algebra, but *do* have nontrivial boundaries. So assume that for every element x of X we are given an element $f(x)$ of $\mathrm{Free}^{\mathsf{Assoc}}(X)_{\deg_{\mathsf{Ch}}(x)-1}$. Then we use the notation

$$\mathrm{Free}^{\mathsf{Assoc}}\big(X \,\big|\, \partial(x) = f(x)\big)$$

for the differential graded k-algebra with underlying $\mathbb{Z}$-graded k-algebra $\mathrm{Free}^{\mathsf{Assoc}}(X)$ and boundary operator (uniquely) extended by k-linearity and the Leibniz rule from the prescription $\partial(x) = f(x)$ for every element x of X. This does not in general actually define a differential graded algebra, as in general there is no reason for the boundary operator to square to 0, so if we use this notation we will need to check that $\partial(\partial(x)) = 0$ for every element x of X.

Sometimes we will omit $\partial(x)$ in this notation for some elements x of X, in which case this is to be interpreted as $\partial(x) = 0$. $\diamondsuit$

5.1.2.2. Construction of A as a directed colimit

In this section we construct a sequence of cocommutative bialgebras

$$A_0 \to A_1 \to A_2 \to \ldots$$

in $\mathsf{Ch}(k)$ and describe its colimit.

[9] Ultimately we want to define differential graded algebras generated by X, and in this differential graded algebra the chain degree of an element x of X will of course be exactly what we (prematurely, to avoid introducing more temporary notation) call $\deg_{\mathsf{Ch}}(x)$ here, making this notation in the end compatible with the notation in Definition 4.1.1.1.

[10] If $n = 0$ we interpret the product as 1.

Construction 5.1.2.2. We will construct a cocommutative bialgebra in chain complexes A_n for every integer $n \geq 0$. Using Notation 5.1.2.1, we define the underlying differential graded k-algebra of A_n as

$$A_n := \mathrm{Free}^{\mathsf{Assoc}}\left(y_1, \dots, y_n \,\middle|\, \partial(y_k) = \sum_{i+j=k} y_i y_j\right)$$

where $\deg_{\mathsf{Ch}(k)}(y_i) = -1$ and where the sum should of course be interpreted to only be taken over those i and j for which y_i and y_j are defined[11]. For this to actually define a differential graded algebra structure the definition of ∂ needs to satisfy $\partial\big(\partial(y_l)\big) = 0$ for any $1 \leq l \leq n$, which is the case as the following basic calculation shows.

$$\begin{aligned}
\partial\big(\partial(y_l)\big) &= \partial\left(\sum_{i+j=l} y_i y_j\right) \\
&= \sum_{i+j=l} \partial(y_i) y_j - \sum_{i+j=l} y_i \partial(y_j) \\
&= \sum_{i+j+k=l} y_i y_j y_k - \sum_{i+j+k=l} y_i y_j y_k = 0
\end{aligned}$$

We next define a cocommutative coalgebra structure on A_n. As the underlying graded k-algebra of A_n is free, we can define the counit $\epsilon\colon A_n \to k$ as well as the comultiplication $\Delta\colon A_n \to A_n \otimes A_n$ to be the morphisms of graded k-algebras determined by

$$\begin{aligned}
\epsilon(y_k) &= 0 \\
\Delta(y_k) &= 1 \otimes y_k + y_k \otimes 1
\end{aligned}$$

for $1 \leq k \leq n$. By definition comultiplication and counit are morphisms of algebras, so if this defines a cocommutative coalgebra structure in $\mathsf{Ch}(k)$, then this will make A_n into a cocommutative bialgebra in $\mathsf{Ch}(k)$ as claimed.

As counit and unit of the presumptive coalgebra structure are morphisms of algebras, it suffices to check compatibility of ϵ and Δ with ∂, coassociativity, counitality, and cocommutativity on multiplicative generators. For example for the comultiplication being a morphism of chain complexes we can calculate

$$\begin{aligned}
\Delta\big(\partial(y_k)\big) &= \Delta\left(\sum_{i+j=k} y_i y_j\right) \\
&= \sum_{i+j=k} (1 \otimes y_i + y_i \otimes 1) \cdot \big(1 \otimes y_j + y_j \otimes 1\big) \\
&= \sum_{i+j=k} 1 \otimes y_i y_j - y_j \otimes y_i + y_i \otimes y_j + y_i y_j \otimes 1
\end{aligned}$$

[11] So in particular, $\partial(y_1) = 0$ and $\partial(y_2) = y_1^2$.

$$\begin{aligned} &= \sum_{i+j=k} 1 \otimes y_i y_j + y_i y_j \otimes 1 \\ &= \partial(1 \otimes y_k + y_k \otimes 1) \\ &= \partial(\Delta(y_k)) \end{aligned}$$

and as another example the following calculation verifies coassociativity.

$$(\mathrm{id} \otimes \Delta)\big(\Delta(y_k)\big) = 1 \otimes 1 \otimes y_k + 1 \otimes y_k \otimes 1 + y_k \otimes 1 \otimes 1 = (\Delta \otimes \mathrm{id})\big(\Delta(y_k)\big)$$

Compatibility of ϵ with ∂, counitality, and cocommutativity are similarly immediate.

We can completely analogously define a cocommutative bialgebra A in $\mathsf{Ch}(k)$ as

$$A \coloneqq \mathrm{Free}^{\mathsf{Assoc}}\left(y_1, y_2, \ldots \;\middle|\; \partial(y_k) = \sum_{i+j=k} y_i y_j \right)$$

with counitality and comultiplication defined exactly as for A_n. ◊

Remark 5.1.2.3. There is a commutative diagram of cocommutative bialgebras in $\mathsf{Ch}(k)$ as follows

$$\begin{tikzcd} A_0 \arrow[r] \arrow[drr] & A_1 \arrow[r] \arrow[dr] & A_2 \arrow[r] \arrow[d] & \cdots \arrow[dl] \\ & & A & \end{tikzcd}$$

where all morphisms are the obvious inclusions. This diagram exhibits A as the colimit of the directed system of inclusions in the top row, as can be seen using that directed colimits of cocommutative bialgebras in $\mathsf{Ch}(k)$ are calculated on underlying chain complexes by [HA, 3.2.2.5] and [HA, 3.2.3.1] in combination with [HTT, 5.5.8.3]. ◊

5.1.2.3. Homology of A

As described in the introduction to Section 5.1 we will later construct a morphism from the object in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$ represented by A to $R^\vee$, the dual of a commutative bialgebra in $\mathcal{D}(k)$ with prescribed homology. From the construction it will be clear that the induced morphism on homology is surjective, and we will want to conclude that the morphism is an equivalence, or equivalently that the induced morphism on homology is an isomorphism. In order to do this we should calculate the homology of A, which we do in this section.

Proposition 5.1.2.4. *The chain complex A constructed in Construction 5.1.2.2 has homology*

$$\mathrm{H}_n(A) \cong \begin{cases} k & \text{if } n = 0 \text{ or } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

and the unit 1 of A and y_1 are representatives of elements forming a basis of $\mathrm{H}_0(A)$ and $\mathrm{H}_1(A)$. ♡

Proof. A is freely generated as a $\mathbb{Z}$-graded k-module by words in the multiplicative generators y_i, i. e. by elements of the form

$$y_{i_1} \cdots y_{i_n}$$

with $n \geq 0$ (for $n = 0$ we interpret the product as 1) and i_j elements of $\mathbb{Z}_{\geq 1}$. For $m \geq 0$, let $A(m)$ be the sub $\mathbb{Z}$-graded k-module generated by elements of this form with $\sum_{j=1}^{n} i_j = m$. It follows directly from the definitions that $A(m)$ is in fact a subcomplex of A, and that furthermore

$$A \cong \bigoplus_{m \geq 0} A(m)$$

in $\mathsf{Ch}(k)$.

Note that $A(0)$ and $A(1)$ are both concentrated in a single degree and of rank 1, with $A(0)$ having a basis formed by 1 in degree 0 and $A(1)$ having a basis formed by y_1 in degree -1. To finish the proof it thus suffices to show that $A(m)$ is acyclic for $m > 1$.

For this, we fix $m > 1$ and define a chain homotopy h on $A(m)$ by extending k-linearly from the following definition on the basis.

$$h(y_{i_1} \cdots y_{i_n}) = \begin{cases} y_{i_2+1} y_{i_3} \cdots y_{i_n} & \text{if } n > 1 \text{ and } i_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

We can now check that h is indeed a contracting homotopy by checking on basis elements. For this we distinguish three cases. First, the only basis element for which $n \leq 1$ is y_m, and for it we have the following calculation.

$$(\partial h + h\partial)(y_m) = \partial(0) + h\left(\sum_{i+j=m} y_i y_j\right) = y_m$$

Next, for those basis elements for which $n > 1$ and $i_1 = 1$, we obtain the following.

$$\begin{aligned}
&(\partial h + h\partial)(y_1 y_{i_2} \cdots y_{i_n}) \\
&= \partial(y_{i_2+1} \cdots y_{i_n}) + h\left(-y_1\left(\sum_{k+l=i_2} y_k y_l y_{i_3} \cdots y_{i_n}\right) + y_1 y_{i_2} \partial(y_{i_3} \cdots y_{i_n})\right) \\
&= \left(\sum_{k+l=i_2+1} y_k y_l y_{i_3} \cdots y_{i_n}\right) - y_{i_2+1}\partial(y_{i_3} \cdots y_{i_n}) \\
&\quad - \left(\sum_{k+l=i_2} y_{k+1} y_l y_{i_3} \cdots y_{i_n}\right) + y_{i_2+1}\partial(y_{i_3} \cdots y_{i_n})
\end{aligned}$$

$$
\begin{aligned}
&= y_1 y_{i_2} \cdots y_{i_n} + \left(\sum_{k+l=i_2} y_{k+1} y_l y_{i_3} \cdots y_{i_n} \right) - y_{i_2+1} \partial(y_{i_3} \cdots y_{i_n}) \\
&\quad - \left(\sum_{k+l=i_2} y_{k+1} y_l y_{i_3} \cdots y_{i_n} \right) + y_{i_2+1} \partial(y_{i_3} \cdots y_{i_n}) \\
&= y_1 y_{i_2} \cdots y_{i_n}
\end{aligned}
$$

Finally, for the other basis elements, i. e. those with $n > 1$ and $i_1 \neq 1$, we have the following calculation.

$$
\begin{aligned}
&(\partial h + h\partial)(y_{i_1} y_{i_2} \cdots y_{i_n}) \\
&= h\left(\left(\sum_{k+l=i_1} y_k y_l y_{i_2} \cdots y_{i_n} \right) - y_{i_1} \partial(y_{i_2} \cdots y_{i_n}) \right) \\
&= y_{i_1-1+1} y_{i_2} \cdots y_{i_n} + \sum_{k+l=i_1, k>1} 0 + 0 \\
&= y_{i_1} y_{i_2} \cdots y_{i_n}
\end{aligned}
$$

□

5.1.2.4. Construction of A_{n+1} from A_n

In order to be able to lift a morphism from A_{n-1} to a morphism from A_n, we will describe A_n as a pushout of A_{n-1} in this section. We start by constructing the relevant commutative square, and show that this square is a pushout square at the end of this section.

Construction 5.1.2.5. Let $n \geq 1$. Using Notation 5.1.2.1 we define a morphism of differential graded algebras as

$$
\underline{B}_n = \mathrm{Free}^{\mathsf{Assoc}}\left(\underline{y}_n\right) \to \mathrm{Free}^{\mathsf{Assoc}}\left(\underline{y}_n, y_n \,\middle|\, \partial(y_n) = \underline{y}_n\right) = B_n
$$

with $\deg_{\mathsf{Ch}}(y_n) = -1$ and $\deg_{\mathsf{Ch}}(\underline{y}_n) = -2$.

We can upgrade this morphism of differential graded k-algebras to a morphism of cocommutative bialgebras in $\mathsf{Ch}(k)$, by defining counit ϵ and comultiplication Δ as follows on the multiplicative basis.

$$
\begin{aligned}
\epsilon(y_n) &= 0 \\
\epsilon\left(\underline{y}_n\right) &= 0 \\
\Delta(y_n) &= 1 \otimes y_n + y_n \otimes 1 \\
\Delta\left(\underline{y}_n\right) &= 1 \otimes \underline{y}_n + \underline{y}_n \otimes 1
\end{aligned}
$$

Checking that ϵ and Δ are compatible with ∂ as well as coassociativity, counitality, and cocommutativity are similar to Construction 5.1.2.2.

We can define a morphism of differential graded algebras

$$B_n \to A_n$$

by sending y_n to y_n and $\underline{y}_n$ to $\partial(y_n)$. It is easy to check that this is also compatible with the coalgebra structure, making this a morphism of cocommutative coalgebras.

Finally, the restriction to $\underline{B}_n$ factors through A_{n-1}, so that we obtain a commutative diagram

$$\begin{tikzcd} \underline{B}_n \arrow[r] \arrow[d] & B_n \arrow[d] \\ A_{n-1} \arrow[r] & A_n \end{tikzcd} \tag{5.2}$$

in $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$. ♢

In order to show that (5.2) is a pushout square, we will need to two preliminary results that allow us to detect colimits in $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$ on underlying algebras in $\mathsf{Ch}(k)$.

Proposition 5.1.2.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $\mathcal{O}$ be a reduced*[12] *∞-operad with $\mathfrak{o}$ the essentially unique object in the underlying ∞-category $\mathcal{O}$. Assume that $\mathcal{C}$ is cocomplete and the tensor product preserves colimits separately in each variable.*

Then $\mathrm{coAlg}_{\mathcal{O}}(\mathcal{C})$ is cocomplete and the induced symmetric monoidal structure on $\mathrm{coAlg}_{\mathcal{O}}(\mathcal{C})$ is also compatible with colimits. ♡

Proof. $\mathrm{coAlg}_{\mathcal{O}}(\mathcal{C})$ is cocomplete by [HA, 3.2.2.5]. Furthermore, the forgetful functor

$$\mathrm{ev}_{\mathfrak{o}}\colon \mathrm{coAlg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$$

is symmetric monoidal by Proposition E.4.2.3, conservative by [HA, 3.2.2.6] and preserves colimits by [HA, 3.2.2.5]. It thus follows that the symmetric monoidal structure on $\mathrm{coAlg}_{\mathcal{O}'}(\mathcal{C})$ is also compatible with colimits. □

Proposition 5.1.2.7. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $\mathcal{O}$ and $\mathcal{O}'$ be ∞-operads. Assume that $\mathcal{O}'$ is reduced and let $\mathfrak{o}$ be the essentially unique object in $\mathcal{O}'$.*

Then the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\mathfrak{o}})\colon \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{coAlg}_{\mathcal{O}'}(\mathcal{C})\big) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \tag{5.3}$$

is conservative, i. e. reflects equivalences.

Assume additionally that $\mathcal{C}$ is cocomplete and the tensor product preserves colimits separately in each variable. Then $\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\mathfrak{o}})$ preserves colimits. In particular, also being conservative, $\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\mathfrak{o}})$ detects colimits. ♡

[12] See [HA, 2.3.4.1] for a definition.

Proof. The symmetric monoidal forgetful functor[13]

$$\mathrm{ev}_{\mathfrak{o}}\colon \mathrm{coAlg}_{\mathcal{O}'}(\mathcal{C}) \to \mathcal{C}$$

is by [HA, 3.2.2.6] conservative and preserves colimits by [HA, 3.2.2.5]. It thus follows from Proposition E.3.4.1 and Proposition E.7.3.1[14] that the forgetful functor (5.3) is also conservative and colimit-preserving, and hence detects colimits. □

Proposition 5.1.2.8. *The commutative square* (5.2) *constructed in Construction 5.1.2.5 is a pushout diagram in* $\mathrm{Alg}\big(\mathrm{coCAlg}(\mathsf{Ch}(k))\big)$ ♡

Proof. It follows from Proposition 5.1.2.7[15] that the forgetful functor from cocommutative bialgebras to underlying algebras

$$\mathrm{Alg}\big(\mathrm{ev}_{\langle 1\rangle}\big)\colon \mathrm{Alg}\big(\mathrm{coCAlg}(\mathsf{Ch}(k))\big) \to \mathrm{Alg}\big(\mathsf{Ch}(k)\big)$$

detects colimits. It thus suffices to show that the underlying square of differential graded k-algebras is a pushout square.

The functor from chain complexes of k-modules to $\mathbb{Z}$-graded k-modules is conservative, symmetric monoidal, and preserves colimits. It thus follows from Proposition E.3.4.1 and Proposition E.7.3.1 just as in the proof of Proposition 5.1.2.7 that the forgetful functor from differential graded k-algebras to $\mathbb{Z}$-graded k-algebras detects colimits, so it actually suffices to show that the underlying commutative square of $\mathbb{Z}$-graded k-algebras is a pushout square.

There is a pushout diagram of $\mathbb{Z}$-graded k-modules

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & k \cdot \{\, y_n \,\} \arrow[d] \\
k \cdot \left\{\, \underline{y}_n \,\right\} \arrow[r] & k \cdot \left\{\, \underline{y}_n, y_n \,\right\}
\end{tikzcd}$$

where all morphisms are the obvious inclusions, which induces the pushout diagram of $\mathbb{Z}$-graded k-algebras at the top of the following commutative diagram

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & \mathrm{Free}^{\mathsf{Assoc}}(y_n) \arrow[d] \\
\mathrm{Free}^{\mathsf{Assoc}}\left(\underline{y}_n\right) \arrow[r] \arrow[d] & \mathrm{Free}^{\mathsf{Assoc}}\left(\underline{y}_n, y_n\right) \arrow[d] \\
\mathrm{Free}^{\mathsf{Assoc}}(y_1, \ldots, y_{n-1}) \arrow[r] & \mathrm{Free}^{\mathsf{Assoc}}(y_1, \ldots, y_n)
\end{tikzcd}$$

[13] See Proposition E.4.2.3.

[14] $\mathrm{coAlg}_{\mathcal{O}'}(\mathcal{C})$ is cocomplete and its symmetric monoidal structure is compatible with colimits by Proposition 5.1.2.6.

[15] The tensor product of $\mathsf{Ch}(k)$ preserves colimits in each variable separately as the symmetric monoidal structure is closed by Definition 4.1.2.1.

where all morphisms are the obvious inclusions. We have to show that the bottom square is a pushout square. As the top square is a pushout square, it suffices to show that the big outer square is a pushout.

But the big outer square is $\mathrm{Free}^{\mathsf{Assoc}}$ applied to the following pushout diagram of $\mathbb{Z}$-graded k-modules

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & k \cdot \{ y_n \} \arrow[d] \\
k \cdot \{ y_1, \dots, y_{n-1} \} \arrow[r] & k \cdot \{ y_1, \dots, y_n \}
\end{tikzcd}$$

and is thus a pushout diagram. □

5.1.2.5. Identification of B_n up to quasiisomorphism

In this section we show that the cocommutative bialgebras B_n defined in Construction 5.1.2.5 are quasiisomorphic to k. We start by remarking that k is a zero object in $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$.

Remark 5.1.2.9. Let $\mathcal{C}$ be a cocomplete and complete symmetric monoidal ∞-category such that the tensor product is compatible with colimits in each variable. By [HA, 3.2.2.4 and 3.2.3.1], $\mathrm{coCAlg}(\mathcal{C})$ is complete and cocomplete, and the induced symmetric monoidal structure is again compatible with colimits by Proposition 5.1.2.6. Another application of [HA, 3.2.2.4 and 3.2.3.1] yields that $\mathrm{Alg}\big(\mathrm{coCAlg}(\mathcal{C})\big)$ is complete and cocomplete.

By [HA, 3.2.1.8], an initial object is given by the monoidal unit. We want to show that this object is also final and thus a zero object in $\mathrm{Alg}\big(\mathrm{coCAlg}(\mathcal{C})\big)$. As the forgetful functor

$$\mathrm{ev}_{\mathfrak{a}} \colon \mathrm{Alg}\big(\mathrm{coCAlg}(\mathcal{C})\big) \to \mathrm{coCAlg}(\mathcal{C})$$

detects limits by [HA, 3.2.2.4] and is also symmetric monoidal by Proposition E.4.2.3, it suffices to show that the monoidal unit is a final object in $\mathrm{coCAlg}(\mathcal{C})$, which again follows from [HA, 3.2.1.8] (and passing to opposite categories twice). ◇

Proposition 5.1.2.10. *Let $n \geq 1$. The unique morphism in $\mathrm{Alg}(\mathrm{coCAlg}(\mathsf{Ch}(k)))$ from the monoidal unit k (see Remark 5.1.2.9) to B_n is a quasi-isomorphism.* ♡

Proof. The forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\langle 1 \rangle})$ is symmetric monoidal and detects colimits by Proposition 5.1.2.7. By [HA, 3.2.1.8] it thus suffices to show that the unique morphism in $\mathrm{Alg}(\mathsf{Ch}(k))$ from the monoidal unit k to $\mathrm{Free}^{\mathrm{Alg}}(B'_n)$ is a quasiisomorphism, where B'_n is the chain complex which as a $\mathbb{Z}$-graded k-module is $k \cdot \Big\{ \underline{y}_n, y_n \Big\}$, with $\deg_{\mathsf{Ch}}(y_n) = -1$ and $\deg_{\mathsf{Ch}}(\underline{y}_n) = -2$, and with boundary operator defined by $\partial(y_n) = \underline{y}_n$.

But the left adjoint $\mathrm{Free}^{\mathrm{Alg}}$ to the forgetful functor $\mathrm{ev}_{\mathfrak{a}}$ preserves initial objects, so this morphism is $\mathrm{Free}^{\mathrm{Alg}}$ applied to the unique morphism of chain complexes $0 \to B'_n$.

By Proposition E.7.2.1 we can thus identify $k \to \mathrm{Free}^{\mathrm{Alg}}(B'_n)$ with the following inclusion of the summand indexed by 0

$$k = B_n'^{\otimes 0} \to \bigoplus_{i \geq 0} B_n'^{\otimes i}$$

As the tensor product of a contractible chain complex with another chain complex is again contractible it hence suffices to show that B'_n is contractible, which is clear. □

5.1.2.6. The resolution in $\mathcal{D}(k)$

In this section we describe the image of the constructions discussed in Section 5.1.2.2 and Section 5.1.2.4 under the symmetric monoidal functor $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$. The important point is that the pushout diagram (5.2) is in fact a homotopy pushout and thus mapped under γ to a pushout in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$, and likewise for the colimit of $A_0 \to A_1 \to A_2 \to \ldots$.

Proposition 5.1.2.11. *The underlying differential graded k-algebras of A_n and A from Construction 5.1.2.2 and of $\underline{B}_n$ and B_n from Construction 5.1.2.5 are cofibrant. Furthermore the pushout square (see Construction 5.1.2.5 and Proposition 5.1.2.8)*

$$\begin{tikzcd} \underline{B}_n \arrow[r] \arrow[d] & B_n \arrow[d] \\ A_{n-1} \arrow[r] & A_n \end{tikzcd}$$

is a homotopy pushout in $\mathrm{Alg}(\mathsf{Ch}(k))$ and the colimit of the directed system (see Remark 5.1.2.3)

$$A_0 \to A_1 \to A_2 \to \ldots$$

is a homotopy colimit in $\mathrm{Alg}(\mathsf{Ch}(k))$. ♡

Proof. For the cofibrancy statements it suffices to show that A_0 and $\underline{B}_n$ (for $n \geq 1$) are cofibrant and that the morphisms $\underline{B}_n \to B_n$ are generating cofibrations. The former is the case as $A_0 \cong \mathrm{Free}^{\mathrm{Alg}}(0)$ and $\underline{B}_n \cong \mathrm{Free}^{\mathrm{Alg}}(k \cdot \{\underline{y_n}\})$, and the chain complexes 0 and $k \cdot \{\underline{y_n}\}$ are cofibrant. The latter is the case as the morphism in question is isomorphic to $\mathrm{Free}^{\mathrm{Alg}}$ applied to a generating cofibration in $\mathsf{Ch}(k)$, see [Hov99, 2.3.3], Fact 4.1.3.1, and Theorem 4.2.2.1 (2).

That the pushout square is a homotopy pushout now follows from [HTT, A.2.4.4], and that the directed colimit is a homotopy colimit follows from [HTT, A.2.9.24 (i)][16]. □

Notation 5.1.2.12. Recall from Proposition 4.3.2.1 that we denote the symmetric monoidal functor from $\mathsf{Ch}(k)^{\mathrm{cof}}$ to $\mathcal{D}(k)$ by γ.

[16] The reference shows that the diagram is cofibrant in the projective model structure on $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathsf{Ch}(k))$ if and only if A_0 is cofibrant and $A_n \to A_{n+1}$ is a cofibration for every $n \geq 0$.

Let $\mathcal{O}$ and $\mathcal{O}'$ be ∞-operads. Then we denote the induced functor on $\mathcal{O}$-algebras of $\mathcal{O}'$-coalgebras as follows.

$$\gamma_{\mathcal{O}}^{\mathcal{O}'}\colon \operatorname{Alg}_{\mathcal{O}}\Big(\operatorname{coAlg}_{\mathcal{O}'}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\Big) \to \operatorname{Alg}_{\mathcal{O}}\Big(\operatorname{coAlg}_{\mathcal{O}'}\big(\mathcal{D}(k)\big)\Big) \qquad \diamondsuit$$

Remark 5.1.2.13. As all involved objects have cofibrant underlying chain complexes by Proposition 5.1.2.11 in combination with Proposition 4.2.2.12, the commutative squares and directed system constructed in Construction 5.1.2.5 and Remark 5.1.2.3 are mapped by $\gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}$ to commutative diagrams in $\operatorname{Alg}(\operatorname{coCAlg}(\mathcal{D}(k)))$. $\diamondsuit$

Corollary 5.1.2.14. *For $n \geq 1$, the commutative square*

$$\begin{tikzcd}
\gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(\underline{B}_n) \arrow[r] \arrow[d] & \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(B_n) \arrow[d] \\
\gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_{n-1}) \arrow[r] & \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_n)
\end{tikzcd}$$

in $\operatorname{Alg}(\operatorname{coCAlg}(\mathcal{D}(k))))$ *is a pushout diagram and the morphisms*

$$\gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_n) \to \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A)$$

exhibit $\gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A)$ *as a colimit of*

$$\gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_0) \to \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_1) \to \gamma_{\mathsf{Assoc}}^{\mathsf{Comm}}(A_2) \to \dots$$

in $\operatorname{Alg}(\operatorname{coCAlg}(\mathcal{D}(k)))$. $\heartsuit$

Proof. As $\mathcal{D}(k)$ is presentable symmetric monoidal by Proposition 4.3.2.1, it suffices by Proposition 5.1.2.7 to show that the underlying diagrams in $\operatorname{Alg}(\mathcal{D}(k))$ are colimit diagrams.

By Proposition 5.1.2.11 the diagrams of differential graded algebras are pointwise cofibrant (not just with cofibrant underlying chain complexes) as well as homotopy colimit diagrams, so the claim follows from combining this with Proposition 4.4.2.1 and [HA, 1.3.4.24]. $\square$

5.1.2.7. Free generation of certain associative algebras

In order to be able work with morphisms out of $\gamma_{\mathbb{E}_0}^{\mathbb{E}_\infty}(\underline{B}_n)$, we will show in this section that $\gamma_{\mathbb{E}_0}^{\mathbb{E}_\infty}(\underline{B}_n)$ is the free associative algebra on an object in $\operatorname{Alg}_{\mathbb{E}_0}\big(\operatorname{coCAlg}(\mathcal{D}(k))\big)$.

We start by constructing the morphism that exhibits $\gamma_{\mathbb{E}_0}^{\mathbb{E}_\infty}(\underline{B}_n)$ as a free associative algebra.

Construction 5.1.2.15. Let $n \geq 1$. We define $\underline{B}'_n$ to be the sub $\mathbb{Z}$-graded k-module of $\underline{B}_n$ (see Construction 5.1.2.5) generated by 1 and $\underline{y}_n$. Note that $\underline{B}'_n$ is closed under ∂ as well as Δ, and the unique morphism $k \to \underline{B}_n$ in $\operatorname{Alg}(\operatorname{coCAlg}(\mathsf{Ch}(k)))$ (see Remark 5.1.2.9) factors over $\underline{B}'_n$.

We can thus consider $\underline{B}'_n$ as an object of $\mathrm{coCAlg}(\mathsf{Ch}(k))_{k/}$. The underlying chain complexes of $\underline{B}'_n$ and $\underline{B}_n$ are cofibrant by [Hov99, 2.3.6], so we can consider the inclusion of $\underline{B}'_n$ into $\underline{B}_n$ as a morphism in $\mathrm{coCAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})_{k/}$.

By [HA, 2.1.3.10] there is an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_0}\Big(\mathrm{coCAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)\Big) \xrightarrow{\simeq} \mathrm{coCAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)_{k/}$$

under which we can consider the inclusion

$$\underline{B}'_n \to \underline{B}_n \tag{5.4}$$

as a morphism in $\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coAlg}_{\mathbb{E}_\infty}(\mathsf{Ch}(k)^{\mathrm{cof}}))$.

Completely analogously we define A'_1 to be the sub $\mathbb{Z}$-graded k-module of A_1 (see Construction 5.1.2.2) spanned by 1 and y_1 and consider the inclusion $A'_1 \to A_1$ as a morphism in $\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coAlg}_{\mathbb{E}_\infty}(\mathsf{Ch}(k)^{\mathrm{cof}}))$. ♢

Remark 5.1.2.16. By [HA, 2.1.3.9] there is a unique morphism of ∞-operads

$$\mathbb{E}_0^{\otimes} \to \mathsf{Assoc}^{\otimes}$$

which can be interpreted as follows. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the induced forgetful functor

$$\mathrm{Alg}_{\mathsf{Assoc}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C}) \simeq \mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$$

(where the equivalence is the one from [HA, 2.1.3.10]) sends an associative algebra A to the unit morphism $\mathbb{1}_{\mathcal{C}} \to A$. ♢

Notation 5.1.2.17. By [HA, 3.1.3.5][17], the forgetful functor

$$\mathrm{Alg}_{\mathsf{Assoc}}\Big(\mathrm{coCAlg}\big(\mathcal{D}(k)\big)\Big) \to \mathrm{Alg}_{\mathbb{E}_0}\Big(\mathrm{coCAlg}\big(\mathcal{D}(k)\big)\Big)$$

from Remark 5.1.2.16 has a left adjoint that we will denote as follows.

$$\mathrm{Free}^{\mathrm{Alg}(\mathrm{coCAlg})}_{\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg})}\colon \mathrm{Alg}_{\mathbb{E}_0}\Big(\mathrm{coCAlg}\big(\mathcal{D}(k)\big)\Big) \to \mathrm{Alg}_{\mathsf{Assoc}}\Big(\mathrm{coCAlg}\big(\mathcal{D}(k)\big)\Big)$$

We use the analogous notation $\mathrm{Free}^{\mathrm{Alg}}_{\mathrm{Alg}_{\mathbb{E}_0}}$ for the left adjoint of the forgetful functor $\mathrm{Alg}_{\mathsf{Assoc}}(\mathcal{D}(k)) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{D}(k))$. ♢

Proposition 5.1.2.18. *In this proposition we use Notation 5.1.2.12.*

Let $n \geq 1$. The morphism

$$\gamma^{\mathbb{E}_\infty}_{\mathbb{E}_0}(\underline{B}'_n) \to \gamma^{\mathbb{E}_\infty}_{\mathbb{E}_0}(\underline{B}_n)$$

induced by the inclusion (5.4) in $\mathrm{Alg}_{\mathbb{E}_0}\big(\mathrm{coCAlg}(\mathcal{D}(k))\big)$ induces a morphism

$$\mathrm{Free}^{\mathrm{Alg}(\mathrm{coCAlg})}_{\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg})}\Big(\gamma^{\mathbb{E}_\infty}_{\mathbb{E}_0}(\underline{B}'_n)\Big) \to \gamma^{\mathbb{E}_\infty}_{\mathsf{Assoc}}(\underline{B}_n) \tag{5.5}$$

[17] Using Proposition 5.1.2.6 and Proposition 4.3.2.1 (1).

in $\mathrm{Alg}\big(\mathrm{coAlg}_{\mathbb{E}_\infty}(\mathcal{D}(k))\big)$. *This morphism is an equivalence.*

The analogously defined morphism

$$\mathrm{Free}^{\mathrm{Alg(coCAlg)}}_{\mathrm{Alg}_{\mathbb{E}_0}\mathrm{(coCAlg)}}\left(\gamma^{\mathbb{E}_\infty}_{\mathbb{E}_0}(A_1')\right) \to \gamma^{\mathbb{E}_\infty}_{\mathsf{Assoc}}(A_1)$$

is also an equivalence. ♡

Proof. We only discuss the case of $\underline{B}_n$, as the case of A_1 is completely analogous.

By Proposition 5.1.2.7 the functor

$$\mathrm{Alg}\big(\mathrm{ev}_{\langle 1\rangle}\big)\colon\ \mathrm{Alg}\big(\mathrm{coCAlg}(\mathcal{D}(k))\big) \to \mathrm{Alg}\big(\mathcal{D}(k)\big)$$

is conservative, so it suffices to show that the underlying morphism in $\mathrm{Alg}(\mathcal{D}(k))$ of (5.5) is an equivalence.

The functor $\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{coCAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$ is symmetric monoidal and preserves colimits[18], so we can apply Proposition E.7.2.2 to conclude that the underlying morphism in $\mathrm{Alg}(\mathcal{D}(k))$ of morphism (5.5) is the morphism[19]

$$\mathrm{Free}^{\mathrm{Alg}}_{\mathrm{Alg}_{\mathbb{E}_0}}\big(\gamma_{\mathbb{E}_0}(\underline{B}_n')\big) \to \gamma_{\mathsf{Assoc}}(\underline{B}_n)$$

adjoint to the morphism $\gamma_{\mathbb{E}_0}(\underline{B}_n') \to \gamma_{\mathbb{E}_0}(\underline{B}_n)$.

Now consider the subcomplex $\underline{B}_n''$ of $\underline{B}_n'$ generated as a free $\mathbb{Z}$-graded k-module by $\underline{y}_n$. This complex is cofibrant and the morphism $\underline{B}_n'' \to \underline{B}_n'$ in $\mathsf{Ch}(k)^{\mathrm{cof}}$ exhibits $\underline{B}_n'$ as the free $\mathbb{E}_0$-algebra generated by $\underline{B}_n''$, see Proposition E.7.2.1.

The symmetric monoidal functor $\gamma\colon \mathsf{Ch}(k) \to \mathcal{D}(k)$ preserves coproducts by Proposition 4.3.2.1 (3) so by Proposition E.7.2.2 variant (3) we can identify $\gamma_{\mathbb{E}_0}(\underline{B}_n')$ with $\mathrm{Free}^{\mathrm{Alg}_{\mathbb{E}_0}}\big(\gamma(\underline{B}_n'')\big)$, and the equivalence

$$\mathrm{Free}^{\mathrm{Alg}_{\mathbb{E}_0}}\big(\gamma(\underline{B}_n'')\big) \xrightarrow{\simeq} \gamma_{\mathbb{E}_0}(\underline{B}_n')$$

is adjoint to the inclusion $\gamma(\underline{B}_n'') \to \gamma(\underline{B}_n')$. Using composability of adjoints [HTT, 5.2.2.6] we can identify $\mathrm{Free}^{\mathrm{Alg}}_{\mathrm{Alg}_{\mathbb{E}_0}} \circ \mathrm{Free}^{\mathrm{Alg}_{\mathbb{E}_0}}$ with $\mathrm{Free}^{\mathrm{Alg}}$, and under this identification the morphism

$$\mathrm{Free}^{\mathrm{Alg}}\big(\gamma(\underline{B}_n'')\big) \simeq \mathrm{Free}^{\mathrm{Alg}}_{\mathrm{Alg}_{\mathbb{E}_0}}\big(\gamma_{\mathbb{E}_0}(\underline{B}_n')\big) \to \gamma_{\mathsf{Assoc}}(\underline{B}_n) \tag{5.6}$$

which we need to show is an equivalence, is adjoint to the inclusion $\gamma(\underline{B}_n'') \to \gamma(\underline{B}_n)$. We finish by invoking Proposition E.7.2.2 again, this time variant (2) (using that γ preserves coproducts by Proposition 4.3.2.1 (3)), and noting that $\underline{B}_n'' \to \underline{B}_n$ indeed exhibits $\underline{B}_n$ as the free differential graded algebra generated by $\underline{B}_n''$ by definition. □

[18] See the proof of Proposition 5.1.2.6.

[19] We are also using that the various functors induced by γ are compatible with the forgetful functors here, to e. g. identify the underlying associative algebra of $\gamma^{\mathbb{E}_\infty}_{\mathsf{Assoc}}(\underline{B}_n)$ with $\gamma_{\mathsf{Assoc}}(\underline{B}_n)$.

5.1.3. Formality of certain associative algebras

Let $\mathcal{C}$ be a monoidal ∞-category and C an associative algebra in $\mathcal{C}$. By [HA, 3.2.1.8] (see also [HA, 3.2.1.4]) C is an initial object in $\mathrm{Alg}_{/\mathsf{Assoc}}(\mathcal{C})$ if and only if the unit morphism $\mathbb{1}_{\mathcal{C}} \to C$ is an equivalence. In this section we show that this is the case if and only if there exists *any* equivalence $\mathbb{1}_{\mathcal{C}} \simeq C$ in $\mathcal{C}$. In particular, this implies that any two associative algebras in $\mathcal{C}$ whose underlying objects in $\mathcal{C}$ are equivalent to $\mathbb{1}_{\mathcal{C}}$ are already equivalent as associative algebras.

Notation 5.1.3.1. Let $\mathcal{C}$ be a monoidal ∞-category and $\mathbb{1}$ a unit of $\mathcal{C}$. We will use the following notation in this section.

As part of the monoidal structure on $\mathcal{C}$, there are equivalences, natural in X,

$$\lambda_{\mathbb{1},X} \colon \mathbb{1} \otimes X \xrightarrow{\simeq} X$$

and

$$\rho_{X,\mathbb{1}} \colon X \otimes \mathbb{1} \xrightarrow{\simeq} X$$

for $\mathbb{1}$ any unit object in $\mathcal{C}$ and X any object in $\mathcal{C}$, called the *left unitor* and *right unitor*, respectively.

The reason why we let $\mathbb{1}$ be part of the notation is that we will consider morphisms between two unit objects that might not (a priori) be equivalences, so it will be important to distinguish them. ♢

Proposition 5.1.3.2. *Let $\mathcal{C}$ be a monoidal ∞-category, let $\mathbb{1}$ be a unit object in $\mathcal{C}$, and f and g two endomorphisms of $\mathbb{1}$. Then $f \circ g$ and $g \circ f$ are homotopic.* ♡

Proof. Two morphisms in an ∞-category are homotopic if and only if their images in the homotopy category are equal. It thus suffices to show that the monoid structure induced by composition on $\pi_0(\mathrm{Map}_{\mathcal{C}}(\mathbb{1},\mathbb{1})) = \mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1},\mathbb{1})$ is commutative.

Note that the monoidal structure on the ∞-category $\mathcal{C}$ induces the structure of an ordinary monoidal category on the homotopy category $\mathrm{Ho}(\mathcal{C})$, see [HA, 4.1.1.12]. We can define a binary operation $\star$ on $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1},\mathbb{1})$ by letting $f \star g$ for f and g in $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1},\mathbb{1})$ be given by conjugating $f \otimes g$ with the left unitor $\lambda_{\mathbb{1},\mathbb{1}}$ as depicted below.

$$\begin{tikzcd}
\mathbb{1} \arrow[d, dashed, "f\star g"'] & \mathbb{1}\otimes\mathbb{1} \arrow[l, "\lambda_{\mathbb{1},\mathbb{1}}"', "\cong"] \arrow[d, "f\otimes g"] \\
\mathbb{1} & \mathbb{1}\otimes\mathbb{1} \arrow[l, "\cong"', "\lambda_{\mathbb{1},\mathbb{1}}"]
\end{tikzcd}$$

Naturality of $\lambda_{\mathbb{1},-}$ immediately implies that $\mathrm{id}_{\mathbb{1}}$ is a left unit for the binary operation $\star$. We could similarly define $\star'$ using the right unitor $\rho_{\mathbb{1},\mathbb{1}}$, for which $\mathrm{id}_{\mathbb{1}}$ would be a *right* unit. As the composition

$$\mathbb{1} \xrightarrow{\lambda_{\mathbb{1},\mathbb{1}}^{-1}} \mathbb{1}\otimes\mathbb{1} \xrightarrow{\rho_{\mathbb{1},\mathbb{1}}} \mathbb{1}$$

is the identity[20], so $\star = \star'$, and hence we can conclude that $\mathrm{id}_{\mathbb{1}}$ is a two-sided unit for the binary operation $\star$ on $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1}, \mathbb{1})$.

As $(f \otimes g) \circ (h \otimes i) = (f \circ h) \otimes (g \circ i)$ in $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1}, \mathbb{1})$ by functoriality of the tensor product for f, g, h, and i endomorphisms of $\mathbb{1}$, we have $(f \star g) \circ (h \star i) = (f \circ h) \star (g \circ i)$ and can thus apply the Eckmann-Hilton argument to conclude that composition is commutative in $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{1}, \mathbb{1})$. □

Proposition 5.1.3.3. *Let $\mathcal{C}$ be a monoidal ∞-category and R an* Assoc*-algebra in $\mathcal{C}$ such that the underlying object in $\mathcal{C}$ is a monoidal unit. Let $\mathbb{1}$ be another, fixed, unit object. Then the unit morphism $\iota\colon \mathbb{1}_{\mathcal{C}} \to R$, that is part of the data of R as an* Assoc*-algebra, is an equivalence.* ♡

Proof. As part of the data of R as an Assoc-algebra there is also a multiplication morphism $\mu\colon R \otimes R \to R$, as well as a commutative diagram exhibiting (part of) unitality for R, depicted in the top half of the following diagram.

$$\begin{tikzcd}
R \arrow[rrr, bend left, "\mathrm{id}_R"] \arrow[dr, "\mathrm{id}_R"'] & \mathbb{1} \otimes R \arrow[l, "\lambda_{\mathbb{1},R}", "\simeq"'] \arrow[r, "\iota \otimes \mathrm{id}_R"] \arrow[d, "\lambda_{\mathbb{1},R}", "\simeq"'] & R \otimes R \arrow[r, "\mu"] \arrow[d, "\lambda_{R,R}"', "\simeq"] & R \\
& R \arrow[r, dashed, "\varphi"'] & R \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

The morphisms φ and ψ are defined as the induced morphisms that make the diagrams commute.

There is also a commutative diagram by naturality of $\rho_{-,R}$ as follows.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\iota"] & R \\
\mathbb{1} \otimes R \arrow[u, "\rho_{\mathbb{1},R}", "\simeq"'] \arrow[r, "\iota \otimes \mathrm{id}_R"'] & R \otimes R \arrow[u, "\rho_{R,R}"', "\simeq"]
\end{tikzcd}$$

Thus ι is an equivalence if and only if $\iota \otimes \mathrm{id}_R$ is, which is an equivalence if and only if φ is. But for φ we already have a left inverse ψ, i. e. $\psi \circ \varphi$ is homotopic to id_R. It follows from Proposition 5.1.3.2 that $\varphi \circ \psi$ is then also homotopic to id_R, so φ is an equivalence. □

5.1.4. Group homology

Let G be a (discrete) group. The goal of this section is to discuss how to calculate orbits of G-objects in $\mathcal{D}(k)$ and discuss the relation to classical notions. The category of

[20] In [Mac98, VII.1] this is required as an axiom for the definition of monoidal categories, but Kelly showed in [Kel64, Theorems 6 and 7] that this in fact follows from the now usual list of axioms.

G-objects in $\mathcal{D}(k)$ is defined as

$$\mathcal{D}(k)^{\mathrm{B}\,G} := \mathrm{Fun}\big(\mathrm{B}\,G, \mathcal{D}(k)\big)$$

where $\mathrm{B}\,G$ is the 1-groupoid with a single object $*$ and $\mathrm{Aut}_{\mathrm{B}\,G} := G$. If $F\colon\ \mathrm{B}\,G \to \mathcal{D}(k)$ is a functor that we think of as an object in $\mathcal{D}(k)$ with G-action, then we will often not distinguish notationally between F and $F(*)$.

Let X be a G-object in $\mathcal{D}(k)$. Then the *G-orbits* X_G of X is the colimit of X considered as a functor $\mathrm{B}\,G \to \mathcal{D}(k)$.

We want to relate the construction of orbits of G-objects in $\mathcal{D}(k)$ to classical notions of homological algebra. To start we note that by [HA, 1.3.4.25] every G-object in $\mathcal{D}(k)$ is represented by a G-object in $\mathsf{Ch}(k)$ that is cofibrant in the projective model structure on $\mathrm{Fun}(\mathrm{B}\,G, \mathsf{Ch}(k))$. Let X be a G-object in $\mathsf{Ch}(k)$ with cofibrant underlying chain complex. We can then apply [HA, 1.3.4.24] to conclude that $\gamma(X)_G \simeq \mathrm{hocolim}_{\mathrm{B}\,G} X$.

The category of G-objects in $\mathsf{Ch}(k)$ can be identified with $\mathsf{Ch}(kG)$, where kG is the group ring of G over k, see [Wei94, Section 6.1]. This isomorphism of categories is compatible with the respective forgetful functors to $\mathsf{Ch}(k)$, from which it immediately follows that the respective weak equivalences and projective fibrations coincide[21], so that this is even an equivalence of combinatorial model categories.

The colimit functor $\mathrm{Fun}(\mathrm{B}\,G, \mathsf{Ch}(k)) \to \mathsf{Ch}(k)$ is a left Quillen functor that is left adjoint to the functor const, the homotopy colimit functor is its derived functor. Under the equivalence $\mathrm{Fun}(\mathrm{B}\,G, \mathsf{Ch}(k)) \cong \mathsf{Ch}(kG)$, the functor const corresponds to the restriction of scalars functor $\mathsf{Ch}(k) \to \mathsf{Ch}(kG)$ that is induced by restriction along the ring homomorphism $kG \to k$ that maps every element of G to 1. The left adjoint of this functor is given by extension of scalars, so $k \otimes_{kG} -$, see also the discussion in [Wei94, Exercise 6.1.1 2 and Lemma 6.1.1].

The upshot is the following: If X is is a G-object in $\mathsf{Ch}(k)$, then there is an equivalence

$$\gamma(X)_G \simeq \gamma\Big(k \otimes^{\mathrm{L}}_{kG} X'\Big)$$

where on the right we take the derived tensor product and X' is the object in $\mathsf{Ch}(kG)$ associated to X.

The homology k-modules of this derived tensor product is by definition given by Tor, and this particular case this is what is called the *group homology* of G with coefficients in X (or X'), and denoted by $\mathrm{H}_*(G; X)$, see [Wei94, Definition 6.1.2 and Exercise 6.1.2]. We can summarize the discussion as follows, using Proposition 4.3.3.2.

Proposition 5.1.4.1. *Let G be a discrete group and X a G-object in $\mathsf{Ch}(k)$. Then there are isomorphisms*

$$\mathrm{H}_i\big(\gamma(X)_G\big) \cong \mathrm{Tor}_i^{kG}(k, X') \cong \mathrm{H}_i(G; X)$$

[21] For the projective model structure on $\mathrm{Fun}(\mathrm{B}\,G, \mathsf{Ch}(k))$, which we take with respect to the projective model structure on $\mathsf{Ch}(k)$, see [HTT, A.2.8.2], and for the projective model structure on $\mathsf{Ch}(kG)$ see Fact 4.1.3.1 – while we did not specifically mention it there, the assumption that the ring over which we take chain complexes is commutative is unnecessary for merely obtaining a combinatorial model category (commutativity is needed if we want to talk about the symmetric monoidal structure).

for every integer i, where X' is the kG-chain complex associated to X under the isomorphism discussed above. These isomorphisms are natural in X. ♡

We can conclude the following from this.

Proposition 5.1.4.2. *Let G be a discrete group and X a G-object in $\mathcal{D}(k)$. Assume that n is an integer such that the homology of X vanishes in degrees below n. Then the homology of X_G also vanishes below degree n, and $\mathrm{H}_n(X_G) \cong \mathrm{H}_n(X)_G$.* ♡

Proof. One way to prove this is to use represent X by a G-object in $\mathsf{Ch}(k)$ concentrated in degrees n and above, and then the statement follows from Proposition 5.1.4.1.

Another way would be to note that $\mathcal{D}(k)_{\geq n}$ is by [HA, 1.2.1.6] closed under colimits, from which it follows that the homology vanishes below degree n, and use Proposition 4.3.3.8 for homology in degree n. □

5.1.5. Formality of certain commutative algebras

The goal of Section 5.1 is to show that any two commutative bialgebras in $\mathcal{D}(k)$ whose homology is concentrated in degrees 0 and 1, where it is isomorphic to k, are equivalent. As a stepping stone we show in this section the analogous and significantly easier statement for commutative algebras, so forgetting the coalgebra structure.

We start in the following construction by constructing a comparison morphism from a "standard" commutative algebra with the prescribed homology (that the homology is the correct one will be shown below in Proposition 5.1.5.3). We will later show that this morphism is an equivalence of commutative algebras.

Construction 5.1.5.1. Let R be an object of $\mathrm{CAlg}(\mathcal{D}(k))$ and $\vartheta\colon \mathrm{ev}_{\langle 1\rangle}(R) \xrightarrow{\simeq} k \oplus k[n]$ an equivalence for some $n > 0$. Note that this equivalence is not assumed to have anything to do with the algebra structure on R, this is only an assumption on the equivalence class of the underlying object in $\mathcal{D}(k)$ of R.

As the underlying object of R is in $(\mathcal{D}(k)_{\geq 0})_{\leq n}$ we can by Proposition 4.3.4.1 (7) and (8) consider R as an object of $\mathrm{CAlg}((\mathcal{D}(k)_{\geq 0})_{\leq n})$.

Denote the inclusions that are part of $k \oplus k[n]$ being a coproduct by $\iota_0\colon k \to k \oplus k[n]$ and $\iota_n\colon k[n] \to k \oplus k[n]$, and let $g\colon k[n] \to \mathrm{ev}_{\langle 1\rangle}(R)$ be $g := \vartheta^{-1} \circ \iota_n$.

By [HA, 1.2.1.6], [HTT, 1.2.13.7], Proposition 4.3.2.1 (1), and [HA, 3.1.3.5], the forgetful functor

$$\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CAlg}\big(\mathcal{D}(k)_{\geq 0}\big) \to \mathcal{D}(k)_{\geq 0}$$

admits a left adjoint $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}$. We thus obtain an induced map of commutative algebras in $\mathcal{D}(k)_{\geq 0}$

$$f'\colon \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}\big(k[n]\big) \to R$$

that is adjoint to g.

Note that as the inclusion $\iota_{\geq 0}\colon \mathcal{D}(k)_{\geq 0} \to \mathcal{D}(k)$ is symmetric monoidal (Proposition 4.3.4.1 (3)) and also preserves colimits ([HA, 1.2.1.6] with [HTT, 1.2.13.7]), we can use Proposition E.7.2.2 to identify $\mathrm{CAlg}\big(\iota_{\geq 0}\big)(f')$ with the morphism

$$f''\colon \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}\big(k[n]\big) \to R$$

that is adjoint to g.

Finally, as R lies in $\mathrm{CAlg}((\mathcal{D}(k)_{\geq 0})_{\leq n})$, the morphism f' is by Proposition 4.3.4.1 (8) adjoint to a morphism

$$f\colon \mathrm{CAlg}(\tau_{\leq n})\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}(k[n])\Big) \to R$$

of commutative algebras in $(\mathcal{D}(k)_{\geq 0})_{\leq n}$. $\diamondsuit$

The equivalence $\vartheta^{-1}\colon k \oplus k[n] \xrightarrow{\simeq} \mathrm{ev}_{\langle 1\rangle}(R)$ in Construction 5.1.5.1 could be anything on the summand k. However, we already have a candidate morphism $k \to \mathrm{ev}_{\langle 1\rangle}(R)$ – the unit morphism of the commutative algebra structure of R. In the next proposition we show that we can replace ϑ^{-1} on the first summand by the unit morphism without losing the property of being an equivalence.

Proposition 5.1.5.2. *In the situation of Construction 5.1.5.1, the morphism*

$$\iota \amalg g\colon k \oplus k[n] \to \mathrm{ev}_{\langle 1\rangle}(R)$$

is an equivalence in $\mathcal{D}(k)$, where ι is the unit morphism of the algebra structure on R. $\heartsuit$

Proof. It suffices to show that the composition $\vartheta \circ (\iota \amalg g)$ is an equivalence. Using the definition of g we can write this morphism as

$$k \oplus k[n] \xrightarrow{\begin{bmatrix} \iota' & 0 \\ \iota'' & \mathrm{id}_{k[n]} \end{bmatrix}} k \oplus k[n]$$

for some morphisms $\iota'\colon k \to k$ and $\iota''\colon k \to k[n]$. It thus suffices to show that ι' is an equivalence, as then

$$\begin{bmatrix} \iota'^{-1} & 0 \\ -\iota''\iota'^{-1} & \mathrm{id}_{k[n]} \end{bmatrix}$$

will be an inverse.

While we do not need this, we note that ι'' must actually be nullhomotopic, as

$$\pi_0\Big(\mathrm{Map}_{\mathcal{D}(k)}(k, k[n])\Big) \cong H_0(k[n]) \cong 0$$

by Proposition 4.3.2.1 (5) and (4).

Applying the natural transformation $\mathrm{id}_{\mathcal{D}(k)} \to \iota_{\leq 0} \circ \tau_{\leq 0}$ (see Section 4.3.4) we obtain a commuting diagram as follows[22]

$$\begin{tikzcd}
k \arrow[r, "\vartheta\circ\iota"] \arrow[d] \arrow[rrrr, bend left, "\iota'"] & k\oplus k[n] \arrow[rrr, "\mathrm{pr}_0"] \arrow[d] & & & k \arrow[d] \\
\tau(k) \arrow[r, "\tau(\vartheta\circ\iota)"] \arrow[dr, "\tau(\iota)"'] & \tau\big(k\oplus k[n]\big) \arrow[rr, "\tau(\mathrm{pr}_0)\times\tau(\mathrm{pr}_n)"] & & \tau(k)\oplus\tau\big(k[n]\big) \arrow[r, "\mathrm{pr}_0"] & \tau(k) \\
& \tau\big(\mathrm{ev}_{\langle 1\rangle}(R)\big) \arrow[u, "\tau(\vartheta)"'] & & &
\end{tikzcd}$$

in $\mathcal{D}(k)$ where the morphisms pr_0 and pr_n are the projections onto the first and second factor, respectively.

We have to show that ι' is an equivalence. As k is in $\mathcal{D}(k)_{\leq 0}$, the leftmost and rightmost vertical morphisms are equivalences. It thus suffices to show that the composite from left to right in the middle row is an equivalence.

As a left adjoint $\tau_{\leq 0}$ preserves colimits and hence finite biproducts, and $\iota_{\leq 0}$ preserves finite biproducts as well by Proposition 4.3.4.2. Thus the morphism

$$\iota_{\leq 0}\tau_{\leq 0}(\mathrm{pr}_0) \times \iota_{\leq 0}\tau_{\leq 0}(\mathrm{pr}_n)$$

in the middle is an equivalence. The morphism pr_0 on the right (in the middle row) is an equivalence as $\tau_{\leq 0}(k[n]) \simeq 0$[23]. As ϑ is an equivalence, $\iota_{\leq 0}\tau_{\leq 0}(\vartheta)$ is also an equivalence.

It thus remains to show that $\iota_{\leq 0}\tau_{\leq 0}(\iota)$ is an equivalence. As we have already seen that domain and codomain of this morphism is equivalent to k and hence in $\mathcal{D}(k)_{\geq 0}$, this morphism is equivalent to $\iota_{\geq 0}\tau_{\geq 0}\iota_{\leq 0}\tau_{\leq 0}(\iota)$, which by [HA, 1.2.1.10] can be identified with $\iota_{\geq 0}\iota_{\geq 0,\leq 0}\tau_{\leq 0}\tau_{\geq 0}(\iota)$. As all four involved functors are lax symmetric monoidal by Proposition 4.3.4.1, this is the unit morphism of a commutative algebra in $\mathcal{D}(k)$ whose underlying object is equivalent to k. We can thus apply Proposition 5.1.3.3 to conclude that $\iota_{\leq 0}\tau_{\leq 0}(\iota)$ is an equivalence. □

Before we can show that the morphism f from Construction 5.1.5.1 is an equivalence, we need to determine the homology of $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$ in low degrees. We do this in the

[22] To save space we write τ instead of $\iota_{\leq 0}\tau_{\leq 0}$.

[23] This can be easily seen using the fiber sequence

$$\iota_{\geq 1}\tau_{\geq 1}(k[n]) \to k[n] \to \iota_{\leq 0}\tau_{\leq 0}(k[n])$$

from [HA, 1.2.1.8] in which the first morphism is an equivalence as $k[n]$ lies in $\mathcal{D}(k)_{\geq 1}$.

following proposition, where we actually calculate the homology in a wider range than would be necessary in this section – the calculations in the extra degrees will be used in later sections.

Proposition 5.1.5.3. *Let $n \geq 1$ and let*

$$\varphi\colon k[n] \to \mathrm{ev}_{\langle 1\rangle}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right)$$

be the morphism in $\mathcal{D}(k)$ exhibiting $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$ as the free commutative algebra generated by $k[n]$ and let

$$i\colon k \to \mathrm{ev}_{\langle 1\rangle}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right)$$

be the unit morphism.

Then the following holds for the homology of $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$.

$$\mathrm{H}_i\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right) \cong \begin{cases} 0 & \text{if } i < 0 \\ k & \text{if } i = 0 \\ 0 & \text{if } 0 < i < n \\ k & \text{if } i = n \\ 0 & \text{if } n < i < 2n \\ k & \text{if } i = 2n \text{ and } n \text{ is even} \\ k/(2) & \text{if } i = 2n \text{ and } n \text{ is odd} \end{cases}$$

Furthermore, a basis of the homology in degrees 0 and n is given by i and φ, i. e. $i \amalg \varphi\colon k \oplus k[n] \to \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$ induces an isomorphism on homology in degrees smaller than $2n$. ♡

Proof. Using [HA, 3.1.3.13] and unpacking the definition of the relevant ∞-groupoids $\mathcal{P}(m)$ for $\mathcal{O}^{\otimes} = \mathsf{Comm}^{\otimes}$[24] we obtain that there is an equivalence[25]

$$\mathrm{ev}_{\langle 1\rangle}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right) \simeq \coprod_{m\geq 0}\left(k[n]^{\otimes m}\right)_{\Sigma_m} \simeq k \amalg k[n] \amalg \coprod_{m\geq 2}\left(k[n]^{\otimes m}\right)_{\Sigma_m}$$

in $\mathcal{D}(k)$ and under this equivalence the unit morphism and the morphism φ exhibiting it as the free commutative algebra generated by $k[n]$ are the inclusions of the summands indexed by 0 and 1, respectively.

By Proposition 4.3.3.5 H_i preserves coproducts, so it suffices to show the following.

(1) $\mathrm{H}_i((k[n]^{\otimes m})_{\Sigma_m}) \cong 0$ for $m \geq 2$ and $i < nm$.

(2) $\mathrm{H}_{2n}((k[n]^{\otimes 2})_{\Sigma_2}) \cong k$ if n is even and $\mathrm{H}_{2n}((k[n]^{\otimes 2})_{\Sigma_2}) \cong k/(2)$ if n is odd.

[24] We get an equivalence of ∞-groupoids $\mathcal{P}(m) \simeq \mathrm{B}\,\Sigma_m$, where $\mathrm{B}\,\Sigma_m$ is the 1-groupoid with a single object and the symmetric group on m elements as automorphism group.

[25] The subscript Σ_m denotes a (homotopy) orbit, i. e. a colimit of a functor from $\mathrm{B}\,\Sigma_m$.

Proof of Claim (1): Note that if $m \geq 2$ then $k[n]^{\otimes m} \simeq k[nm]$ has homology concentrated in degree nm and is hence in $\mathcal{D}(k)_{\geq nm}$. As $\mathcal{D}(k)_{\geq nm}$ is stable under colimits in $\mathcal{D}(k)$ (see [HA, 1.2.1.6]) we can conclude that $(k[n]^{\otimes m})_{\Sigma_m}$ is also in $\mathcal{D}(k)_{\geq nm}$ and hence has vanishing homology in degrees smaller than nm.

Proof of Claim (2): Going through [HA, 3.1.3.13] and [HA, 3.1.3.9] to identify the action of Σ_2 on $k[n] \otimes k[n]$, we see that the nontrivial element acts via the symmetry equivalence that is part of the structure of $\mathcal{D}(k)$ as a symmetric monoidal ∞-category, and which is induced by the symmetry isomorphism of the symmetric monoidal structure on $\mathsf{Ch}(k)$, see Proposition 4.3.2.1 (1) and Definition 4.1.2.1. We can thus represent the Σ_2-object $k[n] \otimes k[n]$ in $\mathcal{D}(k)$ by the Σ_2-object $k[n] \otimes k[n]$ in $\mathsf{Ch}(k)$ where the non-trivial element acts via the symmetry isomorphism. There is an isomorphism $k[n] \otimes k[n] \cong k[2n]$ mapping $1 \otimes 1$ to 1, and we obtain an induced Σ_2-action on $k[2n]$. If n is odd, then the non-trivial element of Σ_2 acts as $-\mathrm{id}$, which reflects the fact that if x is an element in odd degree of a commutative differential graded algebra, then we have $x^2 = -x^2$. If n is even, then the non-trivial element acts as id.

The claim now follows from Proposition 5.1.4.2. □

Proposition 5.1.5.4. *In the situation of Construction 5.1.5.1, the morphism f is an equivalence.*

In particular, if R' is another commutative algebra in $\mathcal{D}(k)$ such that the underlying objects $\mathrm{ev}_{\langle 1\rangle}(R')$ and $\mathrm{ev}_{\langle 1\rangle}(R)$ are equivalent, then R and R' are also equivalent as commutative algebras. ♡

Proof. The adjoint f of f' is by definition given by the composition

$$\mathrm{CAlg}(\tau_{\leq n})\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}(k[n])\Big) \xrightarrow{\mathrm{CAlg}(\tau_{\leq n})(f')} \mathrm{CAlg}(\tau_{\leq n})\Big(\mathrm{CAlg}(\iota_{\geq 0, \leq n})(R)\Big) \to R$$

where the second morphism the the counit of the adjunction $\mathrm{CAlg}(\tau_{\leq n}) \dashv \mathrm{CAlg}(\iota_{\geq 0,\leq n})$. This counit is homotopic to the identity by construction[26], so it suffices to show that $\mathrm{CAlg}(\tau_{\leq n})(f')$ is an equivalence. As $\iota_{\geq 0, \leq n}$ and $\iota_{\geq 0}$ are fully faithful and hence conservative, and $\mathrm{ev}_{\langle 1\rangle}$ is also conservative [HA, 3.2.2.6], it suffices to show that

$$\begin{aligned}
&\Big(\iota_{\geq 0} \circ \iota_{\geq 0,\leq n} \circ \mathrm{ev}_{\langle 1\rangle} \circ \mathrm{CAlg}(\tau_{\leq n})\Big)(f') \\
\simeq\ &\big(\iota_{\geq 0} \circ \iota_{\geq 0,\leq n} \circ \tau_{\leq n} \circ \mathrm{ev}_{\langle 1\rangle}\big)(f') \\
\simeq\ &\big(\iota_{\leq n} \circ \tau_{\leq n} \circ \iota_{\geq 0} \circ \mathrm{ev}_{\langle 1\rangle}\big)(f') \\
\simeq\ &\Big(\iota_{\leq n} \circ \tau_{\leq n} \circ \mathrm{ev}_{\langle 1\rangle} \circ \mathrm{CAlg}(\iota_{\geq 0})\Big)(f') \\
\simeq\ &\big(\iota_{\leq n} \circ \tau_{\leq n} \circ \mathrm{ev}_{\langle 1\rangle}\big)(f'')
\end{aligned}$$

is an equivalence.

Recall from Construction 5.1.5.1 that

$$f''\colon\ \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]) \to R$$

[26] See [HA, 1.2.1.5] and [HTT, 5.2.7.6, 5.2.7.7, and 5.2.7.8].

is the morphism in $\mathrm{CAlg}(\mathcal{D}(k))$ adjoint to g. There is thus a commutative diagram

$$\begin{tikzcd} & k[n] \arrow[dl, "\varphi"'] \arrow[dr, "g"] & \\ \mathrm{ev}_{\langle 1\rangle}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right) \arrow[rr, "\mathrm{ev}_{\langle 1\rangle}(f'')"'] & & \mathrm{ev}_{\langle 1\rangle}(R) \end{tikzcd}$$

where φ exhibits $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$ as the free commutative algebra generated by $k[n]$. If we let i be the unit morphism of $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$ and ι the unit morphism of R, then $f'' \circ i \simeq \iota$ as f'' is a morphism of commutative algebras. We can thus extend this commutative diagram to a commutative diagram as follows.

$$\begin{tikzcd} & k \oplus k[n] \arrow[dl, "i \amalg \varphi"'] \arrow[dr, "\iota \amalg g"] & \\ \mathrm{ev}_{\langle 1\rangle}\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\right) \arrow[rr, "\mathrm{ev}_{\langle 1\rangle}(f'')"'] & & \mathrm{ev}_{\langle 1\rangle}(R) \end{tikzcd}$$

The morphism on the right is an equivalence by Proposition 5.1.5.2. We have to show that $\tau_{\leq n}$ of the bottom morphism is an equivalence, so it suffices to show that $\tau_{\leq n}$ of the left morphism is an equivalence. But this follows from Proposition 5.1.5.3. $\square$

5.1.6. Identification of some mapping spaces

As explained in the introduction to Section 5.1, it will be important for us to show that

$$\pi_0\left(\mathrm{Map}_{\mathrm{Alg}\left(\mathrm{coCAlg}(\mathcal{D}(k))\right)}\left(\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n), R^\vee\right)\right)$$

is trivial for certain commutative bialgebras R. We saw in Section 5.1.2.7 that $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n)$ is free on the pointed cocommutative algebra $\gamma^{\mathsf{Comm}}_{\mathbb{E}_0}(\underline{B}'_n)$, so we are led to consider path components of mapping spaces in $\mathrm{Alg}_{\mathbb{E}_0}\left(\mathrm{coCAlg}(\mathcal{D}(k))\right) \simeq \mathrm{coCAlg}(\mathcal{D}(k))_{k/}$, and after dualizing of mapping spaces in $\mathrm{CAlg}(\mathcal{D}(k))_{/k}$.

This section concerns the steps needed to show that the sets of path components of such mapping spaces that are of interest to us are indeed trivial. In Section 5.1.6.1 we will show that the relevant mapping spaces in $\mathrm{CAlg}(\mathcal{D}(k))_{/k}$ can be calculated as the mapping spaces between the underlying objects in $\mathrm{CAlg}(\mathcal{D}(k))$. In Section 5.1.6.3 we will then show that π_0 of the relevant mapping spaces in $\mathrm{CAlg}(\mathcal{D}(k))$ are trivial. In order to do so, we will need to construct a commutative algebra with prescribed homology. We will define such a commutative algebra as a pushout of free commutative algebras and show that its homology has the required description in Section 5.1.6.2.

5.1.6.1. Identification of a mapping space in an overcategory

In this section we show that, under certain assumptions, mapping spaces in the ∞-category $\mathrm{CAlg}(\mathcal{D}(k))_{/k}$ are equivalent to the mapping spaces between the respective underlying objects in $\mathrm{CAlg}(\mathcal{D}(k))$.

Proposition 5.1.6.1. *Let $R \to k$ and $S \to k$ be objects of $\mathrm{CAlg}(\mathcal{D}(k))_{/k}$, and assume that there is an equivalence $\tau_{\leq 0}(\mathrm{ev}_{\langle 1\rangle}(R)) \simeq k$ in $\mathcal{D}(k)$.*

Then the map induced by the canonical forgetful functor on mapping spaces

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))_{/k}}(R, S) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R, S)$$

is an equivalence. ♡

Proof. By (the dual of) Proposition D.1.3.2 there is a pullback diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))_{/k}}(R, S) \arrow[r] \arrow[d] & \{R \to k\} \arrow[d] \\
\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R, S) \arrow[r, "(S\to k)_*"'] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R, k)
\end{tikzcd}$$

in $\mathcal{S}$, where the left vertical map is the one induced by the forgetful functor. It thus suffices to prove that $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R, k)$ is contractible.

k as well as the underlying object $\mathrm{ev}_{\langle 1\rangle}(R)$ of R are in $\mathcal{D}(k)_{\geq 0}$[27], so using that by Proposition 4.3.4.1 (7) $\mathrm{CAlg}(\iota_{\geq 0})$ is fully faithful with essential image spanned by those commutative algebras whose underlying object is in $\mathcal{D}(k)_{\geq 0}$, it suffices to show that

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k)_{\geq 0})}(R, k) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R, k)$$

is contractible.

As k actually lies in $\left(\mathcal{D}(k)_{\geq 0}\right)_{\leq 0}$ we can use the adjunction $\mathrm{CAlg}(\tau_{\leq 0}) \dashv \mathrm{CAlg}(\iota_{\geq 0,\leq 0})$ with fully faithful right adjoint discussed in Proposition 4.3.4.1 (8) to obtain equivalences

$$\begin{aligned}
&\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k)_{\geq 0})}(R, k) \\
\simeq\ &\mathrm{Map}_{\mathrm{CAlg}\left(\left(\mathcal{D}(k)_{\geq 0}\right)_{\leq 0}\right)}\left(\mathrm{CAlg}(\tau_{\leq 0})(R), k\right) \\
\simeq\ &\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\left(\mathrm{CAlg}(\iota_{\leq 0})\,\mathrm{CAlg}(\tau_{\leq 0})(R), k\right)
\end{aligned}$$

[27] By [HA, 1.2.1.8] there is a cofiber sequence

$$\iota_{\geq 0}\tau_{\geq 0} R \to R \to \iota_{\leq -1}\tau_{\leq -1} R$$

and

$$\tau_{\leq -1} R \simeq \tau_{\leq -1}\tau_{\leq 0} R \simeq \tau_{\leq -1} k \simeq 0$$

so $\iota_{\geq 0}\tau_{\geq 0} R \simeq R$ is in $\mathcal{D}(k)_{\geq 0}$.

By assumption the underlying object

$$\big(\mathrm{ev}_{\langle 1\rangle} \circ \mathrm{CAlg}(\iota_{\leq 0}) \circ \mathrm{CAlg}(\tau_{\leq 0})\big)(R) \simeq \big(\iota_{\leq 0} \circ \tau_{\leq 0} \circ \mathrm{ev}_{\langle 1\rangle}\big)(R)$$

of $\mathrm{CAlg}(\iota_{\leq 0})\,\mathrm{CAlg}(\tau_{\leq 0})(R)$ is equivalent to k, so by Proposition 5.1.3.3 the unit morphism $k \to \mathrm{CAlg}(\iota_{\leq 0})\,\mathrm{CAlg}(\tau_{\leq 0})(R)$ is an equivalence. [HA, 3.2.1.9] then implies that $\mathrm{CAlg}(\iota_{\leq 0})\,\mathrm{CAlg}(\tau_{\leq 0})(R)$ is an initial object of $\mathrm{CAlg}(\mathcal{D}(k))$, so the mapping space

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\big(\mathrm{CAlg}(\iota_{\leq 0})\,\mathrm{CAlg}(\tau_{\leq 0})(R), k\big)$$

is contractible. □

5.1.6.2. The homology of a pushout of commutative algebras

Let $n > 0$ be an integer, and let R be a commutative algebra in $\mathcal{D}(k)$ with homology concentrated in degree 0 and n, where it is isomorphic to k. In Section 5.1.6.3 we want to show that the mapping space in $\mathrm{CAlg}(\mathcal{D}(k))$ from R to another commutative algebra S with certain restrictions on its homology is contractible. To do so, we construct a commutative algebra for which it is easier to calculate mapping spaces out of, and such that its homology is isomorphic to that of R in degrees smaller than or equal to $2n$. We can start with the free commutative algebra generated by one generator in degree n. We calculated the homology in the relevant degrees in Proposition 5.1.5.3, and it is already nearly as we want, except that the homology might not vanish in degree $2n$, where it is generated by a single element. To divide out that unwanted element we can form a pushout over the free commutative algebra with a generator in degree $2n$.

We will start by carrying out this construction in Construction 5.1.6.2, and then spend the remainder of this section proving that the homology is as we require in Proposition 5.1.6.3. One way to do this calculation would be to use the Tor spectral sequence, see [HA, 7.2.1.19], but we have opted for a more direct approach with a concrete resolution that suffices in order to calculate the homology groups in the necessary degrees.

Construction 5.1.6.2. Let $n > 0$ be an integer. In Proposition 5.1.5.3 we showed that $\mathrm{H}_{2n}(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])) \cong k$ if n is even and $\mathrm{H}_{2n}(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])) \cong k/(2)$ if n is odd. In both cases, this k-module can be generated by a single element. Let

$$f'\colon k[2n] \to \mathrm{ev}_{\langle 1\rangle}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])\Big)$$

be a morphism in $\mathcal{D}(k)$ representing a generator[28] of $\mathrm{H}_{2n}(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]))$. We obtain an induced morphism

$$f\colon\ \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]) \to \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$$

in $\mathrm{CAlg}(\mathcal{D}(k))$ that is adjoint to f'.

[28]If 2 is invertible in k and n is odd, then we have $k/(2) \cong 0$, which is of course still generated by a single element 0, so we can carry out this construction also in this case, even though the construction is not really necessary for applications. However, we would like to avoid special handling of this one case.

The zero morphism $k[2n] \to k$ similarly induces a morphism of commutative algebras $p\colon \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]) \to k$.

Define P to be the pushout in $\mathrm{CAlg}(\mathcal{D}(k))$ as in the following diagram.

$$\begin{tikzcd}
\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]) \arrow[r, "p"] \arrow[d, "f"'] & k \arrow[d, "i"] \\
\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]) \arrow[r, "j"'] & P
\end{tikzcd}$$

We will use the notation P, f, p, i, and j elsewhere where we explicitly refer to this construction. $\diamondsuit$

Proposition 5.1.6.3. *Let $n > 0$ be an integer. For P as in Construction 5.1.6.2, the following holds for the homology of P.*

$$\mathrm{H}_i(P) \cong \begin{cases} 0 & \text{if } i < 0 \\ k & \text{if } i = 0 \\ 0 & \text{if } 0 < i < n \\ k & \text{if } i = n \\ 0 & \text{if } n < i \leq 2n \end{cases}$$

Furthermore, the morphism $j\colon \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]) \to P$ from Construction 5.1.6.2 induces an isomorphism on H_i for $i < 2n$. $\heartsuit$

Proof. To improve readability in the formulas we will use the following shorthand notation in this proof. We write F_{2n} for $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n])$ and F_n for $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n])$. Furthermore, we will omit writing forgetful functors and will instead always make explicit in which ∞-category objects and morphisms are considered. We will also use the notation from Construction 5.1.6.2.

The strategy of this calculation is as follows. By construction P is a pushout of commutative algebras, so by Proposition E.8.0.5 can be calculated as a relative tensor product. We thus resolve k as a left-F_{2n}-module in a manner that suffices to extract the homology groups we are interested in from the long exact sequences in homology that we obtain.

Let $g\colon k[2n] \to F_{2n}$ be the morphism in $\mathcal{D}(k)$ that exhibits F_{2n} as the free commutative algebra generated by $k[2n]$. We first consider the following composition in $\mathrm{LMod}_{F_{2n}}(\mathcal{D}(k))$

$$\begin{tikzcd}
F_{2n}[2n] \arrow[r, "\simeq"] \arrow[rrr, bend left, "g'"] & F_{2n} \otimes k[2n] \arrow[r, "\mathrm{id}_{F_{2n}} \otimes g"] & F_{2n} \otimes F_{2n} \arrow[r, "\mu"] & F_{2n} \arrow[r, "p"] & k
\end{tikzcd} \qquad (*)$$

where μ is the multiplication, g' is defined as the composition indicated in the diagram, and F_{2n} acts on $F_{2n} \otimes F_{2n}$ and $F_{2n} \otimes k[2n]$ via the the left tensor factor, and on k via p[29].

We claim that the composition pg' from $F_{2n}[2n]$ to k in ($*$) is nullhomotopic as a morphism in $\mathrm{LMod}_{F_{2n}}(\mathcal{D}(k))$. In fact, every morphism of F_{2n}-algebras $F_{2n}[2n] \to k$ is nullhomotopic, as we have by [HA, 4.2.4.6] an equivalence

$$\mathrm{Map}_{\mathrm{LMod}_{F_{2n}}(\mathcal{D}(k))}\big(F_{2n}[2n], k\big) \simeq \mathrm{Map}_{\mathcal{D}(k)}\big(k[2n], k\big)$$

which is contractible as $k[2n]$ is concentrated in degree $2n > 0$ and k is concentrated in degree 0.

The nullhomotopy of g' induces a morphism in $\mathrm{LMod}_{F_{2n}}(\mathcal{D}(k))$ from the cofiber of g' to k and a commutative triangle as in the following diagram

$$\begin{tikzcd} F_{2n}[2n] \arrow[r, "g'"] & F_{2n} \arrow[r, "\varphi"] \arrow[dr, "p"'] & C \arrow[d, dashed, "\psi"] \\ & & k \end{tikzcd} \tag{$**$}$$

where the top row is a cofiber sequence.

Note that the forgetful functor $\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod}_{F_{2n}}(\mathcal{D}(k)) \to \mathcal{D}(k)$ preserves colimits by [HA, 4.2.3.5]. Using the long exact homology sequence for the cofiber sequence in $\mathcal{D}(k)$ underlying the one from ($**$), together with the calculation of the lower homology groups

[29] Here are some more details on obtaining these morphisms as morphisms in $\mathrm{LMod}_{F_{2n}}(\mathcal{D}(k))$.

There is a commutative diagram in $\mathrm{CAlg}(\mathcal{D}(k))$

$$\begin{tikzcd} & F_{2n} \otimes F_{2n} \arrow[dr, "\mu"] & & \\ F_{2n} \arrow[ur, "\mathrm{id}_{F_{2n}} \otimes 1"] \arrow[rr, "\mathrm{id}_{F_{2n}}"'] & & F_{2n} \arrow[r, "p"'] & k \end{tikzcd}$$

where $\mathrm{id}_{F_{2n}} \otimes 1$ is the composition $F_{2n} \simeq F_{2n} \otimes k$ with the identity tensor the unit of F_{2n} – this is the inclusion of the first summand of the coproduct $F_{2n} \otimes F_{2n} \simeq F_{2n} \amalg F_{2n}$ in $\mathrm{CAlg}(\mathcal{D}(k))$.

We can now forget down to associative algebras and then use the section $\mathrm{Alg}(\mathcal{D}(k)) \to \mathrm{LMod}(\mathcal{D}(k))$ from [HA, 4.2.1.17] that carries an algebra to the underlying object as a module over the algebra itself. We can then restrict the actions to obtain a commutative diagram of F_{2n}-modules. This constructs the morphisms μ and p in ($*$). See also Construction E.8.0.4 for more details for this kind of construction.

The morphism

$$k[2n] \xrightarrow{1 \otimes g} F_{2n} \otimes F_{2n}$$

in $\mathcal{D}(k)$ is adjoint to a morphism of left-F_{2n}-modules $F_{2n} \otimes k[2n] \to F_{2n} \otimes F_{2n}$ (here $F_{2n} \otimes k[2n]$ is the free left-F_{2n}-module generated by $k[2n]$, see [HA, 4.2.4]). The morphism of $\mathcal{D}(k)$ underlying this morphism is then by definition given by the composition

$$F_{2n} \otimes k[2n] \xrightarrow{\mathrm{id}_{F_{2n}} \otimes 1 \otimes g} F_{2n} \otimes F_{2n} \otimes F_{2n} \xrightarrow{\mu \otimes \mathrm{id}_{F_{2n}}} F_{2n} \otimes F_{2n}$$

which is homotopic to $\mathrm{id}_{F_{2n}} \otimes g$.

of F_{2n} from Proposition 5.1.5.3, we obtain that φ induces an isomorphism

$$k \cong \mathrm{H}_0(F_{2n}) \xrightarrow{\mathrm{H}_0(\varphi)} \mathrm{H}_0(C)$$

and that for $i < 4n$ with $i \neq 0$ the homology group $\mathrm{H}_i(C)$ is zero[30]. As $\mathrm{H}_0(p)$ is an isomorphism (p underlies a morphism of commutative algebras and hence preserves the unit morphism) it follows that $\mathrm{H}_0(\psi)$ must be an isomorphism as well.

We now take the fiber of ψ to we obtain another cofiber sequence of left-F_{2n}-modules in $\mathcal{D}(k)$ as follows.

$$D \xrightarrow{\theta} C \xrightarrow{\psi} k \tag{$***$}$$

Again using the long exact sequence in homology we can conclude that $\mathrm{H}_i(D) \cong 0$ for $i < 4n$.

Let us now get back to what we actually need to do, calculate the homology of P in low degrees. As $\mathcal{D}(k)$ is presentable symmetric monoidal by Proposition 4.3.2.1 (1), we can apply Proposition E.8.0.5, which tells us that P is equivalent to the relative tensor product[31] $F_n \otimes_{F_{2n}} k$, where we consider F_n and k as right and left modules over F_{2n}, which is considered as an associative algebra in $\mathrm{CAlg}(\mathcal{D}(k))$. The forgetful functor $\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{CAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$ is symmetric monoidal and preserves $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits by [HA, 3.2.3.2]. We can thus apply Proposition E.8.0.1 to conclude that the underlying object of $F_n \otimes_{F_{2n}} k$ in $\mathcal{D}(k)$ is equivalent to the relative tensor product $F_n \otimes_{F_{2n}} k$, where we consider F_{2n} as just an associative algebra in $\mathcal{D}(k)$.

Tensoring cofiber sequence ($***$) with the right-F_{2n}-module F_n we obtain by [HA, 4.4.2.15] a cofiber sequence in $\mathcal{D}(k)$ as follows.

$$F_n \otimes_{F_{2n}} D \xrightarrow{\mathrm{id} \otimes_{\mathrm{id}} \theta} F_n \otimes_{F_{2n}} C \xrightarrow{\mathrm{id} \otimes_{\mathrm{id}} \psi} F_n \otimes_{F_{2n}} k$$

As $\mathrm{H}_i(P) \cong \mathrm{H}_i(F_n \otimes_{F_{2n}} k)$ for any integer i, we can use the long exact homology sequence associated to the above cofiber sequence to evaluate the homology groups of P. As remarked before, D lies in $\mathcal{D}(k)_{\geq 4n}$, and as F_n and F_{2n} are both in $\mathcal{D}(k)_{\geq 0}$ and taking colimits can only increase connectivity [HA, 1.2.1.6], it follows that

$$F_n \otimes_{F_{2n}} D \simeq \left| F_n \otimes F_{2n}^{\otimes \bullet} D \right|$$

is an object of $\mathcal{D}(k)_{\geq 4n}$ as well[32].

[30] The only nonzero homology groups of $F_{2n}[2n]$ and F_{2n} in degrees smaller than $4n$ are $\mathrm{H}_0(F_{2n}) \cong k$, $\mathrm{H}_{2n}(F_{2n}[2n])$, and $\mathrm{H}_{2n}(F_{2n})$, so the only thing that needs to be done is check that $\mathrm{H}_{2n}(g')$ is an isomorphism. By Proposition 5.1.5.3 the homology group $\mathrm{H}_{2n}(F_{2n}[2n])$ has a basis represented by the morphism $k[2n] \xrightarrow{1 \otimes \mathrm{id}_{k[2n]}} F_{2n} \otimes k[2n]$. Composing this morphism with g' we obtain by definition the morphism

$$\mu \circ (\mathrm{id}_{F_{2n}} \otimes g) \circ \left(1 \otimes \mathrm{id}_{k[2n]}\right) \simeq \mu \circ (1 \otimes g) \simeq g$$

which also by Proposition 5.1.5.3 forms a basis of $\mathrm{H}_{2n}(F_{2n})$.

[31] See Construction E.8.0.4 for an explanation of the relevant module structures.

[32] See [HA, 4.4.2.8] for this description of the relative tensor product. That the bar construction really looks like this in the individual levels follows from unpacking the definition [HA, 4.4.2.7].

We can thus conclude that for $i \leq 2n$ the morphism $\mathrm{id}_{F_n} \otimes_{\mathrm{id}_{F_{2n}}} \psi$ induces an isomorphism as follows.

$$\mathrm{H}_i(F_n \otimes_{F_{2n}} C) \xrightarrow{\cong} \mathrm{H}_i(F_n \otimes_{F_{2n}} k) \cong \mathrm{H}_i(P)$$

To evaluate the homology groups of $F_n \otimes_{F_{2n}} C$ we can use the long exact homology sequence associated to the cofiber sequence

$$F_n \otimes_{F_{2n}} F_{2n}[2n] \xrightarrow{\mathrm{id}_{F_n} \otimes_{\mathrm{id}_{F_{2n}}} g'} F_n \otimes_{F_{2n}} F_{2n} \xrightarrow{\mathrm{id}_{F_n} \otimes_{\mathrm{id}_{F_{2n}}} \varphi} F_n \otimes_{F_{2n}} C$$

which we obtain by applying $F_n \otimes_{F_{2n}} -$ to the cofiber sequence in the top row of ($**$). Using unitality of the relative tensor product [HA, 4.4.3.16] we can identify this cofiber sequence with the top row in the following commutative diagram[33] in $\mathcal{D}(k)$

$$\begin{tikzcd}
F_n[2n] \simeq F_n \otimes k[2n] \arrow[r, "\mu' \circ \left(\mathrm{id}_{F_n} \otimes (f \circ g)\right)"] & F_n \arrow[r, "\lambda"] \arrow[d, "j"'] & F_n \otimes_{F_{2n}} C \arrow[d, "\mathrm{id}_{F_n} \otimes_{\mathrm{id}_{F_{2n}}} \psi"] \\
& P \arrow[r, dash, "\simeq"] & F_n \otimes_{F_{2n}} k
\end{tikzcd}$$

where f and j are as in Construction 5.1.6.2, μ' is the multiplication morphism for F_n, and λ is a newly introduced name. It thus suffices to show that $\mathrm{H}_i(\lambda)$ is an isomorphism for $i < 2n$ and that additionally $\mathrm{H}_{2n}(F_n \otimes_{F_{2n}} C) \cong 0$.

In the range we are interested in $F_n[2n]$ has only homology in degree $2n$ (see Proposition 5.1.5.3), so that it immediately follows using the long exact sequence in homology that $\mathrm{H}_i(\lambda)$ is an isomorphism for $i < 2n$, and the statements for the homology of P in this range now follow from the calculation of the homology in low degrees of F_n, see Proposition 5.1.5.3.

It remains to show that $\mathrm{H}_{2n}(F_n \otimes_{F_{2n}} C) \cong 0$. By the long exact sequence in homology we have to show for this that $\mu' \circ \left(\mathrm{id}_{F_n} \otimes (f \circ g)\right)$ induces a surjection on H_{2n}. Let $\iota\colon k \to F_n$ be the unit morphism. Then by Proposition 5.1.5.3 there is an isomorphism $\mathrm{H}_{2n}(F_n \otimes k[2n]) \cong k$, and this homology group has a basis formed by by $(\iota \otimes \mathrm{id}_{k[2n]}) \circ \eta$, where $\eta\colon k[2n] \simeq k \otimes k[2n]$ is the unitality equivalence of $\mathcal{D}(k)$. Composing with $\mu' \circ \left(\mathrm{id}_{F_n} \otimes (f \circ g)\right)$ we obtain[34]

$$\begin{aligned}
&\mu' \circ \left(\mathrm{id}_{F_n} \otimes (f \circ g)\right) \circ \left(\iota \otimes \mathrm{id}_{k[2n]}\right) \circ \eta \\
\simeq\ &\mu' \circ \left(\iota \otimes (f \circ g)\right) \circ \eta \\
\simeq\ &f \circ g \\
\simeq\ &f'
\end{aligned}$$

which by definition is a generator of $\mathrm{H}_{2n}(F_n)$. □

[33] The identification of the top left morphism arises from unpacking the definitions. For j fitting into the commutative diagram, note that the composition $F_{2n} \xrightarrow{\varphi} C \xrightarrow{\psi} k$ is by definition homotopic to p, and then use the identification of the pushout diagram from Construction 5.1.6.2 with the one from Proposition E.8.0.5.

[34] The last step is by definition, see Construction 5.1.6.2.

5.1.6.3. On a mapping space of commutative algebras

In this section we show that a mapping space relevant in Section 5.1.7 has only a single path component.

Proposition 5.1.6.4. *Let $n > 0$ be an integer. Let R and S be commutative algebras in $\mathcal{D}(k)$, and assume that the homology of R is concentrated in degrees 0 and n, where it is isomorphic to k, that the homology of S is concentrated in degrees i with $0 \leq i \leq 2n$, and that $\mathrm{H}_n(S) \cong 0$.*

Then

$$\pi_0\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R, S)\Big) \cong * \tag{5.7}$$

So up to homotopy, there is a unique morphism of commutative algebras $R \to S$. ♡

Proof. Consider the commutative algebra P constructed in Construction 5.1.6.2. Proposition 5.1.6.3 implies that $\tau_{\leq 2n}(P)$ has the same homology as R. As the homology is free (as a $\mathbb{Z}$-graded k-module) it follows from Proposition 4.3.3.7 that $\tau_{\leq 2n}(P)$ and R are equivalent as objects of $\mathcal{D}(k)$. It then follows from Proposition 5.1.5.4 that $\tau_{\leq 2n}(P)$ and R are even equivalent as commutative algebras in $\mathcal{D}(k)$.

We thus obtain an equivalence as follows.

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(R, S) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\big(\tau_{\leq 2n}(P), S\big)$$

$\tau_{\leq 2n}(P)$ and S both lie in $(\mathrm{CAlg}(\mathcal{D}(k))_{\geq 0})_{\leq 2n}$ by Proposition 4.3.4.1 (7) and (8), and as the inclusion is fully faithful we obtain another equivalence as follows.

$$\simeq \mathrm{Map}_{(\mathrm{CAlg}(\mathcal{D}(k))_{\geq 0})_{\leq 2n}}\big(\tau_{\leq 2n}(P), S\big)$$

We can now continue with the adjunction from Proposition 4.3.4.1 (8).

$$\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))_{\geq 0}}(P, S)$$

Finally, we use that $\mathrm{CAlg}(\iota_{\geq 0})$ is fully faithful and obtain the following equivalence.

$$\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(P, S)$$

As P was defined as a pushout in $\mathrm{CAlg}(\mathcal{D}(k))$, we obtain a pullback diagram in $\mathcal{S}$ (using notation from Construction 5.1.6.2) as follows.

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(P, S) \arrow[r, "j^*"] \arrow[d, "i^*"'] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]), S\Big) \arrow[d, "f^*"] \\
\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(k, S) \arrow[r, "p^*"'] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]), S\Big)
\end{tikzcd}$$

k is initial as a commutative algebra by [HA, 3.2.1.9], so $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(k, S)$ is contractible. This implies that[35]

$$\mathrm{Map}_{\mathrm{CAlg}}(P, S) \xrightarrow{j^*} \mathrm{Map}_{\mathrm{CAlg}}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]), S\Big) \xrightarrow{f^*} \mathrm{Map}_{\mathrm{CAlg}}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]), S\Big)$$

[35] We shorten $\mathrm{CAlg}(\mathcal{D}(k))$ as CAlg.

is a homotopy fiber sequence of which we can take the long exact sequence of homotopy groups. To show that $\pi_0\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(P,S)\Big) \cong *$ it then suffices to show that both

$$\pi_0\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]),S\Big)\Big)$$

and

$$\pi_1\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]),S\Big)\Big)$$

are trivial.

We can use the adjunction $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)} \dashv \mathrm{ev}_{\langle 1\rangle}$ to rewrite these homotopy groups as follows.

$$\begin{aligned}
\pi_0\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[n]),S\Big)\Big) &\cong \pi_0\big(k[n],\mathrm{ev}_{\langle 1\rangle}(S)\big) \cong \mathrm{H}_n(S) \cong 0\\
\pi_1\Big(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\Big(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[2n]),S\Big)\Big) &\cong \pi_1\big(k[2n],\mathrm{ev}_{\langle 1\rangle}(S)\big)\\
&\cong \pi_0\big(k[2n+1],\mathrm{ev}_{\langle 1\rangle}(S)\big) \cong \mathrm{H}_{2n+1}(S) \cong 0 \;\square
\end{aligned}$$

5.1.7. Formality of certain $\mathbb{E}_\infty,\mathbb{E}_1$-bialgebras

In this section we finally put together the various results from sections Sections 5.1.1, 5.1.2, 5.1.3, 5.1.4, 5.1.5 and 5.1.6 and show formality of commutative bialgebras with homology concentrated in degrees 0 and 1, where it is isomorphic to k.

Proposition 5.1.7.1. *Let R be on object of $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ such that*

$$\mathrm{H}_i(R) \cong \begin{cases} k & \text{for } i=0 \text{ and } i=1\\ 0 & \text{otherwise}\end{cases}$$

Then the underlying object of R in $\mathcal{D}(k)$ is dualizable[36]*.*

Let furthermore[37] *$f_1\colon \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_1) \to R^\vee$ be a morphism in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$ where A_1 is as in Construction 5.1.2.2*[38]*, and $R^\vee$ is the dual of R, see Remark 5.1.1.9. Then f_1 can be extended to a morphism $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A) \to R^\vee$, where A is as in Construction 5.1.2.2.* ♡

Proof. That the underlying object of R is dualizable follows immediately from the assumptions on the homology together with the formality statement Proposition 4.3.3.7, see also Example 5.1.1.8.

By Corollary 5.1.2.14 the morphisms $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_n) \to \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A)$ exhibit $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A)$ as a colimit of

$$\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_1) \to \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_2) \to \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_3) \to \dots$$

[36] See Definition 5.1.1.1.
[37] Recall Notation 5.1.2.12.
[38] A_1 is cofibrant as a chain complex by Proposition 5.1.2.11 Proposition 4.2.2.12.

in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$. It hence suffices to prove inductively that given an integer $n > 1$ and a morphism $f_{n-1}\colon \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_{n-1}) \to R^\vee$ there exists an extension to a morphism $f_n\colon \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_n) \to R^\vee$. Also by Corollary 5.1.2.14, it suffices for this to construct a commutative square

$$\begin{tikzcd}
\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n) \arrow[r] \arrow[d] & \gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(B_n) \arrow[d, dashed] \\
\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(A_{n-1}) \arrow[r, "f_{n-1}"'] & R^\vee
\end{tikzcd}$$

in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$, where the morphism on the left and top are the ones constructed in Construction 5.1.2.5. Proposition 5.1.2.10 and Remark 5.1.2.9 imply that $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(B_n)$ is a zero object in $\mathrm{Alg}(\mathrm{coCAlg}(\mathcal{D}(k)))$, so there is an essentially unique morphism $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(B_n) \to R^\vee$ we can fill in on the right.

What remains is to construct a homotopy between the two possible composites from $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n)$ to $R^\vee$ in the diagram. For this it suffices to show that *any* two morphisms from $\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n)$ to $R^\vee$ are homotopic, i. e. that

$$\pi_0\left(\mathrm{Map}_{\mathrm{Alg}\left(\mathrm{coCAlg}(\mathcal{D}(k))\right)}\left(\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n), R^\vee\right)\right) \cong *$$

In Proposition 5.1.2.18 it was shown that

$$\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n) \simeq \mathrm{Free}^{\mathrm{Alg}(\mathrm{coCAlg})}_{\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{coCAlg})}\left(\gamma^{\mathsf{Comm}}_{\mathbb{E}_0}(\underline{B}'_n)\right)$$

where $\gamma^{\mathsf{Comm}}_{\mathbb{E}_0}(\underline{B}'_n)$ is an object in

$$\mathrm{Alg}_{\mathbb{E}_0}\left(\mathrm{coCAlg}\left(\mathcal{D}(k)\right)\right)$$

with underlying object equivalent to $k \oplus k[-2]$, see Construction 5.1.2.15. We thus obtain an isomorphism as follows.

$$\begin{aligned}
&\pi_0\left(\mathrm{Map}_{\mathrm{Alg}\left(\mathrm{coCAlg}(\mathcal{D}(k))\right)}\left(\gamma^{\mathsf{Comm}}_{\mathsf{Assoc}}(\underline{B}_n), R^\vee\right)\right)\\
\cong\ &\pi_0\left(\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_0}\left(\mathrm{coCAlg}(\mathcal{D}(k))\right)}\left(\gamma^{\mathsf{Comm}}_{\mathbb{E}_0}(\underline{B}'_n), R^\vee\right)\right)
\end{aligned}$$

By [HA, 2.1.3.10], the ∞-category of $\mathbb{E}_0$-algebras in a monoidal ∞-category $\mathcal{C}$ can be identified with $\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$, so applying this and dualizing (see Fact 5.1.1.3), we obtain the following isomorphisms.

$$\begin{aligned}
&\pi_0\left(\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_0}\left(\mathrm{coCAlg}(\mathcal{D}(k))\right)}\left(\gamma^{\mathsf{Comm}}_{\mathbb{E}_0}(\underline{B}'_n), R^\vee\right)\right)\\
\cong\ &\pi_0\left(\mathrm{Map}_{\mathrm{coCAlg}(\mathcal{D}(k))_{k/}}\left(\gamma^{\mathsf{Comm}}(\underline{B}'_n), R^\vee\right)\right)
\end{aligned}$$

$$\cong \pi_0\left(\mathrm{Map}_{(\mathrm{CAlg}(\mathcal{D}(k)))_{/k}}\left(R, \gamma^{\mathsf{Comm}}(\underline{B}'_n)^\vee\right)\right)$$

By the assumptions on R, the truncation $\tau_{\leq 0}(R)$ has homology groups concentrated in degree 0 and $\mathrm{H}_0(R)$ is free of rank 1. Using Proposition 4.3.3.7 we can thus apply Proposition 5.1.6.1 to obtain the following isomorphism.

$$\begin{aligned}&\pi_0\left(\mathrm{Map}_{(\mathrm{CAlg}(\mathcal{D}(k)))_{/k}}\left(R, \gamma^{\mathsf{Comm}}(\underline{B}'_n)^\vee\right)\right)\\ \cong\ &\pi_0\left(\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}\left(R, \gamma^{\mathsf{Comm}}(\underline{B}'_n)^\vee\right)\right)\end{aligned}$$

As the dual of $k[l]$ is $k[-l]$, the underlying object in $\mathcal{D}(k)$ of $\gamma^{\mathsf{Comm}}(\underline{B}'_n)^\vee$ is equivalent to $k \oplus k[2]$. Now we can apply Proposition 5.1.6.4 to conclude that this set has exactly one element. □

Proposition 5.1.7.2. *Let R be on object of $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ such that*

$$\mathrm{H}_i(R) \cong \begin{cases} k & \textit{for } i = 0 \textit{ and } i = 1 \\ 0 & \textit{otherwise}\end{cases}$$

and let $g\colon k[1] \to R$ be a morphism in $\mathcal{D}(k)$ representing a basis of $\mathrm{H}_1(R)$. Let x be an element of k. Then there exists a morphism[39]

$$\varphi\colon R \to \gamma(A)^\vee$$

in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ that induces an isomorphism on H_0 and is such that $\mathrm{H}_1(\varphi)$ maps the element represented by g to $x \cdot y_1^\vee$ (see Proposition 5.1.2.4). ♡

Proof. Consider the commutative algebra $\gamma(A'_1)^\vee$. (see Construction 5.1.2.15 for a definition of A'_1). The underlying object of $\gamma(A'_1)$ in $\mathcal{D}(k)$ is by definition equivalent to $k \oplus k[-1]$, so

$$\mathrm{H}_i\left(\gamma(A'_1)^\vee\right) \cong \begin{cases} k & \text{for } i = 0 \text{ and } i = 1 \\ 0 & \text{otherwise}\end{cases}$$

with the homology group in degree 1 generated by $y_1^\vee$.

Define a morphism $\varphi''_1\colon \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}(k[1]) \to \gamma(A'_1)^\vee$ such that composing the morphism $k[1] \to \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[1])$ exhibiting $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[1])$ as the free commutative algebra generated by $k[1]$ with φ''_1 represents the element $x \cdot y_1^\vee$ in $\mathrm{H}_1(\gamma(A'_1)^\vee)$. As a morphism of commutative algebras, the unit morphisms must be preserved, so φ''_1 induces an isomorphism on H_0 by Proposition 5.1.3.3.

We obtain an induced morphism

$$\varphi'_1\colon R \simeq \mathrm{CAlg}(\tau_{\leq 1})\left(\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)_{\geq 0}}(k[1])\right) \xrightarrow{\mathrm{CAlg}(\tau_{\leq 1})(\varphi''_1)} \mathrm{CAlg}(\tau_{\leq 1})\left(\gamma(A'_1)^\vee\right) \simeq \gamma(A'_1)^\vee$$

[39]For a definition of A, see Construction 5.1.2.2. For the duality functor see Fact 5.1.1.3.

where the first equivalence is the one from Proposition 5.1.5.4[40] and the second equivalence is the one arising from $\gamma(A_1')^\vee$ already being concentrated in degrees 0 and 1. φ_1' then induces an isomorphism on H_0 and satisfies $\mathrm{H}_1(\varphi_1')(g) = x \cdot y_1^\vee$.

Applying [HA, 2.1.3.10] and Proposition 5.1.6.1 we can upgrade φ_1' to a morphism in $\mathrm{BiAlg}_{\mathsf{Comm},\mathbb{E}_0}(\mathcal{D}(k))$. Next, applying Proposition 5.1.2.18 and dualizing, we can lift this morphism to a morphism

$$\varphi_1\colon R \to \gamma(A_1)^\vee$$

in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ such that the triangle

$$\begin{tikzcd} R \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_1'"'] & \gamma(A_1)^\vee \arrow[d] \\ & \gamma(A_1')^\vee \end{tikzcd}$$

of underlying morphisms of commutative algebras commutes, with the vertical morphism being the dual of γ applied to the inclusion $A_1' \to A_1$. Applying Proposition 5.1.7.1 (and dualizing twice), we can further lift φ_1 to a morphism φ that fits into a commuting triangle in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ as follows.

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "\varphi_1"'] & \gamma(A)^\vee \arrow[d] \\ & \gamma(A_1)^\vee \end{tikzcd}$$

By Proposition 5.1.2.4 (and dualizing) the homology of $\gamma(A)^\vee$ is k in degrees 0 and 1 and 0 in other degrees, and a basis is formed by $1^\vee$ in degree 0 and by $y_1^\vee$ in degree 1. As the inclusion $A_1' \to A$ sends 1 to 1 and y_1 to y_1, it follows that the induced morphisms $H_i(\gamma(A)^\vee) \to H_i(\gamma(A_1')^\vee)$ send $1^\vee$ to $1^\vee$ and $y_1^\vee$ to $y_1^\vee$ and are thus in particular isomorphisms. That φ satisfies the required properties now follows from this together with the description of φ_1' discussed above. $\square$

Proposition 5.1.7.3. *Let R and S be objects in* $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ *such that*

$$\mathrm{H}_i(R) \cong \begin{cases} k & \text{for } i = 0 \text{ and } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{H}_i(S) \cong \begin{cases} k & \text{for } i = 0 \text{ and } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

[40] We choose this equivalence to be such that the morphism $k[1] \to \mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[1])$ exhibiting $\mathrm{Free}^{\mathrm{CAlg}}_{\mathcal{D}(k)}(k[1])$ as the free commutative algebra generated by $k[1]$ composed with the equivalence is homotopic to g.

and let $\{g_R\}$ *and* $\{g_S\}$ *be a basis of* $\mathrm{H}_1(R)$ *and* $\mathrm{H}_1(S)$*, respectively. Let* x *be an element of* k*.*

Then there exists a morphism

$$\varphi\colon R \to S$$

in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ *such that* $\mathrm{H}_0(\varphi)$ *is an isomorphism and* $\mathrm{H}_1(\varphi)(g_R) = x \cdot g_S$*.*

In particular, φ *is an equivalence if and only if* x *is a invertible in* k*.* ♡

Proof. By Proposition 5.1.7.2 we can construct morphisms

$$R \xrightarrow{\varphi_R} \gamma(A)^\vee \xleftarrow{\varphi_S} S$$

in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ such that both φ_R and φ_S induce an isomorphism on H_0 and

$$\mathrm{H}_1(\varphi_R)(g_R) = x \cdot y_1^\vee \qquad \text{and} \qquad \mathrm{H}_1(\varphi_S)(g_S) = y_1^\vee$$

It follows from Proposition 5.1.2.4 and [HA, 3.2.2.6] that φ_S is an equivalence and φ_R is an equivalence if and only if x is invertible. We now define φ as the composition $(\varphi_S)^{-1} \circ \varphi_R$. □

5.2. The k-linear circle as an $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebra

The goal of this section is to define the circle group $\mathbb{T}$ as well as its k-linear version $k \boxtimes \mathbb{T}$ as commutative and cocommutative bialgebras, for $\mathbb{T}$ in $\mathcal{S}$, and for $k \boxtimes \mathbb{T}$ in $\mathcal{D}(k)$.

$\mathbb{T}$ will be defined in Section 5.2.1. We will then discuss the linearization functor $k \boxtimes -\colon \mathcal{S} \to \mathcal{D}(k)$ in Section 5.2.2, and apply it to define $k \boxtimes \mathbb{T}$ in the very short Section 5.2.3.

5.2.1. The circle group

Let W be the class of weak equivalences in the model structure on sSet discussed in [Hov99, Chapter 3] and [HTT, After A.2.7.3] – these are the morphisms whose geometric realization is a homotopy equivalence of topological spaces. The infinity category of spaces $\mathcal{S}$ can then be defined by inverting those weak equivalences of simplicial sets, so as

$$\mathcal{S} \coloneqq \mathsf{sSet}[W^{-1}]$$

see [HTT, 1.2.16.1] in combination with [HA, 1.3.4.20]. The canonical functor $\mathsf{sSet} \to \mathcal{S}$ preserves finite products, as finite products in sSet are automatically homotopy products[41]. The functor $\mathrm{Sing}\colon \mathsf{Top} \to \mathsf{sSet}$ also preserves products as a right adjoint, so that the composition $\mathsf{Top} \to \mathcal{S}$ also preserves finite products. Giving both involved

[41] As the geometric realization functor $|-|\colon \mathsf{sSet} \to \mathsf{Top}$ is the left adjoint of a Quillen equivalence, this follows from every object in sSet being cofibrant, $|-|$ preserving products [Hov99, 3.1.8], and every object in Top being fibrant.

∞-categories the cartesian symmetric monoidal structure [HA, 2.4.1] upgrades this functor to a symmetric monoidal functor, and so induces an (again symmetric monoidal) functor of ∞-categories of commutative algebras $\mathrm{CAlg}(\mathsf{Top}) \to \mathrm{CAlg}(\mathcal{S})$. This allows us to construct commutative algebras in $\mathcal{S}$ by giving an explicit commutative topological monoid, which we will use in the following construction.

Construction 5.2.1.1. We let the *circle group* $\mathbb{T}$ refer to the object in $\mathrm{CAlg}(\mathcal{S})$ obtained by applying the above functor $\mathrm{CAlg}(\mathsf{Top}) \to \mathrm{CAlg}(\mathcal{S})$ to the (multiplicative) commutative submonoid $\left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\}$ of $\mathbb{C}$.

Note that every commutative topological monoid can be upgraded to a commutative and cocommutative topological bimonoid, with comultiplication given by the diagonal map. This phenomenon is in fact more general, as we saw in Proposition 3.3.1.2 that any commutative algebra in a cartesian symmetric monoidal ∞-category can be upgraded in an essentially unique way to a commutative and cocommutative bialgebra.

In particular, we can upgrade $\mathbb{T}$ in an essentially unique way to an $\mathbb{E}_\infty, \mathbb{E}_\infty$-bialgebra in spaces. ♢

5.2.2. The linearization functor

In Section 5.2.1 we considered $\mathcal{S}$ as a symmetric monoidal ∞-category via the cartesian symmetric monoidal structure. There is also a different way of defining the symmetric monoidal structure on $\mathcal{S}$, as we discuss in the following remark.

Remark 5.2.2.1. The ∞-category $\mathcal{S}$ is the unit object in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ by [HA, 4.8.1.20], and hence can be upgraded to a presentable symmetric monoidal ∞-category that is initial in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ by [HA, 3.2.1.9] in combination with [HA, 4.8.1.9 and 4.8.1.15].

To show that the so obtained symmetric monoidal structure is equivalent to the cartesian symmetric monoidal structure, it suffices in light of [HA, 4.8.1.12] to show that the product functor $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ preserves colimits separately in each variable, which is shown in [HTT, 6.1.3.14]. ♢

The characterization of $\mathcal{S}$ as an initial object in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ allows the following definition.

Definition 5.2.2.2. Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category. Then we obtain an essentially unique colimit preserving symmetric monoidal functor that we denote as follows.

$$\mathbb{1}_{\mathcal{C}} \boxtimes - \colon \mathcal{S} \to \mathcal{C}$$

As $\mathcal{D}(k)$ is a presentable symmetric monoidal ∞-category by Proposition 4.3.2.1 (1), we hence obtain a colimit preserving symmetric monoidal functor

$$k \boxtimes - \colon \mathcal{S} \to \mathcal{D}(k)$$

that we sometimes call the *k-linearization functor*. ♢

Remark 5.2.2.3. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then universality of the functors defined in Definition 5.2.2.2 imply that we obtain a commuting triangle

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "k\boxtimes-"'] \arrow[dr, "k'\boxtimes-"] & \\ \mathcal{D}(k) \arrow[rr, "k'\otimes_k-"'] & & \mathcal{D}(k') \end{tikzcd}$$

where $k' \otimes_k -$ is the colimit-preserving symmetric monoidal functor discussed in Remark 4.3.2.2. ♢

Let X be an object of $\mathcal{S}$. In Section 4.3.3 we discussed the homology functors H_n on $\mathcal{D}(k)$, which we could thus apply to $k \boxtimes X$. In the rest of this section we show that this is compatible with the classical notions of homology of spaces. We begin by reviewing the definition of homology of simplicial sets.

Construction 5.2.2.4. We construct a functor

$$k \cdot -\colon \mathsf{sSet} \to \mathsf{Ch}(k)$$

as follows. There is a functor, which we also call $k \cdot -$, from Set to $\mathrm{LMod}_k(\mathsf{Ab})$ that maps a set X to the free k-module on the basis X. This functor induces a functor as follows.

$$\mathsf{sSet} \cong \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set}) \xrightarrow{(k\cdot -)_*} \mathrm{Fun}\big(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{LMod}_k(\mathsf{Ab})\big)$$

The functor $k \cdot -\colon \mathsf{sSet} \to \mathsf{Ch}(k)$ is then to be the composition of this functor with the functor

$$C\colon \mathrm{Fun}\big(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{LMod}_k(\mathsf{Ab})\big) \to \mathsf{Ch}(k)$$

that maps a functor F to the chain complex $C(F)$ for which $C(F)_n := F([n])$ and $\partial_n^{C(F)} := \sum_{i=0}^{n} F(\delta_i)$. ♢

Classically, one defines homology for simplicial sets X with coefficients in the commutative ring k as $\mathrm{H}_n(X, k) := \mathrm{H}_n(k \cdot X)$. For topological spaces one then defines homology as the homology of their singular simplicial set.

What we would like to show is that there is a commutative diagram

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, "k\cdot-"] \arrow[d] & \mathsf{Ch}(k) \arrow[d, "\gamma"'] \arrow[dr, "\mathrm{H}_n"] & \\ \mathcal{S} \arrow[r, "k\boxtimes-"'] & \mathcal{D}(k) \arrow[r, "\mathrm{H}_n"'] & \mathrm{LMod}_k(\mathsf{Ab}) \end{tikzcd}$$

where the left vertical functor is the canonical one. That there is a filler for the right triangle was shown in Proposition 4.3.3.2. It thus remains to show that there is a filler for the left square. The strategy will be to use that colimit-preserving functors out of

$\mathcal{S}$ are determined by their value on the one-point-space $*$. So we will show that $k \cdot -$ induces a colimit-preserving functor on underlying ∞-categories that maps $*$ to k. This functor will then by definition fit into such a commutative square but also be homotopic to $k \boxtimes -$.

Proposition 5.2.2.5. *The functor*

$$k \cdot - \colon \mathsf{sSet} \to \mathsf{Ch}(k)$$

from Construction 5.2.2.4 preserves weak equivalences as well as cofibrations, where sSet *carries the model structure discussed in [Hov99, Chapter 3] and [HTT, After A.2.7.3], and* $\mathsf{Ch}(k)$ *carries the projective model structure from Fact 4.1.3.1.* ♡

Proof. Weak equivalences in sSet are those maps whose geometric realization is a homotopy equivalence of spaces, and that singular homology maps homotopy equivalences to isomorphisms is classical[42].

Now let $f \colon X \to Y$ be a cofibration in sSet, i. e. the map of sets $f_n \colon X_n \to Y_n$ is injective for every $n \geq 0$. To show that $k \cdot f$ is a cofibration we have by [Hov99, 2.3.9] to show that $k \cdot f$ is a levelwise split injection and that $k \cdot f$ has cofibrant cokernel.

But the morphism $(k \cdot f)_n$ is a morphism of free k-modules induced by an injection among the basis sets, so is a split injection. The cokernel can then be identified with a chain complex that is concentrated in nonnegative degrees and that in level $n \geq 0$ is given by the free k-module with basis $Y_n \setminus f_n(X_n)$. Thus the cokernel of $k \cdot f$ is cofibrant by [Hov99, 2.3.6]. □

Definition 5.2.2.6. By Proposition 5.2.2.5 the functor $k \cdot -$ from Construction 5.2.2.4 induces a functor

$$k \cdot - \colon \mathsf{sSet}^{\mathrm{cof}} \to \mathsf{Ch}(k)^{\mathrm{cof}}$$

preserving weak equivalences and thus a functor on underlying ∞-categories[43]

$$\mathcal{S} \simeq \mathsf{sSet}[W^{-1}] \to \mathsf{Ch}(k)^{\mathrm{cof}}[W'^{-1}] \simeq \mathcal{D}(k)$$

that we also call $k \cdot -$.

By construction this functor comes with a commutative square

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, "k\cdot -"] \arrow[d] & \mathsf{Ch}(k)^{\mathrm{cof}} \arrow[d, "\gamma"] \\ \mathcal{S} \arrow[r, "k\cdot -"'] & \mathcal{D}(k) \end{tikzcd} \tag{5.8}$$

of ∞-categories, where the left vertical functor is the canonical one. ♢

[42] For a discussion in a textbook see for example [Bre93, 16.5]

[43] W is to be the class of weak equivalences in sSet and W' the class of weak equivalences in $\mathsf{Ch}(k)$.

Proposition 5.2.2.7. *The functor*

$$k \cdot - \colon \mathsf{sSet} \to \mathsf{Ch}(k)$$

from Construction 5.2.2.4 preserves small colimits. ♡

Proof. Colimits in both sSet as well as $\mathsf{Ch}(k)$ are calculated levelwise. The statement thus boils down to the functor $k \cdot - \colon \mathsf{Set} \to \mathrm{LMod}_k(\mathsf{Ab})$ preserving colimits. But this functor is left adjoint to the forgetful functor. □

Proposition 5.2.2.8. *The functor*

$$k \cdot - \colon \mathcal{S} \to \mathcal{D}(k)$$

from Definition 5.2.2.6 preserves small colimits. ♡

Proof. By Fact 4.1.3.1 and [HTT, After A.2.7.3] sSet and $\mathsf{Ch}(k)$ are combinatorial model categories. Furthermore, by Proposition 5.2.2.7, [HTT, 5.5.2.9][44], and Proposition 5.2.2.5, the functor

$$k \cdot - \colon \mathsf{sSet} \to \mathsf{Ch}(k)$$

is a left Quillen functor between combinatorial model categories.

The claim thus follows from [HA, 1.3.4.26]. □

Proposition 5.2.2.9. *The functors $k \cdot -$ from Definition 5.2.2.6 and $k \boxtimes -$ from Definition 5.2.2.2 are homotopic as functors of infinity categories from $\mathcal{S}$ to $\mathcal{D}(k)$.* ♡

Proof. $k \boxtimes -$ preserves small colimits by definition and $k \cdot -$ by Proposition 5.2.2.8. Then [HTT, 5.1.5.6] implies that it suffices to check that $k \boxtimes * \simeq k \cdot *$, where $*$ is the one-point-space.

As $k \boxtimes -$ is by definition symmetric monoidal, it maps the monoidal unit $*$ of $\mathcal{S}$ to the monoidal unit k of $\mathcal{D}(k)$.

As $\gamma \colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ is also symmetric monoidal it thus suffices to show that the chain complex[45] $k \cdot *$ is quasiisomorphic to $k[0]$. But it can easily be seen from the definition that $k \cdot *$ is the chain complex[46]

$$\cdots \leftarrow 0 \leftarrow k \xleftarrow{0} k \xleftarrow{\mathrm{id}} k \xleftarrow{0} k \xleftarrow{\mathrm{id}} \cdots$$

and the obvious inclusion of $k[0]$ is a quasiisomorphism. □

We can now put everything together and summarize the previous results as follows.

[44] As both sSet and $\mathsf{Ch}(k)$ have combinatorial model structures they are presentable.

[45] Here $*$ is the simplicial set $\mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ that is constant with value $*$. As pointed out in the introduction to Section 5.2.1, the canonical functor $\mathsf{sSet} \to \mathcal{S}$ preserves finite products, so this simplicial set $*$ maps to the space $*$ in $\mathcal{S}$.

[46] The leftmost k is in level 0.

Proposition 5.2.2.10. *There is a commutative diagram*

$$\begin{tikzcd}
\mathsf{sSet} \arrow[r, "k\cdot -"] \arrow[d] & \mathsf{Ch}(k) \arrow[d, "\gamma"] \arrow[dr, "\mathrm{H}_n"] & \\
 & & \mathrm{LMod}_k(\mathsf{Ab}) \\
\mathcal{S} \arrow[r, "k\boxtimes -"'] & \mathcal{D}(k) \arrow[ur, "\mathrm{H}_n"'] &
\end{tikzcd}$$

where the left vertical functor is the canonical one. ♡

Proof. For the left commutative square combine Proposition 5.2.2.9 with the commutative square (5.8) from Definition 5.2.2.6. The right commutative triangle was constructed in Proposition 4.3.3.2. □

5.2.3. Definition of the k-linear circle

We can now define the k-linear circle as a bialgebra in $\mathcal{D}(k)$.

Definition 5.2.3.1. The *k-linear circle* is the $\mathbb{E}_\infty, \mathbb{E}_\infty$-bialgebra $k \boxtimes \mathbb{T}$ in $\mathcal{D}(k)$. ◇

5.2.4. Formality of the k-linear circle as an $\mathbb{E}_\infty, \mathbb{E}_1$-bialgebra

In this section we apply the main result of Section 5.1, Proposition 5.1.7.3, to the commutative bialgebra $k \boxtimes \mathbb{T}$ that we defined in Section 5.2.3. We start by recording the homology of $k \boxtimes \mathbb{T}$.

Proposition 5.2.4.1. *The following holds for the homology of $k \boxtimes \mathbb{T}$ as defined in Definition 5.2.3.1.*

$$\mathrm{H}_i(\mathbb{T} \boxtimes k) \cong \begin{cases} k & \textit{for } i = 0 \textit{ and } i = 1 \\ 0 & \textit{otherwise} \end{cases}$$

♡

Proof. By Proposition 5.2.2.10 and using the definition of $\mathbb{T}$ in Construction 5.2.1.1 there is an isomorphism

$$\mathrm{H}_*(k \boxtimes \mathbb{T}) \cong \mathrm{H}_*\Big(\big\{\, z \in \mathbb{C} \;\big|\; |z| = 1 \,\big\}\,; k\Big) \cong \mathrm{H}_*\big(S^1; k\big)$$

where on the right we have the usual singular homology of the topological 1-sphere with coefficients in k. □

We can now put all the work of Section 5.1 to use to obtain an equivalence of commutative bialgebras between $k \boxtimes \mathbb{T}$ and D.

Proposition 5.2.4.2. *Let g be a basis element of $\mathrm{H}_1(k \boxtimes \mathbb{T})$. Then there exists an equivalence*[47]

$$\varphi \colon \mathrm{D} \to k \boxtimes \mathbb{T}$$

in $\mathrm{BiAlg}_{\mathsf{Comm},\mathsf{Assoc}}(\mathcal{D}(k))$ *that sends the element* d *of* $\mathrm{H}_1(\mathrm{D})$ *to the element* g *in* $\mathrm{H}_1(k \boxtimes \mathbb{T})$. ♡

Proof. Follows directly from Proposition 5.2.4.1 and Proposition 5.1.7.3. □

From Proposition 5.2.4.2 we obtain an equivalence $\mathrm{D} \simeq k \boxtimes \mathbb{T}$ as commutative bialgebras. This equivalence is however not canonically determined – not even the induced isomorphism on homology is, it depends on the choice of a element g of $\mathrm{H}_1(k \boxtimes \mathbb{T})$ that forms a basis. If g_0 is one element that forms a basis, then the set of all elements forming a basis is given by the products $x \cdot g_0$ where x is an invertible element of k. So which element should we choose?

We can reduce the indeterminacy by varying the ground ring. It follows from Construction 4.2.1.1, Remark 4.3.2.2, and Remark 5.2.2.3 that an equivalence of commutative bialgebras $\mathrm{D}_{\mathbb{Z}} \simeq \mathbb{Z} \boxtimes \mathbb{T}$ in $\mathcal{D}(\mathbb{Z})$ induces an equivalence of commutative bialgebras as follows

$$\mathrm{D}_k \simeq k \otimes_{\mathbb{Z}} \mathrm{D}_{\mathbb{Z}} \simeq k \otimes_{\mathbb{Z}} \mathbb{Z} \boxtimes \mathbb{T} \simeq k \boxtimes \mathbb{T}$$

where the first equivalence is the one obtained from combining Construction 4.2.1.1 with Remark 4.3.2.2, the middle equivalence arises from applying $k \otimes \mathbb{Z}-$ to the equivalence $\mathrm{D}_{\mathbb{Z}} \simeq \mathbb{Z} \boxtimes \mathbb{T}$, and the last equivalence is the one from Remark 5.2.2.3. By choosing this equivalence for k, we have thus reduced the indeterminacy of the isomorphism on H_1 to choosing one of the two generators of $\mathrm{H}_1(\mathbb{Z} \boxtimes \mathbb{T}) \cong \mathbb{Z}$.

So which generator of $\mathrm{H}_1(\mathbb{Z} \boxtimes \mathbb{T})$ should we choose? We will in Section 6.1.1 define a 1-category $\mathbf{\Lambda}$ and call functors from $\mathbf{\Lambda}^{\mathrm{op}}$ into an ∞-category *cyclic objects* in that ∞-category. We will consider two relevant constructions on cyclic objects. We will define a functor

$$|-|_{\mathsf{Mixed}} \colon \operatorname{Fun}\Big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathsf{Mixed}_{\mathrm{cof}} = \mathrm{LMod}_{\mathrm{D}}(\mathsf{Ch}^{\mathrm{cof}})$$

in Section 6.3.1.2 and a functor

$$|-| \colon \operatorname{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \to \mathcal{D}(k)^{\mathrm{B}\mathbb{T}}$$

in Section 6.1.3. Note that there are automorphisms of D and $\mathbb{T}$ that introduce a sign. For D we can describe this automorphism by $\mathrm{d} \mapsto -\mathrm{d}$, and the automorphism of $\mathbb{T}$ is given by $z \mapsto z^{-1}$. These reflect choices that are made when defining the two functors we just mentioned – for example for $|-|_{\mathsf{Mixed}}$ there is no intrinsic reason to define d the way it is done rather than adding an extra sign. But in any case, there are choices that have been made for both $|-|_{\mathsf{Mixed}}$ and $|-|$.

[47] See Notation 4.4.0.2 and Construction 4.2.1.1 for a definition of D and Definition 5.2.3.1 for a definition of $k \boxtimes \mathbb{T}$.

The result [Hoy18, 2.3] can now be phrased as follows: There is a generator of $\mathrm{H}_1(\mathbb{Z}\boxtimes\mathbb{T})$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rrr, "|-|_{\mathsf{Mixed}}"] \arrow[d, "\gamma_*"'] & & & \mathrm{LMod}_{\mathrm{D}}\big(\mathsf{Ch}^{\mathrm{cof}}\big) \arrow[d, "\gamma_{\mathsf{Mixed}}"] \\
\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \arrow[r, "|-|"'] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[r, "\simeq"'] & \mathrm{LMod}_{k\boxtimes\mathbb{T}}\big(\mathcal{D}(k)\big) \arrow[r, "\simeq"'] & \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)
\end{tikzcd} \tag{5.9}$$

where the middle bottom horizontal equivalence is one we will construct in Section 5.3, the right bottom horizontal equivalence is the one induced by the equivalence $\mathrm{D} \simeq k\boxtimes\mathbb{T}$ arising as discussed above from the choice of generator of $\mathrm{H}_1(\mathbb{Z}\boxtimes\mathbb{T})$, and γ_{Mixed} is the functor $\mathsf{Mixed}_{\mathrm{cof}} \to \mathcal{M}\mathrm{ixed}$ from Notation 4.4.1.2. We thus make the following convention.

Convention 5.2.4.3. From now on, when we refer to *the* equivalence of commutative bialgebras in $\mathcal{D}(k)$

$$\mathrm{D} \xrightarrow{\simeq} k\boxtimes\mathbb{T}$$

then this is to be the equivalence that arises in the manner discussed above from the generator of $\mathrm{H}_1(\mathbb{Z}\boxtimes\mathbb{T})$ that is such that there is a commutative diagram (5.9). ♢

Remark 5.2.4.4. The equivalence of bialgebras from Convention 5.2.4.3 induces via the functor LMod from Definition 3.4.2.1 an equivalence of monoidal ∞-categories

$$\mathrm{LMod}_{k\boxtimes\mathbb{T}}\big(\mathcal{D}(k)\big) \simeq \mathrm{LMod}_{\mathrm{D}}\big(\mathcal{D}(k)\big)$$

that is compatible with the forgetful functors to $\mathcal{D}(k)$.

Furthermore, if $\varphi\colon k \to k'$ is a morphism of commutative rings, then there is a commutative diagram[48]

$$\begin{tikzcd}
\mathrm{LMod}_{k\boxtimes\mathbb{T}} \arrow[rr, dash, "\simeq"] \arrow[dr, "\mathrm{ev}_{\mathrm{m}}"] \arrow[ddd, "k'\otimes_k -"'] & & \mathrm{LMod}_{\mathrm{D}_k}\big(\mathcal{D}(k)\big) \arrow[dl, "\mathrm{ev}_{\mathrm{m}}"'] \arrow[ddd, "k'\otimes_k -"] \\
& \mathcal{D}(k) \arrow[d, "k'\otimes_k -"] & \\
& \mathcal{D}(k') & \\
\mathrm{LMod}_{k'\boxtimes\mathbb{T}} \arrow[rr, dash, "\simeq"'] \arrow[ur, "\mathrm{ev}_{\mathrm{m}}"] & & \mathrm{LMod}_{\mathrm{D}_{k'}}\big(\mathcal{D}(k)\big) \arrow[ul, "\mathrm{ev}_{\mathrm{m}}"']
\end{tikzcd}$$

[48] There is also supposed to be a filler for the outer diagram that is compatible with the forgetful functors, i. e. this is a three-dimensional diagram that we are looking at from the top.

of monoidal functors, where the horizontal equivalences are the ones just mentioned and the vertical functors are induced by the symmetric monoidal functor

$$k' \otimes_k -\colon\ \mathcal{D}(k) \to \mathcal{D}(k')$$

from Remark 4.3.2.2. ◊

5.3. Group actions and modules over group rings

Let G be a grouplike[49] associative monoid in $\mathcal{S}$. One important class of examples is supplied by pointed spaces X by taking the loop space ΩX, which has a multiplication arising from composition of loops. The details of this construction are discussed in [HA, Introduction to 5.2.6], where a functor

$$\beta_1\colon \mathcal{S}_* \to \mathrm{Mon}^{\mathrm{gp}}_{\mathsf{Assoc}}(\mathcal{S})$$

is constructed that implements this idea. It turns out that there are no other examples, and that the restriction of β_1 to the full subcategory $\mathcal{S}_*^{\geq 1}$ of $\mathcal{S}_*$ spanned by the path connected spaces is an equivalence

$$\beta_1\colon \mathcal{S}_*^{\geq 1} \xrightarrow{\simeq} \mathrm{Mon}^{\mathrm{gp}}_{\mathsf{Assoc}}(\mathcal{S})$$

as shown in [HA, 5.2.6.10]. The inverse functor of this equivalence will be called B. If we interpret $\mathrm{B}\,G$ as an ∞-groupoid, then $\mathrm{B}\,G$ has (up to equivalence) a unique object, and that object's automorphism space is equivalent to $\Omega\,\mathrm{B}\,G \simeq G$.

Now if $\mathcal{C}$ is an ∞-category, then we can consider the ∞-category of objects with G-action in $\mathcal{C}$, which is defined as[50] follows.

$$\mathcal{C}^{\mathrm{B}\,G} \coloneqq \mathrm{Fun}(\mathrm{B}\,G, \mathcal{C})$$

If $\mathcal{C}$ carries a symmetric monoidal structure, then $\mathcal{C}^{\mathrm{B}\,G}$ can be given the induced pointwise symmetric monoidal structure.

On the other hand, if $\mathcal{C}$ is presentable symmetric monoidal, then we can form out of the Assoc-algebra[51] G in $\mathcal{S}$ the Assoc-algebra $\mathbb{1}_{\mathcal{C}} \boxtimes G$ in $\mathcal{C}$ (see Remark 5.2.2.1), and hence consider the ∞-category $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C})$ of left-$\mathbb{1}_{\mathcal{C}} \boxtimes G$-modules in $\mathcal{C}$. In fact, G can be upgraded essentially uniquely to an object in $\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Comm}}(\mathcal{S})$ by Proposition 3.3.1.2, with comultiplication given by the diagonal map $G \xrightarrow{\mathrm{id}_G \times \mathrm{id}_G} G \times G$. We hence also obtain an $\mathsf{Assoc}, \mathsf{Comm}$-bialgebra structure on $\mathbb{1}_{\mathcal{C}} \boxtimes G$, and thus an induced symmetric monoidal structure on $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C})$ by Definition 3.4.2.1.

[49]See [HA, 5.2.6.2] for a definition.
[50]See for example [HA, 6.1.6.2] for this definition.
[51]By [HA, 2.4.2.5] the ∞-categories of Assoc-monoids in $\mathcal{S}$ and Assoc-algebras in $\mathcal{S}$ are equivalent, as the symmetric monoidal structure on $\mathcal{S}$ is cartesian (see Remark 5.2.2.1).

Let us remark that the diagonal map is also used behind the scenes when defining the pointwise symmetric monoidal structure on $\mathcal{C}^{\mathrm{B}\,G}$ – the pointwise tensor product of two functors F and G can be written as the composition

$$\mathrm{B}\,G \xrightarrow{\mathrm{id}_{\mathrm{B}\,G} \times \mathrm{id}_{\mathrm{B}\,G}} \mathrm{B}\,G \times \mathrm{B}\,G \xrightarrow{F \times G} \mathcal{C} \times \mathcal{C} \xrightarrow{- \otimes -} \mathcal{C}$$

and the diagonal functor of $\mathrm{B}\,G$ can on automorphism spaces be identified with the diagonal map of G.

We can now ask the question whether $\mathcal{C}^{\mathrm{B}\,G}$ and $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C})$ are equivalent as symmetric monoidal ∞-categories, which Proposition 5.3.0.8, which is the goal of this section, will answer affirmatively.

As technical input we need to start by discussing compatibility of the tensor product of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see [HA, 4.8.1.15]) with functor categories. We will need two natural comparison functors, one for presentable symmetric monoidal ∞-categories, and one for presentable ∞-categories, but we will show in Proposition 5.3.0.4 that these constructions are compatible with the forgetful functor $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$. We will then show in Proposition 5.3.0.6 that these comparison functors are equivalences.

Construction 5.3.0.1. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable symmetric monoidal ∞-categories and $\mathcal{I}$ and $\mathcal{J}$ small ∞-categories. By [HA, 4.8.1.9] we can interpret $\mathcal{C}$ and $\mathcal{D}$ as objects in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

The symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ induces a symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ such that the forgetful functor $\mathrm{ev}_{\langle 1 \rangle}$ can be upgraded to a symmetric monoidal functor (see [HA, 3.2.4.4]). By [HA, 3.2.4.10] this symmetric monoidal structure is cocartesian.

The functor categories $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ and $\mathrm{Fun}(\mathcal{J}, \mathcal{D})$ and $\mathrm{Fun}(\mathcal{I} \times \mathcal{J}, \mathcal{C} \otimes \mathcal{D})$ can be given the induced pointwise symmetric monoidal structures (see [HA, 2.1.3.4]). By [HTT, 5.5.3.6] the underlying ∞-categories are presentable again and as both the tensor products as well as colimits are calculated pointwise (see [HTT, 5.1.2.3]), the tensor products again preserve colimits pointwise in each variable [52].

Let $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{D}$ and $\iota_{\mathcal{D}} \colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{D}$ be the two morphisms in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ exhibiting $\mathcal{C} \otimes \mathcal{D}$ as a coproduct of $\mathcal{C}$ and $\mathcal{D}$. Using that $\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \otimes \mathrm{Fun}(\mathcal{J}, \mathcal{D})$ is a coproduct of

[52] To be precise (considering the case of $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$): The pointwise symmetric monoidal structure comes with symmetric monoidal evaluation functors for every object I of $\mathcal{I}$. This means we have commutative diagrams as follows

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \times \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \arrow[r, "- \otimes -"] \arrow[d, "\mathrm{ev}_I \times \mathrm{ev}_I"'] & \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \arrow[d, "\mathrm{ev}_I"] \\
\mathcal{C} \times \mathcal{C} \arrow[r, "- \otimes -"] & \mathcal{C}
\end{tikzcd}$$

where the horizontal functors are the respective tensor product functors. The left vertical functor preserves colimits in each component, and the bottom horizontal functor preserves colimits separately in each variable by assumption. It follows that the composition from top left to the bottom right along the top right preserves colimits separately in each variable, and as this is the case for every object I in $\mathcal{I}$, it follows that this is also the case for the top horizontal functor.

$\mathrm{Fun}(\mathcal{I},\mathcal{C})$ and $\mathrm{Fun}(\mathcal{J},\mathcal{D})$ in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ we can then define a morphism $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ as follows.

$$\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}\colon \mathrm{Fun}(\mathcal{I},\mathcal{C})\otimes\mathrm{Fun}(\mathcal{J},\mathcal{D}) \xrightarrow{(\iota_{\mathcal{C}}\circ-\circ\mathrm{pr}_1)\amalg(\iota_{\mathcal{D}}\circ-\circ\mathrm{pr}_2)} \mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C}\otimes\mathcal{D}) \qquad \diamondsuit$$

We next construct a functor of presentable ∞-categories very analogous to $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ (and with the same name, which will be justified by Proposition 5.3.0.4), where we however do not consider any symmetric monoidal structures.

Construction 5.3.0.2. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories and $\mathcal{I}$ and $\mathcal{J}$ small ∞-categories.

Consider the following diagram, which will be explained below.

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I},\mathcal{C})\times\mathrm{Fun}(\mathcal{J},\mathcal{D}) \arrow[r, "-\times-"] \arrow[d, "\psi'"'] & \mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C}\times\mathcal{D}) \arrow[d, "\psi_*"] \\
\mathrm{Fun}(\mathcal{I},\mathcal{C})\otimes\mathrm{Fun}(\mathcal{J},\mathcal{D}) \arrow[r, dashed, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}"'] & \mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C}\otimes\mathcal{D})
\end{tikzcd} \tag{5.10}$$

First, as already mentioned in Construction 5.3.0.1 are by [HTT, 5.5.3.6] the various functor categories appearing in the diagram diagram representable again. ψ' is to be the functor exhibiting $\mathrm{Fun}(\mathcal{I},\mathcal{C})\otimes\mathrm{Fun}(\mathcal{J},\mathcal{D})$ as the tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of $\mathrm{Fun}(\mathcal{I},\mathcal{C})$ and $\mathrm{Fun}(\mathcal{J},\mathcal{D})$, and likewise ψ is to be the functor exhibiting $\mathcal{C}\otimes\mathcal{D}$ as the tensor product[53]. We claim that the composite from the top left over the top right to the bottom right preserves colimits in each variable separately. For this it suffices by [HTT, 5.1.2.3] to check that the composition with $\mathrm{ev}_{(I,J)}$ preserves colimits in each variable separately for every object I of $\mathcal{I}$ and J of $\mathcal{J}$. But as there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I},\mathcal{C})\times\mathrm{Fun}(\mathcal{J},\mathcal{D}) \arrow[r, "-\times-"] \arrow[d, "\mathrm{ev}_I\times\mathrm{ev}_J"'] & \mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C}\times\mathcal{D}) \arrow[r, "\psi_*"] \arrow[d, "\mathrm{ev}_{(I,J)}"] & \mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C}\otimes\mathcal{D}) \arrow[d, "\mathrm{ev}_{(I,J)}"] \\
\mathcal{C}\times\mathcal{D} \arrow[r, "\mathrm{id}"'] & \mathcal{C}\times\mathcal{D} \arrow[r, "\psi"'] & \mathcal{C}\otimes\mathcal{D}
\end{tikzcd}$$

this follows from ev_I and ev_J preserving colimits by [HTT, 5.1.2.3] and ψ by definition preserving colimits separately in each variable.

It now follows from the universal property[54] of the tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ that there is an essentially unique way to complete (5.10) to a commutative diagram with a colimit preserving dashed functor $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$. $\diamondsuit$

Remark 5.3.0.3. The functors φ from Construction 5.3.0.2 are compatible with colimit preserving functors of presentable ∞-categories and functors of the indexing ∞-categories as we will argue now. Let $f\colon \mathcal{I}'\to\mathcal{I}$ and $g\colon \mathcal{J}'\to\mathcal{J}$ be functors of small

[53] Again see [HA, 4.8.1.2, 4.8.1.3, 4.8.1.4, and 4.8.1.15].
[54] See [HA, 4.8.1.2, 4.8.1.3, 4.8.1.4, and 4.8.1.15].

∞-categories and $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{D} \to \mathcal{D}'$ colimit preserving functors between presentable ∞-categories.

Then consider the following diagram

$$\begin{tikzcd}
 & \mathcal{C}^{\mathcal{I}} \times \mathcal{D}^{\mathcal{J}} \arrow[rr, "- \times -"] \arrow[dl, "(F\circ - \circ f)\times(G\circ - \circ g)"'] \arrow[dd] & & (\mathcal{C}\times\mathcal{D})^{\mathcal{I}\times\mathcal{J}} \arrow[dl, "(F\times G)\circ - \circ(f\times g)"] \arrow[dd] \\
\mathcal{C}'^{\mathcal{I}'} \times \mathcal{D}'^{\mathcal{J}'} \arrow[rr, "- \times -"] \arrow[dd] & & (\mathcal{C}'\times\mathcal{D}')^{\mathcal{I}'\times\mathcal{J}'} \arrow[dd] & \\
 & \mathcal{C}^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}} \arrow[rr, dashed, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}"'] \arrow[dl, "(F\circ - \circ f)\otimes(G\circ - \circ g)"] & & (\mathcal{C}\otimes\mathcal{D})^{\mathcal{I}\times\mathcal{J}} \arrow[dl, "(F\otimes G)\circ - \circ(f\otimes g)"] \\
\mathcal{C}'^{\mathcal{I}'} \otimes \mathcal{D}'^{\mathcal{J}'} \arrow[rr, dashed, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C}',\mathcal{D}'}"'] & & (\mathcal{C}'\otimes\mathcal{D}')^{\mathcal{I}'\times\mathcal{J}'} &
\end{tikzcd}$$

where the vertical functors are (induced) by the various canonical functors exhibiting a presentable ∞-category as a tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The top, left, and right sides commute by the respective naturalities, and the front and back commute by construction. The claim we want to show is that there is an essentially unique filler for the bottom side and the cube. But this follows immediately from the universal property of the back left vertical functor using the fact that all functors on the bottom preserve colimits.

The functors φ from Construction 5.3.0.1 satisfy an analogous naturality property, which one can deduce directly from the definition using the universal property of coproducts. $\diamondsuit$

The next proposition justifies the overloading of notation in Construction 5.3.0.1 and Construction 5.3.0.2.

Proposition 5.3.0.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable symmetric monoidal ∞-categories and $\mathcal{I}$ and $\mathcal{J}$ small ∞-categories.*

As the forgetful functor $\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is symmetric monoidal, we can identify the underlying presentable ∞-categories of the domain and codomain of $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ from Construction 5.3.0.1 with the domain and codomain of $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ from Construction 5.3.0.2.

Under this identification there is an essentially unique homotopy of morphisms in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ between the underlying functor of $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ from Construction 5.3.0.1 and $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ from Construction 5.3.0.2. $\heartsuit$

Proof. Let $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ be the underlying functor of the symmetric monoidal functor defined in Construction 5.3.0.1. By the universal property of the tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ it suffices to show that $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ fits into a commutative diagram as depicted in (5.10).

For this we ponder the following commutative diagram in $\mathcal{C}\mathrm{at}_\infty$[55].

$$\begin{tikzcd}
\mathcal{C}^{\mathcal{I}} \times \mathcal{D}^{\mathcal{J}} \arrow[rr, "\psi'"] \arrow[dr, "(\iota_{\mathcal{C}} \circ - \circ \mathrm{pr}_1) \times (\iota_{\mathcal{D}} \circ - \circ \mathrm{pr}_2)"] \arrow[dd, "- \times -"'] & & \mathcal{C}^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}} \arrow[rr, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}"] \arrow[dr, "(\iota_{\mathcal{C}} \circ - \circ \mathrm{pr}_1) \otimes (\iota_{\mathcal{D}} \circ - \circ \mathrm{pr}_2)"] & & (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \\
& (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \times (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \arrow[rr, "\psi''"] \arrow[drr, "- \times -"] & & (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \otimes (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \arrow[ur, "- \otimes -"] & \\
(\mathcal{C} \times \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \arrow[rrr, "(\iota_{\mathcal{C}} \times \iota_{\mathcal{D}})_*"] \arrow[dr, "\simeq"] \arrow[drrrr, bend right, "\mathrm{id}"'] & & & ((\mathcal{C} \otimes \mathcal{D}) \times (\mathcal{C} \otimes \mathcal{D}))^{\mathcal{I} \times \mathcal{J}} \arrow[uur, "(- \otimes -)_*"] & \\
& ((\mathcal{C} \times *) \times (* \times \mathcal{D}))^{\mathcal{I} \times \mathcal{J}} \arrow[rr, "(\mathrm{id} \times 1 \times 1 \times \mathrm{id})_*"'] & & ((\mathcal{C} \times \mathcal{D}) \times (\mathcal{C} \times \mathcal{D}))^{\mathcal{I} \times \mathcal{J}} \arrow[u, "(\psi \times \psi)_*"'] \arrow[r, "(- \otimes -)_*"'] & (\mathcal{C} \times \mathcal{D})^{\mathcal{I} \times \mathcal{J}} \arrow[uuu, bend right, "\psi_*"']
\end{tikzcd}$$

The composite outer diagram is the one that we are after. All the morphisms ψ with some decoration are to be the canonical morphisms exhibiting some presentable ∞-category as a tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (one could also say: these are the functors arising from lax symmetric monoidality of the inclusion of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ into $\mathcal{C}\mathrm{at}_\infty$), and $1\colon * \to \mathcal{C}$ is to be the unit morphism of the commutative algebra $\mathcal{C}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, i. e. the functor with image $\mathbb{1}_{\mathcal{C}}$, and similarly for $1\colon * \to \mathcal{D}$. The morphisms $\iota_{\mathcal{C}}$ and $\iota_{\mathcal{D}}$ are to be as in Construction 5.3.0.1. Finally, the functors $- \otimes -$ are the internal tensor product functors of the various symmetric monoidal ∞-categories.

Let us now explain how the individual pieces of the above diagram arise. The top right triangle uses that the tensor product functor is the coproduct $\mathrm{id} \amalg \mathrm{id}$ in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. The top left square arises from naturality of the functors denoted by ψ with a decoration – the functor on the right is in fact defined as the essentially unique colimit preserving functor fitting into a square like this. In the middle square below the two already discussed ones we can (again[56]) identify the composition of the top two functors with the tensor product functor of $(\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}}$, and then commutativity of the square arises from the definition of the symmetric monoidal structure on $(\mathcal{C} \otimes \mathcal{D})^{\mathcal{I} \times \mathcal{J}}$ as the pointwise one. The square on the right arises from $\psi\colon \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ being a symmetric monoidal functor, which

[55] To save space we write e. g. $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ as $\mathcal{C}^{\mathcal{I}}$.

[56] One can think of it like this: The lax symmetric monoidal inclusion of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ into $\mathcal{C}\mathrm{at}_\infty$ induces a functor on commutative algebras, which is why a presentable symmetric monoidal ∞-category $\mathcal{E}$ comes with a commutative triangle

$$\begin{tikzcd}
\mathcal{E} \times \mathcal{E} \arrow[d] \arrow[dr, "- \otimes -"] & \\
\mathcal{E} \otimes \mathcal{E} \arrow[r, "- \otimes -"'] & \mathcal{E}
\end{tikzcd}$$

where the left vertical functor is the canonical one exhibiting $\mathcal{E} \otimes \mathcal{E}$ as a tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and where both functors $- \otimes -$ can be thought of as "the tensor product functor" – the one on the bottom encodes that colimits are preserved in each variable separately.

is the case because the functor $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty})$ induced by the lax symmetric monoidal inclusion of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ into $\mathcal{C}\mathrm{at}_{\infty}$ is again lax symmetric monoidal, see [HA, 4.8.1.4] and Proposition E.4.2.3 (7). The upper square on the left comes from functoriality of taking products of functors. The irregularly shaped square at the very bottom arises from unitality of the tensor product functors on $\mathcal{C}$ and $\mathcal{D}$ and the fact that the tensor product on $\mathcal{C} \times \mathcal{D}$ is defined componentwise. Finally, the bottom left square is constructed from the definitions of $\iota_{\mathcal{C}}$ and $\iota_{\mathcal{D}}$. For example for $\iota_{\mathcal{C}}$, the unit morphism $1\colon * \to \mathcal{C}$ induces a colimit preserving functor $1\colon \mathbb{1}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}} \simeq \mathcal{S} \to \mathcal{C}$ and we then obtain the dashed functor in the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, dotted, "\iota_{\mathcal{C}}"] \arrow[d, "\simeq"'] & & \mathcal{C} \otimes \mathcal{D} \\
\mathcal{C} \times * \arrow[r, "\mathrm{id} \times 1"] \arrow[d, "\mathrm{id} \times 1"'] & \mathcal{C} \times \mathcal{D} \arrow[ur, "\psi"] & \\
\mathcal{C} \times \mathcal{S} \arrow[ur, "\mathrm{id} \times 1"'] \arrow[rr, "\psi''''"'] & & \mathcal{C} \otimes \mathcal{S} \arrow[uu, dashed]
\end{tikzcd}$$

The dotted functor $\iota_{\mathcal{C}}$ is then defined as the composition along the outside of the diagram, i. e. making the outer diagram commute, which obviously also implies that there also exists a filler for the top square. □

Notation 5.3.0.5. Given ∞-categories $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{D}$, with $\mathcal{C}$ and $\mathcal{C}'$ admitting all small colimits, we write $\mathrm{Fun}^{\mathrm{colim}}(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the colimit-preserving functors. We write $\mathrm{Fun}^{\mathrm{colim} \times \mathrm{colim}}(\mathcal{C} \times \mathcal{C}', \mathcal{D})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{C}', \mathcal{D})$ of functors preserving colimits in each variable separately. ◊

Proposition 5.3.0.6. *In both the situation of Construction 5.3.0.1 as well as the situation of Construction 5.3.0.2 is the functor $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ an equivalence of presentable (symmetric monoidal) ∞-categories.* ♡

Proof. This proof will follow ideas of [HA, Proof of 4.8.1.15].

By [HA, 2.1.3.8] is a symmetric monoidal functor is equivalence of symmetric monoidal ∞-categories if and only if the underlying functor of ∞-categories is an equivalence. In light of Proposition 5.3.0.4 it thus suffices to discuss the case of Construction 5.3.0.2.

By [HTT, 5.5.1.1, 5.4.2.7, 5.5.4.2, and 5.5.4.15] any presentable ∞-category is equivalent to a localization $S^{-1}\,\mathrm{Fun}(\mathcal{K}, \mathcal{S})$ for some small ∞-category $\mathcal{K}$ and small set of morphisms S in $\mathrm{Fun}(\mathcal{K}, \mathcal{S})$. It will thus suffice to show the following claims.

(1) $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{S},\mathcal{S}}$ is an equivalence for all small ∞-categories $\mathcal{I}$ and $\mathcal{J}$.

(2) Suppose $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ is an equivalence for fixed presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, but arbitrary small ∞-categories $\mathcal{I}$ and $\mathcal{J}$. Then $\varphi^{\mathcal{I},\mathcal{J}}_{\mathrm{Fun}(\mathcal{I}',\mathcal{C}),\mathrm{Fun}(\mathcal{J}',\mathcal{D})}$ is an equivalence for all small ∞-categories $\mathcal{I}'$, $\mathcal{J}'$, $\mathcal{I}$, and $\mathcal{J}$.

(3) Suppose $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ is an equivalence for fixed presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$ and all small ∞-categories $\mathcal{I}$ and $\mathcal{J}$. Let S be a small set of morphisms of $\mathcal{C}$. Then $\varphi^{\mathcal{I},\mathcal{J}}_{S^{-1}\mathcal{C},\mathcal{D}}$ is also an equivalence.

(4) Suppose $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ is an equivalence for fixed presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$ and small ∞-categories $\mathcal{I}$ and $\mathcal{J}$. Then $\varphi^{\mathcal{J},\mathcal{I}}_{\mathcal{D},\mathcal{C}}$ is an equivalence as well.

Proof of claim (1): It suffices to show that the composition

$$\theta\colon \operatorname{Fun}(\mathcal{I},\mathcal{S}) \times \operatorname{Fun}(\mathcal{J},\mathcal{S}) \xrightarrow{-\times-} \operatorname{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{S}\times\mathcal{S}) \xrightarrow{\psi_*} \operatorname{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{S}\otimes\mathcal{S})$$

exhibits $\operatorname{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{S}\otimes\mathcal{S})$ as the tensor product of $\operatorname{Fun}(\mathcal{I},\mathcal{S})$ and $\operatorname{Fun}(\mathcal{J},\mathcal{S})$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, i. e. we have to show that for any ∞-category $\mathcal{E}$ admitting all colimits the induced functor

$$\operatorname{Fun}^{\mathrm{colim}}\big(\operatorname{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{S}\otimes\mathcal{S}),\mathcal{E}\big) \xrightarrow{\theta^*} \operatorname{Fun}^{\mathrm{colim}\times\mathrm{colim}}\big(\operatorname{Fun}(\mathcal{I},\mathcal{S})\times\operatorname{Fun}(\mathcal{J},\mathcal{S}),\mathcal{E}\big)$$

is an equivalence.

Using that mapping spaces in products of ∞-categories are the products of the respective mapping spaces we obtain the following commutative diagram of ∞-categories.

$$\begin{tikzcd}
\mathcal{I}^{\mathrm{op}}\times\mathcal{J}^{\mathrm{op}} \arrow[rr, dash, "\simeq"] \arrow[d] & & (\mathcal{I}\times\mathcal{J})^{\mathrm{op}} \arrow[d] \\
\mathcal{P}(\mathcal{I}^{\mathrm{op}})\times\mathcal{P}(\mathcal{J}^{\mathrm{op}}) \arrow[d, equal] & & \mathcal{P}\big((\mathcal{I}\times\mathcal{J})^{\mathrm{op}}\big) \arrow[d, equal] \\
\operatorname{Fun}(\mathcal{I},\mathcal{S})\times\operatorname{Fun}(\mathcal{J},\mathcal{S}) \arrow[d, "-\times-"'] \arrow[drr, "\theta"] & & \operatorname{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{S}) \\
\operatorname{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{S}\times\mathcal{S}) \arrow[rr, "\psi_*"'] & & \operatorname{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{S}\otimes\mathcal{S}) \arrow[u, "(-\times-)_*"']
\end{tikzcd}$$

where the two top vertical functors are (products of) Yoneda embeddings [HTT, 5.1.3], the top horizontal one is the canonical equivalence witnessing that $-^{\mathrm{op}}$ preserves products, and $-\times-\colon \mathcal{S}\otimes\mathcal{S}\to\mathcal{S}$ is the tensor product of the *cartesian* presentable symmetric monoidal structure on $\mathcal{S}$, see Remark 5.2.2.1.

By applying $\operatorname{Fun}(-,\mathcal{E})$ and passing to appropriate full subcategories we obtain a commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{I}^{\mathrm{op}}\times\mathcal{J}^{\mathrm{op}},\mathcal{E}) \arrow[rr, dash, "\simeq"] & & \operatorname{Fun}\big((\mathcal{I}\times\mathcal{J})^{\mathrm{op}},\mathcal{E}\big) \\
\operatorname{Fun}^{\mathrm{colim}\times\mathrm{colim}}\big(\mathcal{P}(\mathcal{I}^{\mathrm{op}})\times\mathcal{P}(\mathcal{J}^{\mathrm{op}}),\mathcal{E}\big) \arrow[u] & & \operatorname{Fun}^{\mathrm{colim}}\Big(\mathcal{P}\big((\mathcal{I}\times\mathcal{I})^{\mathrm{op}}\big),\mathcal{E}\Big) \arrow[u] \arrow[d, "\big((-\times-)_*\big)^*"] \\
 & & \operatorname{Fun}^{\mathrm{colim}}\big(\operatorname{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{S}\otimes\mathcal{S}),\mathcal{E}\big) \arrow[ull, "\theta^*"]
\end{tikzcd}$$

The top horizontal functor is an equivalence as it is induced by one. The top left and right vertical functors are equivalences by [HTT, 5.1.5.6][57]. Finally, the bottom right vertical functor is an equivalence because it is induced by the equivalence $\mathcal{S} \otimes \mathcal{S} \to \mathcal{S}$ (see [HA, 4.8.1.20]). It follow that θ^* is an equivalence as well.

Proof of claim (2): Let $\mathcal{C}$ and $\mathcal{D}$ be as in the claim and $\mathcal{I}$, $\mathcal{I}'$, $\mathcal{J}$, $\mathcal{J}'$ small ∞-categories. We have to show that $\varphi^{\mathcal{I},\mathcal{J}}_{\mathrm{Fun}(\mathcal{I}',\mathcal{C}),\mathrm{Fun}(\mathcal{J}',\mathcal{D})}$ is an equivalence. For this, consider the following diagram where the unlabeled functors are induced by the unit and counit of the product-Fun-adjunction and symmetry equivalences, and the functors ψ, ψ', ψ'', and ψ''' are the various functors exhibiting a presentable ∞-category as a tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

$$\begin{tikzcd}
\mathcal{C}^{\mathcal{I}\times\mathcal{I}'} \times \mathcal{D}^{\mathcal{J}\times\mathcal{J}'} \arrow[r, "\psi''"] \arrow[d, "\simeq"'] \arrow[dddd, bend right=60, "-\times-"'] & \mathcal{C}^{\mathcal{I}\times\mathcal{I}'} \otimes \mathcal{D}^{\mathcal{J}\times\mathcal{J}'} \arrow[d, "\simeq"] \arrow[dddd, dashed, bend left=60, "\varphi^{\mathcal{I}\times\mathcal{I}',\mathcal{J}\times\mathcal{J}'}_{\mathcal{C},\mathcal{D}}"] \\
\left(\mathcal{C}^{\mathcal{I}'}\right)^{\mathcal{I}} \times \left(\mathcal{D}^{\mathcal{J}'}\right)^{\mathcal{J}} \arrow[r, "\psi'''"] \arrow[d, "-\times-"'] & \left(\mathcal{C}^{\mathcal{I}'}\right)^{\mathcal{I}} \otimes \left(\mathcal{D}^{\mathcal{J}'}\right)^{\mathcal{J}} \arrow[d, "\phi^{\mathcal{I},\mathcal{J}}_{\mathcal{C}^{\mathcal{I}'},\mathcal{D}^{\mathcal{J}'}}"] \\
\left(\mathcal{C}^{\mathcal{I}'} \times \mathcal{D}^{\mathcal{J}'}\right)^{\mathcal{I}\times\mathcal{J}} \arrow[r, "\psi'_*"] \arrow[d, "(-\times-)_*"'] & \left(\mathcal{C}^{\mathcal{I}'} \otimes \mathcal{D}^{\mathcal{J}'}\right)^{\mathcal{I}\times\mathcal{J}} \arrow[d, "\left(\varphi^{\mathcal{I}',\mathcal{J}'}_{\mathcal{C},\mathcal{D}}\right)_*"] \\
\left((\mathcal{C}\times\mathcal{D})^{\mathcal{I}'\times\mathcal{J}'}\right)^{\mathcal{I}\times\mathcal{J}} \arrow[r, "(\psi_*)_*"] \arrow[d, "\simeq"'] & \left((\mathcal{C}\otimes\mathcal{D})^{\mathcal{I}'\times\mathcal{J}'}\right)^{\mathcal{I}\times\mathcal{J}} \arrow[d, "\simeq"] \\
(\mathcal{C}\times\mathcal{D})^{\mathcal{I}\times\mathcal{I}'\times\mathcal{J}\times\mathcal{J}'} \arrow[r, "\psi_*"] & (\mathcal{C}\otimes\mathcal{D})^{\mathcal{I}\times\mathcal{I}'\times\mathcal{J}\times\mathcal{J}'}
\end{tikzcd}$$

The two middle squares commute by definition of $\phi^{\mathcal{I},\mathcal{J}}_{\mathcal{C}^{\mathcal{I}'},\mathcal{D}^{\mathcal{J}'}}$ and $\varphi^{\mathcal{I}',\mathcal{J}'}_{\mathcal{C},\mathcal{D}}$, and the top and left square arise from respective naturalities. As the left rectangle on the left commutes we obtain from the universal property of ψ'' that the colimit preserving vertical composite on the right must be homotopic to $\varphi^{\mathcal{I}\times\mathcal{I}',\mathcal{J}\times\mathcal{J}'}_{\mathcal{C},\mathcal{D}}$. That $\phi^{\mathcal{I},\mathcal{J}}_{\mathcal{C}^{\mathcal{I}'},\mathcal{D}^{\mathcal{J}'}}$ is an equivalence now follows from all other functors in the commuting right long rectangle being equivalences.

Proof of claim (3): Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{I}$, $\mathcal{J}$, and S be as in the statement of the claim. We will write $\overline{S}$ for the strongly saturated collection of morphisms of $\mathcal{C}$ generated by S, see [HTT, 5.5.4.5 and 5.5.4.7]. By [HTT, 5.5.4.15] $S^{-1}\mathcal{C} \simeq (\overline{S})^{-1}\mathcal{C}$ is again presentable, so $\varphi^{\mathcal{I},\mathcal{J}}_{S^{-1}\mathcal{C},\mathcal{D}}$ is defined. We have to show that it is an equivalence.

Before we do so we need to discuss how localizations commute with tensor products in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and with $\mathrm{Fun}(\mathcal{K}, -)$ for small ∞-categories $\mathcal{K}$.

[57] For the top left functor, note that by passing to adjoints $\mathrm{Fun}^{\mathrm{colim}\times\mathrm{colim}}\big(\mathcal{P}(\mathcal{I}^{\mathrm{op}}) \times \mathcal{P}(\mathcal{J}^{\mathrm{op}}), \mathcal{E}\big)$ is equivalent to $\mathrm{Fun}^{\mathrm{colim}}\Big(\mathcal{P}(\mathcal{I}^{\mathrm{op}}), \mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{J}^{\mathrm{op}}), \mathcal{E})\Big)$, and now one can apply [HTT, 5.1.5.6] twice and then pass back to adjoints again.

For interaction with tensor products we note the following, which is taken from the proof of [HA, 4.8.1.15]. Let $\mathcal{E}$ and $\mathcal{F}$ be any presentable ∞-categories, and T a strongly saturated class of small generation of morphisms of $\mathcal{E}$. Let W be the collection of morphisms of the form $s \otimes \mathrm{id}_F$ in $\mathcal{E} \otimes \mathcal{F}$ for any s in S and object F of $\mathcal{F}$. Then $\overline{W}$ is of small generation, as shown in [HA, Proof of 4.8.1.15]. Now consider the following diagram

$$\begin{tikzcd}
T^{-1}\mathcal{E} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \times \mathcal{F} \arrow[r] & \mathcal{E} \otimes \mathcal{F} \arrow[d] \\
T^{-1}\mathcal{E} \otimes \mathcal{F} \arrow[rr, dashed] & & W^{-1}(\mathcal{E} \otimes \mathcal{F})
\end{tikzcd}$$

where the top left horizontal functor is induced by the inclusion $T^{-1}\mathcal{E} \to \mathcal{E}$, the top right horizontal functor and the left vertical functor are the canonical functors exhibiting the respective targets as tensor products in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, and the right vertical functor is the localization functor. $W^{-1}(\mathcal{E} \otimes \mathcal{F})$ is representable by [HTT, 5.5.4.15], and the composite functor from the top left to the bottom right preserves colimits in each variable separately[58]. We hence obtain the induced dashed colimit preserving functor that is an equivalence by [HA, Proof of 4.8.1.15].

We now turn to the interaction of localizations with taking functor categories. For this, let $\mathcal{E}$ be a presentable ∞-category, $\mathcal{K}$ a small ∞-category, and T a strongly saturated class of morphisms of $\mathcal{E}$ of small generation. Let $L\colon \mathcal{E} \to T^{-1}\mathcal{E}$ be the localization functor. Then by Proposition D.2.2.1 and $\mathrm{Fun}(\mathcal{K}, -)$ preserving fully faithful functors by Proposition B.3.0.1 it follows that the induced functor

$$L_*\colon\ \mathrm{Fun}(\mathcal{K}, \mathcal{E}) \to \mathrm{Fun}\big(\mathcal{K}, T^{-1}\mathcal{E}\big)$$

is a localization functor again. Furthermore, $\mathrm{Fun}(\mathcal{K}, T^{-1}\mathcal{E})$ is presentable again by [HTT, 5.5.3.6]. Let W be the class of morphisms in $\mathrm{Fun}(\mathcal{K}, \mathcal{E})$ that are pointwise in T. By combining [HTT, 5.5.4.15], [HTT, 5.5.4.2], and Proposition A.3.2.1 we see that W consists precisely of those morphisms that are mapped to equivalences by L_*. It then follows from [HTT, 5.5.4.2] that there is a canonical equivalence

$$\mathrm{Fun}\big(\mathcal{K}, T^{-1}\mathcal{E}\big) \simeq W^{-1}\,\mathrm{Fun}(\mathcal{K}, \mathcal{E})$$

that is compatible with the localization functors.

We now return to showing that $\varphi^{\mathcal{I},\mathcal{J}}_{S^{-1}\mathcal{C},\mathcal{D}}$ is an equivalence. Let T be the strongly generated class of morphisms in $\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \otimes \mathrm{Fun}(\mathcal{J}, \mathcal{D})$ that is generated by morphisms of the form $\eta \otimes \mathrm{id}_G$ for any object G in $\mathrm{Fun}(\mathcal{J}, \mathcal{D})$ and any morphism η in $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ such that $\eta(I)$ is in S for all objects I of $\mathcal{I}$. Let W be the strongly generated class of morphisms in $\mathrm{Fun}(\mathcal{I} \times \mathcal{J}, \mathcal{C} \otimes \mathcal{D})$ that is generated by those morphisms for which for every object I of $\mathcal{I}$ and J of $\mathcal{J}$ the evaluation at (I, J) is equivalent to a morphism of the form $s \otimes \mathrm{id}_D$ for s in S and D and object of $\mathcal{D}$.

[58]One can see this using that by [HTT, 5.2.7.5] a diagram $p\colon K^{\triangleright} \to T^{-1}\mathcal{E}$ is a colimit if and only if the induced morphism from the colimit taken in $\mathcal{E}$ to the cone object, $\operatorname{colim} p|_K \to p(\infty)$, is a T-equivalence.

Consider the following commutative diagram

$$\begin{tikzcd}
 & T^{-1}\big(\mathcal{C}^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}}\big) \arrow[d, no head, "\simeq"] \\
\mathcal{C}^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}} \arrow[ur] \arrow[r, "L_* \otimes \mathrm{id}_*"'] \arrow[d, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}"'] & \big(S^{-1}\mathcal{C}\big)^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}} \arrow[d, "\varphi^{\mathcal{I},\mathcal{J}}_{S^{-1}\mathcal{C},\mathcal{D}}"] \\
(\mathcal{C} \otimes \mathcal{D})^{\mathcal{I}\times\mathcal{J}} \arrow[r, "(L\otimes \mathrm{id})_*"] \arrow[dr] & \big(S^{-1}\mathcal{C} \otimes \mathcal{D}\big)^{\mathcal{I}\times\mathcal{J}} \arrow[d, no head, "\simeq"] \\
 & W^{-1}\Big((\mathcal{C} \otimes \mathcal{D})^{\mathcal{I}\times\mathcal{J}}\Big)
\end{tikzcd}$$

where L denotes the localization functor $\mathcal{C} \to S^{-1}\mathcal{C}$, the middle square arises from naturality of the functors φ with respect to the colimit preserving functor L (see Remark 5.3.0.3), and the top and bottom triangles use the compatibility of the tensor product and functor categories with localization as discussed above, with the top and bottom functors being the respective localization functors.

By assumption $\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}$ is an equivalence, and it is clear from the definitions that the strongly saturated classes of morphisms T and W correspond under this equivalence, i. e. $\overline{\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}(T)} = W$. It then follows from [HTT, 5.5.4.20] that $\varphi^{\mathcal{I},\mathcal{J}}_{S^{-1}\mathcal{C},\mathcal{D}}$ is also an equivalence.

Proof of claim (4): One can show in a manner analogous to Remark 5.3.0.3 that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^{\mathcal{I}} \otimes \mathcal{D}^{\mathcal{J}} \arrow[r, "\varphi^{\mathcal{I},\mathcal{J}}_{\mathcal{C},\mathcal{D}}"] \arrow[d, "\tau"', "\simeq"] & (\mathcal{C} \otimes \mathcal{D})^{\mathcal{I}\times\mathcal{J}} \arrow[d, "\simeq"', "\tau'_*"] \\
\mathcal{D}^{\mathcal{J}} \otimes \mathcal{C}^{\mathcal{I}} \arrow[r, "\varphi^{\mathcal{J},\mathcal{I}}_{\mathcal{D},\mathcal{C}}"'] & (\mathcal{D} \otimes \mathcal{C})^{\mathcal{J}\times\mathcal{I}}
\end{tikzcd}$$

where τ and τ' are the symmetry equivalences of the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The claim immediately follows from this. □

The proof of Proposition 5.3.0.8 below is also sketched in [Rak20, 2.2.9]. We need a small prerequisite before stating the result.

Proposition 5.3.0.7. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category, $\mathcal{O}$ an ∞-operad, and $\mathcal{O}'$ a reduced ∞-operad*[59]*. Then the unit of the induced symmetric monoidal structure on* $\mathrm{BiAlg}_{\mathcal{O},\mathcal{O}'}(\mathcal{C})$ *is a final object.* ♡

[59] See [HA, 2.3.4.1] for a definition.

Proof. By definition there is an equivalence as follows.

$$\mathrm{BiAlg}_{\mathcal{O},\mathcal{O}'}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}'}\big(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{op}\big)^{\mathrm{op}}$$

The unit is an initial object in $\mathrm{Alg}_{\mathcal{O}'}\big(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{op}\big)$ by [HA, 3.2.1.8] and hence final in $\mathrm{BiAlg}_{\mathcal{O},\mathcal{O}'}(\mathcal{C})$. $\square$

Proposition 5.3.0.8 ([Rak20, 2.2.9])**.** *Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category and G an object in $\mathrm{Mon}^{\mathrm{gp}}_{\mathsf{Assoc}}(\mathcal{S})$. Consider G as a cocommutative bialgebra in $\mathcal{S}$, and give $\mathbb{1}_{\mathcal{C}} \boxtimes G$ the induced cocommutative bialgebra structure, as discussed in the introduction to Section 5.3.*

Then there is a commutative diagram of presentable symmetric monoidal ∞-categories and colimit preserving symmetric monoidal functors[60] as follows

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{B}\,G} \arrow[rr, "\Psi^G_{\mathcal{C}}", "\simeq"'] \arrow[dr, "\mathrm{ev}_*"'] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C}) \arrow[dl, "\mathrm{ev}_{\mathfrak{m}}"] \\
& \mathcal{C} &
\end{tikzcd} \tag{5.11}$$

where $\mathcal{C}^{\mathrm{B}\,G}$ carries the pointwise symmetric monoidal structure discussed in the introduction to Section 5.3 and $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C})$ the one from Definition 3.4.2.1. As indicated in the diagram, $\Psi^G_{\mathcal{C}}$ is an equivalence of presentable symmetric monoidal ∞-categories.

Furthermore, these equivalences can be chosen in such a way as to be compatible with morphisms $f\colon G \to H$ in $\mathrm{Mon}^{\mathrm{gp}}_{\mathsf{Assoc}}(\mathcal{S})$ and $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, in the sense that for such f and F there is a commutative diagram in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ as follows.

$$\begin{tikzcd}
& \mathcal{C}^{\mathrm{B}\,H} \arrow[rr, "\Psi^H_{\mathcal{C}}"] \arrow[dl, "F\circ-\circ\mathrm{B}\,f"'] \arrow[ddr, "\mathrm{ev}_*"] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes H}(\mathcal{C}) \arrow[dl, "\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes f}(F)"'] \arrow[ddl, "\mathrm{ev}_{\mathfrak{m}}"] \\
\mathcal{D}^{\mathrm{B}\,G} \arrow[rr, "\Psi^G_{\mathcal{D}}"] \arrow[ddr, "\mathrm{ev}_*"'] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{D}) \arrow[ddl, "\mathrm{ev}_{\mathfrak{m}}"'] & \\
& & \mathcal{C} \arrow[dl, "F"] & \\
& \mathcal{D} & &
\end{tikzcd} \tag{5.12}$$

$\heartsuit$

[60] In other words, a commutative diagram in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

Remark 5.3.0.9. In the situation of Proposition 5.3.0.8, let $f\colon G \to *$ be the essentially unique morphism of grouplike associative monoids in $\mathcal{S}$. The induced morphism of cocommutative bialgebras in $\mathcal{C}$ given by $\mathbb{1}_{\mathcal{C}} \boxtimes f\colon \mathbb{1}_{\mathcal{C}} \boxtimes G \to \mathbb{1}_{\mathcal{C}} \boxtimes * \simeq \mathbb{1}_{\mathcal{C}}$ is also the essentially unique one, see Proposition 5.3.0.7.

Then there is a commutative diagram by Proposition 5.3.0.8 as follows

$$\begin{tikzcd}
& \mathcal{C} & \\
\mathcal{C}^{\mathrm{B}\,*} \arrow[ur, "\mathrm{ev}_*"] \arrow[rr, "\Psi^*_{\mathcal{C}}"] \arrow[d, "(\mathrm{B}\,f)^*"'] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}}(\mathcal{C}) \arrow[ul, "\mathrm{ev}_{\mathfrak{m}}"'] \arrow[d, "\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes f}(\mathcal{C})"] \\
\mathcal{C}^{\mathrm{B}\,G} \arrow[rr, "\Psi^G_{\mathcal{C}}"'] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes G}(\mathcal{C})
\end{tikzcd}$$

Note that the functors ev_* and $\mathrm{ev}_{\mathfrak{m}}$ are equivalences[61], and we can interpret the composites from the top to the bottom left and bottom right as the functors that map an object of $\mathcal{C}$ to that same object equipped with the trivial action by G. ♢

Proof of Proposition 5.3.0.8. We start by noting that ignoring the horizontal functors, the rest of diagrams (5.11) and (5.12) are indeed diagrams in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. The ∞-category $\mathcal{C}^{\mathrm{B}\,G}$ with the pointwise symmetric monoidal structure is indeed presentable symmetric monoidal, as is explained in Construction 5.3.0.1. That $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes G}(\mathcal{C})$ is presentable symmetric monoidal is by construction, see Definition 3.4.2.1 and the propositions referenced there. $(F \circ - \circ \mathrm{B}\,f)\colon\ \mathrm{Fun}(\mathrm{B}\,H, \mathcal{C}) \to \mathrm{Fun}(\mathrm{B}\,G, \mathcal{D})$ can be upgraded to a symmetric monoidal functor and preserves colimits as both the symmetric monoidal structure as well as colimits are pointwise. Similarly, the evaluation functor ev_* is symmetric monoidal and preserves colimits. $\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}\boxtimes f}(F)$ as well as $\mathrm{ev}_{\mathfrak{m}}$ are symmetric monoidal and colimit preserving by construction, see Definition 3.4.2.1. Finally, the left and right squares in (5.12) arise from naturality of the respective evaluation functors.

The commutative triangle we have to construct will be given as the composite outer triangle in a commutative diagram in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ as indicated below; we will individually construct each part together with the relevant compatibility with respect to $f\colon G \to H$

[61] See [HA, 4.2.4.9] for $\mathrm{ev}_{\mathfrak{m}}$ being an equivalence.

and $F\colon \mathcal{C} \to \mathcal{D}$.

$$\begin{tikzcd}
\mathcal{C}^{BG} \arrow[rr, dash, "\simeq"] \arrow[rrrrrr, bend left, "\Psi_{\mathcal{C}}^{G}"] \arrow[dddrrr, "\mathrm{ev}_*"'] & & \mathcal{C} \otimes \mathcal{S}^{BG} \arrow[rr, dash, "\simeq"] \arrow[dr, "\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_*"] & & \mathcal{C} \otimes \mathrm{LMod}_G(\mathcal{S}) \arrow[rr, dash, "\simeq"] \arrow[dl, "\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_{\mathrm{m}}"'] & & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C}) \arrow[dddlll, "\mathrm{ev}_{\mathrm{m}}"] \\
& & & \mathcal{C} \otimes \mathcal{S} \arrow[dd, "\rho_{\mathcal{C}}", "\simeq"'] & & & \\
& & & & & & \\
& & & \mathcal{C} & & &
\end{tikzcd} \tag{5.13}$$

The tensor product is the tensor product induced on $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ by the tensor product of presentable ∞-categories[62]. The bottom vertical equivalence $\rho_{\mathcal{C}}$ is the right unitor, using that $\mathcal{S}$ is the monoidal unit in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see [HA, 4.8.1.20]).

Construction of the left square: The square arises as the composite outer square in the following commutative diagram in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{B}G} \arrow[d, "\mathrm{ev}_*"] & (\mathcal{C} \otimes \mathcal{S})^{* \times \mathrm{B}G} \arrow[l, "\rho_{\mathcal{C}} \circ - \circ (\mathrm{pr}_2)^{-1}"'] \arrow[d, "\mathrm{ev}_{(*,*)}"] & \mathcal{C}^* \otimes \mathcal{S}^{\mathrm{B}G} \arrow[l, "\varphi_{\mathcal{C},\mathcal{S}}^{*,\mathrm{B}G}"'] \arrow[r, "\mathrm{ev}_* \otimes \mathrm{id}_{\mathcal{S}^{\mathrm{B}G}}"] \arrow[d, "\mathrm{ev}_* \otimes \mathrm{ev}_*"] & \mathcal{C} \otimes \mathcal{S}^{\mathrm{B}G} \arrow[d, "\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_*"] \\
\mathcal{C} & \mathcal{C} \otimes \mathcal{S} \arrow[l, "\rho_{\mathcal{C}}"] & \mathcal{C} \otimes \mathcal{S} \arrow[l, "\mathrm{id}_{\mathcal{C} \otimes \mathcal{S}}"] \arrow[r, "\mathrm{id}_{\mathcal{C} \otimes \mathcal{S}}"'] & \mathcal{C} \otimes \mathcal{S}
\end{tikzcd}$$

Here, the left square is induced by the unitality equivalences $\mathrm{pr}_2\colon * \times \mathrm{B}G \to \mathrm{B}G$ (in $\mathcal{C}\mathrm{at}_\infty$) and $\rho_{\mathcal{C}}\colon \mathcal{C} \otimes \mathcal{S} \to \mathcal{C}$ (in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$), which is clearly compatible with f and F. The equivalence $\varphi_{\mathcal{C},\mathcal{S}}^{*,\mathrm{B}G}$ is the one from Construction 5.3.0.1, and the middle square as well as the commutative cube for compatibility with f and F can be constructed directly using the definition and the universal property of coproducts in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. Finally, the right square arises directly from functoriality of the tensor product of $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, and ev_* is clearly an equivalence.

Construction of the right square: This square arises as the composite outer square obtained by combining the following two commutative diagrams in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

$$\begin{tikzcd}
\mathcal{C} \otimes \mathrm{LMod}_G(\mathcal{S}) \arrow[d, "\mathrm{id}_{\mathcal{C}} \otimes \mathrm{ev}_{\mathrm{m}}"] & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}}}(\mathcal{C}) \otimes \mathrm{LMod}_G(\mathcal{S}) \arrow[l, "\mathrm{ev}_{\mathrm{m}} \otimes \mathrm{id}"'] \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_{\mathrm{m}} \otimes \mathrm{ev}_{\mathrm{m}}"] & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \otimes G}(\mathcal{C} \otimes \mathcal{S}) \arrow[d, "\mathrm{ev}_{\mathrm{m}}"] \\
\mathcal{C} \otimes \mathcal{S} & \mathcal{C} \otimes \mathcal{S} \arrow[l, "\mathrm{id}_{\mathcal{C} \otimes \mathcal{S}}"] \arrow[r, "\mathrm{id}"'] & \mathcal{C} \otimes \mathcal{S}
\end{tikzcd}$$

[62]See [HA, 4.8.1.15]

$$\begin{tikzcd}
\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \otimes G}(\mathcal{C} \otimes \mathcal{S}) \arrow[r, "\mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \otimes G}(\rho_{\mathcal{C}})"] \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"'] & \mathrm{LMod}_{\mathbb{1}_{\mathcal{C}} \boxtimes G}(\mathcal{C}) \arrow[d, "\mathrm{ev}_{\mathfrak{m}}"] \\
\mathcal{C} \otimes \mathcal{S} \arrow[r, "\rho_{\mathcal{C}}"'] & \mathcal{C}
\end{tikzcd}$$

The left square of the first diagram arises from functoriality of the tensor product, and $\mathrm{ev}_{\mathfrak{m}}$ is an equivalence by [HA, 4.2.4.9 and 2.1.3.8]. Compatibility with f and F follows from $\mathrm{ev}_{\mathfrak{m}}$ being a natural transformation, see Definition 3.4.2.1. The right square of the first diagram as well as its compatibility with f and F is the one arising from $\mathrm{ev}_{\mathfrak{m}} \colon \mathrm{LMod} \to \mathrm{pr}$ being a natural transformation of *symmetric monoidal* functors

$$\mathrm{LMod} \colon \mathcal{B}i\mathcal{A}lg\mathcal{O}p^{\mathcal{P}r}_{\mathsf{Comm}} \to \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}})$$

by Remark 3.4.2.2. Finally, the second diagram as well as its compatibility with f and F arises from the naturality of the right unitor ρ and $\mathrm{ev}_{\mathfrak{m}}$. That there is an equivalence $\rho_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}} \otimes G) \simeq \mathbb{1}_{\mathcal{C}} \boxtimes G$ that is compatible with f and F follows immediately from $\mathcal{S}$ being initial in $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}})$ (see Remark 5.2.2.1), so that there is a essentially unique natural equivalence between the composition of the inclusion[63] $\mathcal{S} \to \mathcal{C} \otimes \mathcal{S}$, which sends G to $\mathbb{1}_{\mathcal{C}} \otimes G$, with $\rho_{\mathcal{C}}$, and $\mathbb{1}_{\mathcal{C}} \boxtimes -$.

Construction of the middle triangle: It suffices to construct a commutative triangle

$$\begin{tikzcd}
\mathcal{S}^{\mathrm{B}\,G} \arrow[rr, dashed, "\Psi^G_{\mathcal{S}}", "\simeq"'] \arrow[dr, "\mathrm{ev}_*"'] & & \mathrm{LMod}_G(\mathcal{S}) \arrow[dl, "\mathrm{ev}_{\mathfrak{m}}"] \\
& \mathcal{S} &
\end{tikzcd} \tag{5.14}$$

in $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}})$ that is compatible with f, as the middle triangle in Equation (5.13) we need to construct can then be obtained by tensoring with $\mathcal{C}$.

As both ev_* and $\mathrm{ev}_{\mathfrak{m}}$ are symmetric monoidal as well as limit preserving and detecting[64], it follows from the symmetric monoidal structure on $\mathcal{S}$ being cartesian that the symmetric monoidal structures on $\mathcal{S}^{\mathrm{B}\,G}$ and $\mathrm{LMod}_G(\mathcal{S})$ are cartesian as well[65]. By [HA, 2.4.1.8], any filler for the horizontal functor and the triangle (5.14) in $\mathcal{P}r^{\mathrm{L}}$ such that the horizontal functor is an equivalence[66], can then be lifted in an essentially unique way to a filler for the triangle as a diagram in $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}})$. It thus suffices to construct a commuting triangle (5.14) in $\mathcal{P}r^{\mathrm{L}}$ in which the horizontal functor is an equivalence.

In [BP21, 3.9] an equivalence $\mathcal{S}^{\mathrm{B}\,G} \simeq \mathrm{LMod}_{\beta_1 \mathrm{B}\,G}(\mathcal{S})$ is constructed as a sequence of equivalences[67]. See the introduction of Section 5.3 for a discussion of β_1 – the underlying

[63] This is also the functor we could call $\mathbb{1}_{\mathcal{C} \otimes \mathcal{S}} \boxtimes -$, see Definition 5.2.2.2.

[64] See [HTT, 5.1.2.3] for ev_* and [HA, 4.2.3.3] for $\mathrm{ev}_{\mathfrak{m}}$.

[65] See [HA, 2.4.0.1] for the definition.

[66] Note that ev_* and $\mathrm{ev}_{\mathfrak{m}}$ are known to preserve products as already noted, so if the horizontal functor is an equivalence and hence also preserves products, (5.14) will be a commutative triangle of product preserving functors.

[67] [BP21, 3.9] contains an unnecessary use of $\mathrm{B}\,G^{\mathrm{op}} \simeq \mathrm{B}\,G$, which likely stems from a misreading of the definition of $\mathcal{P}(\mathrm{B}\,G)$ used in [HTT, 5.1.5.6], which is defined as $\mathrm{Fun}(\mathrm{B}\,G^{\mathrm{op}}, \mathcal{S})$ in [HTT, 5.1.0.1], not $\mathrm{Fun}(\mathrm{B}\,G, \mathcal{S})$.

space of $\beta_1 \mathrm{B}\, G$ is $\Omega \mathrm{B}\, G$. As B is defined as the inverse functor to (the appropriately restricted) β_1, there is a canonical equivalence $\beta_1 \mathrm{B}\, G \simeq G$, so that we obtain an equivalence $\mathrm{LMod}_{\beta_1 \mathrm{B}\, G}(\mathcal{S}) \simeq \mathrm{LMod}_G(\mathcal{S})$.

Let us now go through the individual steps to say something about compatibility with forgetful functors to $\mathcal{S}$ and compatibility with f.

For the first step, let $j\colon \mathrm{B}\, G \to \mathrm{Fun}(\mathrm{B}\, G^{\mathrm{op}}, \mathcal{S})$ be the Yoneda embedding[68], and consider the commutative diagram

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{B}\, G, \mathcal{S}) \arrow[dr, "\mathrm{ev}_*"'] & & \mathrm{Fun}^{\mathrm{colim}}\big(\mathrm{Fun}(\mathrm{B}\, G^{\mathrm{op}}, \mathcal{S}), \mathcal{S}\big) \arrow[ll, "j^*"', "\simeq"] \arrow[dl, "\mathrm{ev}_{j(*)}"] \\ & \mathcal{S} & \end{tikzcd}$$

where j^* is an equivalence by [HTT, 5.1.5.6]. Compatibility with f follows from naturality of the Yoneda embedding.

Before we discuss the second step, we first need to note something regarding right fibrations over ∞-groupoids[69]. Let X be an object of $\mathcal{S}$ and consider it as an ∞-groupoid. The ∞-category $\mathcal{RF}\mathrm{ib}(X)$ of right fibrations over X is the full subcategory of $\mathcal{CF}\mathrm{ib}(X)$ spanned by those cartesian fibrations whose fibers are ∞-groupoids. $\mathcal{CF}\mathrm{ib}(X)$ in turn is the subcategory of $\mathcal{C}\mathrm{at}_{\infty/X}$ spanned by the cartesian fibrations and morphisms of cartesian fibrations. Note that by [HTT, 2.4.2.4], if $p\colon \mathcal{E} \to X$ is a right fibration, then every morphism of $\mathcal{E}$ is p-cocartesian, so morphisms among cartesian fibrations over X (i. e. morphisms in $\mathcal{C}\mathrm{at}_{\infty/X}$) are automatically morphisms of cartesian fibrations. $\mathcal{RF}\mathrm{ib}(X)$ is thus the full subcategory of $\mathcal{C}\mathrm{at}_{\infty/X}$ spanned by the right fibrations. That X is an ∞-groupoid together with [HTT, 2.4.2.4 and 2.4.1.5] implies that a functor of ∞-categories $\mathcal{E} \to X$ is a right fibration if and only if $\mathcal{E}$ is an ∞-groupoid.

The inclusion $\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$ is also fully faithful, so induces by Proposition D.1.2.1 a fully faithful functor $\mathcal{S}_{/X} \to \mathcal{C}\mathrm{at}_{\infty/X}$ with the same essential image. We thus obtain a canonical equivalence $\mathcal{RF}\mathrm{ib}(X) \simeq \mathcal{S}_{/X}$, see Proposition B.4.3.1.

Now we can tackle the second step, for which we consider the following composite equivalence

$$\mathrm{Fun}(\mathrm{B}\, G^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{Gr}} \mathcal{RF}\mathrm{ib}(\mathrm{B}\, G) \simeq \mathcal{S}_{/\mathrm{B}\, G}$$

where the first equivalence is the Grothendieck construction. This equivalence is natural in G[70] and hence induces a commutative triangle

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{colim}}\big(\mathrm{Fun}(\mathrm{B}\, G^{\mathrm{op}}, \mathcal{S}), \mathcal{S}\big) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_{j(*)}"'] & & \mathrm{Fun}^{\mathrm{colim}}\big(\mathcal{S}_{/\mathrm{B}\, G}, \mathcal{S}\big) \arrow[dl, "\mathrm{ev}_{\mathrm{Gr}(j(*))}"] \\ & \mathcal{S} & \end{tikzcd}$$

[68]See [HTT, Introduction of 5.1.3] for a definition and discussion of j – it can be described as the functor $\mathrm{Map}_{\mathcal{S}}(\bullet, -)$.

[69]See also [HTT, 5.1.1.1] for a related discussion.

[70]For naturality of the Grothendieck construction see [GHN17, A.32].

that is compatible with f.

$\mathrm{Gr}(j(*))\colon X \to \mathrm{B}\,G$ is the right fibration classified by $j(*)$. By [HTT, 4.4.4.5] the ∞-groupoid X has a final object and is thus contractible, so that we can identify $\mathrm{Gr}(j(*))$ with the inclusion of the basepoint of $\mathrm{B}\,G$.

For the third step the equivalence

$$\mathcal{S}_{/\mathrm{B}\,G} \xrightarrow{\simeq} \mathrm{RMod}_{\beta_1\,\mathrm{B}\,G}(\mathcal{S})$$

is used that is described in [HTT, 5.2.6.28 and 5.2.6.29], and which is compatible with f. By [HTT, 5.2.6.29] this equivalence fits into a commutative diagram

$$\begin{tikzcd}
 & \mathcal{S} \arrow[ddr, "\mathrm{Free}"] & \\
\mathcal{S}_{/*} \arrow[ur, "\mathrm{pr}", "\simeq"'] \arrow[d, "(*\to\mathrm{B}\,G)_*"'] & & \\
\mathcal{S}_{/\mathrm{B}\,G} \arrow[rr, "\simeq"] & & \mathrm{RMod}_{\beta_1\,\mathrm{B}\,G}(\mathcal{S})
\end{tikzcd}$$

where $* \to \mathrm{B}\,G$ refers to the inclusion of the basepoint. It follows that $* \to \mathrm{B}\,G$ is mapped to the free right-$\beta_1\,\mathrm{B}\,G$-module generated by $*$, so to $\beta_1\,\mathrm{B}\,G$ considered as a right module over itself, under the equivalence $\mathcal{S}_{/\mathrm{B}\,G} \simeq \mathrm{RMod}_{\beta_1\,\mathrm{B}\,G}(\mathcal{S})$. By definition of B we also have a canonical equivalence $\beta_1\,\mathrm{B}\,G \simeq G$. We thus obtain a commuting triangle, compatible with f, as follows.

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{colim}}\big(\mathcal{S}_{/\mathrm{B}\,G}, \mathcal{S}\big) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_{(*\to\mathrm{B}\,G)}"'] & & \mathrm{Fun}^{\mathrm{colim}}\big(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}\big) \arrow[dl, "\mathrm{ev}_G"] \\
 & \mathcal{S} &
\end{tikzcd}$$

For the fourth step, it is explained in [BP21, 3.9] that the forgetful functor

$$\mathrm{LinFun}^{\mathrm{colim}}_{\mathcal{S}}\big(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}\big) \to \mathrm{Fun}^{\mathrm{colim}}\big(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}\big)$$

is an equivalence, so that we obtain a commutative triangle

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{colim}}\big(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}\big) \arrow[dr, "\mathrm{ev}_G"'] & & \mathrm{LinFun}^{\mathrm{colim}}_{\mathcal{S}}\big(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}\big) \arrow[ll, "\simeq"'] \arrow[dl, "\mathrm{ev}_G"] \\
 & \mathcal{S} &
\end{tikzcd}$$

that is compatible with f.

Finally, for the fifth step, [HA, 4.8.4.1] is used, where it is shown that there is an equivalence as indicated by the top horizontal functor in the following diagram.

$$\begin{tikzcd}
\mathrm{LinFun}^{\mathrm{colim}}_{\mathcal{S}}\big(\mathrm{RMod}_G(\mathcal{S}), \mathcal{S}\big) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_G"'] & & \mathrm{LMod}_G(\mathcal{S}) \arrow[dl, "\mathrm{ev}_{\mathfrak{m}}"] \\
 & \mathcal{S} &
\end{tikzcd}$$

That there also is a commutative triangle as indicated follows from unpacking the definition of the top horizontal equivalence, from which one also sees that this commutative triangle is also compatible with f, see [HA, 4.8.4.1 and 4.6.2.9].

Combining everything yields a commutative triangle (5.14) in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ in a manner compatible with f. □

5.4. The monoidal equivalence $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$

We can now combine the main result of Section 5.3 with the equivalence between the bialgebras $k \boxtimes \mathbb{T}$ and D in $\mathcal{D}(k)$ to obtain an equivalence as follows.

$$\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \simeq \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$$

This equivalence is only (Assoc-)monoidal, not $\mathbb{E}_2$-monoidal or even symmetric monoidal, see Warning 5.4.0.2 below.

Construction 5.4.0.1. The ∞-category $\mathcal{D}(k)$ is a presentable symmetric monoidal ∞-category by Proposition 4.3.2.1 (1), and as the circle group $\mathbb{T}$ is path connected, it follows from [HA, 5.2.6.4] that $\mathbb{T}$ is grouplike as an associative monoid in $\mathcal{S}$. Hence we can apply Proposition 5.3.0.8 and Remark 5.3.0.9 to obtain a commutative diagram in $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ as follows

$$\begin{tikzcd}
\mathcal{D}(k)^{*} \arrow[d, "(\mathrm{B}\,\mathbb{T}\to *)^{*}"'] & \mathcal{D}(k) \arrow[l, "(\mathrm{ev}_{*})^{-1}"'] \arrow[r, "(\mathrm{ev}_{\mathrm{m}})^{-1}"] & \mathrm{LMod}_{\mathbb{1}_{\mathcal{D}(k)}}(\mathcal{D}(k)) \arrow[d, "\mathrm{LMod}_{\left(k\boxtimes\mathbb{T}\to\mathbb{1}_{\mathcal{D}(k)}\right)}(\mathcal{D}(k))"] \\
\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_{*}"'] & & \mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \arrow[dl, "\mathrm{ev}_{\mathrm{m}}"] \\
& \mathcal{D}(k) &
\end{tikzcd} \tag{5.15}$$

where the middle horizontal morphism is an equivalence and the morphisms of bialgebras $\mathrm{B}\,\mathbb{T} \to *$ and $\mathrm{D} \to \mathbb{1}_{\mathcal{D}(k)}$ are the essentially unique ones, see Proposition 5.3.0.7.

Proposition 5.2.4.2 and Convention 5.2.4.3 provide us with an equivalence of bialgebras in $\mathcal{D}(k)$

$$\varphi\colon \mathrm{D} \to k \boxtimes \mathbb{T}$$

and as k is a final object in $\mathrm{BiAlg}_{\mathsf{Assoc},\mathsf{Assoc}}(\mathcal{D}(k))$ by Proposition 5.3.0.7, we can extend this to a commutative triangle of bialgebras in $\mathcal{D}(k)$ as follows.

$$\begin{tikzcd}
\mathrm{D} \arrow[rr, "\varphi"] \arrow[dr] & & k \boxtimes \mathbb{T} \arrow[dl] \\
& k &
\end{tikzcd}$$

Applying the functor LMod from Definition 3.4.2.1 we obtain a commutative diagram in $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$

$$\begin{tikzcd}
 & \mathrm{LMod}_{\mathbb{1}_{\mathcal{D}(k)}}(\mathcal{D}(k)) \arrow[dl, "\mathrm{LMod}_{(k\boxtimes\mathbb{T}\to\mathbb{1}_{\mathcal{D}(k)})}(\mathcal{D}(k))"'] \arrow[dr, "\mathrm{LMod}_{(\mathrm{D}\to\mathbb{1}_{\mathcal{D}(k)})}(\mathcal{D}(k))"] & \\
\mathrm{LMod}_{k\boxtimes\mathbb{T}}(\mathcal{D}(k)) \arrow[rr, "\mathrm{LMod}_{\varphi}(\mathcal{D}(k))", "\simeq"'] \arrow[dr, "\mathrm{ev}_{\mathfrak{m}}"'] & & \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) \arrow[dl, "\mathrm{ev}_{\mathfrak{m}}"] \\
 & \mathcal{D}(k) &
\end{tikzcd} \tag{5.16}$$

where the top triangle is the one induced by the previous diagram, and the bottom one uses that $\mathrm{ev}_{\mathfrak{m}}$ is a natural transformation.

Combining (5.15) and (5.16) we obtain a commutative diagram in $\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, i. e. of presentable monoidal ∞-categories with monoidal colimit preserving functors, as follows

$$\begin{tikzcd}
\mathcal{D}(k)^{*} \arrow[d, "(\mathrm{B}\,\mathbb{T}\to *)^{*}"'] & \mathcal{D}(k) \arrow[l, "(\mathrm{ev}_{*})^{-1}"'] \arrow[r, "(\mathrm{ev}_{\mathfrak{m}})^{-1}"] & \mathrm{LMod}_{\mathbb{1}_{\mathcal{D}(k)}}(\mathcal{D}(k)) \arrow[d, "\mathrm{LMod}_{(\mathrm{D}\to\mathbb{1}_{\mathcal{D}(k)})}(\mathcal{D}(k))"] \\
\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_{*}"'] & & \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k)) = \mathcal{M}\mathrm{ixed} \arrow[dl, "\mathrm{ev}_{\mathfrak{m}}"] \\
 & \mathcal{D}(k) &
\end{tikzcd}$$

such that the middle horizontal functor is an equivalence. ♢

Warning 5.4.0.2. While both $\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$ and $\mathcal{M}\mathrm{ixed} = \mathrm{LMod}_{\mathrm{D}}(\mathcal{D}(k))$ carry a symmetric monoidal structure, the equivalence between them is only Assoc-monoidal.

For this reason one should be careful to distinguish between "objects of $\mathcal{D}(k)$ with $\mathbb{T}$-action" (or "$\mathbb{T}$-objects in $\mathcal{D}(k)$") on the one hand and "mixed complexes" on the other hand whenever the symmetric monoidal structures might be relevant. ♢

Remark 5.4.0.3. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Combining the compatibility statement with colimit preserving symmetric monoidal functors between presentable symmetric monoidal ∞-categories that is part of Proposition 5.3.0.8 with Remark 5.2.4.4 we obtain a commutative diagram of monoidal colimit preserving functors[71]

$$\begin{tikzcd}
\mathcal{D}(k)^{\mathrm{B}\mathbb{T}} \arrow[rr, dash, "\simeq"] \arrow[dr, "\mathrm{ev}_*"'] \arrow[ddd, "(k'\otimes_k -)_*"'] & & \mathrm{LMod}_{\mathrm{D}_k}\big(\mathcal{D}(k)\big) \arrow[dl, "\mathrm{ev}_{\mathrm{m}}"] \arrow[ddd, "k'\otimes_k -"] \\
& \mathcal{D}(k) \arrow[d, "k'\otimes_k -"] & \\
& \mathcal{D}(k') & \\
\mathcal{D}(k')^{\mathrm{B}\mathbb{T}} \arrow[rr, dash, "\simeq"'] \arrow[ur, "\mathrm{ev}_*"] & & \mathrm{LMod}_{\mathrm{D}_{k'}}\big(\mathcal{D}(k)\big) \arrow[ul, "\mathrm{ev}_{\mathrm{m}}"']
\end{tikzcd}$$

where the horizontal equivalences are the ones from Construction 5.4.0.1. ◊

[71] Like with the diagram in Remark 5.2.4.4, there is also supposed to be a filler for the outer diagram that is compatible with the forgetful functors.

Chapter 6.

Hochschild homology

In this chapter we introduce the main object of study of this text, *Hochschild homology*. We will give both a modern account, in which the main construction is a functor

$$\mathrm{HH}_{\mathbb{T}}\colon \operatorname{Alg}\big(\mathcal{D}(k)\big) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

called *Hochschild homology* that will be defined and discussed in Section 6.2, as well as a description of the classical constructions, where one considers a functor

$$\mathrm{C}\colon \operatorname{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathsf{Mixed}_{\mathrm{cof}}$$

called *standard Hochschild complex*. The latter construction will be discussed in Section 6.3, where we will also show that the two constructions are related – the standard Hochschild complex can be considered as a model for Hochschild homology. For both the definitions the first step is to apply the *cyclic bar construction*, which takes an associative algebra in an some monoidal ∞-category $\mathcal{C}$, and produces a *cyclic object* in $\mathcal{C}$, i. e. a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$, where $\mathbf{\Lambda}$ is Connes' cyclic category. For this reason, we start this chapter in Section 6.1 with a discussion of the cyclic bar construction as well as the geometric realization of cyclic objects.

6.1. The cyclic bar construction and geometric realization of cyclic objects

In this section we discuss the *cyclic bar construction*. Given a presentable symmetric monoidal ∞-category $\mathcal{C}$, this is a (symmetric monoidal) functor

$$\mathrm{B}^{\mathrm{cyc}}\colon \operatorname{Alg}(\mathcal{C}) \to \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$$

that constructs an object in $\mathcal{C}$ with $\mathbb{T}$-action out of every (associative) algebra in $\mathcal{C}$.

The construction proceeds in two main steps. Starting with an algebra R in $\mathcal{C}$, one first constructs a *cyclic object* in $\mathcal{C}$, denoted by $\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)$, and also called the cyclic bar construction[1], which is a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$, where $\mathbf{\Lambda}$ is Connes' cyclic category. We will review $\mathbf{\Lambda}$ in Section 6.1.1, and define the symmetric monoidal functor

$$\mathrm{B}^{\mathrm{cyc}}_{\bullet}\colon \operatorname{Alg}(\mathcal{C}) \to \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

[1]In fact, we will almost exclusively refer to *this* construction as the cyclic bar construction in the remainder of the text.

in Section 6.1.2.

Given a cyclic object X in $\mathcal{C}$, one can then take the *geometric realization* $|X|$, which yields an object in $\mathcal{C}$ with $\mathbb{T}$-action, as will be discussed in Section 6.1.3. The cyclic bar construction $\mathrm{B}^{\mathrm{cyc}}$ of an associative algebra R can then be defined as $\mathrm{B}^{\mathrm{cyc}}(R) \coloneqq |\mathrm{B}^{\mathrm{cyc}}_{\bullet}(R)|$.

As main references for the material below we use [NikSch], [Hoy18], and [Lod98].

6.1.1. Connes' cyclic category $\boldsymbol{\Lambda}$

In this section we discuss Connes' cyclic category $\boldsymbol{\Lambda}$, which has the simplex category $\boldsymbol{\Delta}$ as a subcategory and is mainly of interest because it encodes circle actions. More concretely, if $\mathcal{C}$ is a presentable ∞-category and $X\colon \boldsymbol{\Lambda}^{\mathrm{op}} \to \mathcal{C}$ a diagram, then the geometric realization (i. e. colimit) of the restriction of X to $\boldsymbol{\Delta}^{\mathrm{op}}$ naturally acquires the action of the circle group[2] $\mathbb{T}$, as we will see as Fact 6.1.3.6 in Section 6.1.3.2.

We will start by reviewing the two different approaches towards defining the simplex category $\boldsymbol{\Delta}$ (one via generators and relations, one more abstract) in Section 6.1.1.1, before discussing analogous definitions of the cyclic category $\boldsymbol{\Lambda}$ in Sections 6.1.1.2 and 6.1.1.3. We will show that the two definitions we give for $\boldsymbol{\Lambda}$ are equivalent in Section 6.1.1.4. Finally, we will introduce the notion of *cyclic objects* in Section 6.1.1.5 and describe the self-duality functor of $\boldsymbol{\Lambda}$ in Section 6.1.1.6, which will be relevant for the definition of the cyclic bar construction in Section 6.1.2.

6.1.1.1. The simplex category $\boldsymbol{\Delta}$

Recall that there are two approaches towards defining the simplex category $\boldsymbol{\Delta}$.

- $\boldsymbol{\Delta}$ can be defined as the category of totally ordered non-empty finite sets together with (weakly) order-preserving maps.

- $\boldsymbol{\Delta}$ can be constructed as the category with objects $[n]$ for $n \geq 0$ and morphisms generated by $\delta_i\colon [n-1] \to [n]$ (for $n \geq 1$ and $0 \leq i \leq n$) and $\sigma_i\colon [n+1] \to [n]$ (for $n \geq 0$ and $0 \leq i \leq n$) satisfying the *simplicial identities*[3].

If we temporarily refer to the second definition as $\boldsymbol{\Delta}'$, then we can relate $\boldsymbol{\Delta}'$ and $\boldsymbol{\Delta}$ with a functor $\boldsymbol{\Delta}' \to \boldsymbol{\Delta}$ that can be described as follows.

- $[n]$ is mapped to the totally ordered set $\{0 < 1 < \cdots < n\}$.

- $\delta_i\colon [n-1] \to [n]$ is mapped to the injective order-preserving map that does not have i in the image.

- $\sigma_i\colon [n+1] \to [n]$ is mapped to the order-preserving map that is surjective and maps both i and $i+1$ to i.

[2] $\mathbb{T}$ was defined in Construction 5.2.1.1.

[3] They can be found for example in [Lod98, B.3] or [Mac98, Page 177]. See also Remark 6.1.1.8 below.

This functor is an equivalence of categories, as shown in [Mac98, Proposition 2 on page 178][4]. We will thus usually identify $\mathbf{\Delta}$ and $\mathbf{\Delta}'$ and use whatever description is most appropriate for the occasion.

Notation 6.1.1.1. Let $\mathcal{C}$ be an ∞-category. A functor

$$X\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$$

will be called a *simplicial object* in $\mathcal{C}$. We will write X_n instead of $X([n])$ and accordingly often also use $X_\bullet$ for X if we want to emphasize X being a simplicial object. We will refer to the morphism induced by the opposite of δ_i as d_i, and to the morphism induced by the opposite of σ_i as s_i. ♢

Completely analogously to the situation for the simplex category, there are two approaches to Connes' cyclic category $\mathbf{\Lambda}$. We will discuss an abstract definition first in Section 6.1.1.2 and then discuss a definition using generators and relations in Section 6.1.1.3, before showing that they are equivalent in Section 6.1.1.4.

6.1.1.2. Definition of $\mathbf{\Lambda}$ via posets

Definition 6.1.1.2 ([NikSch, page 380])**.** We denote by PoSet the category of partially ordered sets with (weakly) order preserving maps. We furthermore define

$$\mathbb{Z}\mathsf{PoSet} \coloneqq \mathrm{Fun}(\mathrm{B}\,\mathbb{Z}, \mathsf{PoSet})$$

to be the category of objects in PoSet with $\mathbb{Z}$-action.

An example for an object in $\mathbb{Z}\mathsf{PoSet}$ is $(1/n)\cdot\mathbb{Z}$ for $n \geq 1$; as a subset of $\mathbb{Q}$ this set inherits a partial order, and an integer k acts by addition.

We now define $\mathbf{\Lambda}_\infty$ to be the full subcategory of $\mathbb{Z}\mathsf{PoSet}$ spanned by the objects isomorphic to $(1/n)\cdot\mathbb{Z}$ for $n \geq 1$. The category $\mathbf{\Lambda}_\infty$ is called the *paracyclic category*. ♢

Recall the equivalence

$$\mathcal{S}_*^{\geq 1} \underset{\mathrm{B}}{\overset{\beta_1}{\rightleftarrows}} \mathrm{Mon}^{\mathrm{gp}}_{\mathsf{Assoc}}(\mathcal{S})$$

from [HA, 5.2.6, in particular 5.2.6.10] that was discussed in Section 5.3. The functors β_1 and B induce mutually inverse equivalences on the respective ∞-categories of commutative monoids, so as $\mathbb{Z}$ is commutative $\mathrm{B}\,\mathbb{Z}$ acquires an induced commutative monoid structure. $\mathrm{B}\,\mathbb{Z}$ can in fact be identified, as an object of $\mathrm{CMon}(\mathcal{S}_*^{\geq 1})$, with the circle group $\mathbb{T}$ (see Construction 5.2.1.1). To see this it suffices to check that $\beta_1(\mathbb{T}) \simeq \mathbb{Z}$ as commutative monoids in $\mathcal{S}$, but as the underlying spaces are discrete this is just a classical exercise using the Eckmann-Hilton argument[5].

[4] What is referred to as $\mathbf{\Delta}$ in [Mac98] is not what we refer to as $\mathbf{\Delta}$, but also includes the empty set. What we refer to as $\mathbf{\Delta}$ is denoted by $\mathbf{\Delta}^+$ in [Mac98] and discussed in [Mac98, Bottom of page 178]. But while the statement of [Mac98, Proposition 2 on page 178] does not directly deal with our $\mathbf{\Delta}$, it nevertheless directly implies the result, as there are no maps from a non-empty set to an empty set.

[5] The underlying space of $\beta_1(\mathbb{T})$ is $\Omega\mathbb{T}$. This loop space has two monoid structures – an associative via composition of loops, and a commutative one via pointwise multiplication using the commutative monoid structure on $\mathbb{T}$.

As $\mathbb{T}$ is path connected, it is grouplike as a monoid in $\mathcal{S}$ by [HA, 5.2.6.4], so we can form $\mathrm{B}\,\mathbb{T}$ and consider objects with $\mathbb{T}$-action in some ∞-category $\mathcal{D}$, i.e. functors $\mathrm{B}\,\mathbb{T} \to \mathcal{D}$ – see the introduction to Section 5.3. The ∞-groupoid $\mathrm{B}\,\mathbb{T} \simeq \mathrm{B}\,\mathrm{B}\,\mathbb{Z}$ can be interpreted as the ∞-groupoid with a unique object $*$, unique morphism, and with $\mathbb{Z}$ being the space of 2-morphisms $\mathrm{id}_* \to \mathrm{id}_*$. A $\mathbb{T} \simeq \mathrm{B}\,\mathbb{Z}$-action on an ∞-category $\mathcal{C}$, i.e. a functor $\mathrm{B}\,\mathbb{T} \to \mathcal{C}\mathrm{at}_\infty$ mapping $*$ to $\mathcal{C}$, then essentially consists of a natural equivalence $\mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ corresponding to the generator 1 of $\mathbb{Z}$.

If $\mathcal{C} = \mathsf{C}$ is a 1-category, then this amounts to giving an automorphism $\varphi_X \colon X \to X$ for every object X of C in such a way that these automorphisms are compatible with every morphism of C, i.e. for every morphism $f \colon X \to Y$ of C it must hold $\varphi_Y \circ f = f \circ \varphi_X$. This data is in turn equivalent to a natural action of $\mathbb{Z}$ on the morphism sets of C: We can let n act on $\mathrm{Mor}_{\mathsf{C}}(X, Y)$ by $\varphi_Y^n \circ -$. If instead we have a natural action of $\mathbb{Z}$ on the morphism sets given, then we can recover the automorphisms φ_X as the result of letting 1 act on the element id_X in $\mathrm{Mor}_{\mathsf{C}}(X, X)$.

We can now state the definition of the cyclic category $\mathbf{\Lambda}$ as it is given in [NikSch, page 380].

Definition 6.1.1.3 ([NikSch, page 380])**.** There is an action of $\mathbb{Z}$ on the morphisms spaces of $\mathbf{\Lambda}_\infty$ such that the action of an integer k on a morphism f yields the morphism $f(-) + k = f(- + k)$.

Dividing out this action, i.e. identifying a morphism f with $f + k$ for any integer k, we obtain a category that we denote by $\mathbf{\Lambda}$ and call *Connes' cyclic category.* ♢

Notation 6.1.1.4. We will use the notation $[n]_{\mathbf{\Lambda}}$ for $(1/(n+1)) \cdot \mathbb{Z}$ when considered as an object in $\mathbf{\Lambda}$ as described in Definition 6.1.1.2. Up to isomorphism, the objects of $\mathbf{\Lambda}$ are thus given by $[n]_{\mathbf{\Lambda}}$ for $n \geq 0$. ♢

Warning 6.1.1.5. Notation 6.1.1.4 deviates from the notation in [NikSch], where $[n]_{\mathbf{\Lambda}}$ is defined to be $1/n \cdot \mathbb{Z}$.

The notation we use is chosen to be more consistent with the notation used for objects of $\mathbf{\Delta}$ – it also matches the notation used in [Lod98], see [Lod98, 6.1.1]. ♢

The category $\mathbf{\Lambda}$ contains $\mathbf{\Delta}$ as a subcategory, as we note next.

Construction 6.1.1.6 ([NikSch, page 382])**.** Consider $\mathbf{\Delta}$ as the category of totally ordered non-empty finite sets. We can then define a functor $\mathbf{\Delta} \to \mathbb{Z}\mathsf{PoSet}$ by mapping a totally ordered non-empty finite set S to $\mathbb{Z} \times S$, equipped with the lexicographic order and action by $\mathbb{Z}$ via addition on the first component. If $S = \{s_0 < s_1 < \cdots < s_n\}$, then there is an isomorphism $\mathbb{Z} \times S \cong (1/(n+1)) \cdot \mathbb{Z}$ in $\mathbb{Z}\mathsf{PoSet}$ that maps (k, s_i) to $k + (i/(n+1))$, so the functor factors through $\mathbf{\Lambda}_\infty$.

Following [NikSch, page 382], we will denote the resulting functor $\mathbf{\Delta} \to \mathbf{\Lambda}_\infty$ by j_∞^{op}[6] and the composition

$$\mathbf{\Delta} \to \mathbf{\Lambda}_\infty \to \mathbf{\Lambda}$$

by j^{op}. It is not difficult to check that j_∞^{op} and j^{op} are faithful and induce bijections on isomorphism classes of objects. ♢

[6] The reason for the $^{\mathrm{op}}$ is that the opposite of this functor is more important (or at least more often used) and hence gets to have the name with least decorations.

6.1.1.3. Definition of $\mathbf{\Lambda}$ via generators and relations

We now describe $\mathbf{\Lambda}$ with generators and relations.

Construction 6.1.1.7 ([Lod98, 6.1.1])**.** We define the 1-category $\mathbf{\Lambda}'$ to have objects $[n]_{\mathbf{\Lambda}'}$ for integers $n \geq 0$, and morphisms generated by

$$
\begin{array}{ll}
\delta_i : [n-1]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'} & \text{for } n \geq 1 \text{ and } 0 \leq i \leq n \\
\sigma_i : [n+1]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'} & \text{for } n \geq 0 \text{ and } 0 \leq i \leq n \\
\tau : [n]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'} & \text{for } n \geq 0
\end{array}
$$

subject to the following relations[7].

$$
\begin{array}{ll}
\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1} & \text{for } i < j \\
\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1} & \text{for } i \leq j \\
\sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1} & \text{for } i < j \\
\sigma_j \circ \delta_i = \mathrm{id} & \text{for } i = j \text{ or } i = j+1 \\
\sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j & \text{for } i > j+1 \\
\tau \circ \delta_i = \delta_{i-1} \circ \tau & \text{for } i > 0 \\
\tau \circ \delta_0 = \delta_n & \text{where } \tau\colon [n]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'} \\
\tau \circ \sigma_i = \sigma_{i-1} \circ \tau & \text{for } i > 0 \\
\tau \circ \sigma_0 = \sigma_n \circ \tau^2 & \text{where } \sigma_0\colon [n+1]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'} \\
\tau^{n+1} = \mathrm{id}_{[n]_{\mathbf{\Lambda}'}} & \text{where } \tau\colon [n]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}
\end{array}
$$
◊

Remark 6.1.1.8. If we remove the morphisms τ as generators in Construction 6.1.1.7 (as well as the relations involving them), then we obtain precisely the definition of $\mathbf{\Delta}$ via generators and relations. We thus obtain a functor $j'^{\mathrm{op}}\colon \mathbf{\Delta} \to \mathbf{\Lambda}'$. ◊

6.1.1.4. Comparison of the two definitions of $\mathbf{\Lambda}$

To show that $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ are equivalent, we first construct a comparison functor.

Proposition 6.1.1.9. *There is a functor $\Phi\colon \mathbf{\Lambda}' \to \mathbf{\Lambda}$ defined as follows.*

- *$[n]_{\mathbf{\Lambda}'}$ is mapped to to $[n]_{\mathbf{\Lambda}}$.*
- *$\delta_i\colon [n-1]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}$ is mapped to the unique morphism that sends 0 to 0 and has $\frac{0}{n+1}, \ldots, \frac{i-1}{n+1}, \frac{i+1}{n+1}, \ldots, \frac{n}{n+1}$ in its image.*
- *$\sigma_i\colon [n+1]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}$ is mapped to the unique morphism that sends 0 to 0, is surjective, and sends $\frac{i}{n+2}$ and $\frac{i+1}{n+2}$ to $\frac{i}{n+1}$.*

[7] As we do not specify the n as part of the notation of the three types of morphisms, notation like δ_i refers to more than a single morphism. The relations below are to be satisfied for all choices where the morphisms can be composed as indicated and both sides of the equation have same domain and codomain.

- *$\tau\colon [n]_{\mathbf{\Lambda}'} \to [n]_{\mathbf{\Lambda}'}$ is mapped to the unique morphism that is surjective and sends $\frac{1}{n+1}$ to $\frac{0}{n+1}$.*

Furthermore, this functor fits into a commutative square

$$\begin{tikzcd} \mathbf{\Delta}' \arrow[r, "j'^{\mathrm{op}}"] \arrow[d, "\Phi_{\mathbf{\Delta}}"'] & \mathbf{\Lambda}' \arrow[d, "\Phi"] \\ \mathbf{\Delta} \arrow[r, "j^{\mathrm{op}}"'] & \mathbf{\Lambda} \end{tikzcd}$$

that commutes up to natural isomorphism $\varphi\colon j^{\mathrm{op}} \circ \Phi_{\mathbf{\Delta}} \to \Phi \circ j'^{\mathrm{op}}$ where $\Phi_{\mathbf{\Delta}}$ is the equivalence from Section 6.1.1.1, and j'^{op} and j^{op} are as in Remark 6.1.1.8 and Construction 6.1.1.6. The components of the natural isomorphism φ are to be the isomorphisms

$$\varphi_{[n]}\colon j^{\mathrm{op}}([n]) = \mathbb{Z} \times [n] \xrightarrow{\cong} (1/(n+1)) \cdot \mathbb{Z} = [n]_{\mathbf{\Lambda}}$$

that were discussed in Construction 6.1.1.6. ♡

Proof. Easy but a bit tedious exercise checking the relations. □

For both $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ one can show that morphisms decompose uniquely as the composition of a power of τ with a morphism in the image of the inclusion of $\mathbf{\Delta}$, as we will see next. This is what will imply that the functor $\mathbf{\Lambda}' \to \mathbf{\Lambda}$ from Proposition 6.1.1.9 is an equivalence.

Proposition 6.1.1.10. *Let $f\colon [n]_{\mathbf{\Lambda}'} \to [m]_{\mathbf{\Lambda}'}$ be a morphism in $\mathbf{\Lambda}'$. Then there exists a unique morphism $g\colon [n] \to [m]$ in $\mathbf{\Delta}$ and integer k with $0 \leq k \leq n$ such that $f = j'^{\mathrm{op}}(f) \circ \tau^k$.*

An analogous statement also holds for $\mathbf{\Lambda}$. Let $f\colon [n]_{\mathbf{\Lambda}} \to [m]_{\mathbf{\Lambda}}$ be a morphism in $\mathbf{\Lambda}$. Then there is a unique morphism $g\colon [n] \to [m]$ in $\mathbf{\Delta}$ and integer k with $0 \leq k \leq n$ such that $f = \varphi_{[m]} \circ j^{\mathrm{op}}(f) \circ \varphi_{[n]}^{-1} \circ \Phi(\tau)^k$, where we use notation from Proposition 6.1.1.9. ♡

Proof. The statement for $\mathbf{\Lambda}'$ is precisely [Lod98, 6.1.3].

For $\mathbf{\Lambda}$ note that there is a unique $0 \leq k \leq n$ and morphism $f'\colon [n]_{\mathbf{\Lambda}} \to [m]_{\mathbf{\Lambda}}$ such that $f = f' \circ \Phi(\tau)^k$ and such that f' maps $\mathbb{Z}$ to $\mathbb{Z}$. The claim now follows from the observation that a morphism $\mathbb{Z} \times [n] \to \mathbb{Z} \times [m]$ in $\mathbf{\Lambda}_\infty$ that maps $(0,0)$ to $(0,0)$ must be of the form $\mathrm{id}_{\mathbb{Z}} \times g$ for a unique morphism $g\colon [n] \to [m]$ in $\mathbf{\Delta}$. □

Corollary 6.1.1.11. *The functor Φ from Proposition 6.1.1.9 is an equivalence.* ♡

Proof. Φ is by definition essentially surjective. That Φ is also fully faithful follows immediately from Proposition 6.1.1.10. □

We will from now on not distinguish between $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ and use the description best adapted for each individual situation.

6.1.1.5. Cyclic objects

Notation 6.1.1.12 ([Lod98, 6.1.2.1])**.** Let $\mathcal{C}$ be an ∞-category. We call a functor

$$X\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$$

a *cyclic object* in $\mathcal{C}$. We will use the same notational conventions as explained in Notation 6.1.1.1 for simplicial objects, and will refer to the image of $[n]_{\mathbf{\Lambda}}$ under X as X_n (and sometimes write $X_\bullet$ for X), to the morphism induced by the opposite of δ_i as d_i, to the morphism induced by the opposite of σ_i as s_i, and to the morphism induced by the opposite of τ as t. $\diamondsuit$

6.1.1.6. Self-duality of $\mathbf{\Lambda}$

We record that $\mathbf{\Lambda}$ has a self-duality functor, which will be needed later.

Fact 6.1.1.13 ([Lod98, 6.1.11])**.** *There is an equivalence* $-^\circ\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathbf{\Lambda}$ *that maps*

- $[n]_{\mathbf{\Lambda}}$ *to* $[n]_{\mathbf{\Lambda}}$,
- δ_i^{op} *to* σ_i,
- σ_i^{op} *to* δ_{i+1},
- τ^{op} *to* τ^{-1}.

where $\sigma_{n+1}\colon [n+1]_{\mathbf{\Lambda}} \to [n]_{\mathbf{\Lambda}}$ *is the* extra degeneracy *defined as* $\sigma_{n+1} = \sigma_0 \circ \tau^{-1}$. $\clubsuit$

The above is also proven in [NikSch, page 381] using the definition of $\mathbf{\Lambda}$ via posets[8], and one can check that the two functors agree by unpacking the definitions.

6.1.2. The cyclic bar construction as a cyclic object

In this section we discuss the cyclic bar construction of associative algebras. Let C be a symmetric monoidal 1-category and A an associative algebra in C. Then one can construct a simplicial object in C

$$\cdots \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} A\otimes A\otimes A \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} A\otimes A \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} A$$

where the structure morphisms $d_i\colon A^{\otimes n} \to A^{\otimes(n-1)}$ and $s_i\colon A^{\otimes n} \to A^{\otimes(n+1)}$ can be described as follows[9]:

1. If $i \leq n-2$, then d_i is $\mathrm{id}_A^{\otimes i} \otimes \mu \otimes \mathrm{id}_A^{\otimes(n-2-i)}$, where $\mu\colon A\otimes A \to A$ is the multiplication morphism.

[8] Whereas [Lod98, 6.1.11] uses the definition via generators and relations.

[9] We omit making explicit any associativity or unitality isomorphisms from the symmetric monoidal structure on C.

2. d_{n-1} is the postcomposition of the symmetry isomorphism that brings the last tensor factor to the front with $\mu \otimes \mathrm{id}_A^{\otimes(n-2)}$.

3. s_i is $\mathrm{id}_A^{i+1} \otimes \iota \otimes \mathrm{id}_A^{\otimes(n-i-1)}$, where $\iota\colon \mathbb{1}_{\mathsf{C}} \to A$ is the unit morphism.

Making use of cyclic permutations of the tensor factors, we can even extend the above simplicial object to a cyclic object

$$\begin{tikzcd}
\cdots \arrow[r, shift left=6] \arrow[r, <-, shift left=4] \arrow[r, shift left=2] \arrow[r, <-] \arrow[r, shift right=2] \arrow[r, <-, shift right=4] \arrow[r, shift right=6] & A\otimes A\otimes A \arrow[loop above] \arrow[r, shift left=4] \arrow[r, <-, shift left=2] \arrow[r] \arrow[r, <-, shift right=2] \arrow[r, shift right=4] & A\otimes A \arrow[loop above] \arrow[r, shift left=2] \arrow[r, <-] \arrow[r, shift right=2] & A \arrow[loop above]
\end{tikzcd}$$

where the structure morphism $t\colon A^{\otimes n} \to A^{\otimes n}$ is the symmetry isomorphism moving the last tensor factor to the front.

The goal of this section is to rigorously define a cyclic object implementing this idea for associative algebras in any symmetric monoidal ∞-category $\mathcal{C}$. Furthermore, we will also show that the resulting functor $\mathrm{B}_\bullet^{\mathrm{cyc}}\colon \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ can be upgraded to a symmetric monoidal functor, where $\mathrm{Alg}(\mathcal{C})$ carries the induced symmetric monoidal structure from Proposition E.4.2.3 and $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ the pointwise symmetric monoidal structure from [HA, 2.1.3.4].

$\mathrm{B}_\bullet^{\mathrm{cyc}}$ will be defined as a composition

$$\begin{aligned}
\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) &\xrightarrow{A} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big) \\
&\xrightarrow{(\otimes)_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \\
&\xrightarrow{\left(V\circ(-^\circ)\right)^*} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})
\end{aligned}$$

and we will define individual ingredients one by one[10].

Let us now give a brief overview over the subsections below. We will start in Section 6.1.2.1 by discussing the symmetric monoidal envelope of an ∞-operad, which will explain what symmetric monoidal structure we consider on $\mathcal{C}^{\otimes}_{\mathrm{act}}$. In Section 6.1.2.2 we will then construct the first row (i. e. the first two functors) in the composition above that will define $\mathrm{B}_\bullet^{\mathrm{cyc}}$, and show that the composition of those two functor is lax symmetric monoidal. We will then define the symmetric monoidal functor $\otimes\colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ in Section 6.1.2.3 and show that the composition of the lax symmetric monoidal functor

$$\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)$$

from Section 6.1.2.2 with the symmetric monoidal functor $(\otimes)_*$ is not just lax symmetric monoidal, but symmetric monoidal. For the last step in the definition of $\mathrm{B}_\bullet^{\mathrm{cyc}}$, we have already defined the functor $(-)^\circ$, in Fact 6.1.1.13, and will define the remaining functor

[10] We warn though that while $\mathrm{B}_\bullet^{\mathrm{cyc}}$ will be shown to be symmetric monoidal, we do not claim that the individual functors in the above composition are symmetric monoidal functors of symmetric monoidal ∞-categories.

$V\colon \mathbf{\Lambda} \to \mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ in Section 6.1.2.4. This will be the last ingredient that we need to define $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$, and we will put everything together in Section 6.1.2.5. We will end this section by giving a more direct description for $\mathrm{CAlg}(\mathrm{B}^{\mathrm{cyc}}_{\bullet})$, the functor induced by $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ on commutative algebras, in Section 6.1.2.6, and showing that $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ preserves sifted colimits in Section 6.1.2.7.

6.1.2.1. The symmetric monoidal envelope

Let $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. In [HA, 2.2.4] the *symmetric monoidal envelope* of $\mathcal{O}$ is discussed[11], which is defined in [HA, 2.2.4.1] as

$$\mathrm{Env}(\mathcal{O})^{\otimes} := \mathcal{O}^{\otimes} \times_{\mathsf{Fin}_*} \mathrm{Act}(\mathsf{Fin}_*) \tag{6.1}$$

where the functor $\mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ is given by $p_{\mathcal{O}}$, the ∞-category $\mathrm{Act}(\mathsf{Fin}_*)$ is defined as the full subcategory of $\mathrm{Fun}([1], \mathsf{Fin}_*)$ spanned by the active morphisms[12], and the functor $\mathrm{Act}(\mathsf{Fin}_*) \to \mathsf{Fin}_*$ is ev_0.

Like [NikSch, page 366] and [HA, 2.2.4.3], we will use the notation $\mathcal{O}^{\otimes}_{\mathrm{act}}$ to refer to the subcategory of $\mathcal{O}^{\otimes}$ spanned by all objects and the active morphisms[13], i. e. those morphisms mapped by $p_{\mathcal{O}}$ to an active morphism in Fin_*. Note that the inclusion

$$\mathcal{O}^{\otimes}_{\mathrm{act}} \to \mathcal{O}^{\otimes}$$

can be identified with the functor

$$\begin{tikzcd} \mathcal{O}^{\otimes} \times_{\mathsf{Fin}_*} (\mathsf{Fin}_*)_{\mathrm{act}} \arrow[r] \arrow[rr, bend left, "\mathrm{pr}_1"] & \mathcal{O}^{\otimes} \times_{\mathsf{Fin}_*} \mathsf{Fin}_* \arrow[r, "\simeq", "\mathrm{pr}_1"'] & \mathcal{O}^{\otimes} \end{tikzcd}$$

where the left functor is the one induced by the inclusion $(\mathsf{Fin}_*)_{\mathrm{act}} \to \mathsf{Fin}_*$ – this follows from Proposition B.5.2.1 and Proposition B.4.3.1, see also Remark B.6.0.1.

Let $p_{\mathrm{Env}(\mathcal{O})}\colon \mathrm{Env}(\mathcal{O})^{\otimes} \to \mathsf{Fin}_*$ be defined as $\mathrm{ev}_1 \circ \mathrm{pr}_2$. Unpacking the definition of $\mathrm{Env}(\mathcal{O})^{\otimes}$, we can then interpret an object lying over $\langle n \rangle$ as a pair (O, α) with O an object of $\mathcal{O}^{\otimes}$ and α an active morphism $p_{\mathcal{O}}(O) \to \langle n \rangle$ in Fin_* – see [HA, 2.2.4.2]. In particular, as there is a unique active morphism from any object of Fin_* to $\langle 1 \rangle$, one can identify $\mathrm{Env}(\mathcal{O})^{\otimes}_{\langle 1 \rangle}$ with $\mathcal{O}^{\otimes}_{\mathrm{act}}$ – see [HA, 2.2.4.3].

One important result about $\mathrm{Env}(\mathcal{O})$ that we will need is the following.

Fact 6.1.2.1 ([HA, 2.2.4.4 and 2.2.4.15])**.** *Let $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. Then $p_{\mathrm{Env}(\mathcal{O})}\colon \mathrm{Env}(\mathcal{O})^{\otimes} \to \mathsf{Fin}_*$ is a cocartesian fibration of ∞-operads, i. e. exhibits $\mathcal{O}^{\otimes}_{\mathrm{act}}$ as a symmetric monoidal ∞-category.*

Furthermore, a morphism in $\mathrm{Env}(\mathcal{O})^{\otimes}$ is $p_{\mathrm{Env}(\mathcal{O})}$-cocartesian if and only if pr_1 maps that morphism to an inert morphism in $\mathcal{O}^{\otimes}$. ♧

[11] The definitions in [HA, 2.2.4] are more general, but we only need the symmetric monoidal case.
[12] So those morphisms for which the preimage of $*$ has a single element, see [HA, 2.1.2.1].
[13] See [HA, 2.1.2.1 and 2.1.2.3] for a definition.

Let us describe $p_{\mathrm{Env}(\mathcal{O})}$-cocartesian lifts a bit more concretely. Let O be an object of $\mathcal{O}^{\otimes}$, $\alpha\colon \langle n\rangle \to \langle m\rangle$ an active morphism in Fin_*, and consider (O,α) as an object of $\mathrm{Env}(\mathcal{O})^{\otimes}_{\langle m\rangle}$. Let $\beta\colon \langle m\rangle \to \langle k\rangle$ be a morphism of Fin_*. Then we can factor $\beta\circ\alpha$ as a composition of an inert morphism $\gamma\colon \langle n\rangle \to \langle l\rangle$ and an active morphism $\delta\colon \langle l\rangle \to \langle k\rangle$ in a unique way, see [HA, 2.1.2.2]. We can then interpret the commutative diagram

$$\begin{tikzcd} \langle n\rangle \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & \langle l\rangle \arrow[d, "\delta"] \\ \langle m\rangle \arrow[r, "\beta"'] & \langle k\rangle \end{tikzcd} \tag{6.2}$$

as a morphism from α to δ in $\mathrm{Act}(\mathsf{Fin}_*)$. Let $\overline{\gamma}\colon O \to O'$ be a $p_{\mathcal{O}}$-cocartesian lift of γ. Then $\overline{\gamma}$ together with (6.2) determine a $p_{\mathrm{Env}(\mathcal{O})}$-cocartesian morphism

$$(O,\alpha) \to (O',\delta)$$

in $\mathrm{Env}(\mathcal{O})$ lying over β. One implication of this discussion is that if O and O' are two objects of $\mathcal{O}^{\otimes}_{\mathrm{act}}$, then their tensor product is given by $O\oplus O'$, see also [HA, 2.2.4.6]. The monoidal unit of $\mathcal{O}^{\otimes}_{\mathrm{act}}$ is given by the essentially unique object in $\mathcal{O}^{\otimes}_{\langle 0\rangle}$.

The identity functor of $\mathcal{O}^{\otimes}$ together with the functor $\mathcal{O}^{\otimes} \to \mathrm{Act}(\mathsf{Fin}_*)$ that maps an object O to the active morphism $\mathrm{id}_{p_{\mathcal{O}}(O)}$[14] define a functor[15] $\mathcal{O}^{\otimes} \to \mathrm{Env}(\mathcal{O})^{\otimes}$ over Fin_*. Using Fact 6.1.2.1 it follows immediately that this functor is a morphism of ∞-operads. We are now ready to state the crucial result concerning $\mathrm{Env}(\mathcal{O})^{\otimes}$.

Fact 6.1.2.2 ([HA, 2.2.4.9])**.** *Let $\mathcal{O} \to \mathsf{Fin}_*$ be an ∞-operad and $\mathcal{D}$ a symmetric monoidal ∞-category. Then restriction along the functor $\mathcal{O}^{\otimes} \to \mathrm{Env}(\mathcal{O})^{\otimes}$ discussed above induces an equivalence*

$$\mathrm{Fun}^{\otimes}\big(\mathrm{Env}(\mathcal{O}),\mathcal{D}\big) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$$

between the ∞-category of symmetric monoidal functors $\mathrm{Env}(\mathcal{O}) \to \mathcal{D}$ and the ∞-category of morphisms of ∞-operads $\mathcal{O} \to \mathcal{D}$. ♧

Remark 6.1.2.3. Let $\alpha\colon \mathcal{O}' \to \mathcal{O}$ be a morphism of ∞-operads and $G\colon \mathcal{D} \to \mathcal{D}'$ a symmetric monoidal functor between symmetric monoidal ∞-categories.

It follows from Fact 6.1.2.1 that the morphism of ∞-categories α induces a symmetric monoidal functor

$$\mathrm{Env}(\alpha)\colon\ \mathrm{Env}(\mathcal{O}') \to \mathrm{Env}(\mathcal{O})$$

fitting into a commutative square of morphisms of ∞-operads as in the left of the following diagram, where the left horizontal functors are the morphisms of ∞-operads

[14] More rigorously, we consider the functor $(p_{\mathcal{O}})_*\circ\mathrm{const}\colon \mathcal{O}^{\otimes} \to \mathrm{Fun}([1],\mathsf{Fin}_*)$ that is adjoint to the composition

$$[1]\times\mathcal{O}^{\otimes} \xrightarrow{\mathrm{pr}_2} \mathcal{O}^{\otimes} \xrightarrow{p_{\mathcal{O}}} \mathsf{Fin}_*$$

and remark that it factors through $\mathrm{Act}(\mathsf{Fin}_*)$.

[15] This functor is also discussed in [HA, Before 2.2.4.9].

discussed above.

$$\begin{tikzcd}
\mathcal{O} \arrow[r] \arrow[rr, bend left, "F"] & \mathrm{Env}(\mathcal{O}) \arrow[r, "\widetilde{F}"'] & \mathcal{D} \arrow[d, "G"] \\
\mathcal{O}' \arrow[u, "\alpha"] \arrow[r] & \mathrm{Env}(\mathcal{O}') \arrow[u, "\mathrm{Env}(\alpha)"'] & \mathcal{D}'
\end{tikzcd}$$

The symmetric monoidal functor $\widetilde{F}$ in the above diagram is to be the one corresponding to F via the equivalence from Fact 6.1.2.2, i. e. making the triangle at the top commute.

It then follows from commutativity of the above diagram and Fact 6.1.2.2 that there is an equivalence

$$(\widetilde{G \circ F \circ \alpha}) \simeq G \circ \widetilde{F} \circ \mathrm{Env}(\alpha)$$

where $(\widetilde{G \circ F \circ \alpha})$ is the symmetric monoidal functor $\mathrm{Env}(\mathcal{O}') \to \mathcal{D}'$ corresponding to $G \circ F \circ \alpha$ under the equivalence of Fact 6.1.2.2. ♢

6.1.2.2. From associative algebras to active diagrams

Let us denote by $p_{\mathsf{Assoc}} \colon \mathsf{Assoc}^{\otimes} \to \mathsf{Fin}_*$ the canonical morphism of ∞-operads and let $p_{\mathcal{C}} \colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category. Recall from Proposition E.4.2.3 that $\mathrm{Alg}(\mathcal{C})$ inherits an induced symmetric monoidal structure $p_{\mathrm{Alg}(\mathcal{C})} \colon \mathrm{Alg}(\mathcal{C})^{\otimes} \to \mathsf{Fin}_*$. This comes with a canonical inclusion

$$\iota_{\mathrm{Alg}} \colon \mathrm{Alg}(\mathcal{C})^{\otimes} \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \tag{6.3}$$

where the functors with respect to which the pullback is taken are $(p_{\mathcal{C}})_*$ and the functor[16] adjoint to $\mathsf{Fin}_* \times \mathsf{Assoc}^{\otimes} \xrightarrow{\mathrm{id}_{\mathsf{Fin}_*} \times p_{\mathsf{Assoc}}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{-\wedge -} \mathsf{Fin}_*$. The functor $p_{\mathrm{Alg}(\mathcal{C})}$ is then given by the composition $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$.

The functor ι_{Alg} will be the first step in the definition of the symmetric monoidal functor $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$.

We next recall that the pointwise symmetric monoidal structure on $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}})$ is given by the cocartesian fibration of ∞-operads

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes} = \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\big) \times_{\mathrm{Fun}(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*)} \mathsf{Fin}_* \xrightarrow{\mathrm{pr}_2} \mathsf{Fin}_* \tag{6.4}$$

that exhibits $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}})$ as a symmetric monoidal ∞-category, where the pullback is formed with respect to the functors $(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}})_*$ and the functor const[17].

We are now ready to construct a functor

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes}$$

[16] See also Proposition E.6.0.1.

[17] In other words the functor adjoint to $\mathrm{pr}_1 \colon \mathsf{Fin}_* \times \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \to \mathsf{Fin}_*$.

over Fin_* whose composition with ι_{Alg} will be a lax symmetric monoidal functor. To be able to understand what this functor does it will later turn out to be helpful to also construct a certain natural transformation $\mu\colon A^{\mathrm{const}} \to \mathrm{pr}_1 \circ A^{\otimes}$.

Construction 6.1.2.4. Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category, and let us use notation as above. We will construct a functor

$$A^{\otimes}\colon \operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right) \times_{\operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \to \operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\right)^{\otimes}$$

over Fin_*, as well as a functor

$$A^{\mathrm{const}}\colon \operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right) \times_{\operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \to \operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\right)$$

together with a natural transformation[18] $\mu\colon A^{\mathrm{const}} \to \mathrm{pr}_1 \circ A^{\otimes}$ such that the natural transformation[19] $(\mathrm{pr}_1)_* \circ \mu$ is a natural equivalence. The names μ, A^{const} and $A^{\otimes}$ will only be used where we directly refer to this construction. The letter A has been chosen as a reference to the word *active*, and $A^{\otimes}$ has the superscript $\otimes$ as its composition with ι_{Alg} will be shown in Proposition 6.1.2.5 below to be a morphism of ∞-operads, whereas A^{const} is not even a functor over Fin_*. The reason why A^{const} has superscript const and the natural transformation is called μ will become clear during the construction. We will later also use the notation $A^{\otimes}$ for the functor obtained by composing $A^{\otimes}$ as constructed here with ι_{Alg}, see Proposition 6.1.2.5.

By the definition of $\operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\right)^{\otimes}$ (see the introduction of Section 6.1.2.2) and the universal property of pullbacks, constructing $A^{\otimes}$, A^{const}, and μ as stated above is equivalent to constructing a diagram as follows

$$\begin{tikzcd}
\operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right) \times_{\operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \arrow[r, "\mathrm{pr}_2"] \arrow[d, dashed, bend right, "A^{\mathrm{const}}"'] \arrow[d, dashed, bend left, "A'"] \arrow[d, phantom, "\overset{\mu}{\Rightarrow}"] & \mathsf{Fin}_* \arrow[d, "\mathrm{const}"] \\
\operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\right) \arrow[r, "\left(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}\right)_*"'] & \operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\right)
\end{tikzcd}$$

[18]The functor pr_1 appearing in $\mathrm{pr}_1 \circ A^{\otimes}$ is the following functor.

$$\begin{aligned}
\operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\right)^{\otimes} &= \operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\right) \times_{\operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \\
&\xrightarrow{\mathrm{pr}_1} \operatorname{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\right)
\end{aligned}$$

See (6.4).

[19]The functor pr_1 appearing in $(\mathrm{pr}_1)_*$ is the following functor.

$$(\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \operatorname{Act}(\mathsf{Fin}_*) \xrightarrow{\mathrm{pr}_1} \mathcal{C}^{\otimes}$$

See Section 6.1.2.1 and in particular (6.1).

where the ∞-category in the upper left is the pullback from (6.3), the two functors on the right and bottom are as explained around (6.4), and the square on the right[20] is to be a commutative square, while μ is a natural transformation from A^{const} to A' such that $(\mathrm{pr}_1)_* \circ \mu$ is a natural equivalence. Using the $\times$-Fun-adjunction and plugging in the definition of the symmetric monoidal envelope $\mathrm{Env}(\mathcal{C})^{\otimes} = (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}$ from (6.1) this is in turn equivalent to constructing a diagram

$$\begin{tikzcd}
\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \arrow[r, "\mathrm{id}_{\mathsf{Assoc}^{\otimes}_{\mathrm{act}}} \times \mathrm{pr}_2"] \arrow[d, dashed, bend right, "A''^{\mathrm{const}}"'] \arrow[d, dashed, bend left, "A''"] \arrow[d, phantom, "\overset{\mu''}{\Rightarrow}"] & \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathsf{Fin}_* \arrow[d, "\mathrm{pr}_2"] \\
\mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathrm{Act}(\mathsf{Fin}_*) \arrow[r, "p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}"'] & \mathsf{Fin}_*
\end{tikzcd} \tag{6.5}$$

where again the square is to come with a filler exhibiting it as a commutative square, while μ'' is merely a natural transformation such that $\mathrm{pr}_1 \circ \mu''$ is a natural equivalence.

As the composition from the top left along the top right to the bottom right is the projection to the last factor and using the definition of $p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}$ as $\mathrm{ev}_1 \circ \mathrm{pr}_2$, we can finally unpack this to see that we need to construct the following.

(1) A commutative diagram as follows.

$$\begin{tikzcd}
& \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \arrow[dl, dashed, "A''_l"'] \arrow[dr, dashed, "A''_r"] & \\
\mathcal{C}^{\otimes} \arrow[dr, "p_{\mathcal{C}}"'] & & \mathrm{Act}(\mathsf{Fin}_*) \arrow[dl, "\mathrm{ev}_0"] \\
& \mathsf{Fin}_* &
\end{tikzcd} \tag{6.6}$$

This diagram will then encode the functor A'' from (6.5).

(2) A natural transformation

$$\mu''_r \colon A''^{\mathrm{const}}_r \to A''_r$$

such that $\mathrm{ev}_0 \circ \mu''_r$ is an equivalence. Together with A''_l and the filler of the commutative diagram (6.6) this encodes a natural transformation $\mu'' \colon A''^{\mathrm{const}} \to A''$ such that $\mathrm{pr}_1 \circ \mu''$ can be identified with $\mathrm{id}_{A''_l}$.

(3) A natural equivalence $\mathrm{ev}_1 \circ A''_r \simeq \mathrm{pr}_3$, which then encodes a filler for the right square in (6.5).

[20] So involving A', but not A^{const}.

Construction of A''_l: We start by giving a definition of A''_l. This is to be the composition

$$\begin{aligned}&\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \\ &\xrightarrow{\mathrm{pr}_1 \times \mathrm{pr}_2} \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \xrightarrow{\mathrm{ev}} \mathcal{C}^{\otimes}\end{aligned}$$

that maps a tuple $(\langle m\rangle, F, \langle n\rangle)$ to $F(\langle m\rangle)$, which will be an object in $\mathcal{C}^{\otimes}_{\langle n\rangle\wedge\langle m\rangle}$, as we will see properly next. Indeed, the equivalences[21]

$$\begin{aligned} p_{\mathcal{C}} \circ A''_l &= p_{\mathcal{C}} \circ \mathrm{ev} \circ (\mathrm{pr}_1 \times \mathrm{pr}_2) \\ &\simeq \mathrm{ev} \circ \big(\mathrm{pr}_1 \times (p_{\mathcal{C}})_*\big) \circ (\mathrm{pr}_1 \times \mathrm{pr}_2) \\ &\simeq \mathrm{ev} \circ \Big(\mathrm{pr}_1 \times \big((p_{\mathcal{C}})_* \circ \mathrm{pr}_2\big)\Big) \\ &\simeq \mathrm{ev} \circ \left(\mathrm{pr}_1 \times \left(\big(\widehat{\mathrm{id}_{\mathsf{Fin}_*} \wedge p_{\mathsf{Assoc}}}\big) \circ \mathrm{pr}_3\right)\right) \\ &\simeq \mathrm{pr}_3 \wedge (p_{\mathsf{Assoc}} \circ \mathrm{pr}_1) \end{aligned}$$

allows us to identify the composition $p_{\mathcal{C}} \circ A''_l$ with the functor that can be informally described as mapping a tuple $(\langle m\rangle, F, \langle n\rangle)$ to $\langle n\rangle \wedge \langle m\rangle$.

Construction of μ''_r: Let us now think about the functor A''_r. The constraints imposed by (1) and (3) imply that A''_r needs to map a tuple $(\langle m\rangle, F, \langle n\rangle)$ to an active morphism $\langle n\rangle \wedge \langle m\rangle \to \langle n\rangle$. The idea is to use the active morphism

$$\langle n\rangle \wedge \langle m\rangle \xrightarrow{\mathrm{id}_{\langle n\rangle} \wedge \mu_m} \langle n\rangle \wedge \langle 1\rangle \cong \langle n\rangle$$

where μ_m is the unique active morphism $\langle m\rangle \to \langle 1\rangle$ and the isomorphism $\langle n\rangle \wedge \langle 1\rangle \cong \langle n\rangle$ is the unitality isomorphism, see [HA, 2.2.5.2].

For A''^{const}_r we have the same constraint regarding the domain, but no constraint on the codomain. We can thus let A''^{const}_r map a tuple $(\langle m\rangle, F, \langle n\rangle)$ to the active morphism

$$\mathrm{id}_{\langle n\rangle\wedge\langle m\rangle} \colon \langle n\rangle \wedge \langle m\rangle \to \langle n\rangle \wedge \langle m\rangle$$

which also explains why we are using the superscript const in the notation.

The component of μ''_r at $(\langle m\rangle, F, \langle n\rangle)$ is then to be given by the commutative diagram

$$\begin{tikzcd} \langle n\rangle \wedge \langle m\rangle \arrow[r, "\mathrm{id}_{\langle n\rangle\wedge\langle m\rangle}"] \arrow[d, "\mathrm{id}_{\langle n\rangle\wedge\langle m\rangle}"'] & \langle n\rangle \wedge \langle m\rangle \arrow[d, "\mathrm{id}_{\langle n\rangle} \wedge \mu_m"] \\ \langle n\rangle \wedge \langle m\rangle \arrow[r, "\mathrm{id}_{\langle n\rangle} \wedge \mu_m"'] & \langle n\rangle \wedge \langle 1\rangle \end{tikzcd}$$

[21] From the first to the second line we use functoriality of evaluation, from the second to the third functoriality of products of functors, from the third to the fourth the equivalence that is part of the data of the pullback over $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)$, and from the fourth to the fifth the $\times$-Fun-adjunction and functoriality.

considered as a morphism from $\mathrm{id}_{\langle n\rangle\wedge\langle m\rangle}$ to $\mathrm{id}_{\langle n\rangle}\wedge\mu_m$ in $\mathrm{Act}(\mathsf{Fin}_*)$, whose evaluation at 0 is $\mathrm{id}_{\langle n\rangle\wedge\langle m\rangle}$, and whose evaluation at 1 is $\mathrm{id}_{\langle n\rangle}\wedge\mu_m$.

To actually construct such functors and such a natural transformation, we first note that $(\mathsf{Fin}_*)_{\mathrm{act}}$ has a final object $\langle 1\rangle$, so that there exists a section

$$s\colon (\mathsf{Fin}_*)_{\mathrm{act}} \to \big((\mathsf{Fin}_*)_{\mathrm{act}}\big)_{/\langle 1\rangle}$$

of the projection, sending $\langle m\rangle$ to μ_m. We thus obtain a composition

$$(\mathsf{Fin}_*)_{\mathrm{act}} \xrightarrow{s} ((\mathsf{Fin}_*)_{\mathrm{act}})_{/\langle 1\rangle} \xrightarrow{i} \mathrm{Fun}\big([1],(\mathsf{Fin}_*)_{\mathrm{act}}\big)$$

where i is the inclusion. That s is a section means that we have an identification $\mathrm{ev}_0\circ i\circ s\simeq \mathrm{id}_{(\mathsf{Fin}_*)_{\mathrm{act}}}$ As ev_0 is right adjoint[22] to the functor const, we thus obtain a natural transformation

$$\widetilde{\mu}\colon \mathrm{const}\to i\circ s$$

of functors $(\mathsf{Fin}_*)_{\mathrm{act}}\to\mathrm{Fun}([1],(\mathsf{Fin}_*)_{\mathrm{act}})$.

We can now define A''_r as the composition

$$\begin{aligned}
&\mathsf{Assoc}^{\otimes}_{\mathrm{act}}\times\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes},\mathcal{C}^{\otimes}\big)\times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes},\mathsf{Fin}_*\right)}\mathsf{Fin}_*\\
&\xrightarrow{\mathrm{pr}_3\times\mathrm{pr}_1}\mathsf{Fin}_*\times\mathsf{Assoc}^{\otimes}_{\mathrm{act}}\\
&\xrightarrow{\mathrm{id}_{\mathsf{Fin}_*}\times p_{\mathsf{Assoc}}}\mathsf{Fin}_*\times(\mathsf{Fin}_*)_{\mathrm{act}}\\
&\xrightarrow{\mathrm{id}_{\mathsf{Fin}_*}\times(i\circ s)}\mathsf{Fin}_*\times\mathrm{Fun}\big([1],(\mathsf{Fin}_*)_{\mathrm{act}}\big)\\
&\xrightarrow{\mathrm{const}\times i'}\mathrm{Act}(\mathsf{Fin}_*)\times\mathrm{Act}(\mathsf{Fin}_*)\\
&\xrightarrow{-\wedge-}\mathrm{Act}(\mathsf{Fin}_*)
\end{aligned}$$

where i' is the inclusion $\mathrm{Fun}([1],(\mathsf{Fin}_*)_{\mathrm{act}})\to\mathrm{Act}(\mathsf{Fin}_*)$.

We similarly make the following definitions.

$$\begin{aligned}
A''^{\mathrm{const}}_r &:= (\mathrm{const}\circ\mathrm{pr}_3)\wedge\big(i'\circ\mathrm{const}\circ p_{\mathsf{Assoc}}\circ\mathrm{pr}_1\big)\\
\mu''_r &:= (\mathrm{const}\circ\mathrm{pr}_3)\wedge\big(i'\circ\widetilde{\mu}\circ p_{\mathsf{Assoc}}\circ\mathrm{pr}_1\big)
\end{aligned}$$

Construction of the commutative diagram (6.6) *in (1)*: We already obtained an identification

$$p_{\mathcal{C}}\circ A''_l\simeq\mathrm{pr}_3\wedge(p_{\mathsf{Assoc}}\circ\mathrm{pr}_1)$$

above. For $\mathrm{ev}_0\circ A''_r$ we obtain the following sequence of equivalences

$$\begin{aligned}
\mathrm{ev}_0\circ A''_r &= \mathrm{ev}_0\circ\Big((\mathrm{const}\circ\mathrm{pr}_3)\wedge\big(i'\circ i\circ s\circ p_{\mathsf{Assoc}}\circ\mathrm{pr}_1\big)\Big)\\
&\simeq(\mathrm{ev}_0\circ\mathrm{const}\circ\mathrm{pr}_3)\wedge\big(\mathrm{ev}_0\circ i'\circ i\circ s\circ p_{\mathsf{Assoc}}\circ\mathrm{pr}_1\big)\\
&\simeq\mathrm{pr}_3\wedge(\mathrm{ev}_0\circ i\circ s\circ p_{\mathsf{Assoc}}\circ\mathrm{pr}_1)
\end{aligned}$$

[22] Note that as 0 is an initial object of $[0]$, we can identify ev_0 with lim.

$$\simeq \mathrm{pr}_3 \wedge (p_{\mathsf{Assoc}} \circ \mathrm{pr}_1)$$

where from the first to second line we use compatibility of ev_0 with the functor $- \wedge -$, from the second to the third we use the identification $\mathrm{ev}_0 \circ \mathrm{const} \simeq \mathrm{id}$ and compatibility of ev_0 with the inclusion i', and from the third to the fourth we use the identification $\mathrm{ev}_0 \circ i \circ s \simeq \mathrm{id}_{(\mathsf{Fin}_*)_{\mathrm{act}}}$.

On $\mathrm{ev}_0 \circ \mu''_r$ *being a natural equivalence, thereby completing (2)*: Using identifications as just done for $\mathrm{ev}_0 \circ A''_r$ we see that it suffices to show that $\mathrm{ev}_0 \circ \widetilde{\mu}$ is a natural equivalence. But by definition we can identify $\mathrm{ev}_0 \circ \widetilde{\mu}$ with $\mathrm{id}_{(\mathsf{Fin}_*)_{\mathrm{act}}}$.

Construction of a natural equivalence $\mathrm{ev}_1 \circ A''_r \simeq \mathrm{pr}_3$ *as in (3)*: There is a sequence of equivalences as follows

$$\begin{aligned} \mathrm{ev}_1 \circ A''_r &\simeq \mathrm{pr}_3 \wedge (\mathrm{ev}_1 \circ i \circ s \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1) \\ &\simeq \mathrm{pr}_3 \wedge \big(\mathrm{const}_{\langle 1\rangle} \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1\big) \\ &\simeq \mathrm{pr}_3 \wedge \big(\mathrm{const}_{\langle 1\rangle}\big) \\ &\simeq \mathrm{pr}_3 \end{aligned}$$

where the first one is obtained just like for $\mathrm{ev}_0 \circ A''_r$, the equivalence from the first to the second line uses the definition of i as the inclusion of $(\mathsf{Assoc}^{\otimes}_{\mathrm{act}})_{/\langle 1\rangle}$, the equivalence from the second to the third line uses the canonical equivalences for precompositions of constant functors, and the last equivalence uses the natural unitality equivalence [HA, 2.2.5.2][23] $- \wedge \langle 1\rangle \cong \mathrm{id}_{\mathsf{Fin}_*}$. ♢

Proposition 6.1.2.5. *Let* $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ *be a symmetric monoidal* ∞*-category. Then the composition of functors over* Fin_*

$$\begin{aligned} \mathrm{Alg}(\mathcal{C})^{\otimes} &\xrightarrow{\iota_{\mathrm{Alg}}} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \\ &\xrightarrow{A^{\otimes}} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes} \end{aligned}$$

where ι_{Alg} *is as discussed in the introduction to Section 6.1.2.2 in* (6.3) *and* $A^{\otimes}$ *is as in Construction 6.1.2.4, is a lax symmetric monoidal functor.*

We will also denote this lax symmetric monoidal composition by $A^{\otimes}$*.* ♡

Proof. We have to show that the composition sends $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$-cocartesian morphisms over an inert morphism in Fin_* to a pr_2-cocartesian morphism[24]. So let $\varphi\colon R \to S$ be a $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$-cocartesian morphism in $\mathrm{Alg}(\mathcal{C})^{\otimes}$ lying over an inert morphism in Fin_*. We have to show that $(A^{\otimes} \circ \iota_{\mathrm{Alg}})(\varphi)$ is pr_2-cocartesian. By the result [HTT, 2.4.1.3 (2)] regarding cocartesian morphisms and pullbacks it suffices for this to show that $(\mathrm{pr}_1 \circ A^{\otimes} \circ \iota_{\mathrm{Alg}})(\varphi)$ is $(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}})_*$-cocartesian. Applying [HTT, 3.1.2.1] on cocartesian fibrations and functor categories and using that $p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}$ is a cocartesian fibration by Fact 6.1.2.1, we are further reduced

[23] Depending on the definition one takes, this might even by an equality, see [HA, 2.2.5.1].

[24] See the introduction to Section 6.1.2.2 for a discussion of the canonical morphisms of ∞-operads from the two symmetric monoidal ∞-categories to Fin_*. Without looking at the previous pages for reference it may be hard to follow what the various projections etc. in this proof refer to.

to showing that for every object X of $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$, the morphism $(\mathrm{ev}_X \circ \mathrm{pr}_1 \circ A^{\otimes} \circ \iota_{\mathrm{Alg}})(\varphi)$ is $p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}$-cocartesian. Finally, using the description of $p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}$-cocartesian morphisms from Fact 6.1.2.1, we conclude that we need to show that for every object X of $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ the morphism $(\mathrm{pr}_1 \circ \mathrm{ev}_X \circ \mathrm{pr}_1 \circ A^{\otimes} \circ \iota_{\mathrm{Alg}})(\varphi)$ is an inert morphism in $\mathcal{C}^{\otimes}$.

Using notation from Construction 6.1.2.4 we have by construction a sequence of equivalences[25] as follows.

$$
\begin{aligned}
&\mathrm{pr}_1 \circ \mathrm{ev}_X \circ \mathrm{pr}_1 \circ A^{\otimes} \circ \iota_{\mathrm{Alg}}\\
\simeq\ &\mathrm{pr}_1 \circ \mathrm{ev}_X \circ A' \circ \iota_{\mathrm{Alg}}\\
\simeq\ &\mathrm{pr}_1 \circ A'' \circ \big(\mathrm{const}_X \times \iota_{\mathrm{Alg}}\big)\\
\simeq\ &A''_l \circ \big(\mathrm{const}_X \times \iota_{\mathrm{Alg}}\big)\\
\simeq\ &\mathrm{ev} \circ (\mathrm{pr}_1 \times \mathrm{pr}_2) \circ \big(\mathrm{const}_X \times \iota_{\mathrm{Alg}}\big)\\
\simeq\ &\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}
\end{aligned}
$$

The claim now follow directly from Proposition E.4.2.3 (2). □

We will later need the following proposition, which will allow us to deduce statements for $A^{\otimes}$ from A^{const}, for which we will also provide a simpler description in Proposition 6.1.2.7 below.

Proposition 6.1.2.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and X an object of*

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_*$$

i. e. of the domain of $A^{\otimes}$ and A^{const} from Construction 6.1.2.4. Then the morphism

$$\mu_X \colon A^{\mathrm{const}}(X) \to \big(\mathrm{pr}_1 \circ A^{\otimes}\big)(X)$$

in

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\big)$$

is $(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}})_*$-cocartesian. ♡

Proof. Let X be as in the statement. By [HTT, 3.1.2.1] and the description of $p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}$-cocartesian morphisms in Fact 6.1.2.1 it suffices to show that for every object Y in $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ the morphism $(\mathrm{pr}_1 \circ \mathrm{ev}_Y)(\mu_X) = (\mathrm{ev}_Y \circ (\mathrm{pr}_1))(\mu_X)$ is inert. But by Construction 6.1.2.4 that morphism is an equivalence, and hence in particular inert. □

We end this section by giving another, simpler, description for the functor A^{const} from Construction 6.1.2.4.

Proposition 6.1.2.7. *Let $p_{\mathcal{C}} \colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category. Then the functor*

$$A^{\mathrm{const}} \colon\ \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}\big)$$

[25]The pr_1 in the last line corresponds to pr_2 in the second to last line.

constructed in Construction 6.1.2.4 is equivalent to the composition

$$\begin{aligned}
&\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \\
\xrightarrow{\mathrm{pr}_1}\ &\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \\
\xrightarrow{\mathrm{Fun}(\alpha, \iota_{\mathrm{act}})}\ &\mathrm{Fun}\Big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \big(\mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes}\Big)
\end{aligned}$$

where $\alpha\colon \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \to \mathsf{Assoc}^{\otimes}$ *is the inclusion, and* $\iota_{\mathrm{act}}\colon \mathcal{C}^{\otimes} \to (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}$ *is the functor described before Fact 6.1.2.2.* ♡

Proof. In this proof we use notation from Construction 6.1.2.4, as well as the discussions of the relevant definitions at the start of Section 6.1.2.2.

It suffices to check that the adjoint functors

$$\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \to \mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathrm{Act}(\mathsf{Fin}_*)$$

are homotopic. For A^{const} this adjoint functor is by construction A''^{const}. For the composition given in the statement this adjoint is equivalent to the following composition, which we will call $\widetilde{A}''^{\mathrm{const}}$ for now.

$$\begin{aligned}
&\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \\
\xrightarrow{(\alpha \circ \mathrm{pr}_1) \times \mathrm{pr}_2} &\mathsf{Assoc}^{\otimes} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \\
\xrightarrow{\mathrm{ev}} &\mathcal{C}^{\otimes} \\
\xrightarrow{\iota_{\mathrm{act}}} &\mathcal{C}^{\otimes} \times_{\mathsf{Fin}_*} \mathrm{Act}(\mathsf{Fin}_*)
\end{aligned}$$

To show that two such functor are equivalent we need to show that we can identify the two corresponding commutative diagrams of the following form.

$$\begin{tikzcd}
 & \mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathsf{Assoc}^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \arrow[dl] \arrow[dr] & \\
\mathcal{C}^{\otimes} \arrow[dr, "p_{\mathcal{C}}"'] & & \mathrm{Act}(\mathsf{Fin}_*) \arrow[dl, "\mathrm{ev}_0"] \\
 & \mathsf{Fin}_* &
\end{tikzcd} \tag{6.7}$$

To simplify this problem we first notice that $\mathrm{pr}_2 \circ \iota_{\mathrm{act}}$, and hence $\mathrm{pr}_2 \circ \widetilde{A}''^{\mathrm{const}}$, by definition factors though $\mathrm{const}\colon \mathsf{Fin}_* \to \mathrm{Act}(\mathsf{Fin}_*)$. Similarly, we have equivalences as follows.

$$\mathrm{pr}_2 \circ A''^{\mathrm{const}} = A_r''^{\mathrm{const}}$$

By definition we obtain the following.

$$= (\mathrm{const} \circ \mathrm{pr}_3) \wedge \big(i' \circ \mathrm{const} \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1\big)$$

Using functoriality of $- \wedge -$.

$$\simeq \mathrm{const} \circ \big(\mathrm{pr}_3 \wedge (p_{\mathsf{Assoc}} \circ \mathrm{pr}_1)\big)$$

This shows that also $\mathrm{pr}_2 \circ A''^{\mathrm{const}}$ factors through const.

We claim that because of this it actually suffices to construct a homotopy between $\mathrm{pr}_1 \circ \widetilde{A}''^{\mathrm{const}}$ and $\mathrm{pr}_1 \circ A''^{\mathrm{const}}$, as we can then obtain a homotopy between $\mathrm{pr}_2 \circ \widetilde{A}''^{\mathrm{const}}$ and $\mathrm{pr}_2 \circ A''^{\mathrm{const}}$ in such a manner that there is an evident compatible homotopy between the fillers of the commutative squares (6.7) as follows.

$$\mathrm{pr}_2 \circ \widetilde{A}''^{\mathrm{const}}$$

Using that $\mathrm{const} \circ \mathrm{ev}_0 \circ \mathrm{const} \simeq \mathrm{const}$.

$$\simeq \mathrm{const} \circ \mathrm{ev}_0 \circ \mathrm{pr}_2 \circ \widetilde{A}''^{\mathrm{const}}$$

Using the canonical homotopy from the diagram (6.7) associated to $\widetilde{A}''^{\mathrm{const}}$.

$$\simeq \mathrm{const} \circ p_{\mathcal{C}} \circ \mathrm{pr}_1 \circ \widetilde{A}''^{\mathrm{const}}$$

Using the homotopy $\mathrm{pr}_1 \circ \widetilde{A}''^{\mathrm{const}} \simeq \mathrm{pr}_1 \circ A''^{\mathrm{const}}$ that we assume given.

$$\simeq \mathrm{const} \circ p_{\mathcal{C}} \circ \mathrm{pr}_1 \circ A''^{\mathrm{const}}$$

Using the canonical homotopy from the diagram (6.7) associated to A''^{const}.

$$\simeq \mathrm{const} \circ \mathrm{ev}_0 \circ \mathrm{pr}_2 \circ A''^{\mathrm{const}}$$

Using that $\mathrm{const} \circ \mathrm{ev}_0 \circ \mathrm{const} \simeq \mathrm{const}$.

$$\simeq \mathrm{pr}_2 \circ A''^{\mathrm{const}}$$

It thus suffices to show that $\mathrm{pr}_1 \circ \widetilde{A}''^{\mathrm{const}} \simeq \mathrm{pr}_1 \circ A''^{\mathrm{const}}$. But it follows immediately from unpacking the definitions that there is an equivalence as follows.

$$\mathrm{pr}_1 \circ \widetilde{A}''^{\mathrm{const}} = \mathrm{id}_{\mathcal{C}^{\otimes}} \circ \mathrm{ev} \circ \big((\alpha \circ \mathrm{pr}_1) \times \mathrm{pr}_2\big) \simeq \mathrm{ev} \circ \big((\alpha \circ \mathrm{pr}_1) \times \mathrm{pr}_2\big) = \mathrm{pr}_1 \circ A''^{\mathrm{const}} \qquad \square$$

6.1.2.3. Tensoring active diagrams together

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. In Section 6.1.2.1 we discussed the symmetric monoidal structure on the ∞-category $\mathcal{C}^{\otimes}_{\mathrm{act}}$, where the tensor product can be described by $(\bigoplus_{1 \leq i \leq n} X_i) \otimes (\bigoplus_{n+1 \leq i \leq n+m} X_i) \simeq \bigoplus_{1 \leq i \leq n+m} X_i$. In Definition 6.1.2.8 below we will define a symmetric monoidal functor $\otimes \colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$, which can be described as mapping $\bigoplus_{1 \leq i \leq n} X_i$ to $\bigotimes_{1 \leq i \leq n} X_i$. Given the informal description of the symmetric monoidal structure on $\mathcal{C}^{\otimes}_{\mathrm{act}}$ it should be plausible that there is such a symmetric monoidal functor.

Definition 6.1.2.8. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We let

$$\otimes \colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$$

be the symmetric monoidal functor that corresponds to the lax symmetric monoidal functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ (which is actually symmetric monoidal, but we do not use that here) under the equivalence of Fact 6.1.2.2. $\diamondsuit$

Note that by definition, the underlying functor of $\otimes$ from Definition 6.1.2.8 maps objects X of $\mathcal{C}^{\otimes}_{\langle 1\rangle}$ to X, so symmetric monoidality implies that $\bigoplus_{1\leq i\leq n} X_i$ must be mapped to $\bigotimes_{1\leq i\leq n} X_i$.

Remark 6.1.2.9. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor of symmetric monoidal ∞-categories. Combining Remark 6.1.2.3 and $F^*(\mathrm{id}_{\mathcal{D}}) = F_*(\mathrm{id}_{\mathcal{C}})$ yields a commutative diagram of symmetric monoidal functors as follows

$$\begin{tikzcd}
\mathcal{C}^{\otimes}_{\mathrm{act}} \arrow[r, "F^{\otimes}_{\mathrm{act}}"] \arrow[d, "\otimes"'] & \mathcal{D}^{\otimes}_{\mathrm{act}} \arrow[d, "\otimes"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

where the two functors denoted by $\otimes$ are those from Definition 6.1.2.8. ♢

As $\otimes\colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ is a symmetric monoidal functor, it induces a symmetric monoidal functor

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big) \xrightarrow{(\otimes)_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)$$

on functor categories with the pointwise symmetric monoidal structure[26]. Furthermore, the composition $(\otimes_*)^{\otimes} \circ A^{\otimes}$ of the lax symmetric monoidal functor $A^{\otimes}$ from Proposition 6.1.2.5 with this symmetric monoidal functor is not only lax symmetric monoidal, but actually symmetric monoidal, as we see in Proposition 6.1.2.11 below. Before doing so we will use Proposition 6.1.2.6 and Proposition 6.1.2.7 to describe the compositions $\mathrm{ev}_{\langle m\rangle} \circ \otimes_* \circ A$.

Proposition 6.1.2.10. *Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category. Then the composition*[27]

$$\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big) \xrightarrow{\alpha^*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\big) \xrightarrow{(p_{\mathcal{C}})_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\big)$$

is the constant functor with image $p_{\mathsf{Assoc}} \circ \alpha$ and the composition[28]

$$\begin{aligned}
\mathrm{Alg}(\mathcal{C}) \xrightarrow{A} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big) \xrightarrow{\otimes_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \\
\xrightarrow{(\mathcal{C}\to\mathcal{C}^{\otimes})_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\big) \xrightarrow{(p_{\mathcal{C}})_*} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\big)
\end{aligned}$$

is the constant functor with image $\mathrm{const}_{\langle 1\rangle}$.

[26] This follows directly from the definition [HA, 2.1.3.4] together with Proposition C.1.1.1 and [HTT, 3.1.2.1].

[27] The functor $\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big)$ is to be the canonical one, i.e. inclusion into $\mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\big)$ followed by the projection, and α is the inclusion of $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ into $\mathsf{Assoc}^{\otimes}$.

[28] A is the underlying functor of the lax symmetric monoidal functor from Proposition 6.1.2.5, and $\otimes$ is the functor defined in Definition 6.1.2.8.

Let $\mu^{\mathsf{Fin}_*}\colon p_{\mathsf{Assoc}} \circ \alpha \to \mathrm{const}_{\langle 1\rangle}$ be the unique natural transformation of functors $\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \to \mathsf{Fin}_*$ that is pointwise an active morphism. Then there is a homotopy between the composition

$$\mathrm{Alg}(\mathcal{C}) \xrightarrow{\alpha^* \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right)\right)} \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\right)_{p_{\mathsf{Assoc}} \circ \alpha} \xrightarrow{\left(\mu^{\mathsf{Fin}_*}\right)_!} \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\right)_{\mathrm{const}_{\langle 1\rangle}}$$

and the following functor.

$$\left(\mathcal{C} \to \mathcal{C}^{\otimes}\right)_* \circ \otimes_* \circ A\colon\ \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\right)_{\mathrm{const}_{\langle 1\rangle}}$$

In particular, there is a commutative diagram of ∞-categories as follows for every $m \geq 0$

$$\begin{tikzcd}
\mathrm{Alg}(\mathcal{C}) \arrow[r, "A"] \arrow[d] & \mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\right) \arrow[r, "\otimes_*"] & \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\right) \arrow[d, "\mathrm{ev}_{\langle m\rangle}"] \\
\mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right) \arrow[r, "\mathrm{ev}_{\langle m\rangle}"'] & \mathcal{C}^{\otimes}_{\langle m\rangle} \arrow[r, "(\mu_m)_!"'] & \mathcal{C}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{C}
\end{tikzcd} \tag{6.8}$$

where the left vertical functor is the canonical functor and and μ_m is the unique active morphism $\langle m\rangle \to \langle 1\rangle$ in Fin_*.

Now let R be an associative algebra in $\mathcal{C}$. Then we can identify $(\otimes_* \circ A)(R)(\langle m\rangle)$ with $R^{\otimes m}$ and if $f\colon \langle m\rangle \to \langle m'\rangle$ is an active morphism in $\mathsf{Assoc}^{\otimes}$, then we can identify $(\otimes_* \circ A)(R)(f)$ with the morphism $R^{\otimes m} \to R^{\otimes m'}$ induced by f, so for example for f the unique active morphism $\langle 0\rangle \to \langle 1\rangle$ we can identify $(\otimes_* \circ A)(R)(f)$ with the unit morphism $\mathbb{1}_{\mathcal{C}} \to R$. ♡

Proof. In this proof we use notation from Construction 6.1.2.4.

Recall the natural transformation[29] $\mu\colon A^{\mathrm{const}} \to \mathrm{pr}_1 \circ A^{\otimes}$ from Construction 6.1.2.4. We can define a natural transformation

$$\overline{\mu} \coloneqq \left(\otimes^{\otimes}\right)_* \circ \mu \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\right)$$

of functors from $\mathrm{Alg}(\mathcal{C})$ to $\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\right)$.

We first claim that it suffices to show the following.

(1) $\left(\otimes^{\otimes}\right)_* \circ A^{\mathrm{const}} \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\right) \simeq \alpha^* \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right)\right)$

(2) $\left(\otimes^{\otimes}\right)_* \circ \mathrm{pr}_1 \circ A^{\otimes} \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\right) \simeq \left(\mathcal{C} \to \mathcal{C}^{\otimes}\right)_* \circ \otimes_* \circ A$

(3) $(p_{\mathcal{C}})_* \circ \overline{\mu} \simeq \mathrm{const}_{\mu^{\mathsf{Fin}_*}}$

[29] We use A^{const} here as notation for the restriction of what was called A^{const} in Construction 6.1.2.4 to $\mathrm{Alg}(\mathcal{C})^{\otimes}$, and similarly for μ – like we do for $A^{\otimes}$.

(4) For every object R of $\mathrm{Alg}(\mathcal{C})$, the component $\overline{\mu}_R$ of $\overline{\mu}$ is $(p_{\mathcal{C}})_*$-cocartesian.

Let us now explain how the statements we need to prove follow from claims (1), (2), (3) and (4).

The claims regarding the images of the two functors to $\mathrm{Fun}(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*)$ follow directly from claims (1), (2) and (3), and the identification of

$$\left(\mathcal{C} \to \mathcal{C}^{\otimes}\right)_* \circ \otimes_* \circ A$$

then follows from claims (1), (2), (3) and (4)[30]. The inclusion functor $\mathcal{C} \to \mathcal{C}^{\otimes}$ is fully faithful[31], so for construction of a commutative diagram (6.8) it suffices by Proposition B.4.3.1 to show that the two composite functors from the top left to the bottom right become homotopic after composing with the inclusion to $\mathcal{C}^{\otimes}$. But we have a chain of equivalences as follows.

$$\left(\mathcal{C} \to \mathcal{C}^{\otimes}\right) \circ \mathrm{ev}_{\langle m\rangle} \circ \otimes_* \circ A$$

Using compatibility of evaluation with postcomposition.

$$\simeq \mathrm{ev}_{\langle m\rangle} \circ \left(\mathcal{C} \to \mathcal{C}^{\otimes}\right)_* \circ \otimes_* \circ A$$

Postcomposing the already obtained equivalence with $\mathrm{ev}_{\langle m\rangle}$.

$$\simeq \mathrm{ev}_{\langle m\rangle} \circ \left(\mu^{\mathsf{Fin}_*}\right)_! \circ \alpha^* \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right)\right)$$

Using [HTT, 3.1.2.1 (2)].

$$\simeq (\mu_m)_! \circ \mathrm{ev}_{\langle m\rangle} \circ \alpha^* \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right)\right)$$

Finally, compatibility of evaluations with precomposing and (un)making the identification $\mathcal{C}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{C}$.

$$\simeq \left(\mathcal{C} \to \mathcal{C}^{\otimes}\right) \circ (\mu_m)_! \circ \mathrm{ev}_{\langle m\rangle} \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right)\right)$$

Finally, the concrete description of $(\otimes_* \circ A)(R)$ follows directly from the identification of $\left(\mathcal{C} \to \mathcal{C}^{\otimes}\right)_* \circ \otimes_* \circ A$ by unpacking the definitions.

So let us now prove claims (1), (2), (3) and (4).

Proof of claim (1): We have equivalences as follows.

$$\left(\otimes^{\otimes}\right)_* \circ A^{\mathrm{const}} \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\right)$$

Using the description of A^{const} from Proposition 6.1.2.7.

$$\simeq \left(\otimes^{\otimes}\right)_* \circ (\iota_{\mathrm{act}})_* \circ \alpha^* \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right)\right)$$

Using that by definition of the functor $\otimes$ – see Definition 6.1.2.8 – there is an equivalence $\otimes^{\otimes} \circ \iota_{\mathrm{act}} \simeq \mathrm{id}_{\mathcal{C}}$.

$$\simeq \alpha^* \circ \left(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}, \mathcal{C}^{\otimes}\right)\right)$$

[30] We remark that we do not need to worry about the equivalences in claims (1) and (2) lying over non-identity natural isomorphisms of functors to Fin_*, as the unique active morphism $\langle m\rangle \to \langle 1\rangle$ in Fin_* stays unchanged if we pre- and postcompose it by isomorphisms.

[31] This follows from Proposition B.5.3.1 using that $\{\langle 1\rangle\} \to \mathsf{Fin}_*$ is fully faithful.

Proof of claim (2): Follows immediately by using that lax monoidal functors such as A and $\otimes$ are compatible with the inclusion of the underlying ∞-category into the respective ∞-operad.

Proof of claim (3): It suffices to show that the adjoint natural transformations of functors

$$\mathsf{Assoc}^{\otimes}_{\mathrm{act}} \times \mathrm{Alg}(\mathcal{C}) \to \mathsf{Fin}_*$$

are equivalent, i. e. that there is an equivalence between $p_{\mathcal{C}} \circ \breve{\overline{\mu}}$ and $\mu^{\mathsf{Fin}_*} \circ \mathrm{pr}_1$.

We first note that as $\otimes^{\otimes} \colon (\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes} \to \mathcal{C}^{\otimes}$ is a functor over Fin_*, we have an equivalence as follows.

$$p_{\mathcal{C}} \circ \otimes^{\otimes} \simeq p_{\mathcal{C}^{\otimes}_{\mathrm{act}}} = \mathrm{ev}_1 \circ \mathrm{pr}_2$$

Unpacking the definition of μ in Construction 6.1.2.4 we thus obtain equivalences as follows.

$$\begin{aligned}
&p_{\mathcal{C}} \circ \breve{\overline{\mu}}\\
&= p_{\mathcal{C}} \circ \otimes^{\otimes} \circ \breve{\mu} \circ \Big(\mathrm{id}_{\mathsf{Assoc}^{\otimes}_{\mathrm{act}}} \times \Big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\Big)\Big)\\
&\simeq \mathrm{ev}_1 \circ \mathrm{pr}_2 \circ \breve{\mu} \circ \Big(\mathrm{id}_{\mathsf{Assoc}^{\otimes}_{\mathrm{act}}} \times \Big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\Big)\Big)\\
&\simeq \mathrm{ev}_1 \circ \mu''_r \circ \Big(\mathrm{id}_{\mathsf{Assoc}^{\otimes}_{\mathrm{act}}} \times \Big(\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})^{\otimes}\Big)\Big)\\
&\simeq \mathrm{ev}_1 \circ \Big(\big(\mathrm{const}_{\langle 1\rangle}\big) \wedge \big(i' \circ \widetilde{\mu} \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1\big)\Big)\\
&\simeq \mathrm{ev}_1 \circ i' \circ \widetilde{\mu} \circ p_{\mathsf{Assoc}} \circ \mathrm{pr}_1\\
&\simeq \mu^{\mathsf{Fin}_*} \circ \mathrm{pr}_1
\end{aligned}$$

Proof of claim (4): Follows immediately by combining that all components of μ are $\Big(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}}\Big)_*$-cocartesian by Proposition 6.1.2.6, that $\otimes^{\otimes}$ is symmetric monoidal by definition, and [HTT, 3.1.2.1].

□

Proposition 6.1.2.11. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Consider the composition*

$$\mathrm{Alg}(\mathcal{C})^{\otimes} \xrightarrow{A^{\otimes}} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}_{\mathrm{act}}\big)^{\otimes} \xrightarrow{(\otimes_*)^{\otimes}} \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)^{\otimes}$$

of functors over Fin_*, where $A^{\otimes}$ is as in Proposition 6.1.2.5 and $(\otimes_*)^{\otimes}$ is the symmetric monoidal functor induced by $\otimes$ from Definition 6.1.2.8 on functor categories with the pointwise symmetric monoidal structure.

Then this composition is a symmetric monoidal functor. ♡

Proof. We will use notation from Construction 6.1.2.4 in this proof[32], which will be similar to the proof of Proposition 6.1.2.10.

[32]We will though use A^{const} as notation for the restriction of what was called A^{const} in Construction 6.1.2.4 to $\mathrm{Alg}(\mathcal{C})^{\otimes}$, and similarly for μ, as we do for $A^{\otimes}$.

Just like in Proposition 6.1.2.5, it suffices to show that for every object $\langle m\rangle$ in $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$, the composition

$$\mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ (\otimes_*)^{\otimes} \circ A^{\otimes}$$

maps $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$-cocartesian morphisms to $p_{\mathcal{C}}$-cocartesian morphisms. Also like in Proposition 6.1.2.5, we use the definitions of the various functors to rewrite this composition into a more suitable form. We start by using the definition of $(\otimes_*)^{\otimes}$ and compatibility of evaluation with postcomposition of functors to obtain homotopies as follows.

$$\begin{aligned}
&\mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ (\otimes_*)^{\otimes} \circ A^{\otimes}\\
\simeq\ &\mathrm{ev}_{\langle m\rangle} \circ \left(\otimes^{\otimes}\right)_* \circ \mathrm{pr}_1 \circ A^{\otimes}\\
\simeq\ &\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ A^{\otimes}
\end{aligned}$$

Let $f\colon X \to Y$ be a $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$-cocartesian morphism in $\mathrm{Alg}(\mathcal{C})^{\otimes}$. From the natural transformation $\mu\colon A^{\mathrm{const}} \to \mathrm{pr}_1 \circ A^{\otimes}$ we obtain a commutative square as follows.

$$\begin{tikzcd}
\left(\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ A^{\mathrm{const}}\right)(X) \arrow[r, "\left(\otimes^{\otimes}\circ\mathrm{ev}_{\langle m\rangle}\right)(\mu_X)"] \arrow[d, "\left(\otimes^{\otimes}\circ\mathrm{ev}_{\langle m\rangle}\circ A^{\mathrm{const}}\right)(f)"'] & \left(\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ A^{\otimes}\right)(X) \arrow[d, "\left(\otimes^{\otimes}\circ\mathrm{ev}_{\langle m\rangle}\circ\mathrm{pr}_1\circ A^{\otimes}\right)(f)"] \\
\left(\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ A^{\mathrm{const}}\right)(Y) \arrow[r, "\left(\otimes^{\otimes}\circ\mathrm{ev}_{\langle m\rangle}\right)(\mu_Y)"'] & \left(\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ A^{\otimes}\right)(Y)
\end{tikzcd}$$

We need to show that the right vertical morphism is $p_{\mathcal{C}}$-cocartesian. By Proposition 6.1.2.6 we know that μ_X and μ_Y are $(p_{\mathcal{C}^{\otimes}_{\mathrm{act}}})_*$-cocartesian, so it follows from [HTT, 3.1.2.1] and $\otimes^{\otimes}$ being symmetric monoidal by definition that the top and bottom horizontal morphisms in the diagram are $p_{\mathcal{C}}$-cocartesian. It thus suffices by [HTT, 2.4.1.7] to show that the left vertical morphism is $p_{\mathcal{C}}$-cocartesian.

For this we use the description of A^{const} from Proposition 6.1.2.7 and that by definition $\otimes^{\otimes} \circ \iota_{\mathrm{act}} \simeq \mathrm{id}_{\mathcal{C}^{\otimes}}$ to obtain equivalences as follows.

$$\begin{aligned}
&\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ A^{\mathrm{const}}\\
\simeq\ &\otimes^{\otimes} \circ \mathrm{ev}_{\langle m\rangle} \circ \left(\iota_{\mathrm{act}}\right)_* \circ \alpha^* \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\\
\simeq\ &\otimes^{\otimes} \circ \iota_{\mathrm{act}} \circ \mathrm{ev}_{\langle m\rangle} \circ \alpha^* \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\\
\simeq\ &\mathrm{ev}_{\langle m\rangle} \circ \alpha^* \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\\
\simeq\ &\mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}
\end{aligned}$$

So what is left to show is that $\mathrm{ev}_{\langle m\rangle} \circ \mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}$ maps $\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}$-cocartesian morphisms to $p_{\mathcal{C}}$-cocartesian morphisms. But this follows immediately from Proposition E.4.2.3 (4). □

Remark 6.1.2.12. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor of symmetric monoidal ∞-categories. Then going through the constructions and using Remark 6.1.2.3

it is straightforward to see that there is a commutative diagram of symmetric monoidal functors as follows

$$\begin{tikzcd}
\mathrm{Alg}(\mathcal{C})^{\otimes} \arrow[r, "(\otimes_*)^{\otimes}\circ A^{\otimes}\circ\iota_{\mathrm{Alg}}"] \arrow[d, "\mathrm{Alg}(F)^{\otimes}"'] & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)^{\otimes} \arrow[d, "(F_*)^{\otimes}"] \\
\mathrm{Alg}(\mathcal{D})^{\otimes} \arrow[r, "(\otimes_*)^{\otimes}\circ A^{\otimes}\circ\iota_{\mathrm{Alg}}"'] & \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{D}\big)^{\otimes}
\end{tikzcd}$$

where the horizontal functors are the compositions considered in Proposition 6.1.2.11 for $\mathcal{C}$ and $\mathcal{D}$, respectively. Furthermore, if $G\colon \mathcal{D} \to \mathcal{E}$ is another symmetric monoidal functor, then the composite of the compatibility diagrams for F and G as above can be identified with the compatibility diagram for $G \circ F$. ♢

6.1.2.4. The functor $V\colon \mathbf{\Lambda} \to \mathsf{Assoc}^{\otimes}_{\mathrm{act}}$

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. With Proposition 6.1.2.11 we have now constructed a symmetric monoidal functor

$$\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)$$

that is the first[33] step in the symmetric monoidal functor $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$. We already constructed the self-duality functor

$$-^{\circ}\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathbf{\Lambda}$$

in Section 6.1.1.6. We will now introduce a functor

$$V\colon \mathbf{\Lambda} \to \mathsf{Assoc}^{\otimes}_{\mathrm{act}}$$

so that precomposition with $V \circ (-^{\circ})$ induces a symmetric monoidal functor

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

with respect to the pointwise symmetric monoidal structures.

Fact 6.1.2.13 ([NikSch, B.1])**.** *There is a functor*

$$V\colon \mathbf{\Lambda} \to \mathsf{Assoc}^{\otimes}_{\mathrm{act}}$$

that maps

- $[n]_{\mathbf{\Lambda}}$ *to* $\langle n+1\rangle$,
- $\delta_j\colon [n-1]_{\mathbf{\Lambda}} \to [n]_{\mathbf{\Lambda}}$ *to the active map that sends* i *to* i *if* $i < j+1$ *and to* $i+1$ *otherwise*[34],

[33] Or the first two or three, however one wants to count.

[34] For the reader confused by why it is $j+1$ and not j: This arises from the fact that we defined δ_j using elements $\frac{0}{n+1}, \dots, \frac{n}{n+1}$ (i. e. we start counting from 0), whereas the elements of $\langle n+1\rangle$ are $1, \dots, n+1$ (i. e. we start counting from 1).

- *$\sigma_j\colon [n+1]_{\mathbf{\Lambda}} \to [n]_{\mathbf{\Lambda}}$ to the active map that sends i to i if $i \le j+1$ and to $i-1$ otherwise, with ordering on the preimage of $j+1$ given by $j+1 < j+2$,*
- *$\tau\colon [n]_{\mathbf{\Lambda}} \to [n]_{\mathbf{\Lambda}}$ to the active map that sends 1 to $n+1$ and i to $i-1$ for $i > 1$.* ♧

Proposition 6.1.2.14. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the functor*

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathsf{Fin}_*\right)} \mathsf{Fin}_* \xrightarrow{\left(V\circ(-^{\circ})\right)^* \times_{(V\circ(-^{\circ}))^*} \mathrm{id}} \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Fin}_*)} \mathsf{Fin}_*$$

over Fin_* upgrades the functor

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \xrightarrow{\left(V\circ(-^{\circ})\right)^*} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

to a symmetric monoidal functor with respect to the pointwise symmetric monoidal structures (see [HA, 2.1.3.4]). ♡

Proof. Follows directly from the definition of the pointwise symmetric monoidal structures and Proposition C.1.1.1 and [HTT, 3.1.2.1]. □

Remark 6.1.2.15. The symmetric monoidal functor from Proposition 6.1.2.14 is natural in $\mathcal{C}$. In particular, for $F\colon \mathcal{C} \to \mathcal{D}$ a symmetric monoidal functor between symmetric monoidal ∞-categories, we obtain a commutative square

$$\begin{tikzcd}
\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \arrow[r, "\left(V\circ(-^{\circ})\right)^*"] \arrow[d, "F_*"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \arrow[d, "F_*"] \\
\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{D}\big) \arrow[r, "\left(V\circ(-^{\circ})\right)^*"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D})
\end{tikzcd}$$

of symmetric monoidal functors. ♢

6.1.2.5. The definition of the cyclic bar construction as a cyclic object

We are now ready to define the cyclic bar construction $\mathrm{B}^{\mathrm{cyc}}_\bullet$.

Definition 6.1.2.16 ([NikSch, III.2.3])**.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We define the *cyclic bar construction* as the symmetric monoidal functor[35]

$$\mathrm{B}^{\mathrm{cyc}}_\bullet\colon \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

that is given as the composition of the symmetric monoidal functor

$$\mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big)$$

from Proposition 6.1.2.11 and the symmetric monoidal functor

$$\mathrm{Fun}\big(\mathsf{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C}\big) \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

from Proposition 6.1.2.14. ♢

[35] In the codomain with respect to the pointwise symmetric monoidal structure.

Remark 6.1.2.17. $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ is compatible with symmetric monoidal functors. If $F\colon \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor, then there is a commuting diagram

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{C}) \arrow[r, "\mathrm{B}^{\mathrm{cyc}}_{\bullet}"] \arrow[d, "\mathrm{Alg}(F)"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \arrow[d, "F_*"] \\ \mathrm{Alg}(\mathcal{D}) \arrow[r, "\mathrm{B}^{\mathrm{cyc}}_{\bullet}"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}) \end{tikzcd}$$

of symmetric monoidal functors. Furthermore, if $G\colon \mathcal{D} \to \mathcal{E}$ is another symmetric monoidal functor, then the composite of the compatibility squares as above for F and G can be identified with the compatibility square for $G \circ F$. This follows by combining Remark 6.1.2.12 with Remark 6.1.2.15. ♢

6.1.2.6. $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ for cocartesian symmetric monoidal ∞-categories

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. The cyclic bar construction

$$\mathrm{B}^{\mathrm{cyc}}_{\bullet}\colon \ \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

is a symmetric monoidal functor and thus induces a functor as follows.

$$\mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CAlg}\big(\mathrm{Alg}(\mathcal{C})\big) \xrightarrow{\mathrm{CAlg}\left(\mathrm{B}^{\mathrm{cyc}}_{\bullet}\right)} \mathrm{CAlg}\big(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})\big) \simeq \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C})\big) \tag{6.9}$$

In this section we will give a different description of this functor: It is the left adjoint of the forgetful functor $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$.

To prove this we will proceed as follows. We will first show in Proposition 6.1.2.18 that already $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ – so without passing to commutative algebras – is left adjoint to $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$, under the assumption that the symmetric monoidal structure on $\mathcal{C}$ is cocartesian. In order to apply this to the composition Equation (6.9), we will then show in Proposition 6.1.2.19 how we can identify $\mathrm{CAlg}(\mathrm{B}^{\mathrm{cyc}}_{\bullet})$ (where the cyclic bar construction is taken of algebras in $\mathcal{C}$) with the cyclic bar construction for $\mathrm{CAlg}(\mathcal{C})$.

Proposition 6.1.2.18. *Let $\mathcal{C}$ a symmetric monoidal ∞-category and assume that the underlying ∞-category admits finite coproducts and that the symmetric monoidal structure is cocartesian in the sense of [HA, 2.4.0.1]. Under these assumptions the forgetful functor*

$$\mathrm{ev}_{\mathfrak{a}}\colon \ \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$$

is an equivalence by [HA, 2.4.3.9].

Then the composite

$$\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1}\colon \mathcal{C} \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

is left adjoint to the evaluation functor $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$. ♡

Proof. Let $i\colon \{[0]_{\mathbf{\Lambda}}\} \to \mathbf{\Lambda}^{\mathrm{op}}$ be the inclusion. We will identify $\mathcal{C}$ with $\mathrm{Fun}(\{[0]_{\mathbf{\Lambda}}\}, \mathcal{C})$ and consider $\mathrm{ev}_{\mathfrak{a}}$ as a functor to $\mathrm{Fun}(\{[0]_{\mathbf{\Lambda}}\}, \mathcal{C})$. Under this identification, the functor $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$ corresponds to precomposition with i.

We start by noting that we can use Proposition 6.1.2.10 to identify the composition $i^* \circ \mathrm{B}^{\mathrm{cyc}}_{\bullet}$ with $\mathrm{ev}_{\mathfrak{a}}$ and this identification provides for every object R of $\mathcal{C}$ a commutative triangle of ∞-categories as follows.

$$\begin{tikzcd}
\{[0]_{\mathbf{\Lambda}}\} \arrow[d, "i"'] \arrow[dr, "\mathrm{const}_R"] & \\
\mathbf{\Lambda}^{\mathrm{op}} \arrow[r, "(\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)"'] & \mathcal{C}
\end{tikzcd}$$

It now suffices to show that this triangle exhibits $(\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)$ as a left Kan extension of const_R – see [HA, 4.3.2, 4.3.3, and in particular 4.3.3.7][36].

For this we need to show by [HA, 4.3.2.2 and 4.3.1.3] that for every object $[n]_{\mathbf{\Lambda}}$ of $\mathbf{\Lambda}^{\mathrm{op}}$ the induced diagram

$$\begin{tikzcd}
(\mathbf{\Lambda}^{\mathrm{op}})_{/[n]_{\mathbf{\Lambda}}} \times_{\mathbf{\Lambda}^{\mathrm{op}}} \{[0]_{\mathbf{\Lambda}}\} \arrow[r, "\mathrm{pr}"] \arrow[d] & \mathbf{\Lambda}^{\mathrm{op}} \arrow[r, "(\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)"] & \mathcal{C} \\
\left((\mathbf{\Lambda}^{\mathrm{op}})_{/[n]_{\mathbf{\Lambda}}} \times_{\mathbf{\Lambda}^{\mathrm{op}}} \{[0]_{\mathbf{\Lambda}}\}\right)^{\triangleright} \arrow[urr, "G"'] & &
\end{tikzcd}$$

where the left vertical functor is the inclusion and G is the functor that is induced by $(\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)$, exhibits $G(\infty) = (\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)([n]_{\mathbf{\Lambda}})$ as a colimit of $(\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R) \circ \mathrm{pr}$.

Let us start by unpacking what the category $(\mathbf{\Lambda}^{\mathrm{op}})_{/[n]_{\mathbf{\Lambda}}} \times_{\mathbf{\Lambda}^{\mathrm{op}}} \{[0]_{\mathbf{\Lambda}}\}$ looks like. As $[0]_{\mathbf{\Lambda}}$ has no nontrivial endomorphisms the 1-category $(\mathbf{\Lambda}^{\mathrm{op}})_{/[n]_{\mathbf{\Lambda}}} \times_{\mathbf{\Lambda}^{\mathrm{op}}} \{[0]_{\mathbf{\Lambda}}\}$ is actually a discrete category. Objects are morphisms $[0]_{\mathbf{\Lambda}} \to [n]_{\mathbf{\Lambda}}$ in $\mathbf{\Lambda}^{\mathrm{op}}$, so morphisms $[n]_{\mathbf{\Lambda}} \to [0]_{\mathbf{\Lambda}}$ in $\mathbf{\Lambda}$. There are $n+1$ such morphisms, namely f_m for $1 \leq m \leq n+1$, where f_m is the morphism $(1/(n+1)) \cdot \mathbb{Z} \to \mathbb{Z}$ in $\mathbf{\Lambda}$[37] that maps $l/(n+1)$ to 0 for $0 \leq l < m-1$ and to 1 for $m-1 \leq l \leq n$. In terms of the generators of $\mathbf{\Lambda}$[38] we can write f_m as $f_m \coloneqq \sigma_0^n \circ \tau^{m-1}$.

Hence what we need to show is that the morphism

$$\coprod_{1 \leq m \leq n+1} (\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)([0]_{\mathbf{\Lambda}}) \xrightarrow{\amalg_{1 \leq m \leq n+1} (\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)(f_m)} (\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)([n]_{\mathbf{\Lambda}}) \qquad (*)$$

is an equivalence in $\mathcal{C}$.

For this we need to understand what $(\mathrm{B}^{\mathrm{cyc}}_{\bullet} \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)$ maps the morphism f_m to. First we use Fact 6.1.1.13 to see that the self-duality functor $-^{\circ}$ of $\mathbf{\Lambda}$ maps $f_m = \sigma_0^n \circ \tau^{m-1}$

[36] That we only need to check this pointwise for a single (though of course arbitrary) R boils down to the fact that induced natural transformations between left Kan extensions are defined essentially uniquely through the universal property of left Kan extensions and ultimately colimits.

[37] See Section 6.1.1.2.

[38] See Section 6.1.1.3.

to $\tau^{1-m}\delta_1^n$. Next we need to apply the functor V from Fact 6.1.2.13, which maps this to the active morphism $\langle 1\rangle \to \langle n+1\rangle$ in $\mathsf{Assoc}^{\otimes}$ that sends 1 to m. Denote this morphism of $\mathsf{Assoc}^{\otimes}_{\mathrm{act}}$ by f'_m.

We can then identify morphism $(*)$ with the morphism[39]

$$\coprod_{1\le m\le n+1} (\otimes_* \circ A \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)(\langle 1\rangle) \xrightarrow{\amalg_{1\le m\le n+1}(\otimes_*\circ A\circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)(f'_m)} (\otimes_* \circ A \circ \mathrm{ev}_{\mathfrak{a}}^{-1})(R)(\langle n+1\rangle)$$

in $\mathcal{C}$. With Proposition 6.1.2.10 we can further identify this morphism with the morphism

$$\coprod_{1\le m\le n+1} R \xrightarrow{\amalg_{1\le m\le n+1}\left(R\simeq \mathbb{1}_{\mathcal{C}}^{\otimes m-1}\otimes R\otimes \mathbb{1}_{\mathcal{C}}^{\otimes n-m} \xrightarrow{u^{\otimes m-1}\otimes \mathrm{id}_R\otimes u^{\otimes n-m}} R^{\otimes n+1}\right)} R^{\otimes n+1} \qquad (**)$$

where $u\colon \mathbb{1}_{\mathcal{C}} \to R$ is the unit morphism of the associative algebra $\mathrm{ev}_{\mathfrak{a}}^{-1}(R)$. Morphism $(**)$ is an equivalence as the symmetric monoidal structure on $\mathcal{C}$ is cocartesian. $\square$

Proposition 6.1.2.19. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We compare $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ for $\mathrm{CAlg}(\mathcal{C})$ and $\mathcal{C}$ in this proposition, so to distinguish them we will use superscripts such as $\mathrm{B}^{\mathrm{cyc},\mathcal{C}}_{\bullet}$.*

Then there is a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{Alg}(\mathcal{C})) \arrow[r, "\mathrm{CAlg}(\mathrm{B}^{\mathrm{cyc},\mathcal{C}}_{\bullet})"] \arrow[d, "\simeq"', no head] & \mathrm{CAlg}(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})) \arrow[d, "\simeq", no head] \\
\mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \arrow[r, "\mathrm{B}^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}_{\bullet}"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C}))
\end{tikzcd} \qquad (6.10)$$

where the left and right vertical equivalences are the canonical ones[40]*.* ♡

Proof. The symmetric monoidal forgetful functor $\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ induces by Remark 6.1.2.17 a commuting diagram

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \arrow[r, "\mathrm{B}^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}_{\bullet}"] \arrow[d, "\mathrm{Alg}(\mathrm{ev}_{\langle 1\rangle})"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C})) \arrow[d, "(\mathrm{ev}_{\langle 1\rangle})_*"] \\
\mathrm{Alg}(\mathcal{C}) \arrow[r, "\mathrm{B}^{\mathrm{cyc},\mathcal{C}}_{\bullet}"'] & \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})
\end{tikzcd}$$

[39] We use notation like in Proposition 6.1.2.10.

[40] For the left equivalence this is the composition

$$\mathrm{CAlg}(\mathrm{Alg}(\mathcal{C})) \simeq \mathrm{BiFunc}(\mathsf{Comm}, \mathsf{Assoc}; \mathcal{C}) \simeq \mathrm{BiFunc}(\mathsf{Assoc}, \mathsf{Comm}; \mathcal{C}) \simeq \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C}))$$

where the middle equivalence is given by precomposition with the symmetry equivalence and the other two are the ones from Proposition E.5.0.1. For the right vertical equivalence see [HA, 2.1.3.4].

of symmetric monoidal functors. Applying CAlg to this diagram we obtain the bottom commutative square in the commutative diagram of ∞-categories below.

$$\begin{tikzcd}
\mathrm{Alg}\big(\mathrm{CAlg}(\mathcal{C})\big) \arrow[rr, "\mathrm{B}_\bullet^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}"] & & \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C})\big) \\
\mathrm{CAlg}\Big(\mathrm{Alg}\big(\mathrm{CAlg}(\mathcal{C})\big)\Big) \arrow[u, "\mathrm{ev}_{\langle 1\rangle}"] \arrow[d, "\mathrm{CAlg}\left(\mathrm{Alg}\left(\mathrm{ev}_{\langle 1\rangle}\right)\right)"'] \arrow[rr, "\mathrm{CAlg}\left(\mathrm{B}_\bullet^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}\right)"] & & \mathrm{CAlg}\Big(\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C})\big)\Big) \arrow[u, "\mathrm{ev}_{\langle 1\rangle}"'] \arrow[d, "\mathrm{CAlg}\left(\left(\mathrm{ev}_{\langle 1\rangle}\right)_*\right)"] \\
\mathrm{CAlg}\big(\mathrm{Alg}(\mathcal{C})\big) \arrow[rr, "\mathrm{CAlg}\left(\mathrm{B}_\bullet^{\mathrm{cyc},\mathcal{C}}\right)"'] & & \mathrm{CAlg}\big(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})\big)
\end{tikzcd} \tag{$*$}$$

By [HA, 3.2.4.7] the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is cocartesian, so it follows that the induced symmetric monoidal structure on $\mathrm{Alg}(\mathrm{CAlg}(\mathcal{C}))$ is also cocartesian, and hence the left top vertical functor is an equivalence by [HA, 2.4.3.9]. To see that the lower left vertical functor is also an equivalence and that the composite left vertical equivalence can be identified with the one in diagram (6.10), we consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}\big(\mathrm{CAlg}(\mathcal{C})\big) \arrow[d, dash, "\simeq"] & \mathrm{CAlg}\Big(\mathrm{Alg}\big(\mathrm{CAlg}(\mathcal{C})\big)\Big) \arrow[l, "\mathrm{ev}_{\langle 1\rangle}"', "\simeq"] \arrow[d, dash, "\simeq"] \arrow[r, "\mathrm{CAlg}\left(\mathrm{Alg}\left(\mathrm{ev}_{\langle 1\rangle}\right)\right)"] & \mathrm{CAlg}\big(\mathrm{Alg}(\mathcal{C})\big) \arrow[d, dash, "="] \\
\mathrm{CAlg}\big(\mathrm{Alg}(\mathcal{C})\big) & \mathrm{CAlg}\Big(\mathrm{CAlg}\big(\mathrm{Alg}(\mathcal{C})\big)\Big) \arrow[l, "\simeq", "\mathrm{ev}_{\langle 1\rangle}"'] \arrow[r, "\mathrm{CAlg}\left(\mathrm{ev}_{\langle 1\rangle}\right)"'] & \mathrm{CAlg}\big(\mathrm{Alg}(\mathcal{C})\big)
\end{tikzcd}$$

where the middle and left vertical equivalences are (induced by) the canonical equivalence exchanging the "inner" Alg and CAlg. By Proposition E.6.0.1, the bottom right horizontal functor is an equivalence, and the composite equivalence from the bottom left to the bottom right is homotopic to the identity functor. It follows that the bottom left vertical functor in diagram ($*$) is an equivalence and that the composite left vertical equivalence can be identified with the left vertical equivalence in diagram (6.10).

We can argue completely analogously for the two right vertical functors in diagram ($*$) being equivalences and the identification of the composite with the right vertical equivalence in diagram (6.10) – this time we need to exchange the "inner" $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, -)$ and CAlg. $\square$

Proposition 6.1.2.20. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Consider the com-*

position[41]

$$\mathrm{CAlg}(\mathcal{C}) \xrightarrow{\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})^{-1}} \mathrm{CAlg}\big(\mathrm{Alg}(\mathcal{C})\big) \xrightarrow{\mathrm{CAlg}(\mathrm{B}^{\mathrm{cyc}}_{\bullet})} \mathrm{CAlg}\big(\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})\big) \xrightarrow{\simeq} \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C})\big)$$

where the last functor is the canonical equivalence [HA, 2.1.3.4][42]*. This composition is left adjoint to the functor* $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$. ♡

Proof. Using Proposition 6.1.2.19 we can identify the composition in question with the following composition

$$\mathrm{CAlg}(\mathcal{C}) \xrightarrow{\mathrm{ev}_{\mathfrak{a}}^{-1}} \mathrm{Alg}\big(\mathrm{CAlg}(\mathcal{C})\big) \xrightarrow{\mathrm{B}^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}_{\bullet}} \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C})\big)$$

where $\mathrm{B}^{\mathrm{cyc},\mathrm{CAlg}(\mathcal{C})}_{\bullet}$ is the cyclic bar construction with respect to the symmetric monoidal ∞-category $\mathrm{CAlg}(\mathcal{C})$. The claim now follows from Proposition 6.1.2.18, as the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is cocartesian. □

6.1.2.7. $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ and sifted colimits

The following statement concerning $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ and sifted colimits will be helpful later when we want to show that Hochschild homology is compatible with relative tensor products.

Proposition 6.1.2.21. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Let $\mathcal{I}$ be a small sifted ∞-category*[43]*, and assume that the symmetric monoidal structure of $\mathcal{C}$ is compatible with $\mathcal{I}$-indexed colimits in the sense of [HA, 3.1.1.18].*

Then the functor

$$\mathrm{B}^{\mathrm{cyc}}_{\bullet}\colon \mathrm{Alg}(\mathcal{C}) \to \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$

from Definition 6.1.2.16 preserves $\mathcal{I}$-indexed colimits. ♡

Proof. Colimits in functor categories are detected pointwise by [HTT, 5.1.2.3], so it suffices to show that for every $m \geq 1$ the composition $\mathrm{ev}_{[m-1]_{\mathbf{\Lambda}}} \circ \mathrm{B}^{\mathrm{cyc}}_{\bullet}$ preserves $\mathcal{I}$-indexed colimits. Unpacking the definition of $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$, we can identify this composition with $\mathrm{ev}_{\langle m\rangle} \circ \otimes_{*} \circ A$, see Definition 6.1.2.16 and Proposition 6.1.2.11. Using Proposition 6.1.2.10 we can further identify this composition with

$$\mathrm{Alg}(\mathcal{C}) \xrightarrow{\mathrm{ev}_{\langle m\rangle}} \mathcal{C}^{\otimes}_{\langle m\rangle} \xrightarrow{(\mu_m)_!} \mathcal{C}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{C}$$

where $\mu_m\colon \langle m\rangle \to \langle 1\rangle$ is the unique active morphism in Fin_*.

---

[41] $\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})$ can be identified with the composition

$$\mathrm{CAlg}\big(\mathrm{Alg}(\mathcal{C})\big) \simeq \mathrm{Alg}\big(\mathrm{CAlg}(\mathcal{C})\big) \xrightarrow{\mathrm{ev}_{\mathfrak{a}}} \mathrm{CAlg}(\mathcal{C})$$

and is thus an equivalence by [HA, 3.2.4.7 and 2.4.3.9].

[42] This equivalence arises from using that $\mathrm{Fun}(\mathsf{Fin}_*, -)$ preserves pullbacks and the $\times$-Fun-adjunction.

[43] See [HTT, 5.5.8.1] for a definition.

By [HA, 3.2.3.7], the functor $(\mu_m)_!$ appearing above preserves $\mathcal{I}$-indexed colimits, so it remains to show that

$$\mathrm{ev}_{\langle m\rangle}\colon\ \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}^{\otimes}_{\langle m\rangle}$$

also does so. The inert morphisms $\rho^i\colon \langle m\rangle \to \langle 1\rangle$ determine natural transformations $\mathrm{ev}_{\rho^i}\colon \mathrm{ev}_{\langle m\rangle} \to \mathrm{ev}_{\langle 1\rangle}$. By definition of $\mathrm{Alg}(\mathcal{C})$, these natural transformations will be componentwise inert morphisms in $\mathcal{C}^{\otimes}$ lying over ρ^i. It follows[44] that the natural transformation

$$\prod_{1\le i\le m} \mathrm{ev}_{\rho^i}\colon \mathrm{ev}_{\langle m\rangle} \to \prod_{1\le i\le m} \mathrm{ev}_{\langle 1\rangle}$$

is a natural equivalence.

It thus suffices to show that

$$\prod_{1\le i\le m} \mathrm{ev}_{\langle 1\rangle}\colon\ \mathrm{Alg}(\mathcal{C}) \to \prod_{1\le i\le m} \mathcal{C}$$

preserves $\mathcal{I}$-indexed colimits. As colimits in products of ∞-categories are detected componentwise by [HTT, 5.1.2.3], we are left to show that

$$\mathrm{ev}_{\langle 1\rangle}\colon\ \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$$

preserves $\mathcal{I}$-indexed colimits, which is true by [HA, 3.2.3.1 (4)]. □

6.1.3. Geometric realization of cyclic objects

Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category and $X\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}$ a cyclic object in $\mathcal{C}$. Recall from Construction 6.1.1.6 that there is a functor $j\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Lambda}^{\mathrm{op}}$, along which we can precompose X, obtaining a simplicial object j^*X. In this section we discuss how the extra automorphisms in $\mathbf{\Lambda}$ provide the structure of a $\mathbb{T}$-action on the geometric realization $|j^*X| = \operatorname{colim} j^*X$. We follow the approach of [Hoy18], but see also [NikSch, Appendix B].

We will start in Section 6.1.3.1 by briefly reviewing ∞-groupoid completions and the fact that the ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$ is $\mathrm{B}\,\mathbb{T}$, which will be needed to define the geometric realization functor for cyclic objects in Section 6.1.3.2. We will end in Section 6.1.3.3 by discussing monoidality of this construction.

6.1.3.1. The ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$

In this short section we recall that the ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$ is given by $\mathrm{B}\,\mathbb{T}$. We first introduce some notation.

Notation 6.1.3.1. Let $\mathcal{C}$ be an ∞-category. We denote the *∞-groupoid completion* of $\mathcal{C}$ by $\mathcal{C}^{\mathrm{gpd}}$. Concretely $\mathcal{C}^{\mathrm{gpd}}$ is the ∞-groupoid obtained by inverting all morphisms of $\mathcal{C}$, and comes with a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{gpd}}$ that is initial among functors with domain $\mathcal{C}$ and whose codomain is an ∞-groupoid.

[44] See Proposition A.3.2.1 and [HA, 2.1.1.14].

This construction can be made into a functor $-^{\mathrm{gpd}}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ that is left left adjoint to the inclusion, see [HTT, 1.2.5.6 and the preceding discussion] and [HA, 1.3.4.1]. ♢

We can now recall the following result about the ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$. The two references state their results as $\mathbf{\Lambda}^{\mathrm{gpd}} \simeq \mathrm{B}\,\mathbb{T}$, but Fact 6.1.3.2 can be immediately obtained from this by either using that $\mathbf{\Lambda}$ is self-dual by Fact 6.1.1.13 or using that $-^{\mathrm{gpd}}$ is compatible with passing to opposite ∞-categories and that ∞-groupoids are equivalent to their opposites.

Fact 6.1.3.2 ([Hoy18, 1.2], [NikSch, B.4])**.** *There is an equivalence*

$$(\mathbf{\Lambda}^{\mathrm{op}})^{\mathrm{gpd}} \simeq \mathrm{B}\,\mathbb{T}$$

of ∞-groupoids. ♧

6.1.3.2. Definition of the geometric realization

We now come to the definition of the geometric realization of cyclic objects. This will be defined as a left adjoint, so we start by showing that the left adjoint exists.

Proposition 6.1.3.3. *Let $\mathcal{C}$ be an ∞-category. Denote by $\phi\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathrm{B}\,\mathbb{T}$ the canonical functor exhibiting $\mathrm{B}\,\mathbb{T}$ as the ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$, see Fact 6.1.3.2. Then the following hold.*

(1) The functor
$$\phi^*\colon \operatorname{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{C}) \to \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$$
is fully faithful, and its essential image is spanned by those functors that map every morphism in $\mathbf{\Lambda}^{\mathrm{op}}$ to an equivalence in $\mathcal{C}$.

(2) Assume that $\mathcal{C}$ is presentable. Then ϕ^* admits a left adjoint. ♡

Proof. Proof of claim (1): Holds by definition, see [HA, 1.3.4.1].

Proof of claim (2): By [HTT, 5.5.3.6], both $\operatorname{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{C})$ and $\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ are presentable. By the adjoint functor theorem [HTT, 5.5.2.9] it thus suffices to show that ϕ^* is accessible and preserves small limits. This follows immediately from the fact that limits and colimits in functor categories are calculated pointwise[45]. □

We can now make the following definition.

Definition 6.1.3.4 ([Hoy18, Page 2])**.** Let $\mathcal{C}$ be a presentable ∞-category. Then we denote the left adjoint to ϕ^* from Proposition 6.1.3.3 by

$$|-|\colon \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$$

and call it the *geometric realization* functor for cyclic objects. ♢

[45] See [HTT, 5.1.2.3] for the fact that (co)limits are calculated pointwise, and [HTT, 5.4.2.5 and 5.3.4.5] for the definition of accessible functors.

Remark 6.1.3.5. Let
$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C}' \arrow[l, shift left, "G"] \end{tikzcd}$$
be an adjunction of ∞-categories, with $\mathcal{C}$ and $\mathcal{C}'$ both presentable.

Then compatibility of precomposing with postcomposing yields a commutative diagram
$$\begin{tikzcd} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) & \mathcal{C}^{\mathrm{B}\,\mathbb{T}} \arrow[l, "\phi^*"'] \\ \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}') \arrow[u, "G_*"] & \mathcal{C}'^{\mathrm{B}\,\mathbb{T}} \arrow[l, "\phi^*"] \arrow[u, "G_*"'] \end{tikzcd}$$
so that, by passing to left adjoints and using Proposition D.2.2.1 and [HTT, 5.2.6.2] we obtain a commutative diagram
$$\begin{tikzcd} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \arrow[r, "|-|"] \arrow[d, "F_*"'] & \mathcal{C}^{\mathrm{B}\,\mathbb{T}} \arrow[d, "F_*"] \\ \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}') \arrow[r, "|-|"'] & \mathcal{C}'^{\mathrm{B}\,\mathbb{T}} \end{tikzcd}$$
relating the geometric realization functors for $\mathcal{C}$ and $\mathcal{C}'$. ♢

We end this section with the following comparison between geometric realization of cyclic and simplicial objects, which gives a description of the underlying object of $|X|$ for a cyclic object X.

Fact 6.1.3.6 ([Hoy18, 1.1])**.** *Let $\mathcal{C}$ be a presentable ∞-category. Then there is a commutative square of ∞-categories as follows*
$$\begin{tikzcd} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \arrow[r, "|-|"] \arrow[d, "j^*"'] & \mathcal{C}^{\mathrm{B}\,\mathbb{T}} \arrow[d, "\mathrm{ev}_*"] \\ \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}) \arrow[r, "|-|"'] & \mathcal{C} \end{tikzcd}$$
where ϕ is as in Construction 6.1.1.6, $*$ is the basepoint (i. e. the up to equivalence unique object) of $\mathrm{B}\,\mathbb{T}$*, and the lower horizontal functor is the geometric realization functor for simplicial objects, so the functor* $\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}$*.* ♧

6.1.3.3. Monoidality

If $\mathcal{C}$ is a presentable *symmetric monoidal* ∞-category, then both $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ and $\mathcal{C}^{\mathrm{B}\,\mathbb{T}}$ can be given the pointwise symmetric monoidal structure[46], with respect to which the functor ϕ^* from Proposition 6.1.3.3 can be upgraded to a symmetric monoidal functor. In this section we show that the geometric realization functor for cyclic objects can also be upgraded to a symmetric monoidal functor.

[46] See [HA, 2.1.3.4].

Proposition 6.1.3.7. *Let $\mathcal{O}$ be an ∞-operad and let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration of ∞-operads, and assume furthermore that $\mathcal{C}_X$ is presentable for every object X of $\mathcal{O}$, and that the $\mathcal{O}$-monoidal structure on $\mathcal{C}$ is compatible with small colimits in the sense of [HA, 3.1.1.18 and 3.1.1.19].*

Then the adjunctions $|-| \dashv \phi^*$ from Definition 6.1.3.4 for the presentable ∞-categories $\mathcal{C}_X$ for objects X of $\mathcal{O}$ can be upgraded to an adjunction relative to $\mathcal{O}^{\otimes}$ in the sense of [HA, 7.3.2.2]

$$\begin{tikzcd}
\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})^{\otimes} \arrow[rr, shift left, "(|-|)^{\otimes}"] \arrow[dr] & & (\mathcal{C}^{\mathrm{B}\,\mathbb{T}})^{\otimes} \arrow[ll, shift left, "(\phi^*)^{\otimes}"] \arrow[dl] \\
& \mathcal{O}^{\otimes} &
\end{tikzcd}$$

where the functors to $\mathcal{O}^{\otimes}$ are the canonical $\mathcal{O}$-monoidal functors that exhibit $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ and $\mathcal{C}^{\mathrm{B}\,\mathbb{T}}$ as equipped with the pointwise $\mathcal{O}$-monoidal structure.

Furthermore, both $\big(|-|\big)^{\otimes}$ and $(\phi^*)^{\otimes}$ are $\mathcal{O}$-monoidal functors. ♡

Proof. $(\phi^*)^{\otimes}$ is defined as the induced functor

$$\mathrm{Fun}\big(\mathrm{B}\,\mathbb{T}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{O}^{\otimes})} \mathcal{O}^{\otimes} \xrightarrow{\phi^* \times_{\phi^*} \mathrm{id}_{\mathcal{O}^{\otimes}}} \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{O}^{\otimes})} \mathcal{O}^{\otimes}$$

which by [HTT, 3.1.2.1] and Proposition C.1.1.1 preserves pr_2-cocartesian morphisms and is thus $\mathcal{O}$-monoidal. Furthermore, by Proposition 6.1.3.3 (1), the functors

$$\phi^*\colon\ \mathrm{Fun}\big(\mathrm{B}\,\mathbb{T}, \mathcal{C}^{\otimes}\big) \to \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}^{\otimes}\big)$$

and

$$\phi^*\colon\ \mathrm{Fun}\big(\mathrm{B}\,\mathbb{T}, \mathcal{O}^{\otimes}\big) \to \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{O}^{\otimes}\big)$$

are fully faithful, with essential image spanned by those functors that map all morphisms to equivalences. It follows from Proposition B.5.3.1 that $(\phi^*)^{\otimes}$ is also fully faithful, with essential image spanned by those objects which are mapped by pr_1 to functors that invert all morphisms. An object in $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})^{\otimes}$ lying over $X \simeq X_1 \oplus \cdots \oplus X_n$ in $\mathcal{O}^{\otimes}$ is mapped by pr_1 to a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathcal{C}^{\otimes}$ that factors over the conservative inclusion of $\mathcal{C}^{\otimes}_X \simeq \mathcal{C}_{X_1} \times \cdots \times \mathcal{C}_{X_n}$. As morphisms in products of ∞-categories are equivalences if and only if their component morphisms are, we can hence identify the essential image of $(\phi^*)^{\otimes}$ with the the induced ∞-operad structure as defined in [HA, Start of section 2.2.1] on the full subcategory $\mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{C})$ of the the underlying ∞-category $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ of the ∞-operad $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})^{\otimes}$.

The claims will now follow from the conclusion of [HA, 2.2.1.9][47]. To verify the requirements to apply that result, it remains to show that the localization functors

$$\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}_X) \xrightarrow{|-|} \mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{C}_X) \xrightarrow{\phi^*} \mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}_X)$$

[47] That $|-|^{\otimes}_X$ will be given by $|-|\colon \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}_X\big) \to \mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{C}_X)$ for X an object of $\mathcal{O}$ follows from [HA, 7.3.2.5] and [HTT, 5.2.6].

for X an object of $\mathcal{O}$ are compatible with the $\mathcal{O}$-monoidal structure on $\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})^{\otimes}$ in the sense of [HA, 2.2.1.6].

So let $f\colon X_1 \oplus \cdots \oplus X_n \to Y$ be a morphism in $\mathcal{O}^{\otimes}$, with X_i and Y objects of $\mathcal{O}$. We obtain an induced functor on fibers

$$\prod_{1 \leq i \leq n} \operatorname{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}_{X_i}\big) \simeq \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})^{\otimes}_{X_1 \oplus \cdots \oplus X_n} \xrightarrow{f_!} \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})^{\otimes}_{Y} \simeq \operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}_Y)$$

and what we have to show is that if morphisms g_i are an mapped to equivalences by

$$|-|\colon\ \operatorname{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}_{X_i}\big) \to \operatorname{Fun}\big(\mathrm{B}\,\mathbb{T}, \mathcal{C}_{X_i}\big)$$

for each $1 \leq i \leq n$, then so is $f_!(g_1 \oplus \cdots \oplus g_n)$.

Using that the forgetful functor $\mathrm{ev}_*\colon \operatorname{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{C}_Y) \to \mathcal{C}_Y$ detects equivalences by Proposition A.3.2.1, and combining this with Fact 6.1.3.6, this boils down to showing that

$$\big(\mathrm{ev}_* \circ |-|\big)\big(f_!(g_1 \oplus \cdots \oplus g_n)\big) \simeq \left(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \circ j^*\right)\big(f_!(g_1 \oplus \cdots \oplus g_n)\big)$$

is an equivalence if $(\operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \circ j^*)(g_i)$ is for every $1 \leq i \leq n$.

Let us unpack the functor $\operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \circ j^* \circ f_!$. We have natural equivalences as follows, where C_i is an object of $\mathcal{C}_{X_i}$.

$$\left(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \circ j^* \circ f_!\right)(C_1 \oplus \cdots \oplus C_n)$$

Using that j^* is $\mathcal{O}$-monoidal with respect to the pointwise $\mathcal{O}$-monoidal structures on $\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C})$ and $\operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$.

$$\simeq \left(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \circ f_!\right)(j^*C_1 \oplus \cdots \oplus j^*C_n)$$

Using the definition of the pointwise $\mathcal{O}$-monoidal structure.

$$\simeq \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \left(\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\prod_{1 \leq i \leq n} \mathrm{id}_{\mathbf{\Delta}^{\mathrm{op}}}} \prod_{1 \leq i \leq n} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\prod_{1 \leq i \leq n} C_i \circ j} \prod_{1 \leq i \leq n} \mathcal{C}_{X_i} \xrightarrow{f_!} \mathcal{C}_Y\right)$$

Applying [HA, 3.2.3.7], which is applicable as the $\mathcal{O}$-monoidal structure of $\mathcal{C}$ is compatible with small colimits by assumption and $\mathbf{\Delta}^{\mathrm{op}}$ is sifted [HTT, 5.5.8.1 and 5.5.8.4].

$$\simeq f_!\left(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}\left(\prod_{1 \leq i \leq n} C_i \circ j\right)\right)$$

Using that colimits in products are calculated pointwise [HTT, 5.1.2.3].

$$\simeq f_!\left(\bigoplus_{1 \leq i \leq n} \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} C_i \circ j\right)$$

Thus the claim we need to show ultimately boils down to the following: If $g_i\colon C_i \to D_i$ induces an equivalence

$$\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(C_i \circ j) \to \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(D_i \circ j)$$

for every $1 \leq i \leq n$, then the induced morphism

$$f_!\left(\bigoplus_{1\leq i\leq n} \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(C_i \circ j)\right) \to f_!\left(\bigoplus_{1\leq i\leq n} \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(D_i \circ j)\right)$$

is an equivalence as well, which is clear. $\square$

Remark 6.1.3.8. Let $\mathcal{O}$ be an ∞-operad and let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $p_{\mathcal{C}'}\colon \mathcal{C}'^{\otimes} \to \mathcal{O}^{\otimes}$ be cocartesian fibrations of ∞-operads that both satisfy the conditions of Proposition 6.1.3.7. Let

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[rr, shift left, "F^{\otimes}"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p_{\mathcal{C}}"'] & & \mathcal{C}'^{\otimes} \arrow[ll, shift left, "G^{\otimes}"] \arrow[dl, "p_{\mathcal{C}}"] \\
& \mathcal{O}^{\otimes} &
\end{tikzcd}$$

be an adjunction relative to $\mathcal{O}^{\otimes}$ in the sense of [HA, 7.3.2.2 and 7.3.2.3], with both F and G being $\mathcal{O}$-monoidal.

Then proceeding exactly like in Remark 6.1.3.5 and using Proposition 6.1.3.7, we can conclude that the commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}) \arrow[r, "|-|"] \arrow[d, "F_*"'] & \mathcal{C}^{\mathrm{B}\mathbb{T}} \arrow[d, "F_*"] \\
\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{C}') \arrow[r, "|-|"'] & \mathcal{C}'^{\mathrm{B}\mathbb{T}}
\end{tikzcd}$$

can be upgraded to a commutative diagram of $\mathcal{O}$-monoidal functors. $\diamondsuit$

6.2. Hochschild homology

In this section we finally define the functor

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\colon \operatorname{Alg}\big(\mathcal{D}(k)\big) \to \mathcal{M}\mathrm{ixed}$$

that the chapters below will be about, and discuss some crucial first properties[48].

We will start with the definition in Section 6.2.1. In Section 6.2.2 we will then discuss different descriptions of Hochschild homology of *commutative* algebras. Finally, we will show in Section 6.2.3 that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserves relative tensor product, which will later be crucial for calculations.

[48]We will compare $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ with the classical standard Hochschild complex in the next section, Section 6.3.

6.2.1. Definition of Hochschild homology

We can now define Hochschild homology by specializing the general discussion of the cyclic bar construction and geometric realization of cyclic objects of Section 6.1 to the case of $\mathcal{D}(k)$. We can apply the definitions of $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ and $|-|$ to $\mathcal{D}(k)$ as it is a presentable symmetric monoidal ∞-category according to Proposition 4.3.2.1.

Definition 6.2.1.1. We define $\mathrm{HH}_{\mathbb{T}}$ to be the symmetric monoidal functor that is given as the composition

$$\mathrm{HH}_{\mathbb{T}}\colon \operatorname{Alg}\big(\mathcal{D}(k)\big) \xrightarrow{\mathrm{B}^{\mathrm{cyc}}_{\bullet}} \operatorname{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \xrightarrow{|-|} \mathcal{D}(k)^{\mathrm{B}\mathbb{T}}$$

where $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ is the symmetric monoidal functor from Definition 6.1.2.16 and $|-|$ is the symmetric monoidal functor from Definition 6.1.3.4 and Proposition 6.1.3.7.

We furthermore denote by

$$\mathrm{HH}\colon \operatorname{Alg}\big(\mathcal{D}(k)\big) \to \mathcal{D}(k)$$

the symmetric monoidal functor given by composing $\mathrm{HH}_{\mathbb{T}}$ with the symmetric monoidal functor ev_*.

We refer to both $\mathrm{HH}_{\mathbb{T}}$ and HH as the *Hochschild homology functor*. $\diamondsuit$

The reason we use the subscript $\mathbb{T}$ for $\mathrm{HH}_{\mathbb{T}}$ is to distinguish this functor from the composition with the equivalence $\mathcal{D}(k)^{\mathrm{B}\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ from Construction 5.4.0.1, as we will need to refer to both functors in later chapters. We thus also give the latter functor a name.

Definition 6.2.1.2. We define

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\colon \operatorname{Alg}\big(\mathcal{D}(k)\big) \to \mathcal{M}\mathrm{ixed}$$

to be the monoidal functor obtained by composing the symmetric monoidal functor $\mathrm{HH}_{\mathbb{T}}$ from Definition 6.2.1.1 with the monoidal equivalence from Construction 5.4.0.1. $\diamondsuit$

Notation 6.2.1.3. If we evaluate HH, $\mathrm{HH}_{\mathbb{T}}$, or $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ at an object of the form $\operatorname{Alg}(\gamma)(R)$, with R an object of $\operatorname{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})$, then we will often omit γ from the notation and just write e. g. $\mathrm{HH}(R)$ instead of $\mathrm{HH}(\operatorname{Alg}(\gamma)(R))$. $\diamondsuit$

Warning 6.2.1.4. As the equivalence $\mathcal{D}(k)^{\mathrm{B}\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ from Construction 5.4.0.1 is only (associatively) monoidal, not symmetric monoidal, the same is true for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$. $\diamondsuit$

Remark 6.2.1.5. As the monoidal equivalence $\mathcal{D}(k)^{\mathrm{B}\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$ constructed in Construction 5.4.0.1 is compatible with the forgetful functors to $\mathcal{D}(k)$, we obtain a homotopy

$$\mathrm{ev}_{\mathfrak{m}} \circ \mathrm{HH}_{\mathcal{M}\mathrm{ixed}} \simeq \mathrm{ev}_* \circ \mathrm{HH}_{\mathbb{T}} \simeq \mathrm{HH}$$

of monoidal functors. $\diamondsuit$

Remark 6.2.1.6. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then combining Remark 6.1.2.17 with Remark 6.1.3.8 applied to the adjunction from Remark 4.3.2.2 we obtain a commutative diagram of symmetric monoidal functors as follows.

$$\begin{tikzcd}
\mathrm{Alg}\big(\mathcal{D}(k)\big) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"] \arrow[d, "k'\otimes_k -"'] \arrow[rr, bend left, "\mathrm{HH}"] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \arrow[r, "\mathrm{ev}_{\mathrm{m}}"] \arrow[d, "\left(k'\otimes_k -\right)_*"] & \mathcal{D}(k) \arrow[d, "k'\otimes_k -"] \\
\mathrm{Alg}\big(\mathcal{D}(k)'\big) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"'] \arrow[rr, bend right, "\mathrm{HH}"'] & \mathcal{D}(k')^{\mathrm{B}\,\mathbb{T}} \arrow[r, "\mathrm{ev}_{\mathrm{m}}"'] & \mathcal{D}(k')
\end{tikzcd}$$

Combining the above with Remark 5.4.0.3 we also obtain a commutative diagram of monoidal functors as follows.

$$\begin{tikzcd}
\mathrm{Alg}\big(\mathcal{D}(k)\big) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}"] \arrow[d, "k'\otimes_k -"'] & \mathcal{M}\mathrm{ixed}_k \arrow[d, "k'\otimes_k -"] \\
\mathrm{Alg}\big(\mathcal{D}(k')\big) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}"'] & \mathcal{M}\mathrm{ixed}_{k'}
\end{tikzcd}$$

$\diamondsuit$

6.2.2. Hochschild homology and commutative algebras

The functors $\mathrm{HH}_{\mathbb{T}}$ and HH defined in Definition 6.2.1.1 are symmetric monoidal functors and thus induce functors on ∞-categories of commutative algebras. In this section we will give different characterizations of those induced functors that will be of use later.

We will start in Section 6.2.2.1 by mostly fixing notation. In Section 6.2.2.3 we will show that if R is a commutative algebra in $\mathcal{D}(k)$, then $\mathrm{HH}_{\mathbb{T}}(R)$ can essentially be obtained as $R \boxtimes \underline{\mathbb{T}}$, i. e. tensoring R as an object of $\mathrm{CAlg}(\mathcal{D}(k))$ with $\mathbb{T}$, considered as a space with a $\mathbb{T}$-action. To properly discuss this, we will first introduce $- \boxtimes -$ and $\underline{\mathbb{T}}$ in Section 6.2.2.2. As an application of this description, we will show in Section 6.2.2.4 and Section 6.2.2.5 how interpret HH of commutative algebras as pushouts and relative tensor products in $\mathrm{CAlg}(\mathcal{D}(k))$.

6.2.2.1. HH for commutative algebras

As the functors HH and $\mathrm{HH}_{\mathbb{T}}$ from Definition 6.2.1.1 are symmetric monoidal, they induce functors on ∞-categories of commutative algebras as well. By precomposing and postcomposing with canonical equivalences, we arrive at the following definitions.

Definition 6.2.2.1. We denote by $\mathrm{HH}_{\mathbb{T}}$ the composition

$$\mathrm{CAlg}\big(\mathcal{D}(k)\big) \xrightarrow{\simeq} \mathrm{CAlg}\Big(\mathrm{Alg}\big(\mathcal{D}(k)\big)\Big) \xrightarrow{\mathrm{CAlg}(\mathrm{HH}_{\mathbb{T}})} \mathrm{CAlg}\Big(\mathcal{D}(k)^{\mathrm{B}\mathbb{T}}\Big) \xrightarrow{\simeq} \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\mathbb{T}}$$

where the individual functors are as follows.

- The first equivalence is the inverse of the following equivalence[49].

$$\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})\colon\ \mathrm{CAlg}\Big(\mathrm{Alg}\big(\mathcal{D}(k)\big)\Big) \to \mathrm{CAlg}\big(\mathcal{D}(k)\big)$$

- The functor $\mathrm{HH}_{\mathbb{T}}$ appearing in $\mathrm{CAlg}(\mathrm{HH}_{\mathbb{T}})$ refers to the symmetric monoidal functor from Definition 6.2.1.1.

- The second equivalence refers to the canonical equivalence, see [HA, 2.1.3.4].

We furthermore denote by HH the composition of the functor $\mathrm{HH}_{\mathbb{T}}$ above with the functor

$$\mathrm{ev}_*\colon\ \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\mathbb{T}} \to \mathrm{CAlg}\big(\mathcal{D}(k)\big)$$

that is given by evaluation at the basepoint. Equivalently, HH is the composition

$$\mathrm{CAlg}\big(\mathcal{D}(k)\big) \xrightarrow{\simeq} \mathrm{CAlg}\Big(\mathrm{Alg}\big(\mathcal{D}(k)\big)\Big) \xrightarrow{\mathrm{CAlg}(\mathrm{HH})} \mathrm{CAlg}\big(\mathcal{D}(k)\big)$$

where the equivalence is like above and the symmetric monoidal functor HH occurring in $\mathrm{CAlg}(\mathrm{HH})$ is the one from Definition 6.2.1.1. ◊

We next show that the definitions made in Definition 6.2.2.1 are compatible with the definitions from Definition 6.2.1.1 in the appropriate way.

---

[49] This functor can be identified with the composition of the equivalence

$$\mathrm{CAlg}\Big(\mathrm{Alg}\big(\mathcal{D}(k)\big)\Big) \simeq \mathrm{BiFunc}\big(\mathsf{Comm}, \mathsf{Assoc}; \mathcal{D}(k)\big)$$

from Proposition E.5.0.1, the equivalence

$$\mathrm{BiFunc}\big(\mathsf{Comm}, \mathsf{Assoc}; \mathcal{D}(k)\big) \simeq \mathrm{BiFunc}\big(\mathsf{Assoc}, \mathsf{Comm}; \mathcal{D}(k)\big)$$

given by precomposing with the symmetry equivalence

$$\mathsf{Assoc}^{\otimes} \times \mathsf{Comm}^{\otimes} \simeq \mathsf{Comm}^{\otimes} \times \mathsf{Assoc}^{\otimes}$$

the equivalence

$$\mathrm{BiFunc}\big(\mathsf{Assoc}, \mathsf{Comm}; \mathcal{D}(k)\big) \simeq \mathrm{Alg}\Big(\mathrm{CAlg}\big(\mathcal{D}(k)\big)\Big)$$

from Proposition E.5.0.1, and the functor

$$\mathrm{ev}_{\mathfrak{a}}\colon\ \mathrm{Alg}\Big(\mathrm{CAlg}\big(\mathcal{D}(k)\big)\Big) \to \mathrm{CAlg}\big(\mathcal{D}(k)\big)$$

that is an equivalence by [HA, 3.2.4.7 and 2.4.3.9].

Proposition 6.2.2.2. *There is a commutative diagram*

$$\begin{tikzcd} \mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"] \arrow[d, "p^*_{\mathsf{Assoc}}"'] & \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\,\mathbb{T}} \arrow[d, "\left(\mathrm{ev}_{\langle 1\rangle}\right)_*"] \\ \mathrm{Alg}\big(\mathcal{D}(k)\big) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"'] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \end{tikzcd} \tag{6.11}$$

where p_{Assoc} is the canonical morphism of ∞-operads $\mathsf{Assoc}^{\otimes} \to \mathsf{Comm}^{\otimes}$, the top horizontal functor is the one from Definition 6.2.2.1 and the bottom horizontal functor is the one from Definition 6.2.1.1.

Similarly, there is a commutative diagram

$$\begin{tikzcd} \mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[r, "\mathrm{HH}"] \arrow[d, "p^*_{\mathsf{Assoc}}"'] & \mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"] \\ \mathrm{Alg}\big(\mathcal{D}(k)\big) \arrow[r, "\mathrm{HH}"'] & \mathcal{D}(k) \end{tikzcd} \tag{6.12}$$

in $\mathcal{C}\mathrm{at}_\infty$. ♡

Proof. Diagram (6.11) is obtained as the composite outer diagram of the following commutative diagram.

$$\begin{tikzcd} \mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[r, "\mathrm{HH}_{\mathbb{T}}"] \arrow[dd, bend right=60, "p^*_{\mathsf{Assoc}}"'] & \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\,\mathbb{T}} \arrow[dd, bend left=60, "\left(\mathrm{ev}_{\langle 1\rangle}\right)_*"] \\ \mathrm{CAlg}\Big(\mathrm{Alg}\big(\mathcal{D}(k)\big)\Big) \arrow[u, "\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})"] \arrow[r, "\mathrm{CAlg}(\mathrm{HH}_{\mathbb{T}})"] \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"'] & \mathrm{CAlg}\Big(\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}\Big) \arrow[u, "\simeq"'] \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"] \\ \mathrm{Alg}\big(\mathcal{D}(k)\big) \arrow[r, "\mathrm{HH}"'] & \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \end{tikzcd}$$

where the upper right vertical functor is the canonical equivalence. The top square commutes by definition of the top horizontal functor, the bottom square commutes by naturality of $\mathrm{ev}_{\langle 1\rangle}$, and commutativity of the right triangle is clear from the definition. It remains discuss the left triangle, which we obtain as the outer commutative triangle in the following commutative diagram

$$\begin{tikzcd}
\mathrm{CAlg}\big(\mathrm{Alg}(\mathcal{D}(k))\big) \arrow[dr, "\mathrm{ev}_{\langle 1\rangle}"'] \arrow[rr, bend left, "\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})"] & \mathrm{CAlg}\big(\mathrm{CAlg}(\mathcal{D}(k))\big) \arrow[l, "\mathrm{CAlg}(p^*_{\mathsf{Assoc}})"'] \arrow[r, "\mathrm{CAlg}(\mathrm{ev}_{\langle 1\rangle})", "\mathrm{ev}_{\langle 1\rangle}"'] & \mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[dl, "p^*_{\mathsf{Assoc}}"] \\
& \mathrm{Alg}\big(\mathcal{D}(k)\big) &
\end{tikzcd}$$

where we use that $\mathrm{CAlg}(\mathrm{ev}_{\langle 1\rangle})$ and $\mathrm{ev}_{\langle 1\rangle}$ are homotopic and both equivalences by Proposition E.6.0.1 and that $\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})$, and hence $\mathrm{CAlg}(p^*_{\mathsf{Assoc}})$, are equivalences as well.

To obtain commutative diagram (6.12) from (6.11) it suffices to remark that there is an equivalence $\mathrm{ev}_* \circ (\mathrm{ev}_{\langle 1\rangle})_* \simeq \mathrm{ev}_{\langle 1\rangle} \circ \mathrm{ev}_*$. □

6.2.2.2. Circle actions on tensor products with $\mathbb{T}$

There is one object with $\mathbb{T}$-action that is perhaps the most obvious non-trivial example: $\mathbb{T}$ acting on itself. Roughly, this action should be encoded in a functor $\mathrm{B}\,\mathbb{T} \to \mathcal{S}$ that maps the object $*$ to the underlying space of $\mathbb{T}$, and a morphism in $\mathrm{B}\,\mathbb{T}$, corresponding to an element t of $\mathbb{T}$, to the map $t \cdot -\colon \mathbb{T} \to \mathbb{T}$. A bit more rigorously, we could view $\mathbb{T}$ as an object in $\mathrm{LMod}_{\mathbb{T}}(\mathcal{S})$ using the morphism of ∞-operads $\mathsf{LM} \to \mathsf{Assoc}$ from [HA, 4.2.1.5], and then use the equivalence $\mathcal{S}^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{\mathbb{T}}(\mathcal{S})$ from Proposition 5.3.0.8. As yet another alternative approach, one can define the functor $\mathrm{B}\,\mathbb{T} \to \mathcal{S}$ as the left Kan extension along the inclusion $* \to \mathrm{B}\,\mathbb{T}$ of the functor $\mathrm{const}_*\colon * \to \mathcal{S}$, as discussed in [RSV21, Before 2.12]. We will follow [RSV21] in denoting this object of $\mathcal{S}^{\mathrm{B}\,\mathbb{T}}$ by $\underline{\mathbb{T}}$.

That $\underline{\mathbb{T}}$ defined as a left Kan extension is equivalent to the object with $\mathbb{T}$-action obtained from $\mathbb{T}$ as a left module over itself can be seen by using that the left Kan extension functor $\mathcal{S} \simeq \mathrm{Fun}(*, \mathcal{S}) \to \mathrm{Fun}(\mathrm{B}\,\mathbb{T}, \mathcal{S})$ is left adjoint to the forgetful functor ev_* by [HTT, 4.3.3.7], that the left-$\mathbb{T}$-module $\mathbb{T}$ can be described as the free $\mathbb{T}$-module generated by $*$ and so as the image of $*$ under the left adjoint of $\mathrm{ev}_{\mathfrak{m}}$ by [HA, 4.2.4.8], and that the equivalence $\mathcal{S}^{\mathrm{B}\,\mathbb{T}} \simeq \mathrm{LMod}_{\mathbb{T}}(\mathcal{S})$ is shown in Proposition 5.3.0.8 to be compatible with the respective forgetful functors to $\mathcal{S}$, and hence must also be compatible with their left adjoints.

Now let $\mathcal{C}$ be a presentable ∞-category. $\mathcal{S}$ is the unit object in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ by [HA, 4.8.1.20], so there is a unitality equivalence $\mathcal{C} \otimes \mathcal{S} \simeq \mathcal{C}$ that amounts to a functor

$$- \boxtimes -\colon \mathcal{C} \times \mathcal{S} \to \mathcal{C}$$

that preserves small colimits separately in each variable[50]. We thus obtain a colimit-preserving functor

$$- \boxtimes \mathbb{T}\colon \mathcal{C} \to \mathcal{C}$$

[50] Compare with Section 5.2.2 for a more detailed related discussion.

which we should lift to a functor as follows.

$$- \boxtimes \underline{\mathbb{T}} \colon \mathcal{C} \to \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$$

This is indeed the case, and this functor has in fact the following universal property.

Fact 6.2.2.3 ([RSV21, 2.12])**.** *Let $\mathcal{C}$ be a presentable ∞-category. Then there is an adjunction*

$$\mathcal{C} \underset{\mathrm{ev}_*}{\overset{-\boxtimes\underline{\mathbb{T}}}{\rightleftarrows}} \mathcal{C}^{\mathrm{B}\,\mathbb{T}}$$

such that the composition $\mathrm{ev}_* \circ (- \boxtimes \underline{\mathbb{T}})$ is equivalent to $- \boxtimes \mathbb{T}$. ♧

6.2.2.3. HH of commutative algebras as a tensor product with $\mathbb{T}$

$\mathcal{D}(k)$ is a presentable symmetric monoidal ∞-category by Proposition 4.3.2.1, so $\mathrm{CAlg}(\mathcal{D}(k))$ is presentable by [HA, 3.2.3.5 (2)]. We can thus apply Fact 6.2.2.3 and obtain a functor

$$- \boxtimes \underline{\mathbb{T}} \colon \ \mathrm{CAlg}\big(\mathcal{D}(k)\big) \to \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\,\mathbb{T}}$$

which we will now show is equivalent to the functor $\mathrm{HH}_{\mathbb{T}}$ from Definition 6.2.2.1[51].

Proposition 6.2.2.4. *There is an adjunction*

$$\mathrm{CAlg}\big(\mathcal{D}(k)\big) \underset{\mathrm{ev}_*}{\overset{\mathrm{HH}_{\mathbb{T}}}{\rightleftarrows}} \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\,\mathbb{T}}$$

where $\mathrm{HH}_{\mathbb{T}}$ is the functor from Definition 6.2.2.1. Furthermore, there is a homotopy $\mathrm{HH}_{\mathbb{T}} \simeq (- \boxtimes \underline{\mathbb{T}})$ of functors from $\mathrm{CAlg}\big(\mathcal{D}(k)\big)$ to $\mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\,\mathbb{T}}$ as well as $\mathrm{HH} \simeq (- \boxtimes \mathbb{T})$ of endofunctors of $\mathrm{CAlg}\big(\mathcal{D}(k)\big)$. ♡

Proof. It suffices to show the claim that $\mathrm{HH}_{\mathbb{T}}$ is left adjoint to ev_*, as the other two claims then follow immediately from Fact 6.2.2.3 by using uniqueness of left adjoints [HTT, 5.2.6] and the definition of HH as $\mathrm{ev}_* \circ \mathrm{HH}_{\mathbb{T}}$ in Definition 6.2.2.1.

Unpacking the definition of $\mathrm{HH}_{\mathbb{T}}$ in Definition 6.2.2.1 and Definition 6.2.1.1, the functor $\mathrm{HH}_{\mathbb{T}}$ of the statement is given by the composition

$$\begin{aligned}\mathrm{CAlg}\big(\mathcal{D}(k)\big) &\xrightarrow{\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})^{-1}} \mathrm{CAlg}\Big(\mathrm{Alg}\big(\mathcal{D}(k)\big)\Big) \xrightarrow{\mathrm{CAlg}\left(\mathrm{B}_\bullet^{\mathrm{cyc}}\right)} \mathrm{CAlg}\Big(\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big)\Big) \\ &\xrightarrow{\mathrm{CAlg}(|-|)} \mathrm{CAlg}\Big(\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}\Big) \xrightarrow{\simeq} \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\,\mathbb{T}}\end{aligned} \tag{6.13}$$

where the last equivalence is the canonical one.

By Definition 6.1.3.4 the functor

$$|-| \colon \ \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

[51] This claim also appears as Proposition IV.2.2 in [NikSch], but the proof only considers the underlying objects in $\mathcal{D}(k)$.

is left adjoint to ϕ^*, where $\phi\colon \mathbf{\Lambda}^{\mathrm{op}} \to \mathrm{B}\,\mathbb{T}$ is the canonical functor exhibiting $\mathrm{B}\,\mathbb{T}$ as the ∞-groupoid completion of $\mathbf{\Lambda}^{\mathrm{op}}$, see Fact 6.1.3.2. Applying Proposition 6.1.3.7 and Proposition E.3.3.1 we obtain that $\mathrm{CAlg}(|-|)$ is left adjoint to $\mathrm{CAlg}(\phi^*)$. From the the commutative diagram

$$\begin{tikzcd}
\mathrm{CAlg}\Big(\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big)\Big) \arrow[d, "\simeq"'] & \mathrm{CAlg}\Big(\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}\Big) \arrow[l, "\mathrm{CAlg}(\phi^*)"'] \arrow[d, "\simeq"] \\
\mathrm{Fun}\Big(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}\big(\mathcal{D}(k)\big)\Big) & \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\,\mathbb{T}} \arrow[l, "\phi^*"]
\end{tikzcd}$$

where the vertical equivalences are the canonical ones, together with uniqueness of adjoints, we obtain a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{CAlg}\Big(\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big)\Big) \arrow[r, "\mathrm{CAlg}(|-|)"] \arrow[d, "\simeq"'] & \mathrm{CAlg}\Big(\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}\Big) \arrow[d, "\simeq"] \\
\mathrm{Fun}\Big(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}\big(\mathcal{D}(k)\big)\Big) \arrow[r, "|-|"'] & \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\,\mathbb{T}}
\end{tikzcd}$$

We can thus identify the composition (6.13) with the following composition.

$$\begin{aligned}
\mathrm{CAlg}\big(\mathcal{D}(k)\big) &\xrightarrow{\mathrm{CAlg}(\mathrm{ev}_{\mathfrak{a}})^{-1}} \mathrm{CAlg}\Big(\mathrm{Alg}\big(\mathcal{D}(k)\big)\Big) \xrightarrow{\mathrm{CAlg}(\mathrm{B}^{\mathrm{cyc}}_{\bullet})} \mathrm{CAlg}\Big(\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big)\Big) \\
&\xrightarrow{\simeq} \mathrm{Fun}\Big(\mathbf{\Lambda}^{\mathrm{op}}, \mathrm{CAlg}\big(\mathcal{D}(k)\big)\Big) \xrightarrow{|-|} \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{\mathrm{B}\,\mathbb{T}}
\end{aligned} \tag{6.14}$$

By Definition 6.1.3.4, $|-|$ is left adjoint to ϕ^* and by Proposition 6.1.2.20 the composition of the first three functors of (6.14) is left adjoint to $\mathrm{ev}_{[0]_{\mathbf{\Lambda}}}$. It follows from composability of adjoints [HTT, 5.2.2.6] that the composition of all four functors of (6.14) is left adjoint to

$$\mathrm{ev}_{[0]_{\mathbf{\Lambda}}} \circ \phi^* \simeq \mathrm{ev}_{\phi([0]_{\mathbf{\Lambda}})} \simeq \mathrm{ev}_*$$

which is what needed to be shown. □

6.2.2.4. HH of commutative algebras as a pushout

The description of HH for commutative algebras from Proposition 6.2.2.4 allows us to derive the following alternative description that will be useful when comparing it to the classical standard Hochschild complex.

Proposition 6.2.2.5. *The functor*

$$\mathrm{HH}\colon \mathrm{CAlg}\big(\mathcal{D}(k)\big) \to \mathrm{CAlg}\big(\mathcal{D}(k)\big)$$

from Definition 6.2.2.1 is homotopic to the functor that maps a commutative algebra R to the pushout of[52]

$$\begin{tikzcd} R \amalg R \arrow[r, "\mathrm{id}_R \amalg \mathrm{id}_R"] \arrow[d, "\mathrm{id}_R \amalg \mathrm{id}_R"'] & R \\ R & \end{tikzcd} \tag{6.15}$$

in $\mathrm{CAlg}\big(\mathcal{D}(k)\big)$ *– the coproduct in the diagram is also to be taken in* $\mathrm{CAlg}\big(\mathcal{D}(k)\big)$ *and is hence by [HA, 3.2.4.7] given by the tensor product.* ♡

Proof. By Proposition 6.2.2.4 the functor HH is homotopic to $- \boxtimes \mathbb{T}$. The underlying space of $\mathbb{T}$ is a 1-circle, and there is thus a pushout diagram

$$\begin{tikzcd} * \amalg * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathbb{T} \end{tikzcd}$$

in $\mathcal{S}$. As $- \boxtimes -$ preserves colimits in each variable separately (see Section 6.2.2.2), the claim immediately follows using that $- \boxtimes * \simeq \mathrm{id}$. □

[52]Here is how to more rigorously define this functor. Let

$$\mathcal{I} = (\bullet \leftarrow \bullet \rightarrow \bullet) = [1] \amalg_{\{0\}} [1]$$

so that it suffices to construct a functor $\mathrm{CAlg}\big(\mathcal{D}(k)\big) \to \mathrm{Fun}(\mathcal{I}, \mathrm{CAlg}\big(\mathcal{D}(k)\big))$ that maps an object R to the diagram (6.15), for we can then compose this functor with the functor $\mathrm{colim}_{\mathcal{I}}$. Using the $\times$-Fun-adjunction, it suffices to construct a functor

$$\mathcal{I} \times \mathrm{CAlg}\big(\mathcal{D}(k)\big) \to \mathrm{CAlg}\big(\mathcal{D}(k)\big)$$

for which it suffices to produce a commutative diagram as follows.

$$\begin{tikzcd} \{0\} \times \mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[r] \arrow[d] & {[1]} \times \mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[d, dashed] \\ {[1]} \times \mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[r, dashed] & \mathrm{CAlg}\big(\mathcal{D}(k)\big) \end{tikzcd}$$

with the left vertical and top horizontal functor the inclusion. Each of the two other functors are to correspond to the natural transformation that sends R to $R \amalg R \xrightarrow{\mathrm{id} \amalg \mathrm{id}} R$, and taking the same functors there is an obvious filler for the diagram, so it suffices to construct this natural transformation.

The functor mapping R to $R \amalg R$ is the composition

$$\mathrm{CAlg}\big(\mathcal{D}(k)\big) \xrightarrow{\mathrm{const}} \mathrm{CAlg}\big(\mathcal{D}(k)\big)^{* \amalg *} \xrightarrow{\mathrm{colim}} \mathrm{CAlg}\big(\mathcal{D}(k)\big)$$

so as colim is left adjoint to the functor const (see [HTT, 4.2.4.3]) we obtain the required natural transformation as the counit of the adjunction.

6.2.2.5. HH of commutative algebras as a relative tensor product

As pushouts of commutative algebras can be calculated as relative tensor products, we obtain the following corollary of Proposition 6.2.2.5.

Corollary 6.2.2.6. *The functor*

$$\mathrm{HH}\colon\ \mathrm{CAlg}\big(\mathcal{D}(k)\big) \to \mathrm{CAlg}\big(\mathcal{D}(k)\big)$$

from Definition 6.2.2.1 is homotopic to the functor that maps a commutative algebra R to the relative tensor product in $\mathrm{CAlg}(\mathcal{D}(k))$

$$R \otimes_{R\otimes R} R$$

where the structure of R as a left and right $R \otimes R$-module arises from the morphism of commutative algebras

$$R \otimes R \simeq R \amalg R \xrightarrow{\mathrm{id}_R \amalg \mathrm{id}_R} R$$

– see Construction E.8.0.4 for more details on how to construct the necessary data to take the relative tensor product of of this. ♡

Proof. Follows immediately from combining Proposition 6.2.2.5 with Proposition E.8.0.5. □

Remark 6.2.2.7. If R is a commutative algebra in $\mathcal{D}(k)$, then the underlying morphism in $\mathcal{D}(k)$ of the morphism

$$R \otimes R \simeq R \amalg R \xrightarrow{\mathrm{id}_R \amalg \mathrm{id}_R} R$$

in $\mathrm{CAlg}(\mathcal{D}(k))$ can be identified with the multiplication morphism of R. This essentially follows from Proposition E.6.0.1[53]. ♢

6.2.3. Hochschild homology and relative tensor products

In this short section we show that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserves relative tensor products, which will be crucial later for calculating $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of certain quotients.

[53] Denote for the moment the functor

$$\mathrm{ev}_{\langle 1\rangle}\colon\ \mathrm{CAlg}\Big(\mathrm{CAlg}\big(\mathcal{D}(k)\big)\Big) \to \mathrm{CAlg}\big(\mathcal{D}(k)\big)$$

by $\mathrm{ev}'_{\langle 1\rangle}$ to distinguish it from the following functor.

$$\mathrm{ev}_{\langle 1\rangle}\colon\ \mathrm{CAlg}\big(\mathcal{D}(k)\big) \to \mathcal{D}(k)$$

Then the morphism in question is – as a morphism in $\mathrm{CAlg}(\mathcal{D}(k))$ – the multiplication morphism of the object R' in $\mathrm{CAlg}(\mathrm{CAlg}(\mathcal{D}(k)))$ corresponding to R under the equivalence $\mathrm{ev}'_{\langle 1\rangle}$. As $\mathrm{ev}_{\langle 1\rangle}$ is symmetric monoidal, it maps this morphism to the multiplication morphism of the commutative algebra in $\mathcal{D}(k)$ given by $\mathrm{CAlg}(\mathrm{ev}_{\langle 1\rangle})(R')$. We would like to identify this with R, and Proposition E.6.0.1 says that we can.

Proposition 6.2.3.1. *The functors* $\mathrm{HH}_{\mathbb{T}}$, $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$, *and* HH *from Definition 6.2.1.1 and Definition 6.2.1.2 preserve sifted colimits.*

In particular, all three functors being monoidal as well, they also preserve relative tensor products[54]*.* ♡

Proof. As $\mathcal{D}(k)$ is presentable symmetric monoidal by Proposition 4.3.2.1 (1), the symmetric monoidal structure on $\mathcal{D}(k)$ is in particular compatible with sifted colimits, and hence we can apply Proposition 6.1.2.21 to conclude that

$$\mathrm{B}^{\mathrm{cyc}}_{\bullet}\colon \mathrm{Alg}\big(\mathcal{D}(k)\big) \to \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big)$$

preserves sifted colimits. As a left adjoint, the geometric realization functor

$$|-|\colon \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

preserves all colimits, so in particular sifted colimits – see Definition 6.1.3.4 and [HTT, 5.2.3.5]. It thus follows that $\mathrm{HH}_{\mathbb{T}}$ and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserve sifted colimits, and as the forgetful functor $\mathrm{ev}_*\colon \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \to \mathcal{D}(k)$ preserves colimits by [HTT, 5.1.2.3] it also follows that HH preserves sifted colimits.

All three functors are monoidal by definition, so they also preserve relative tensor products by Proposition E.8.0.1. □

6.3. The standard Hochschild complex

In this section we review the classical definitions for Hochschild homology on the level of chain complexes. The main point is that if A is a differential graded algebra, then one can construct a strict mixed complex $\mathrm{C}(A)$ out of A, called the *standard Hochschild complex*, which represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(A)$. Similarly, when A is a *commutative* differential graded algebra, then the underlying chain complex of $\mathrm{C}(A)$ can be upgraded to a commutative differential graded algebra that represents $\mathrm{HH}(A)$.

We will start in Section 6.3.1 by reviewing the standard Hochschild complex for associative algebras, before treating the commutative case in Section 6.3.2. In Section 6.3.3 we will then discuss in what way $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ preserves relative tensor products, which will be relevant when we show that the standard Hochschild complex indeed represents Hochschild homology in Section 6.3.4.

6.3.1. The standard Hochschild complex for associative algebras

In Section 6.2.1 we defined a functor

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\colon \mathrm{Alg}\big(\mathcal{D}(k)\big) \to \mathcal{M}\mathrm{ixed}$$

[54] See Remark E.8.0.2 for a discussion of what the statement that those functors preserve relative tensor products means.

called *Hochschild homology*. This was a definition on the level of the ∞-category $\mathcal{D}(k)$. There is also a classical definition of Hochschild homology constructed on the level of chain complexes, and we will recall the main definitions in this section[55]. We use the book [Lod98] as well as [Hoy18] the main references for this material.

We will start in Section 6.3.1.1 by making concrete how the cyclic bar construction $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ looks like in the case of the symmetric monoidal category $\mathsf{Ch}(k)^{\mathrm{cof}}$. While in the definition of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ the next step would be the geometric realization functor for cyclic objects that would yield an object of $\left(\mathsf{Ch}(k)^{\mathrm{cof}}\right)^{\mathrm{B}\,\mathbb{T}}$, this is not sensible in this setting[56] – as $\mathsf{Ch}(k)^{\mathrm{cof}}$ is a 1-category, any functor $\mathrm{B}\,\mathbb{T} \to \mathsf{Ch}(k)^{\mathrm{cof}}$ factors though $\tau_{\leq 1}(\mathrm{B}\,\mathbb{T}) \simeq *$, so a $\mathbb{T}$-action on an object of $\mathsf{Ch}(k)^{\mathrm{cof}}$ yields no extra information. So in Section 6.3.1.2 we instead give a different construction that produces a strict mixed complex out of a cyclic object in chain complexes. We end in Section 6.3.1.3 by defining the standard Hochschild complex as the composite functor from $\mathrm{Alg}(\mathsf{Ch}(k))$ to Mixed.

6.3.1.1. The cyclic bar construction for chain complexes

$\mathsf{Ch}(k)$ is a symmetric monoidal category, so we can apply Definition 6.1.2.16 to obtain the cyclic bar construction functor $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$. The next proposition makes this functor more concrete.

Proposition 6.3.1.1. *The functor*

$$\mathrm{B}^{\mathrm{cyc}}_{\bullet}\colon\ \mathrm{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathrm{Fun}\Big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\Big)$$

from Definition 6.1.2.16 is given on a differential graded algebra A with cofibrant underlying complex by the following formulas[57]*.*

$$\begin{aligned}
\mathrm{B}^{\mathrm{cyc}}_{n}(A) &= A^{\otimes(n+1)}\\
d_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n \qquad \textit{for } i < n\\
d_n(a_0 \otimes \cdots \otimes a_n) &= (-1)^{\deg_{\mathsf{Ch}}(a_n)\cdot \sum_{i=0}^{n-1} \deg_{\mathsf{Ch}}(a_i)} a_n \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}\\
s_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n\\
t(a_0 \otimes \cdots \otimes a_n) &= (-1)^{\deg_{\mathsf{Ch}}(a_n)\cdot \sum_{i=0}^{n-1} \deg_{\mathsf{Ch}}(a_i)} a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}
\end{aligned}$$

In particular, the restriction of $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ to $\mathrm{Alg}(\mathrm{LMod}_k(\mathsf{Ab}))$ via the inclusion of chain complexes that are concentrated in degree 0[58] *can be identified with the functor defined in [Lod98, 6.1.12]*[59] ♡

[55] We will later see in Section 6.3.4.1 that the classical definition indeed represents the one from Section 6.2.1.

[56] Even without asking for the construction to be compatible with $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$.

[57] See Notation 6.1.1.12 for the notation we use here.

[58] This implies that the signs in the formulas above vanish.

[59] Compare also to [Lod98, 1.6.1, 2.1.0, and 2.5.4] – there are though some differences in the signs, see [Lod98, 6.1.2.2].

Proof. This amounts to unpacking the definition of the functors $-^\circ$ and V in Fact 6.1.1.13 and Fact 6.1.2.13 to see where the generators of $\mathbf{\Lambda}^{\mathrm{op}}$ are taken by $V \circ (-)^\circ$, and then applying Proposition 6.1.2.10[60]. □

6.3.1.2. Geometric realization of cyclic chain complexes

In Definition 6.2.1.2 we defined $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ as the composition of the cyclic bar construction with the geometric realization functor

$$\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \to \mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}}$$

defined in Definition 6.1.3.4 and the equivalence

$$\mathcal{D}(k)^{\mathrm{B}\,\mathbb{T}} \simeq \mathcal{M}\mathrm{ixed}$$

from Construction 5.4.0.1. There is also a classical way of obtaining a strict mixed complex out of a cyclic chain complex, as we recall now.

Construction 6.3.1.2 ([Hoy18, Section 2] and [Lod98, 2.5.10])**.** Let $X_\bullet$ be an object in $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k))$. We then define a number of new operators on $X_\bullet$ as follows.

$$\begin{aligned}
&\partial^X\colon X_n \to X_{n-1}, && \partial^X \coloneqq \sum_{i=0}^{n} (-1)^i d_i\\
&s_{-1}\colon X_n \to X_{n+1}, && s_{-1} \coloneqq t \circ s_n\\
&t'\colon X_n \to X_n, && t' \coloneqq (-1)^n t\\
&N\colon X_n \to X_n, && N \coloneqq \sum_{i=0}^{n} t'^i\\
&\mathrm{d}\colon X_n \to X_{n+1}, && \mathrm{d} \coloneqq \big(\mathrm{id} - t'\big) \circ s_{-1} \circ N
\end{aligned}$$

The operator ∂^X then satisfies $\partial^X \circ \partial^X = 0$ so that we can consider $X_\bullet$ together with ∂^X as a complex in $\mathsf{Ch}(k)$, i. e. a double complex[61], and hence can form the total complex, an object of $\mathsf{Ch}(k)$, by setting

$$\mathrm{Tot}\Big(X_\bullet, \partial^X\Big)_n \coloneqq \bigoplus_{i+j=n} (X_i)_j$$

and for x and element of $(X_i)_j$

$$\partial^{\mathrm{Tot}\left(X_\bullet, \partial^X\right)}(x) \coloneqq \partial^X(x) + (-1)^i \partial^{X_i}(x)$$

[60] The signs arise from the signs in the symmetry isomorphism of the symmetric monoidal structure on $\mathsf{Ch}(k)$, see Definition 4.1.2.1.

[61] To be precise, we set

$$X_n \coloneqq \begin{cases} X_{[n]_{\mathbf{\Lambda}}} & \text{if } n \geq 0\\ 0 & \text{otherwise}\end{cases}$$

so in particular, $(X_\bullet, \partial^X)$ is a upper half plane (or right half plane, depending on which way around one arranges the two indices) double complex.

as the boundary operator[62].

The operator d induces morphisms $\operatorname{Tot}(X_\bullet, \partial^X)_* \to \operatorname{Tot}(X_\bullet, \partial^X)_{*+1}$ that we also denote by d, and the identities holding in $\mathbf{\Lambda}$ (see Construction 6.1.1.7) imply that d makes $\operatorname{Tot}(X_\bullet, \partial^X)$ into a strict mixed complex[63], see for example the arguments in [Lod98, Section 2.1].

This construction is functorial, and we denote the resulting functor

$$\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)) \to \mathsf{Mixed}$$

by $|-|_{\mathsf{Mixed}}$. Composing with the forgetful functor that maps strict mixed complexes to their underlying chain complexes we obtain a functor

$$\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)) \to \mathsf{Ch}(k)$$

that we denote by $|-|_{\mathsf{Ch}}$. ♢

Warning 6.3.1.3. Our notation deviates from the notation used in most previous work. We use ∂ and d instead of b and B, which is the notation used in for example [Lod98] and [Hoy18], which are the sources we have otherwise followed in Construction 6.3.1.2. The notation ∂ is widely used for the boundary operator of a chain complex[64], and d fits better with the relation to the mixed complex of de Rham forms, which will be introduced in Section 7.1.

Apart from the change of notation, the various operators in Construction 6.3.1.2 agree with the definitions in [Hoy18, Section 2]. The definitions also agree with the definitions given in [Lod98, 2.5.10] if we restrict to cyclic objects in $\mathrm{LMod}_k(\mathsf{Ab})$ (via the inclusion as chain complexes concentrated in degree 0). While the formulas in [Lod98, 2.5.10] differ by some signs, those arise from the fact that Loday does not actually define a mixed complex from the input of a cyclic object in chain complexes, but of a *cyclic module* as defined in [Lod98, 2.5.1]. While the data of a cyclic module and a cyclic object in $\mathrm{LMod}_k(\mathsf{Ab})$ are isomorphic, the isomorphism introduces some signs, see [Lod98, 6.1.2.2]. After composing Loday's construction with the isomorphism between cyclic objects in $\mathrm{LMod}_k(\mathsf{Ab})$ and cyclic modules, the signs cancel. ♢

Proposition 6.3.1.4. *If $X_\bullet$ is a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathsf{Ch}(k)$ that is pointwise cofibrant, then $|X_\bullet|_{\mathsf{Ch}}$ is cofibrant as well.*

We thus obtain a commutative diagrams as follows

$$\begin{tikzcd}
\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}) \arrow[r, "|-|_{\mathsf{Mixed}}"] \arrow[d] & \mathsf{Mixed}_{\mathrm{cof}} \arrow[r, "\mathrm{ev}_{\mathrm{m}}"] \arrow[d] & \mathsf{Ch}(k)^{\mathrm{cof}} \arrow[d] \\
\operatorname{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)) \arrow[r, "|-|_{\mathsf{Mixed}}"'] & \mathsf{Mixed} \arrow[r, "\mathrm{ev}_{\mathrm{m}}"'] & \mathsf{Ch}(k)
\end{tikzcd}$$

[62] In the formula for the boundary operator, $\partial^X(x)$ is an element of $(X_{i-1})_j$ and $\partial^{X_i}(x)$ is an element of $(X_i)_{j-1}$.

[63] See Definition 4.2.1.2 and Remark 4.2.1.4 for the definition.

[64] So is d, but this would be very confusing when the mixed complex of de Rham forms shows up in Section 7.1.

where $\mathsf{Mixed}_{\mathrm{cof}}$ *is the full subcategory of* Mixed *spanned by those strict mixed complexes whose underlying chain complex is cofibrant (see Definition 4.2.1.2), and the vertical functors are (induced by) the inclusion of* $\mathsf{Ch}(k)^{\mathrm{cof}}$ *into* $\mathsf{Ch}(k)$*.* ♡

Proof. Let $X_\bullet$ be an object in $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}})$. Define a sequence

$$\ldots \to |X_\bullet|_{\mathsf{Ch}}^{\leq -1} \to |X_\bullet|_{\mathsf{Ch}}^{\leq 0} \to |X_\bullet|_{\mathsf{Ch}}^{\leq 1} \to \ldots$$

of sub-chain-complexes of $|X_\bullet|_{\mathsf{Ch}}$ by letting $|X_\bullet|_{\mathsf{Ch}}^{\leq m}$ be given by[65]

$$\left(|X_\bullet|_{\mathsf{Ch}}^{\leq m}\right)_n := \bigoplus_{i+j=n, i\leq m} (X_i)_j$$

which one should think of as taking the total complex of the brutal truncation of $X_\bullet$ to degrees less than or equal to m.

Note that $|X_\bullet|_{\mathsf{Ch}}^{\leq m} \cong 0$ for $m < 0$, and $|X_\bullet|_{\mathsf{Ch}}$ is the colimit of the above sequence of inclusions. It thus suffices to show that $|X_\bullet|_{\mathsf{Ch}}^{\leq 0}$ is cofibrant and that each inclusion $|X_\bullet|_{\mathsf{Ch}}^{\leq m} \to |X_\bullet|_{\mathsf{Ch}}^{\leq m+1}$ is a cofibration.

That $|X_\bullet|_{\mathsf{Ch}}^{\leq 0}$ is cofibrant follows immediately from the assumption, as there is an obvious isomorphism $|X_\bullet|_{\mathsf{Ch}}^{\leq 0} \cong X_0$. So now let m be a nonnegative integer. Then there is a pushout diagram as follows

$$\begin{tikzcd}
S^m \otimes X_{m+1} \arrow[r, "\partial^X"] \arrow[d, "i\otimes \mathrm{id}_{X_{m+1}}"'] & {|X_\bullet|_{\mathsf{Ch}}^{\leq m}} \arrow[d] \\
D^{m+1} \otimes X_{m+1} \arrow[r] & {|X_\bullet|_{\mathsf{Ch}}^{\leq m+1}}
\end{tikzcd}$$

where S^m and D^{m+1} are as in [Hov99, 2.3.3][66] and i is the inclusion, ∂^X is to be understood as mapping $1 \otimes x$ to $\partial^X(x)$, which is defined as in Construction 6.3.1.2, and the right vertical morphism is the inclusion. As X_{m+1} was assumed to be a cofibrant chain complex and i is a cofibration, it follows from $\mathsf{Ch}(k)$ being a symmetric monoidal model category that the left vertical morphism, and hence also the right vertical morphism, are cofibrations. □

Remark 6.3.1.5. Construction 6.3.1.2 is clearly compatible with respect to extension of scalars. Specifically, let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the symmetric monoidal functor $k' \otimes_k -$ from $\mathsf{Ch}(k)^{\mathrm{cof}}$ to $\mathsf{Ch}(k')^{\mathrm{cof}}$ (see Fact 4.1.5.1) induces an obvious commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}\left(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\right) \arrow[r, "|-|_{\mathsf{Mixed}}"] \arrow[d, "(k'\otimes_k -)_*"'] & \mathsf{Mixed}_{k,\mathrm{cof}} \arrow[d, "k'\otimes_k -"] \\
\mathrm{Fun}\left(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k')^{\mathrm{cof}}\right) \arrow[r, "|-|_{\mathsf{Mixed}}"'] & \mathsf{Mixed}_{k',\mathrm{cof}}
\end{tikzcd}$$

of 1-categories. ♢

[65] The boundary operator of $|X_\bullet|_{\mathsf{Ch}}$ never increases i or j, so this indeed defines a sub-chain-complex.

[66] So S^m is $k[m]$, and D^{m+1} is concentrated in degrees m and $m+1$, with the boundary operator from degree $m+1$ to degree m being id_k.

6.3.1.3. The standard Hochschild complex

Combining Sections 6.3.1.1 and 6.3.1.2 we obtain the following definition.

Definition 6.3.1.6. Composing the cyclic bar construction for associative algebras in $\mathsf{Ch}(k)^{\mathrm{cof}}$[67], with the functor $|-|_{\mathsf{Mixed}}$ from Construction 6.3.1.2 we obtain a functor

$$\mathrm{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathsf{Mixed}_{\mathrm{cof}}$$

that we denote by C and call the *standard Hochschild complex.* ♢

Remark 6.3.1.7. Combining functoriality of $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ (see Remark 6.1.2.17) and $|-|_{\mathsf{Mixed}}$ (see Remark 6.3.1.5) we can deduce that C is functorial in k. Concretely, if $\varphi\colon k \to k'$ is a morphism of commutative rings, then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"] \arrow[d, "k'\otimes_k -"'] & \mathsf{Mixed}_{k,\mathrm{cof}} \arrow[d, "k'\otimes_k -"] \\
\mathrm{Alg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"'] & \mathsf{Mixed}_{k',\mathrm{cof}}
\end{tikzcd}$$

in Cat. ♢

6.3.1.4. C for algebras concentrated in degree 0

In this section we discuss the standard Hochschild complex as defined in Definition 6.3.1.6 for k-algebras R with projective underlying k-module, which we consider as algebras in $\mathsf{Ch}(k)^{\mathrm{cof}}$ concentrated in degree 0.

Remark 6.3.1.8. The restriction of the standard Hochschild complex functor C as we defined it to k-algebras whose underlying k-module is projective agrees with the functor C defined in [Lod98], see [Lod98, Section 1.1, in particular 1.1.3, and section 2.1, in particular 2.1.7]. This follows from Proposition 6.3.1.1 and Warning 6.3.1.3. ♢

Going through the definitions, one obtains the following description.

Proposition 6.3.1.9. *Let R be a k-algebra with projective underlying k-module. Then the strict mixed complex $\mathrm{C}(R)$ is concentrated in nonnegative degrees and for $n \geq 0$ the following hold*[68]*.*

$$\begin{aligned}
\mathrm{C}_n(R) &= R^{\otimes(n+1)} \\
\partial(r_0 \otimes \cdots \otimes r_n) &= (-1)^n r_n \cdot r_0 \otimes r_1 \otimes \cdots \otimes r_{n-1} \\
&\qquad + \sum_{i=0}^{n-1} (-1)^i r_0 \otimes \cdots \otimes r_i \cdot r_{i+1} \otimes \cdots \otimes r_n
\end{aligned}$$

[67] See Proposition 6.3.1.1.
[68] For $n = 0$ we instead have $\partial(r_0) = 0$.

$$\begin{aligned}\mathrm{d}(r_0 \otimes \cdots \otimes r_n) = &\sum_{i=0}^{n} (-1)^{in} 1 \otimes r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1} \\ &+ \sum_{i=0}^{n} (-1)^{in} r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1}\end{aligned}$$

These formulas agree with the definitions used in [Lod98][69]. ♡

Proof. Follows immediately by unpacking the definitions in Proposition 6.3.1.1 and Construction 6.3.1.2. Let us go through the steps for the last formula in a bit more detail. We use that d is defined as $(\mathrm{id} - t') \circ s_{-1} \circ N$, and go through the application of each composition factor individually. $r_0 \otimes \cdots \otimes r_n$ is mapped by N to the following element

$$\sum_{i=0}^{n} (-1)^{in} r_{n+1-i} \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{n-i}$$

where the summand indexed by $i = 0$ is to be interpreted as $r_0 \otimes \cdots \otimes r_n$. Using that $(n+1)n$ is even, we can replace i by $n+1-i$ to rewrite the above expression as

$$\sum_{i=1}^{n+1} (-1)^{in} r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1}$$

which is also equal to the following sum, as the summand for $i = 0$ is equal to the one for $i = n + 1$.

$$\sum_{i=0}^{n} (-1)^{in} r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1}$$

The effect of applying s_{-1} can be described as inserting a tensor factor 1 at the start, so the above expression is mapped by s_{-1} to the following.

$$\sum_{i=0}^{n} (-1)^{in} 1 \otimes r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1}$$

Finally, applying $\mathrm{id} - t'$ we obtain

$$\begin{aligned}&\sum_{i=0}^{n} (-1)^{in} 1 \otimes r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1} \\ -(-1)^{n+1} &\sum_{i=0}^{n} (-1)^{in} r_{i-1} \otimes 1 \otimes r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-2}\end{aligned}$$

[69] For the boundary operator, see [Lod98, 1.1.1]. For the differential a formula is given in [Lod98, 2.1.7.3], which is though differing from our formula by the sign before the second sum, which is presumably due to a typo – the definition given in [Lod98, 2.1.7.1 and 2.1.0] yields the formula we have stated above. That there must be a typo in [Lod98] around this formula can also be seen by comparing with the formulas for $B(a_0)$ and $B(a_0, a_1)$ given just below [Lod98, 2.1.7.3], which are compatible with the sign as in the formula stated above, but not the sign in [Lod98, 2.1.7.3].

and after replacing i by $i-1$ in the second sum to remove the sign due to

$$-(-1)^{n+1}(-1)^n = 1$$

and using that the resulting summands for $i = 0$ and $i = n+1$ are equal we finally obtain the following.

$$\begin{aligned} &= \sum_{i=0}^{n} (-1)^{in} 1 \otimes r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1} \\ &+ \sum_{i=0}^{n} (-1)^{in} r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1} \end{aligned}$$ □

6.3.1.5. The normalized standard Hochschild complex

To simplify formulas it is often useful to divide out a particularly easy to describe acyclic subcomplex of $\mathrm{C}(R)$, spanned by elements of the form $r_0 \otimes \cdots \otimes r_n$ with one of the elements $r_1, \ldots, r_n$ being equal to 1. We only use this for the case where R is concentrated in degree 0 and refer to [Lod98, 1.1.14] for more details.

Proposition 6.3.1.10 ([Lod98, 1.1.14 and 1.1.15])**.** *Let R be a k-algebra with projective underlying k-module. We define $\overline{R}$ to be the quotient $R/(k \cdot 1)$ of k-modules, where $k \cdot 1$ is the k-submodule of R spanned by the unit 1. We will use the notation $\overline{r}$ for the image of an element r of R under the quotient map $R \to \overline{R}$. Define*

$$\overline{\mathrm{C}}_n(R) := \begin{cases} R \otimes \overline{R}^{\otimes n} & \textit{if } n \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

for integers n and note that $\overline{\mathrm{C}}_n(R)$ is a quotient of $\mathrm{C}_n(R)$.

Then the strict mixed complex structure of $\mathrm{C}(R)$ induces a strict mixed complex structure on $\overline{\mathrm{C}}(R)$ such that the following simplified formula holds for the differential.

$$\mathrm{d}(r_0 \otimes \overline{r_1} \otimes \cdots \otimes \overline{r_n}) = \sum_{i=0}^{n} (-1)^{in} 1 \otimes \overline{r_i} \otimes \cdots \otimes \overline{r_n} \otimes \overline{r_0} \otimes \cdots \otimes \overline{r_{i-1}}$$

Furthermore, the morphism of strict mixed complexes

$$\mathrm{C}(R) \to \overline{\mathrm{C}}(R), \qquad r_0 \otimes r_1 \otimes \cdots \otimes r_n \mapsto r_0 \otimes \overline{r_1} \otimes \cdots \otimes \overline{r_n}$$

determines a natural transformation $\mathrm{C} \to \overline{\mathrm{C}}$ of functors from the category of k-algebras with projective underlying k-module to Mixed *that is pointwise a quasiisomorphism.*

We call $\overline{\mathrm{C}}(R)$ the normalized standard Hochschild complex. ♡

Proof. That $\overline{\mathrm{C}}(R)$ obtains an induced chain complex structure is [Lod98, 1.6.4] and that the quotient morphism $\mathrm{C}(R) \to \overline{\mathrm{C}}(R)$ is a quasiisomorphism is shown in [Lod98, 1.1.15 and 1.6.5]. That these quotient morphisms assemble to a natural transformation as claimed follows directly from the definition.

That the kernel of $\mathrm{C}(R) \to \overline{\mathrm{C}}(R)$ is closed under d is clear by looking at the formula given for d in Proposition 6.3.1.9, and the expression for the induced operator d on $\overline{\mathrm{C}}(R)$ also follows immediately. See also [Lod98, 2.1.9]. □

Remark 6.3.1.11. Functoriality of C with respect to change of scalars as discussed in Remark 6.3.1.7 passes to $\overline{\mathrm{C}}$. In particular, for $\varphi\colon k \to k'$ a morphism of commutative rings there exists a dashed natural isomorphism fitting into a commutative diagram

$$\begin{tikzcd} \mathrm{C}(k' \otimes_k -) \arrow[r] \arrow[d, "\cong"'] & \overline{\mathrm{C}}(k' \otimes_k -) \arrow[d, dashed, "\cong"] \\ k' \otimes_k \mathrm{C}(-) \arrow[r] & k' \otimes_k \overline{\mathrm{C}}(-) \end{tikzcd}$$

of functors from the category k-algebras with projective underlying k-module to $\mathsf{Mixed}_{k'}$. The top and bottom natural transformations are (induced by) the ones from Proposition 6.3.1.10 and the left natural isomorphism is the one from Remark 6.3.1.7. ♢

6.3.2. The standard Hochschild complex for commutative algebras

The functor

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\colon\ \mathrm{Alg}\big(\mathcal{D}(k)\big) \to \mathcal{M}\mathrm{ixed}$$

is monoidal and hence induces a functor on ∞-categories of (associative) algebras. Unfortunately, the standard Hochschild complex functor

$$\mathrm{C}\colon\ \mathrm{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathsf{Mixed}_{\mathrm{cof}}$$

that was defined in Definition 6.3.1.6 is *not* monoidal and not even lax or colax monoidal, see [Lod98, 4.3.1] and [Kas87]. To get around this for Künneth-type-formulas, one can employ a weakened notion of morphism between strict mixed complexes that is called *strongly homotopy linear map* in [Kas87] and *S-morphism* in [Lod98] – see [Kas87, 2.2] and [Lod98, 2.5.14]. This is a morphism of underlying chain complexes that need not strictly commute with d, but only up to specified homotopy, which in turn also does not need to strictly commute with d, but up to specified homotopy, and so on. For a more detailed discussion of strongly homotopy linear morphisms see Section 4.2.3.

We take the necessity to consider these kind of sequences of higher homotopies as a hint that if one is interested in both the mixed structure as well as (symmetric) monoidal structure, then one should work at the level of ∞-categories and consider the functor $\mathrm{HH}_{\mathbb{T}}$. From this perspective, that C may not be fully adequate to consider both mixed and multiplicative structures can also be expected from the fact that while $\mathrm{HH}_{\mathbb{T}}$ and HH are symmetric monoidal, $\mathrm{HH}_{\mathrm{Mixed}}$ has only been shown to be (associatively) monoidal – so it would be unexpected for C as a functor to $\mathsf{Mixed}_{\mathrm{cof}}$ to be *symmetric* monoidal[70].

To nevertheless be able to do some calculations on the level of chain complexes regarding multiplicative structures, we forget about the strict mixed complex structure, and only consider C as a functor to $\mathsf{Ch}(k)^{\mathrm{cof}}$.

To bring the standard Hochschild complex functor C as a functor to $\mathsf{Ch}(k)^{\mathrm{cof}}$ into a form that is more amenable for our purposes, we discuss the *bar resolution* $\mathrm{C}^{\mathrm{Bar}}(A)$ of an associative algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$ in Section 6.3.2.1, which will allow us to rewrite $\mathrm{C}(A)$ as a relative tensor product $\mathrm{C}(A) \cong A \otimes_{A\otimes A^{\mathrm{op}}} \mathrm{C}^{\mathrm{Bar}}(A)$ in Section 6.3.2.2. We will also show that as a left-$A \otimes A^{\mathrm{op}}$-module, $\mathrm{C}^{\mathrm{Bar}}(A)$ is a *cofibrant* replacement of A, which will be relevant in Section 6.3.4.2, where we compare the standard Hochschild complex to HH. In Section 6.3.2.3 we then introduce the *shuffle product* on $\mathrm{C}^{\mathrm{Bar}}(A)$, and upgrade all the relevant constructions to commutative algebras – provided that A itself was commutative. This will allow us to describe the standard Hochschild complex of a commutative differential graded algebra with cofibrant underlying chain complex as an object of $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$ in Section 6.3.2.4.

6.3.2.1. The bar resolution

In this section we introduce the bar resolution, that will be used in Section 6.3.2.2 below to give an alternative description of the standard Hochschild complex of Definition 6.3.1.6. We closely follow [Lod98, 1.1.11 to 1.1.13], though we also consider differential graded algebras that are not concentrated in degree 0.

Construction 6.3.2.1. [Lod98, 1.1.11 to 1.1.13] Let A be an associative algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$.

We let $\mathrm{Bar}_A(A,A)_\bullet$ be the chain complex in $\mathsf{Ch}(k)$ (so a double complex) that is determined by the following formulas.

$$\mathrm{Bar}_A(A,A)_n := A \otimes A^{\otimes n} \otimes A$$

$$\partial^{\mathrm{Bar}_A(A,A)_\bullet}(a_0 \otimes \cdots \otimes a_{n+1}) := \sum_{i=0}^{n} (-1)^i (a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_{n+1})$$

We then define $\mathrm{C}^{\mathrm{Bar}}(A)$, called the *bar resolution* of A, to be the total complex of $\mathrm{Bar}_A(A,A)_\bullet$, so we let

$$\mathrm{C}^{\mathrm{Bar}}(A)_n := \bigoplus_{i+j=n} \left(\mathrm{Bar}_A(A,A)_i\right)_j = \bigoplus_{i+j=n} \left(A^{\otimes(i+2)}\right)_j$$

and for a an element of $\left(\mathrm{Bar}_A(A,A)_i\right)_j$ we define the boundary operator as follows.

$$\partial^{\mathrm{C}^{\mathrm{Bar}}(A)}(a) := \partial^{\mathrm{Bar}_A(A,A)_\bullet}(a) + (-1)^i \partial^{A\otimes A^{\otimes i}\otimes A}(a)$$

[70] At least in a homotopically meaningful way that is compatible with $\mathrm{HH}_{\mathbb{T}}$.

Note that if A is concentrated in degree 0, then $\mathrm{C}^{\mathrm{Bar}}(A)$ is precisely the complex C_*^{bar} defined in [Lod98, 1.1.11].

There are two important extra pieces of structure regarding $\mathrm{C}^{\mathrm{Bar}}(A)$ that we will also need.

The first is that there is a natural morphism of chain complexes $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ that is defined by the formula

$$(a_0 \otimes \cdots \otimes a_{i+1}) \mapsto \begin{cases} a_0 \cdot a_1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

an again, this is precisely the augmentation of C_*^{bar} as defined in [Lod98, 1.1.11] if A is concentrated in degree 0.

The second extra piece of structure is that $\mathrm{C}^{\mathrm{Bar}}(A)$ can be given the structure of a left module over $A \otimes A^{\mathrm{op}}$, where A^{op} refers to the opposite algebra of A, i.e. the differential graded algebra with the same underlying chain complex, but if we denote the multiplication in A with $\cdot$ and in A^{op} with $\star$, then $\star$ is defined as $a \star a' := (-1)^{\deg_{\mathsf{Ch}}(a) \cdot \deg_{\mathsf{Ch}}(a')} a' \cdot a$. The left module structure is then defined via the following formula.

$$\begin{gathered}\left(a \otimes a'\right) \cdot \left(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}\right) := \\ (-1)^{\left(\sum_{i=0}^{n+1} \deg_{\mathsf{Ch}}(a_i)\right) \deg_{\mathsf{Ch}}(a')} \left((a \cdot a_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes (a_{n+1} \cdot a')\right)\end{gathered}$$

One can similarly define a left-$A \otimes A^{\mathrm{op}}$-module structure on A, via

$$\left(a \otimes a'\right) \cdot a'' := (-1)^{\deg_{\mathsf{Ch}}(a') \deg_{\mathsf{Ch}}(a'')} a \cdot a'' \cdot a'$$

and this makes the morphism of chain complexes $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ into a morphism of left-$A \otimes A^{\mathrm{op}}$-modules. This structure is again (for A concentrated in degree 0) exactly the one considered in [Lod98, 1.1.13].

The above constructions can be summarized in the following diagram

$$\begin{tikzcd} \mathrm{Alg}\left(\mathsf{Ch}(k)^{\mathrm{cof}}\right) \arrow[rr, bend left, "\mathrm{C}^{\mathrm{Bar}}"] \arrow[rr, bend right, "A \mapsto A"'] \arrow[rr, phantom, "\Downarrow"] \arrow[dr, "A \mapsto A \otimes A^{\mathrm{op}}"'] & & \mathrm{LMod}\left(\mathsf{Ch}(k)\right) \arrow[dl] \\ & \mathrm{Alg}(\mathsf{Ch}(k)) & \end{tikzcd}$$

where the functor on the right is the forgetful functor, the bottom functor at the top maps A to A considered as a left-$A \otimes A^{\mathrm{op}}$-module as described above, and the natural transformation at the top lies over the identity natural transformation of $A \mapsto A \otimes A^{\mathrm{op}}$.
$\diamondsuit$

To show that the terminology "bar *resolution*" is reasonable, we will now prove that $\mathrm{C}^{\mathrm{Bar}}(A)$ is cofibrant as a left-$A \otimes A^{\mathrm{op}}$-module, as well as quasiisomorphic to A.

Proposition 6.3.2.2 ([Lod98, 1.1.12])**.** *Let A be an associative algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$. Then the morphism*

$$\mathrm{C}^{\mathrm{Bar}}(A) \to A$$

of chain complexes constructed in Construction 6.3.2.1 is a quasiisomorphism. ♡

Proof. The proof is an immediate generalization of the proof of [Lod98, 1.1.12], though we need to add some signs to account for elements of A in odd degrees. So let ϕ denote the morphism $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ and let $\psi\colon A \to \mathrm{C}^{\mathrm{Bar}}(A)$ be the morphism of chain complexes that maps a to $1 \otimes a$. Then $\phi \circ \psi = \mathrm{id}_A$, so it suffices to construct a homotopy h between $\mathrm{id}_{\mathrm{C}^{\mathrm{Bar}}(A)}$ and $\psi \circ \phi$. For this, define h via

$$h(a_0 \otimes \cdots \otimes a_{n+1}) := 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$$

by k-linearly extending.

If $n > 0$ we then have

$$\begin{aligned}
&\left(\partial^{\mathrm{C}^{\mathrm{Bar}}(A)} \circ h + h \circ \partial^{\mathrm{C}^{\mathrm{Bar}}(A)}\right)(a_0 \otimes \cdots \otimes a_{n+1}) \\
&= 1 \cdot a_0 \otimes \cdots \otimes a_{n+1} - \sum_{i=0}^{n} (-1)^i (1 \otimes a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_{n+1}) \\
&\quad + (-1)^{n+1} \sum_{i=0}^{n+1} (-1)^{\sum_{j=0}^{i-1} \deg_{\mathsf{Ch}}(a_j)} \left(1 \otimes a_0 \otimes \cdots \otimes \partial^A(a_i) \otimes \cdots \otimes a_{n+1}\right) \\
&\quad + \sum_{i=0}^{n} (-1)^i (1 \otimes a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_{n+1}) \\
&\quad + (-1)^{n} \sum_{i=0}^{n+1} (-1)^{\sum_{j=0}^{i-1} \deg_{\mathsf{Ch}}(a_j)} \left(1 \otimes a_0 \otimes \cdots \otimes \partial^A(a_i) \otimes \cdots \otimes a_{n+1}\right) \\
&= a_0 \otimes \cdots \otimes a_{n+1}
\end{aligned}$$

while for $n = 0$ the third term does not appear, so that we obtain

$$\begin{aligned}
&\left(\partial^{\mathrm{C}^{\mathrm{Bar}}(A)} \circ h + h \circ \partial^{\mathrm{C}^{\mathrm{Bar}}(A)}\right)(a_0 \otimes a_1) \\
&= a_0 \otimes a_1 - 1 \otimes a_0 \cdot a_1 \\
&= (\mathrm{id} - \psi \circ \phi)(a_0 \otimes a_1)
\end{aligned}$$

which shows that h is a homotopy as required. □

Proposition 6.3.2.3. *Let A be an associative algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$. Then $\mathrm{C}^{\mathrm{Bar}}(A)$ as defined in Construction 6.3.2.1 is cofibrant as a left-$A \otimes A^{\mathrm{op}}$-module with respect to the model structure of Theorem 4.2.2.1.*

In particular, the underlying chain complex of $\mathrm{C}^{\mathrm{Bar}}(A)$ is cofibrant. ♡

Proof. Let us begin by noting that the second claim, that the underlying chain complex of $\mathrm{C}^{\mathrm{Bar}}(A)$ is cofibrant, follows from the first claim by applying Theorem 4.2.2.1 (8), which is applicable as the underlying chain complex of A is cofibrant by assumption.

Let $\mathrm{Bar}_A^{\leq m}(A,A)_\bullet$ be the chain complex in $\mathsf{Ch}(k)$ defined as the brutal truncation to degrees smaller or equal to m of $\mathrm{Bar}_A(A,A)_\bullet$ from Construction 6.3.2.1, i. e.

$$\mathrm{Bar}_A^{\leq m}(A,A)_n \coloneqq \begin{cases} A \otimes A^{\otimes n} \otimes A & \text{if } n \leq m \\ 0 & \text{otherwise} \end{cases}$$

and with boundary operator defined by the same formula as in Construction 6.3.2.1.

We then let $\mathrm{C}^{\mathrm{Bar}}_{\leq m}(A)$ be the total complex of $\mathrm{Bar}_A^{\leq m}(A,A)_\bullet$, which concretely means that $\mathrm{C}^{\mathrm{Bar}}_{\leq m}(A)$ is given in level n by $\bigoplus_{i+j=n, i\leq m}(A^{\otimes(i+2)})_j$.

Note that the left-$A\otimes A^{\mathrm{op}}$-module structure restricts from $\mathrm{C}^{\mathrm{Bar}}(A)$ to $\mathrm{C}^{\mathrm{Bar}}_{\leq m}(A)$, and $\mathrm{C}^{\mathrm{Bar}}(A)$ is the colimit of the sequence

$$\mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A) \to \mathrm{C}^{\mathrm{Bar}}_{\leq 1}(A) \to \mathrm{C}^{\mathrm{Bar}}_{\leq 2}(A) \to \dots$$

so that it suffices to show that $\mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A)$ is cofibrant and each of the morphisms

$$\mathrm{C}^{\mathrm{Bar}}_{\leq m}(A) \to \mathrm{C}^{\mathrm{Bar}}_{\leq m+1}(A)$$

is a cofibration.

For $\mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A)$ we note that

$$\mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A) \cong A \otimes A^{\mathrm{op}}$$

as left-$A\otimes A^{\mathrm{op}}$-modules, so $\mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A)$ is isomorphic to the free left-$A\otimes A^{\mathrm{op}}$-module generated by k and hence cofibrant, as k is cofibrant in $\mathsf{Ch}(k)$.

For $m \geq 0$ there is an evident pushout diagram in $\mathsf{Ch}(\mathsf{Ch}(k))$

$$\begin{tikzcd} A^{\otimes m+3} \otimes S^m \arrow[r] \arrow[d, "\mathrm{id}_{A^{\otimes m+3}} \otimes i'"] & \mathrm{Bar}_A^{\leq m}(A,A)_\bullet \arrow[d] \\ A^{\otimes m+3} \otimes D^{m+1} \arrow[r] & \mathrm{Bar}_A^{\leq (m+1)}(A,A)_\bullet \end{tikzcd}$$

where $A^{\otimes m+3}$ is concentrated in degree 0 with respect to the "outer" chain degree, S^m is the chain complex in $\mathsf{Ch}(k)$ that is concentrated in degree m, where it is $k[0]$, the complex D^{m+1} is concentrated in degrees m and $m+1$, where it is $k[0]$, with the boundary operator from degree $m+1$ to degree m the identity morphism, and i' is the inclusion.

As the formation of the total complex preserves pushouts, we obtain a pushout diagram in $\mathsf{Ch}(k)$. It is not difficult to see that that square can be considered as a commutative square of left-$A\otimes A^{\mathrm{op}}$-modules of the following form

$$\begin{tikzcd} \mathrm{Free}^{\mathrm{LMod}_{A\otimes A^{\mathrm{op}}}}\left(A^{\otimes m+1} \otimes S^m\right) \arrow[r] \arrow[d, "\mathrm{Free}^{\mathrm{LMod}_{A\otimes A^{\mathrm{op}}}}\left(\mathrm{id}_{A^{\otimes m+1}} \otimes i\right)"'] & \mathrm{C}^{\mathrm{Bar}}_{\leq m}(A) \arrow[d] \\ \mathrm{Free}^{\mathrm{LMod}_{A\otimes A^{\mathrm{op}}}}\left(A^{\otimes m+1} \otimes D^{m+1}\right) \arrow[r] & \mathrm{C}^{\mathrm{Bar}}_{\leq m+1}(A) \end{tikzcd}$$

where i is the inclusion of $S^m = k[m]$ into the chain complex concentrated in degrees m and $m+1$ that is given by $(D^{m+1})_m = (D^{m+1})_{m+1} = k$, with boundary operator the identity, see [Hov99, 2.3.3]. As we assumed A to have cofibrant underlying complex and i is a cofibration in $\mathsf{Ch}(k)$, the tensor product $\mathrm{id}_{A^{\otimes m+1}} \otimes i$ is a cofibration as well, and it then follows that $\mathrm{Free}^{\mathrm{LMod}_{A\otimes A^{\mathrm{op}}}}(\mathrm{id}_{A^{\otimes m+1}} \otimes i)$ is a cofibration of left-$A \otimes A^{\mathrm{op}}$-modules, and thus so is the inclusion $\mathrm{C}^{\mathrm{Bar}}_{\leq m}(A) \to \mathrm{C}^{\mathrm{Bar}}_{\leq m+1}(A)$. □

6.3.2.2. C as a relative tensor product

Using the bar resolution from Section 6.3.2.1 we can now give a different description of the standard Hochschild complex that we defined in Section 6.3.1.3.

Proposition 6.3.2.4 ([Lod98, 1.1.13]). *The standard Hochschild complex functor*

$$\mathrm{C}\colon \operatorname{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathsf{Ch}(k)^{\mathrm{cof}}$$

as defined in Definition 6.3.1.6 (but postcomposed with the forgetful functor from $\mathsf{Mixed}_{\mathrm{cof}}$ to $\mathsf{Ch}(k)^{\mathrm{cof}}$) is naturally isomorphic to the functor[71]

$$A \mapsto A \otimes_{A\otimes A^{\mathrm{op}}} \mathrm{C}^{\mathrm{Bar}}(A)$$

where $\mathrm{C}^{\mathrm{Bar}}(A)$ is as in Construction 6.3.2.1 and A is a right-$A \otimes A^{\mathrm{op}}$-module via the action defined by $a \cdot (a' \otimes a'') \coloneqq a'' a a'$. ♡

Proof. Follows from unpacking the definitions and using isomorphisms of the following form.

$$\begin{aligned} A \otimes_{A\otimes A^{\mathrm{op}}} \big(A \otimes A^{\otimes n} \otimes A\big) &\cong A \otimes A^{\otimes n} \\ a \otimes (a_0 \otimes \cdots \otimes a_{n+1}) &\mapsto a_{n+1} \cdot a \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_n \end{aligned}$$

□

6.3.2.3. The shuffle product

In this section we assume that A is a *commutative* algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$, and upgrade the bar resolution $\mathrm{C}^{\mathrm{Bar}}(A)$ from Section 6.3.2.1 to a commutative differential graded algebra.

Definition 6.3.2.5 ([Lod98, 4.2.1] and [BACH, 1.2]). Let n and m be nonnegative integers. Then we define

$$\begin{aligned} B_{n,m} &\coloneqq \big\{\, \sigma \in \Sigma_{n+m} \;\big|\; \sigma(1) < \cdots < \sigma(n) \text{ and } \sigma(n+1) < \cdots < \sigma(n+m) \,\big\} \\ &= \big\{\, \sigma \in \Sigma_{n+m} \;\big|\; \sigma \text{ preserves the ordering of } \{1, \ldots, n\} \text{ and } \{n+1, \ldots, n+m\} \,\big\} \end{aligned}$$

where Σ_{n+m} is the symmetric group on $n+m$ elements, see Section 2.3 (34). ◇

[71] We take the relative tensor product in $\mathsf{Ch}(k)$. That the relative tensor product is isomorphic to $\mathrm{C}(A)$ shows that it is indeed cofibrant and can thus be considered as functor to $\mathsf{Ch}(k)^{\mathrm{cof}}$.

Construction 6.3.2.6 ([Lod98, E.4.2.2] and [BACH, 1.2])**.** Let A be a commutative algebra in $\mathsf{Ch}(k)^{\mathrm{cof}}$. We then define a product on $\mathrm{C}^{\mathrm{Bar}}(A)$ from Section 6.3.2.1 by k-linearly extending the following formula

$$\begin{aligned}&(a_l \otimes a_1 \otimes \cdots \otimes a_n \otimes a_r) \cdot (a'_l \otimes a_{n+1} \otimes \cdots \otimes a_{n+m} \otimes a'_r)\\ &\coloneqq \sum_{\sigma \in B_{n,m}} (-1)^s \cdot (a_l \cdot a'_l \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n+m)} \otimes a_r a'_r)\end{aligned}$$

where s is a sign (dependent on σ etc.) defined as follows.

$$\begin{aligned} s = \operatorname{sgn}(\sigma) &+ \left(\deg_{\mathsf{Ch}}(a_r) \cdot \sum_{i=n+1}^{n+m} \deg_{\mathsf{Ch}}(a_i)\right) + \left(\deg_{\mathsf{Ch}}(a'_l) \cdot \left(\sum_{i=1}^{n} \deg_{\mathsf{Ch}}(a_i)\right)\right)\\ &+ \left(\deg_{\mathsf{Ch}}(a_r) \cdot \deg_{\mathsf{Ch}}(a'_l)\right) + \left(\sum_{i=1}^{n+m} \deg_{\mathsf{Ch}}(a_i) \cdot \left(\sum_{i<j, \sigma(j)<\sigma(i)} \deg_{\mathsf{Ch}}(a_j)\right)\right)\end{aligned}$$

To make the formula more intuitive, let us provide the following interpretation. The summand indexed by σ should be thought of as moving a_i, which previously occupied what we might describe as "slot i" in the tensor product to "slot $\sigma(i)$" – this explains why σ^{-1} rather than σ occurs in the indices. Moving the a_i past each other then incurs signs coming from the symmetry isomorphism in $\mathsf{Ch}(k)$ (see Definition 4.1.2.1), and this is how the last summand of s arises. The other three summands of s involving chain degrees arise from moving a_r and a'_l to their correct positions. Finally, $\operatorname{sgn}(\sigma)$ is needed for compatibility with the part of the boundary operator coming from $\partial^{\mathrm{Bar}_A(A,A)_\bullet}$ – see Construction 6.3.2.1.

A tedious, but straightforward, calculation shows that the above multiplication is compatible with the boundary operator as well as associative and commutative, and with unit $1 \otimes 1$, making $\mathrm{C}^{\mathrm{Bar}}(A)$ into an object of $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$ (see Proposition 6.3.2.3 for cofibrancy of the underlying chain complex). Let us just mention one aspect of the required calculations when checking that the multiplication is compatible with the boundary operator. The boundary operator has two summands, with one arising from $\partial^{\mathrm{Bar}_A(A,A)_\bullet}$. With regards to that summand, multiplying first and then applying the boundary operator results in (a priori) extra summands (compared to applying the boundary operator first and then multiplying), where originally non-neighboring elements have been multiplied together. However, these summands always arise in pairs from two elements of $B_{n,m}$ that only differ by a transposition, and using that A is commutative one can see that they always cancel each other out. The rest of the needed calculations are mostly checking that the signs match.

With respect to this commutative algebra structure on $\mathrm{C}^{\mathrm{Bar}}(A)$, it is straightforward to check that the morphism $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ from Construction 6.3.2.1 becomes a morphism in $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$.

Furthermore, the inclusion of $A \otimes A \cong \mathrm{C}^{\mathrm{Bar}}_{\leq 0}(A)$ (see the proof of Proposition 6.3.2.3 for this notation) into $\mathrm{C}^{\mathrm{Bar}}(A)$ becomes a morphism of commutative algebras as well, and the left-$A \otimes A$-module structure[72] on $\mathrm{C}^{\mathrm{Bar}}(A)$ that was discussed in Construction 6.3.2.1 can be identified with the one induced from this morphism of commutative algebras.

The left-$A \otimes A$-module structure on A considered in Construction 6.3.2.1 can similarly be identified with the one arising from the morphism of commutative algebras $A \otimes A \to A$ that is given by the multiplication morphism. That the morphism $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ is a morphism of left-$A \otimes A$-modules is then reflected in the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{Bar}}(A) \arrow[rr] & & A \\
& A \otimes A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

of commutative algebras in $\mathsf{Ch}(k)$.

We can thus summarize these constructions in the commutative diagram

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{Bar}} \arrow[rr] & & \mathrm{id}_{\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})} \\
& A \mapsto A \otimes A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

of natural transformations between endofunctors of $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$.

Finally, note that the *right*-$A \otimes A$-module structure on A considered in Proposition 6.3.2.4 can also be identified with the one arising from the morphism of commutative algebras $A \otimes A \to A$ considered above. ♢

6.3.2.4. C for commutative algebras

Combining the description of the standard Hochschild complex as a relative tensor product with the bar resolution in Section 6.3.2.2 and the commutative algebra structure on the bar resolution constructed in Construction 6.3.2.6, we can now upgrade the standard Hochschild complex for commutative algebras to an object of $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$.

Proposition 6.3.2.7. *The composition of the forgetful functor*

$$\mathrm{CAlg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathrm{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)$$

with the standard Hochschild complex functor

$$\mathrm{C}\colon\ \mathrm{Alg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathsf{Mixed}_{\mathrm{cof}}$$

[72] As A is commutative we have $A = A^{\mathrm{op}}$.

from Definition 6.3.1.6 and the forgetful functor $\mathsf{Mixed}_{\mathrm{cof}} \to \mathsf{Ch}(k)^{\mathrm{cof}}$ *factors through* $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$*, so that we obtain a commutative diagram*

$$\begin{tikzcd} \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr, "\mathrm{C}"] \arrow[d] & & \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[d] \\ \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"'] & \mathsf{Mixed}_{\mathrm{cof}} \arrow[r] & \mathsf{Ch}(k)^{\mathrm{cof}} \end{tikzcd}$$

where we denote the lift by C *as well, and all the unlabeled functors are the respective forgetful functors.*

Furthermore, the functor

$$\mathrm{C}\colon \mathrm{CAlg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathrm{CAlg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)$$

is given by $A \mapsto A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)$*, where the* $A \otimes A$*-module structures on* A *and* $\mathrm{C}^{\mathrm{Bar}}(A)$ *arise via the natural transformations of functors to* $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$ *discussed in Construction 6.3.2.6 and the relative tensor product is taken in* $\mathrm{CAlg}(\mathsf{Ch}(k))$*.* ♡

Proof. Follows immediately from Proposition 6.3.2.4 and Construction 6.3.2.6 using that the symmetric monoidal forgetful functor from $\mathrm{CAlg}(\mathsf{Ch}(k))$ to $\mathsf{Ch}(k)$ preserves relative tensor products[73]. □

Remark 6.3.2.8. Going through the definition, it is straightforward to check that the natural isomorphisms encoding functoriality in k of C of associative algebras as described in Remark 6.3.1.7 are multiplicative after restricting to commutative differential graded algebras. So concretely, if $\varphi\colon k \to k'$ is a morphism of commutative rings, then there is a commutative diagram

$$\begin{tikzcd} \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"] \arrow[d, "k'\otimes_k -"'] & \mathrm{CAlg}\big(\mathsf{Mixed}_{k,\mathrm{cof}}\big) \arrow[d, "k'\otimes_k -"] \\ \mathrm{CAlg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"'] & \mathrm{CAlg}\big(\mathsf{Mixed}_{k',\mathrm{cof}}\big) \end{tikzcd}$$

lifting the commutative diagram from Remark 6.3.1.7. ♢

6.3.2.5. $\overline{\mathrm{C}}$ for commutative algebras concentrated in degree 0

Like in Section 6.3.1.4, we unpack the commutative algebra structure on the standard Hochschild complex $\mathrm{C}(R)$ in the case that R is concentrated in degree 0.

[73] See Proposition E.8.0.1 and [HA, 3.2.3.1 (4)]. Note that in 1-categories, geometric realizations – i. e. colimits over $\mathbf{\Delta}^{\mathrm{op}}$ – are calculated as coequalizers (see [Rie14, 8.3.8]), so that relative tensor products are the “classical” ones.

Definition 6.3.2.9 ([Lod98, 1.3.4])**.** Let R be a commutative k-algebra and $n \geq 0$ an integer. Then we define an action of the symmetric group Σ_n on $\mathrm{C}_n(R)$ as follows. For $r_0, \dots, r_n$ elements of R, we define the action of σ on $r_0 \otimes r_1 \otimes \cdots \otimes r_n$ as

$$\sigma \cdot (r_0 \otimes r_1 \otimes \cdots \otimes r_n) \coloneqq r_0 \otimes r_{\sigma^{-1}(1)} \otimes \cdots \otimes r_{\sigma^{-1}(n)}$$

and extend this k-linearly to an action of Σ_n on $\mathrm{C}_n(R)$. An action of Σ_n on $\overline{\mathrm{C}}_n(R)$ is defined analogously. ♢

Proposition 6.3.2.10. *Let R be a commutative k-algebra with projective underlying k-module. Then the unit 1 of R, considered as an element of $\mathrm{C}(R)_0$, is the unit of the commutative algebra structure on $\mathrm{C}(R)$, and the following formula holds for the multiplication*[74]*.*

$$\begin{aligned}&(r_0 \otimes r_1 \otimes \cdots \otimes r_n) \cdot (r_0' \otimes r_{n+1} \otimes \cdots \otimes r_{n+m})\\ &= \sum_{\sigma \in B_{n,m}} \operatorname{sgn}(\sigma) \cdot \sigma \cdot (r_0 \cdot r_0' \otimes r_1 \otimes \cdots \otimes r_{n+m})\end{aligned}$$ ♡

Proof. Follows directly from Construction 6.3.2.6 and Proposition 6.3.2.7. □

We also obtain an induced multiplication on the normalized standard Hochschild complex.

Proposition 6.3.2.11. *Let R be a commutative k-algebra with projective underlying k-module. Then the commutative algebra structure on $\mathrm{C}(R)$ induces a commutative algebra structure on $\overline{\mathrm{C}}(R)$ that makes the quotient morphism*

$$\mathrm{C}(R) \to \overline{\mathrm{C}}(R)$$

into a morphism in $\mathrm{CAlg}(\mathsf{Ch}(k))$. ♡

Proof. Follows immediately from Proposition 6.3.1.10 and Proposition 6.3.2.10. □

Remark 6.3.2.12. Given a morphism of commutative rings $\varphi\colon k \to k'$, the diagram of natural transformations

$$\begin{tikzcd}
\mathrm{C}(k' \otimes_k -) \arrow[r] \arrow[d, "\cong"'] & \overline{\mathrm{C}}(k' \otimes_k -) \arrow[d, "\cong"] \\
k' \otimes_k \mathrm{C}(-) \arrow[r] & k' \otimes_k \overline{\mathrm{C}}(-)
\end{tikzcd}$$

discussed in Remark 6.3.1.11 can be lifted to a commutative diagram of natural transformations from the category of commutative k-algebras with projective underlying k-module to the category $\mathrm{CAlg}(\mathsf{Mixed}_{k'})$, such that the left natural isomorphism is the one from Remark 6.3.2.8 and the top and bottom natural transformations are the ones from Proposition 6.3.2.11. ♢

[74] We identify $\mathrm{C}(R)_n$ for $n \geq 0$ with the tensor product $R^{\otimes(n+1)}$ for these formulas.

Warning 6.3.2.13. Let R be a commutative k-algebra with projective underlying k-module. While $\mathrm{C}(R)$ has both a strict mixed complex structure as well as the structure of a differential graded algebra, it is *not* in general an algebra in Mixed. To see this, let r and r' be elements of R Then, using the formulas from Propositions 6.3.1.9 and 6.3.2.10 we obtain

$$\begin{aligned}\mathrm{d}(r \cdot r') &= 1 \otimes r \cdot r' + r \cdot r' \otimes 1 \\ \mathrm{d}(r) \cdot r' + r \cdot \mathrm{d}(r') &= \big((1 \otimes r) + (r \otimes 1)\big) \cdot r' + r \cdot \Big(\big(1 \otimes r'\big) + \big(r' \otimes 1\big)\Big) \\ &= \big(r' \otimes r\big) + \big(r \cdot r' \otimes 1\big) + \big(r \otimes r'\big) + \big(r \cdot r' \otimes 1\big)\end{aligned}$$

which shows that, in general, d does not satisfy the Leibniz rule and hence $\mathrm{C}(R)$ does not form an algebra in Mixed – see Remark 4.2.1.12.

The formulas simplify slightly for $\overline{\mathrm{C}}(R)$ so that we get

$$\begin{aligned}\mathrm{d}(r \cdot r') &= 1 \otimes \overline{r \cdot r'} \\ \mathrm{d}(r) \cdot r' + r \cdot \mathrm{d}(r') &= \big(r' \otimes \overline{r}\big) + \Big(r \otimes \overline{r'}\Big)\end{aligned}$$

which is however nevertheless not in general equal.

We can note though that

$$\partial\Big(1 \otimes \overline{r} \otimes \overline{r'}\Big) = r \otimes \overline{r'} - 1 \otimes \overline{r \cdot r'} + r' \otimes \overline{r}$$

so that the Leibniz rule is at least satisfied up to homotopy for elements of degree 0 – which is to be expected, as $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ has the structure of an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, and we will see in Section 6.3.4 that $\mathrm{C}(R)$ represents the underlying mixed complex of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ if we consider it as an object of $\mathsf{Mixed}_{\mathrm{cof}}$, and the underlying algebra in $\mathcal{D}(k)$ of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ if we consider it as an object of $\mathrm{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})$. ♢

Despite Warning 6.3.2.13, we can show instances of the Leibniz rule for the normalized standard Hochschild complex under additional assumptions, as we show next.

Proposition 6.3.2.14. *Let R be a commutative k-algebra with projective underlying k-module. Let $n \geq 1$ and $r, s_1, \ldots, s_n$ elements of $\overline{\mathrm{C}}(R)$ (of arbitrary degree). Then the following partial Leibniz rule identity holds.*

$$\mathrm{d}\big(r \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_n)\big) = \mathrm{d}(r) \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_n) \qquad \heartsuit$$

Proof. We first note that it suffices to prove the case $n = 1$. For suppose we have already proved the statement for all $1 \leq n \leq m$, and that $r, s_1, \ldots, s_{m+1}$ are elements of $\overline{\mathrm{C}}(R)$. Then the following calculation shows how we can deduce the claim for $n = m + 1$.

$$\begin{aligned}&\mathrm{d}\big(r \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_{m+1})\big) \\ &= \mathrm{d}\Big(\big(r \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_m)\big) \cdot \mathrm{d}(s_{m+1})\Big)\end{aligned}$$

Applying the claim for $n = 1$.

$$= \mathrm{d}\big(r \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_m)\big) \cdot \mathrm{d}(s_{m+1})$$

Applying the claim for $n = m$.

$$= \mathrm{d}(r) \cdot \mathrm{d}(s_1) \cdots \mathrm{d}(s_m) \cdot \mathrm{d}(s_{m+1})$$

So now assume that $n, m \geq 0$, that r is an element of $\overline{\mathrm{C}}_n(R)$ and s is an element of $\overline{\mathrm{C}}_m(R)$. We have to show that $\mathrm{d}\big(r \cdot \mathrm{d}(s)\big) = \mathrm{d}(r) \cdot \mathrm{d}(s)$.

Using notation from Section 2.3 (34), the formula from Proposition 6.3.1.10 for the differential $\mathrm{d}(r)$ of an element r in degree n of $\overline{\mathrm{C}}(R)$ can be written in a more concise way as

$$\mathrm{d}(r) = \sum_{\tau \in C_{n+1}} \mathrm{sgn}(\tau) \cdot \tau \cdot (1 \otimes r)$$

where $1 \otimes r$ is to interpreted as notation for $1 \otimes \overline{r_0} \otimes \cdots \otimes \overline{r_n}$ if $r = r_0 \otimes \overline{r_1} \otimes \cdots \otimes \overline{r_n}$ for $r_0, \ldots, r_n$ elements of R, and k-linearly extended for other elements. We now begin by unpacking the definition of $\mathrm{d}\big(r \cdot \mathrm{d}(s)\big)$.

$$\begin{aligned}
&\mathrm{d}\big(r \cdot \mathrm{d}(s)\big) \\
&= \mathrm{d}\left(r \cdot \sum_{\tau_r \in C_{m+1}} \mathrm{sgn}(\tau_r) \cdot \tau_r \cdot (1 \otimes s)\right) \\
&= \mathrm{d}\left(\sum_{\substack{\tau_r \in C_{m+1},\\ \sigma \in B_{n,m+1}}} \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau_r) \cdot \sigma \cdot \big(\mathrm{id}_{\{1,\ldots,n\}} \amalg \tau_r\big) \cdot (r \otimes s)\right) \\
&= \sum_{\substack{\tau_r \in C_{m+1},\\ \sigma \in B_{n,m+1},\\ \tau \in C_{n+m+2}}} \mathrm{sgn}(\tau) \cdot \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau_r) \cdot \tau \cdot \big(\mathrm{id}_{\{1\}} \amalg \sigma\big) \cdot \big(\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r\big) \cdot (1 \otimes r \otimes s) \\
&= \sum_{\substack{\tau_r \in C_{m+1},\\ \sigma \in B_{n,m+1},\\ \tau \in C_{n+m+2}}} \mathrm{sgn}\Big(\tau \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big) \circ \big(\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r\big)\Big) \\
&\qquad\qquad \cdot \Big(\tau \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big) \circ \big(\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r\big)\Big) \cdot (1 \otimes r \otimes s)
\end{aligned}$$

Next we unpack the definition of $\mathrm{d}(r) \cdot \mathrm{d}(s)$.

$$\begin{aligned}
&\mathrm{d}(r) \cdot \mathrm{d}(s) \\
&= \left(\sum_{\tau_l \in C_{n+1}} \mathrm{sgn}(\tau_l) \cdot \tau_l \cdot (1 \otimes r)\right) \cdot \left(\sum_{\tau_r \in C_{m+1}} \mathrm{sgn}(\tau_r) \cdot \tau_r \cdot (1 \otimes s)\right) \\
&= \sum_{\substack{\tau_l \in C_{n+1}\\ \tau_r \in C_{m+1}}} \mathrm{sgn}(\tau_l \amalg \tau_r) \cdot \big(\tau_l \cdot (1 \otimes r)\big) \cdot \big(\tau_r \cdot (1 \otimes s)\big)
\end{aligned}$$

$$= \sum_{\substack{\tau_l \in C_{n+1} \\ \tau_r \in C_{m+1} \\ \sigma \in B_{n+1,m+1}}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau_l \amalg \tau_r) \cdot \sigma \cdot (\tau_l \amalg \tau_r) \cdot (1 \otimes r \otimes s)$$

$$= \sum_{\substack{\tau_l \in C_{n+1} \\ \tau_r \in C_{m+1} \\ \sigma \in B_{n+1,m+1}}} \operatorname{sgn}\big(\sigma \circ (\tau_l \amalg \tau_r)\big) \cdot \big(\sigma \circ (\tau_l \amalg \tau_r)\big) \cdot (1 \otimes r \otimes s)$$

The claim thus boils down to a statement about different decompositions of elements of Σ_{n+m+2} that we now make concrete. We define two maps of sets as follows.

$$\begin{aligned} f\colon C_{m+1} \times B_{n,m+1} \times C_{n+1+m+1} &\to \Sigma_{n+1+m+1} \\ (\tau_r, \sigma, \tau) &\mapsto \tau \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big) \circ \big(\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r\big) \\ g\colon C_{n+1} \times C_{m+1} \times B_{n+1,m+1} &\to \Sigma_{n+1+m+1} \\ (\tau_l, \tau_r, \sigma) &\mapsto \sigma \circ (\tau_l \amalg \tau_r) \end{aligned}$$

To show $\mathrm{d}\big(r \cdot \mathrm{d}(s)\big)$ is equal $\mathrm{d}(r) \cdot \mathrm{d}(s)$ it then suffices to show that for every element ρ of $\Sigma_{n+1+m+1}$ the preimages of ρ under f and g satisfy $|f^{-1}(\rho)| = |g^{-1}(\rho)|$. We will show this by going through the following steps.

(1) Proof that f is injective.

(2) Proof that g is injective

(3) Definition of a subset $C_{n+1,m+1}$ of $\Sigma_{n+1+m+1}$.

(4) Proof that $\mathrm{Im}(g) = C_{n+1,m+1}$.

(5) Proof that $\mathrm{Im}(f) \subseteq C_{n+1,m+1}$.

(6) Proof that $\mathrm{Im}(f) = C_{n+1,m+1}$.

Step (1): Let (τ_r, σ, τ) be an element of $C_{m+1} \times B_{n,m+1} \times C_{n+1+m+1}$, and let ρ be the composition $\rho = \tau \circ (\mathrm{id}_{\{1\}} \amalg \sigma) \circ (\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r)$. What we have to show is that τ_r, σ, and τ are uniquely determined by ρ. First note that $\rho(1) = \tau(1)$. As elements of $C_{n+1+m+1}$ are determined uniquely by their value on a single element, this means that τ is uniquely determined by ρ. As $\mathrm{id}_{\{1\}} \amalg \sigma$ preserves the order of the elements of the subset $\{n+1+1, \ldots, n+1+m+1\}$, we obtain

$$r_{\{n+1+1,\ldots,n+1+m+1\}}\Big(\big(\mathrm{id}_{\{1\}} \amalg \sigma\big) \circ \big(\mathrm{id}_{\{1,\ldots,n+1\}} \amalg \tau_r\big)\Big) = \tau_r$$

which shows the claim.

Step (2): Let σ be an element of $B_{n+1,m+1}$, τ_l an element of C_{n+1} and τ_r an element of C_{m+1}. As σ preserves the order of the elements of the subsets $\{1, \ldots, n+1\}$ as well as $\{n+1+1, \ldots, n+1+m+1\}$, we obtain

$$r_{\{1,\ldots,n+1\}}\big(\sigma \circ (\tau_l \amalg \tau_r)\big) = \tau_l$$

and similarly

$$r_{\{n+1+1,\dots,n+1+m+1\}}\big(\sigma \circ (\tau_l \amalg \tau_r)\big) = \tau_r$$

which implies the claim.

Step (3): We let $C_{n+1,m+1}$ be the subset of $\Sigma_{n+1+m+1}$ consisting of those permutations ρ for which $r_{\{1,\dots,n+1\}}(\rho)$ is an element of C_{n+1} and $r_{\{n+1+1,\dots,n+1+m+1\}}(\rho)$ is an element of C_{m+1}. One should think of $C_{n+1,m+1}$ as a variant of $B_{n+1,m+1}$; The permutations in $B_{n+1,m+1}$ are those that preserve the order of the elements of the two subsets $\{1,\dots,n+1\}$ and $\{n+1,\dots,n+1+m+1\}$,[75] and the permutations in $C_{n+1,m+1}$ are those which *cyclically* preserve the order of the elements of those subsets.

Step (4): The argument used in step (2) shows that $\operatorname{Im}(g) \subseteq C_{n+1,m+1}$. For the other direction, suppose that ρ is an element of $C_{n+1,m+1}$. Then let $\tau_l = r_{\{1,\dots,n+1\}}(\rho)$ and $\tau_r = r_{\{n+1,\dots,n+1+m+1\}}(\rho)$, and define $\sigma \coloneqq \rho \circ \big(\tau_l^{-1} \amalg \tau_r^{-1}\big)$. Then we obtain

$$r_{\{1,\dots,n+1\}}(\sigma) = \tau_l \circ \tau_l^{-1} = \mathrm{id} \qquad \text{and} \qquad r_{\{n+1+1,\dots,n+1+m+1\}}(\sigma) = \tau_r \circ \tau_r^{-1} = \mathrm{id}$$

so that σ is an element of $B_{n+1,m+1}$. This shows that $C_{n+1,m+1} \subseteq \operatorname{Im}(g)$.

Step (5): It follows from the previous step that permutations of the form

$$(\mathrm{id}_{\{1\}} \amalg \sigma) \circ (\mathrm{id}_{\{1,\dots,n+1\}} \amalg \tau_r)$$

for σ an element of $B_{n,m+1}$ and τ_r an element of C_{m+1} lie in $C_{n+1,m+1}$. It thus suffices to show that $C_{n+1,m+1}$ is closed under postcomposition with elements of $C_{n+1+m+1}$. This follows from the fact that if X is a subset of $\{1,\dots,n+1+m+1\}$ and τ an element of $C_{n+1+m+1}$, then $r_X(\tau)$ is an element of $C_{|X|}$.

Step (6): By the previous two steps it suffices to show that

$$\big|\operatorname{Im}(f)\big| = \big|\operatorname{Im}(g)\big|$$

and as both f and g are injective, it suffices to show that

$$|C_{m+1}| \cdot \big|B_{n,m+1}\big| \cdot |C_{n+1+m+1}| = |C_{n+1}| \cdot |C_{m+1}| \cdot \big|B_{n+1,m+1}\big|$$

which is verified by the following calculation.

$$\begin{aligned}
&|C_{m+1}| \cdot \big|B_{n,m+1}\big| \cdot |C_{n+1+m+1}| \\
&= (m+1) \cdot \left(\frac{(n+m+1)!}{n! \cdot (m+1)!}\right) \cdot (n+1+m+1) \\
&= (m+1) \cdot \left(\frac{(n+1) \cdot (n+1+m+1)!}{(n+1)! \cdot (m+1)! \cdot (n+1+m+1)}\right) \cdot (n+1+m+1) \\
&= (n+1) \cdot (m+1) \cdot \left(\frac{(n+1+m+1)!}{(n+1)! \cdot (m+1)!}\right) \\
&= |C_{n+1}| \cdot |C_{m+1}| \cdot \big|B_{n+1,m+1}\big|
\end{aligned}$$

□

[75] So the respective restrictions yield the elements $\mathrm{id}_{\{1,\dots,n+1\}}$ of Σ_{n+1} and $\mathrm{id}_{\{1,\dots,m+1\}}$ of Σ_{m+1}.

6.3.3. Relative tensor products in $\mathsf{Ch}(k)$ and $\mathcal{D}(k)$

The canonical functor $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ is symmetric monoidal – see Proposition 4.3.2.1 – and thus preserves tensor products. In this section we discuss how γ interacts with *relative* tensor products. There is no reason to expect that γ preserves $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits in general, so we can not just apply Proposition E.8.0.1. Instead, we will show that γ preserves relative tensor products if one of the two modules is cofibrant as a module. Cofibrancy is here taken to be with respect to the model structure on $\mathrm{RMod}_R(\mathsf{Ch}(k))$ and $\mathrm{LMod}_R(\mathsf{Ch}(k))$ for an algebra R in $\mathsf{Ch}(k)$ from Theorem 4.2.2.1[76]. Note that as $\mathsf{Ch}(k)$ is a 1-category, geometric realizations – i. e. colimits over $\mathbf{\Delta}^{\mathrm{op}}$ – are calculated as coequalizers[77], so that the relative tensor product in $\mathsf{Ch}(k)$ is the "classical" one.

We begin by noting that there is a canonical comparison map from $\gamma(X) \otimes_{\gamma(R)} \gamma(Y)$ to $\gamma(X \otimes_R Y)$.

Remark 6.3.3.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor of monoidal ∞-categories, and assume that the monoidal structures on $\mathcal{C}$ and $\mathcal{D}$ are compatible with $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits in the sense of [HA, 3.1.1.18]. The relative tensor product induces a functor

$$\mathrm{RMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{LMod}(\mathcal{C}) \xrightarrow{-\otimes_{-}-} \mathcal{C}$$

and similarly for $\mathcal{D}$, see [HA, 4.4.2.10 and 4.4.2.11].

By [HA, 4.4.2.8] this functor can be identified as the functor mapping a triple (M, R, N) to $|\mathrm{Bar}_R(M,N)_\bullet|$, the geometric realization of the simplicial object $\mathrm{Bar}_R(M,N)_\bullet$ which can be described as $M \otimes R^{\otimes\bullet} \otimes N$, see also Section E.8.

As F is monoidal, it follows from the definition of the bar construction [HA, 4.4.2.7] that there is a natural equivalence as follows.

$$\mathrm{Bar}_{F(R)}(F(M), F(N))_\bullet \simeq F \circ \mathrm{Bar}_R(M,N)_\bullet$$

As there is a natural transformation

$$|F \circ X_\bullet| = \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(F \circ X_\bullet) \to F\Big(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet\Big) = F\big(|X_\bullet|\big)$$

for simplicial objects $X_\bullet$ in $\mathcal{C}$, we thus obtain a canonical natural transformation comparing first applying F, and then taking the relative tensor product with first taking the relative tensor product and then applying F.

$$\begin{tikzcd}
\mathrm{RMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{LMod}(\mathcal{C}) \arrow[r, "-\otimes_{-}-"] \arrow[d, "\mathrm{RMod}(F)\otimes_{\mathrm{Alg}(F)}\mathrm{LMod}(F)"'] & \mathcal{C} \arrow[d, "F"] \\
\mathrm{RMod}(\mathcal{D}) \times_{\mathrm{Alg}(\mathcal{D})} \mathrm{LMod}(\mathcal{D}) \arrow[r, "-\otimes_{-}-"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{D}
\end{tikzcd}$$

$\diamondsuit$

[76] $\mathsf{Ch}(k)$ satisfies the assumptions by Fact 4.1.3.1.
[77] See [Rie14, 8.3.8].

Remark 6.3.3.2. We would like to compare relative tensor products of chain complexes with relative tensor products in $\mathcal{D}(k)$. There is a slight issue here that Remark 6.3.3.1 does not directly apply to give us what we want: We can not apply it to $\gamma\colon \mathsf{Ch}(k) \to \mathcal{D}(k)$ as this functor is not monoidal, so the monoidal functor we would want to consider is $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$, but there is no reason for the full subcategory $\mathsf{Ch}(k)^{\mathrm{cof}}$ of $\mathsf{Ch}(k)$ to be closed under $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits.

However, this is not actually a problem. If R is an algebra in $\mathsf{Ch}(k)$, and M and N are right and left modules over R, and such that the underlying chain complexes of R, M, and N are cofibrant, then, because γ is monoidal on $\mathsf{Ch}(k)^{\mathrm{cof}}$, we obtain an equivalence

$$\mathrm{Bar}_{\gamma(R)}(\gamma(M), \gamma(N))_\bullet \simeq \gamma \circ \mathrm{Bar}_R(M, N)_\bullet$$

just like in Remark 6.3.3.1.

We also still obtain a canonical morphism

$$\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}\big(\mathrm{Bar}_{\gamma(R)}(\gamma(M), \gamma(N))_\bullet\big) \simeq \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}\big(\gamma \circ \mathrm{Bar}_R(M, N)_\bullet\big) \to \gamma\Big(\operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}}\big(\mathrm{Bar}_R(M, N)_\bullet\big)\Big)$$

where on the right the colimit is taken in $\mathsf{Ch}(k)$ rather than $\mathsf{Ch}(k)^{\mathrm{cof}}$, and the γ is the functor

$$\gamma\colon \mathsf{Ch}(k) \to \mathcal{D}(k)$$

that is given by postcomposing the other functor called γ with the cofibrant replacement functor.

The upshot is that we still have a canonical comparison transformation as in Remark 6.3.3.1, even if it doesn't *quite* fit into the setup of Remark 6.3.3.1. ♢

Proposition 6.3.3.3. *Let (M, R, N) be an object of*

$$\mathrm{RMod}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \times_{\mathrm{Alg}\left(\mathsf{Ch}(k)^{\mathrm{cof}}\right)} \mathrm{LMod}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)$$

i. e. R is a differential graded algebra, M is a right module over R, N is a left module over R, and all three have cofibrant underlying chain complex.

Assume that one of M and N is cofibrant as a module over R with respect to the model structure of Theorem 4.2.2.1. Then the relative tensor product $M \otimes_R N$, calculated in $\mathsf{Ch}(k)$, is again cofibrant and the canonical comparison morphism (see Remark 6.3.3.2)

$$\gamma(M) \otimes_{\gamma(R)} \gamma(N) \to \gamma(M \otimes_R N)$$

is an equivalence. ♡

Proof. Let R be an object of $\mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$. We will use the notation

$$\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{RMod}_R}\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathrm{RMod}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)$$

as well as $\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}$, $\mathrm{Free}_{\mathcal{D}}^{\mathrm{RMod}_R}$, and $\mathrm{Free}_{\mathcal{D}}^{\mathrm{LMod}_R}$ for the left adjoints to the respective forgetful functors ev_{m}. We also let C be the collection of objects (M, R, N) of

$$\mathrm{RMod}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \times_{\mathrm{Alg}\left(\mathsf{Ch}(k)^{\mathrm{cof}}\right)} \mathrm{LMod}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)$$

and $C^{\simeq}$ the subcollection of those tuples (M, R, N) for which the canonical comparison morphism

$$\gamma(M) \otimes_{\gamma(R)} \gamma(N) \to \gamma(M \otimes_R N)$$

is an equivalence. When we refer to colimits below while talking about objects and morphisms in $\mathsf{Ch}(k)^{\mathrm{cof}}$, those colimits are always to be taken in the category $\mathsf{Ch}(k)$.

We first show the claim regarding cofibrancy of the relative tensor product, and will do the case where N is cofibrant as a module – the other case is analogous. Fix R and M as in the statement. Then it suffices to show that the functor

$$M \otimes_R -\colon \mathrm{LMod}_{\mathsf{Ch}}(k) \to \mathsf{Ch}(k)$$

maps generating cofibrations to cofibrations and preserves colimits. That the functor preserves colimits follows from [HA, 4.4.2.15]. Let $i\colon X \to Y$ be a cofibration in $\mathsf{Ch}(k)$. Then it remains to show that

$$M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(i)\colon M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \to M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y)$$

is again a cofibration. But this morphism can be identified with the morphism

$$M \otimes i\colon M \otimes X \to M \otimes Y$$

which is a cofibration as M is cofibrant and i a cofibration.

Let us now turn towards the claim that $\gamma(M) \otimes_{\gamma(R)} \gamma(N) \to \gamma(M \otimes_R N)$ is an equivalence if one of M and N is cofibrant as a module. By the definition of the model structure on modules[78] and [Hov99, 2.1.18 (b) and 2.1.9] it suffices to show the following.

(1) Let (M, R, N) be in C. Then $(M, R, 0)$ and $(0, R, N)$ are in $C^{\simeq}$.

(2) Let R be an object of $\mathrm{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})$, let M be an object of $\mathrm{RMod}(\mathsf{Ch}(k)^{\mathrm{cof}})$, and let X be an object in $\mathsf{Ch}(k)^{\mathrm{cof}}$.

    Then $(M, R, \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X))$ is in $C^{\simeq}$.

(3) Let (M, R, N) be in $C^{\simeq}$ with N cofibrant as a module, let $i\colon X \to Y$ be a cofibration between cofibrant objects of $\mathsf{Ch}(k)$, and let $f\colon \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \to N$ be a morphism in $\mathrm{LMod}_R\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big)$. Then $(M, R, N \amalg_{\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)} \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y))$ is again in $C^{\simeq}$, where the pushouts are formed with respect to the morphisms f and $\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(i)$.

    The analogous statement holds for pushouts of this form in the first component.

(4) Let R be an object of $\mathrm{Alg}(\mathsf{Ch}(k)^{\mathrm{cof}})$ and M an object of $\mathrm{RMod}(\mathsf{Ch}(k)^{\mathrm{cof}})$.

    Let λ be an ordinal and let $F\colon \lambda \to \mathrm{LMod}(\mathsf{Ch}(k)^{\mathrm{cof}})$ be a λ-sequence[79]. Assume that for every morphism $\alpha \to \alpha+1$ in λ the induced morphism $F(\alpha) \to F(\alpha+1)$ is

[78] Theorem 4.2.2.1
[79] See for example [Hov99, 2.1.1] for a definition.

a cofibration in $\mathrm{LMod}(\mathsf{Ch}(k)^{\mathrm{cof}})$, and that for every object α of λ the left-R-module $F(\alpha)$ is cofibrant and the triple $(M, R, F(\alpha))$ is in $C^{\simeq}$.

Then $(M, R, \operatorname{colim}_{\lambda} F)$ is also in $C^{\simeq}$. The analogous statement holds for transfinite compositions in the first component as well.

As all statements are symmetrical, we will only show the statements with regards to the *last* component.

Proof of claim (1): As both $\gamma(M) \otimes_{\gamma(R)} 0 \simeq 0$ and $M \otimes_R 0 \cong 0$, this follows from γ preserving the zero object by Proposition 4.3.2.1 (3).

Proof of claim (2): Consider the following commutative diagram

$$\begin{tikzcd}
\gamma(M) \otimes \gamma(X) \arrow[r] \arrow[d] & \gamma(M \otimes X) \arrow[d] \\
\gamma(M) \otimes_{\gamma(R)} \gamma\Big(\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)\Big) \arrow[r] & \gamma\Big(M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)\Big)
\end{tikzcd}$$

where the horizontal morphisms are the canonical comparison morphisms, and the vertical morphisms are induced by the morphism[80]

$$(M, k, X) \to (M, R, \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X))$$

in

$$\mathrm{RMod}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \times_{\mathrm{Alg}\left(\mathsf{Ch}(k)^{\mathrm{cof}}\right)} \mathrm{LMod}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big)$$

that is given by the identity of M, the unit morphism $k \to R$, and the morphism from X to the underlying object of $\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)$ that exhibits the latter as a free left-R-module generated by X.

It follows from Proposition E.7.4.1 that the induced morphism

$$\gamma(X) \to \gamma\Big(\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)\Big)$$

exhibits the codomain as the free left-$\gamma(R)$-module generated by $\gamma(X)$, so it follows from associativity [HA, 4.4.3.14] and unitality [HA, 4.4.3.16] of the relative tensor product that both the left and right vertical morphisms in the above diagram are equivalences[81]. As the top horizontal morphism is an equivalence as well, so must be the bottom horizontal morphism.

[80] See [HA, 4.4.2.9] for the identification of the relative tensor product over the unit k with the (non-relative) tensor product.

[81] One can easily see from the definition of free modules that $\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(k) \simeq R$, and that $\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \simeq \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(k) \otimes X$. One thus obtains equivalences

$$M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \simeq M \otimes_R (R \otimes X) \simeq (M \otimes_R R) \otimes X \simeq M \otimes X$$

and similarly for the other relevant relative tensor product in $\mathcal{D}(k)$.

Proof of claim (3): Applying the canonical comparison transformation for the relative tensor products to the commutative square

$$\begin{tikzcd} \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \arrow[r, "f"] \arrow[d, "\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(i)"'] & N \arrow[d] \\ \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y) \arrow[r] & P \end{tikzcd} \tag{$*$}$$

where we write P for the pushout, we obtain the commuting cube

$$\begin{tikzcd}[row sep=small, column sep=small] & \gamma(M) \otimes_{\gamma(R)} \gamma\Big(\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)\Big) \arrow[rr] \arrow[dl] \arrow[dd] & & \gamma\Big(M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X)\Big) \arrow[dl] \arrow[dd] \\ \gamma(M) \otimes_{\gamma(R)} \gamma(N) \arrow[rr, crossing over] \arrow[dd] & & \gamma(M \otimes_R N) & \\ & \gamma(M) \otimes_{\gamma(R)} \gamma\Big(\mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y)\Big) \arrow[rr] \arrow[dl] & & \gamma\Big(M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y)\Big) \arrow[dl] \\ \gamma(M) \otimes_{\gamma(R)} \gamma(P) \arrow[rr] & & \gamma(M \otimes_R P) \arrow[from=uu, crossing over] & \end{tikzcd} \tag{$**$}$$

in $\mathcal{D}(k)$. We need to show that the bottom front horizontal morphism is an equivalence. For this it suffices to show the following.

(a) The right side in diagram $(**)$ is a pushout square.

(b) The left side in diagram $(**)$ is a pushout square.

(c) The horizontal morphism in diagram $(**)$ other than the bottom front one are equivalences.

Proof of claim (a): In the commutative square

$$\begin{tikzcd} M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(X) \arrow[r, "M\otimes_R f"] \arrow[d, "M\otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(i)"'] & M \otimes_R N \arrow[d] \\ M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(Y) \arrow[r] & M \otimes_R P \end{tikzcd}$$

the chain complex $M \otimes_R N$ is cofibrant and $M \otimes_R \mathrm{Free}_{\mathsf{Ch}}^{\mathrm{LMod}_R}(i)$ is a cofibration by what we already showed at the beginning of the proof. As $(*)$ is a pushout square, and $M \otimes_R -$ preserves colimits by [HA, 4.4.2.15], this is again a pushout square, and by [HTT, A.2.4.4] even a homotopy pushout square. The claim thus follows by applying [HA, 1.3.4.24].

Proof of claim (b): Follows from [HA, 1.3.4.24] using that $(*)$ is a homotopy pushout by [HTT, A.2.4.4].

Proof of claim (c): For the two back horizontal morphisms this follows from claim (2), and for the top front horizontal morphism this is by assumption.

Proof of claim (4): Analogous to (3), this time using that transfinite compositions are already homotopy colimits if all morphisms of the form $F(\alpha) \to F(\alpha+1)$ are cofibrations, which follows from [HTT, A.2.9.24 (1)], which shows that such diagrams are cofibrant in the projective model structure on λ-diagrams. □

6.3.4. The standard Hochschild complex as a model for HH

In this section we compare the Hochschild homology functors defined in Section 6.2 with the standard Hochschild complex functors as defined in Sections 6.3.1 and 6.3.2, showing that the latter represent the former.

We first discuss the case where we take into account the mixed complex structure, but not multiplicative structure, in Section 6.3.4.1, and then the case of commutative algebras, where we take into account the commutative algebra structure on Hochschild homology, but not the mixed structure, in Section 6.3.4.2.

6.3.4.1. The mixed case

The following comparison result by Hoyois shows that the standard Hochschild complex of A, considered as a strict mixed complex, is a model for the mixed complex $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\gamma(A))$.

Proposition 6.3.4.1 ([Hoy18, 2.3])**.** *There is a commuting diagram*[82]

$$\begin{tikzcd}
\mathrm{Alg}\left(\mathsf{Ch}(k)^{\mathrm{cof}}\right) \arrow[r, "\mathrm{B}^{\mathrm{cyc}}_{\bullet}"] \arrow[d, "\mathrm{Alg}(\gamma)"'] \arrow[rrr, bend left, "\mathrm{C}"] & \mathrm{Fun}\left(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\right) \arrow[rr, "|-|_{\mathsf{Mixed}}"] \arrow[d, "\gamma_*"] & & \mathsf{Mixed}_{\mathrm{cof}} \arrow[d, "\gamma_{\mathsf{Mixed}}"] \\
\mathrm{Alg}(\mathcal{D}(k)) \arrow[r, "\mathrm{B}^{\mathrm{cyc}}_{\bullet}"'] \arrow[rrr, bend right, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}"'] & \mathrm{Fun}\left(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\right) \arrow[r, "|-|"'] & \mathcal{D}(k)^{\mathrm{B}\mathbb{T}} \arrow[r, "\simeq"'] & \mathcal{M}\mathrm{ixed}
\end{tikzcd}$$

where the horizontal equivalence at the bottom left is the monoidal equivalence from Construction 5.4.0.1. ♡

[82]Here, γ refers to the symmetric monoidal functor $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$. The construction $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$ is defined in Definition 6.1.2.16, $|-|$ is defined in Definition 6.1.3.4, $|-|_{\mathsf{Mixed}}$ is defined in Construction 6.3.1.2, C is defined in Definition 6.3.1.6, and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is defined in Definition 6.2.1.2.

Proof. The top and bottom rectangles commute by definition of C and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$, see Definition 6.3.1.6 and Definition 6.2.1.2. For the left square in the middle see Remark 6.1.2.17.

For $X_\bullet$ a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathsf{Ch}(k)$, the underlying chain complex of $|X_\bullet|_{\mathsf{Mixed}}$ is defined in Construction 6.3.1.2 as the total complex of a certain double complex, which is an upper[83] half plane complex. If $X_{[n]_\mathbf{\Lambda}}$ is acyclic for every $n \geq 0$, then it follows that the rows of the corresponding double complex are all acyclic, so that we can apply the acyclic assembly lemma [Wei94, 2.7.3] to conclude that the total complex $|X_\bullet|_{\mathsf{Mixed}}$ is acyclic. As colimits of (double) complexes as well as functor categories are calculated degreewise, and the construction of the total complex from a double complex preserves colimits, it follows by using the long exact sequence of homology that every morphism $X_\bullet \to Y_\bullet$ in $\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}})$ that is pointwise a quasiisomorphism is mapped under $|-|_{\mathsf{Mixed}}$ to a quasiisomorphism.

The upshot is that $|-|_{\mathsf{Mixed}}$ induces a functor

$$K\colon \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \to \mathcal{M}\mathrm{ixed}$$

of ∞-categories that fits into a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{Fun}(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}) \arrow[r, "|-|_{\mathsf{Mixed}}"] \arrow[d, "\gamma_*"'] & \mathsf{Mixed}_{\mathrm{cof}} \arrow[d, "\gamma_{\mathsf{Mixed}}"] \\
\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \arrow[r, "K"'] & \mathcal{M}\mathrm{ixed}
\end{tikzcd}$$

This is the functor also called K that is defined in [Hoy18, Right before 2.2].

We are thus left to construct a commuting triangle

$$\begin{tikzcd}
\mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \arrow[r, "|-|"] \arrow[dr, "K"'] & \mathcal{D}(k)^{\mathrm{B}\mathbb{T}} \arrow[d, "\simeq"] \\
& \mathcal{M}\mathrm{ixed}
\end{tikzcd}$$

where the vertical equivalence is the one from Construction 5.4.0.1. This is exactly what [Hoy18, 2.3] provides – as long as we chose the correct vertical equivalence. However, the vertical equivalence has been chosen in Construction 5.4.0.1 and Convention 5.2.4.3 in reference to [Hoy18, 2.3] as exactly the one that is required to obtain the above commuting triangle. □

Remark 6.3.4.2. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the symmetric monoidal functor $k' \otimes_k -\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathsf{Ch}(k')^{\mathrm{cof}}$ (see Fact 4.1.5.1) induces a natural

[83] Or right, depending on the convention. We will assume in this proof that we convert a complex of complexes to a double complex such that $X_{i,j} = (X_j)_i$. If $X_\bullet = \mathrm{B}^{\mathrm{cyc}}_\bullet(A)$, then the row indexed by $n \geq 0$ contains $A^{\otimes(n+1)}$, and the rows indexed by $n < 0$ are empty.

transformation from the the commutative diagram from Proposition 6.3.4.1 for k to the one for k'.

To be more precise, functoriality of the cyclic bar construction (see Remark 6.1.2.17) with respect to the commutative diagram

$$\begin{tikzcd} \mathsf{Ch}(k)^{\mathrm{cof}} \arrow[r, "k'\otimes_k -"] \arrow[d, "\gamma"'] & \mathsf{Ch}(k')^{\mathrm{cof}} \arrow[d, "\gamma"] \\ \mathcal{D}(k) \arrow[r, "k'\otimes_k -"'] & \mathcal{D}(k') \end{tikzcd} \tag{$*$}$$

of symmetric monoidal functors from Remark 4.3.2.2 yields a commutative cube

$$\begin{tikzcd} & \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[dl] \arrow[dd] \\ \mathrm{Alg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[rr] \arrow[dd] & & \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[dd] & \\ & \mathrm{Alg}\big(\mathcal{D}(k)\big) \arrow[rr] \arrow[dl] & & \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \arrow[dl] \\ \mathrm{Alg}\big(\mathcal{D}(k')\big) \arrow[rr] & & \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k')\big) & \end{tikzcd}$$

where the horizontal functors are all $\mathrm{B}^{\mathrm{cyc}}_{\bullet}$, the vertical functors are induced by γ, and the functors from the back to the front are induced by $k' \otimes_k -$. Existence of a commutative cube

$$\begin{tikzcd} & \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathsf{Mixed}_{k,\mathrm{cof}} \arrow[dl] \arrow[dd] \\ \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[rr] \arrow[dd] & & \mathsf{Mixed}_{k',\mathrm{cof}} \arrow[dd] & \\ & \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k)\big) \arrow[rr] \arrow[dl] & & \mathcal{M}\mathrm{ixed}_k \arrow[dl] \\ \mathrm{Fun}\big(\mathbf{\Lambda}^{\mathrm{op}}, \mathcal{D}(k')\big) \arrow[rr] & & \mathcal{M}\mathrm{ixed}_{k'} & \end{tikzcd}$$

where the horizontal functors are $|-|$ and $|-|_{\mathsf{Mixed}}$, and the left and right sides are induced by diagram ($*$) is implicit in the proof of [Hoy18, 2.3], though unfortunately not explicitly stated[84]. Combining the two commutative cubes we obtain a commutative cube

$$\begin{tikzcd}
 & \mathrm{Alg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr, "\mathrm{C}"] \arrow[dl] \arrow[dd] & & \mathsf{Mixed}_{k,\mathrm{cof}} \arrow[dl] \arrow[dd] \\
\mathrm{Alg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[rr, "\mathrm{C}" near end] \arrow[dd] & & \mathsf{Mixed}_{k',\mathrm{cof}} \arrow[dd] & \\
 & \mathrm{Alg}\big(\mathcal{D}(k)\big) \arrow[rr, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}" near end] \arrow[dl] & & \mathcal{M}\mathrm{ixed}_k \arrow[dl] \\
\mathrm{Alg}\big(\mathcal{D}(k')\big) \arrow[rr, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}"'] & & \mathcal{M}\mathrm{ixed}_{k'} &
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$, where the front and back sides are the big outer squares in Proposition 6.3.4.1, the left and right sides are induced by diagram $(*)$, the top is the diagram from Remark 6.3.1.7 and the bottom is the diagram from Remark 6.2.1.6. ♢

6.3.4.2. The commutative case

We now compare the standard Hochschild complex C in the commutative case to $\mathrm{HH}\colon \mathrm{CAlg}(\mathcal{D}(k)) \to \mathrm{CAlg}(\mathcal{D}(k))$ from Definition 6.2.2.1, which will be possible because we can write both as a relative tensor product according to Corollary 6.2.2.6 and Proposition 6.3.2.7, and discussed how to compare relative tensor products in $\mathsf{Ch}(k)$ with relative tensor products in $\mathcal{D}(k)$ in Section 6.3.3.

Proposition 6.3.4.3. *There is a commuting diagram*

$$\begin{tikzcd}
\mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[r, "\mathrm{C}"] \arrow[d, "\mathrm{CAlg}(\gamma)"'] & \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[d, "\mathrm{CAlg}(\gamma)"] \\
\mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[r, "\mathrm{HH}"'] & \mathrm{CAlg}\big(\mathcal{D}(k)\big)
\end{tikzcd} \tag{6.16}$$

where C *is the functor from Proposition 6.3.2.7 and* HH *is the functor from Definition 6.2.2.1 and γ is the symmetric monoidal functor $\mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$.* ♡

Proof. By Proposition 6.3.2.7 C is given as the relative tensor product $A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)$ in $\mathrm{CAlg}(\mathsf{Ch}(k))$ – see Construction 6.3.2.6 and Construction E.8.0.4 for a definition of the relevant $A \otimes A$-module structures.

Like in Remark 6.3.3.1 and Remark 6.3.3.2 we obtain a natural comparison transformation

$$\mathrm{CAlg}(\gamma)(A) \otimes_{\mathrm{CAlg}(\gamma)(A\otimes A)} \mathrm{CAlg}(\gamma)\Big(\mathrm{C}^{\mathrm{Bar}}(A)\Big) \to \mathrm{CAlg}(\gamma)\Big(A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)\Big) \qquad (*)$$

[84] See also Remark 5.4.0.3

where we use that we already know that the relative tensor product will have cofibrant underlying chain complex[85]. We want to show that this morphism is an equivalence. As the forgetful functor

$$\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{CAlg}\big(\mathcal{D}(k)\big) \to \mathcal{D}(k)$$

detects equivalences by [HA, 3.2.2.6], it suffices to show that the underlying morphism in $\mathcal{D}(k)$ is an equivalence. By [HA, 3.2.3.1 (4)] and Proposition E.4.2.3 (5) in combination with Proposition E.8.0.1, both forgetful functors

$$\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{CAlg}\big(\mathsf{Ch}(k)\big) \to \mathsf{Ch}(k) \qquad \text{and} \qquad \mathrm{ev}_{\mathfrak{a}}\colon \mathrm{CAlg}\big(\mathcal{D}(k)\big) \to \mathcal{D}(k)$$

preserve relative tensor products, so that we can identify the composition of the natural transformation $(*)$ of functors $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}}) \to \mathrm{CAlg}(\mathcal{D}(k))$ with $\mathrm{ev}_{\mathfrak{a}}$ with the natural comparison transformation

$$\gamma(A) \otimes_{\gamma(A\otimes A)} \gamma\Big(\mathrm{C}^{\mathrm{Bar}}(A)\Big) \to \gamma\Big(A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)\Big)$$

from Remark 6.3.3.2. As $\mathrm{C}^{\mathrm{Bar}}(A)$ is cofibrant as a left-$A\otimes A$-module by Proposition 6.3.2.3, we can apply Proposition 6.3.3.3 to conclude that this is an equivalence.

We have now seen that the composition $\mathrm{CAlg}(\gamma) \circ \mathrm{C}$ in (6.16) is homotopic to the functor that is described by

$$A \mapsto \mathrm{CAlg}(\gamma)(A) \otimes_{\mathrm{CAlg}(\gamma)(A\otimes A)} \mathrm{CAlg}(\gamma)\Big(\mathrm{C}^{\mathrm{Bar}}(A)\Big)$$

---

[85] Here are some more details. $\mathrm{CAlg}(\gamma)\colon \mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}}) \to \mathrm{CAlg}(\mathcal{D}(k))$ is symmetric monoidal, and thus induces a natural equivalence of bar constructions as follows.

$$\mathrm{Bar}_{\mathrm{CAlg}(\gamma)(A\otimes A)}\Big(\mathrm{CAlg}(\gamma)(A), \mathrm{CAlg}(\gamma)(\mathrm{C}^{\mathrm{Bar}}(A))\Big)_{\bullet} \simeq \mathrm{CAlg}(\gamma) \circ \mathrm{Bar}_{A\otimes A}\Big(A, \mathrm{C}^{\mathrm{Bar}}(A)\Big)_{\bullet}$$

The relative tensor product $A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A)$ is given by the colimit

$$\operatorname*{colim}_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Bar}_{A\otimes A}\Big(A, \mathrm{C}^{\mathrm{Bar}}(A)\Big)_{\bullet}$$

calculated in $\mathrm{CAlg}(\mathsf{Ch}(k))$ (see the introduction to Section 6.3.3), so comes with a cocone diagram

$$(\boldsymbol{\Delta}^{\mathrm{op}})^{\triangleright} \to \mathrm{CAlg}(\mathsf{Ch}(k))$$

but as we know that the relative tensor product has cofibrant underlying chain complex in this instance, this functor actually factors over $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$. Postcomposing this cocone diagram (as a diagram in $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$) with $\mathrm{CAlg}(\gamma)$, we obtain a cocone diagram from $\mathrm{CAlg}(\gamma) \circ \mathrm{Bar}_{A\otimes A}(A, \mathrm{C}^{\mathrm{Bar}}(A))_{\bullet}$ to $\mathrm{CAlg}(\gamma)(A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A))$, and hence by the universal property of colim a morphism as follows.

$$\begin{aligned}
&\mathrm{CAlg}(\gamma)(A) \otimes_{\mathrm{CAlg}(\gamma)(A\otimes A)} \mathrm{CAlg}(\gamma)\Big(\mathrm{C}^{\mathrm{Bar}}(A)\Big)\\
\simeq\; &\operatorname*{colim}_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Bar}_{\mathrm{CAlg}(\gamma)(A\otimes A)}\Big(\mathrm{CAlg}(\gamma)(A), \mathrm{CAlg}(\gamma)(\mathrm{C}^{\mathrm{Bar}}(A))\Big)_{\bullet}\\
\simeq\; &\operatorname*{colim}_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{CAlg}(\gamma) \circ \mathrm{Bar}_{A\otimes A}(A, \mathrm{C}^{\mathrm{Bar}}(A))_{\bullet}\\
\to\; &\mathrm{CAlg}(\gamma)(A \otimes_{A\otimes A} \mathrm{C}^{\mathrm{Bar}}(A))
\end{aligned}$$

where the the $A \otimes A$-module structures are as in Construction 6.3.2.6 and Construction E.8.0.4. The natural morphism $\mathrm{C}^{\mathrm{Bar}}(A) \to A$ of left-$A \otimes A$-modules from Construction 6.3.2.6 provides a natural transformation

$$\mathrm{CAlg}(\gamma)(A) \otimes_{\mathrm{CAlg}(\gamma)(A\otimes A)} \mathrm{CAlg}(\gamma)\Big(\mathrm{C}^{\mathrm{Bar}}(A)\Big) \to \mathrm{CAlg}(\gamma)(A) \otimes_{\mathrm{CAlg}(\gamma)(A\otimes A)} \mathrm{CAlg}(\gamma)(A)$$

that is an equivalence by Proposition 6.3.2.2[86]. As γ and $\mathrm{CAlg}(\gamma)$ are symmetric monoidal, we can further identify $\gamma(A \otimes A)$ with $\gamma(A) \otimes \gamma(A)$ and the left and right module structures of $\gamma(A)$ over $\gamma(A \otimes A)$ (which arise from the morphism of commutative algebras $A \otimes A \to A$ given by the multiplication morphism) with the module structures arising from the multiplication morphism $\gamma(A) \otimes \gamma(A) \to \gamma(A)$.

We have thus identified the composition $\mathrm{CAlg}(\gamma) \circ \mathrm{C}$ in (6.16) with the functor described by

$$A \mapsto \gamma(A) \otimes_{\gamma(A)\otimes\gamma(A)} \gamma(A)$$

which is precisely the description of $\mathrm{HH} \circ \mathrm{CAlg}(\gamma)$ one obtains from Corollary 6.2.2.6. $\square$

Remark 6.3.4.4. Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then there is a commutative cube

$$\begin{tikzcd}
& \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[rr, "\mathrm{C}"] \arrow[dl] \arrow[dd] & & \mathrm{CAlg}\big(\mathsf{Ch}(k)^{\mathrm{cof}}\big) \arrow[dl] \arrow[dd] \\
\mathrm{CAlg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[rr, "\mathrm{C}"] \arrow[dd] & & \mathrm{CAlg}\big(\mathsf{Ch}(k')^{\mathrm{cof}}\big) \arrow[dd] & \\
& \mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[rr, "\mathrm{HH}"] \arrow[dl] & & \mathrm{CAlg}\big(\mathcal{D}(k)\big) \arrow[dl] \\
\mathrm{CAlg}\big(\mathcal{D}(k')\big) \arrow[rr, "\mathrm{HH}"'] & & \mathrm{CAlg}\big(\mathcal{D}(k')\big) &
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$, where the top square is the one from Remark 6.3.2.8, the bottom square is induced by the one from Remark 6.2.1.6, the left and right squares are induced by the one from Remark 4.3.2.2, and the front and back squares are the ones from Proposition 6.3.4.3. To see this, one goes through the construction of the fillers for the different sides, which are ultimately constructed from symmetric monoidality of different functors and the universal property of colimits – see Remark 6.3.3.1. Using the universal property of colimits, one is left to check commutativity of a diagram of equivalences of simplicial

[86] Using that equivalences of left-$\mathrm{CAlg}(\gamma)(A \otimes A)$-modules are detected by the composition of the forgetful functors $\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{LMod}_{\mathrm{CAlg}(\gamma)(A\otimes A)}(\mathrm{CAlg}(\mathcal{D}(k))) \to \mathrm{CAlg}(\mathcal{D}(k))$ and $\mathrm{ev}_{\mathfrak{a}}\, \mathrm{CAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$ by [HA, 3.2.3.1 (4)] and [HA, 4.2.3.3 (2)].

objects that looks in level n like the outer diagram of equivalences depicted below.

$$\begin{tikzcd}
& \left(k' \otimes_k \gamma(R)\right)^{\otimes_{k'}(n+1)} \arrow[r, dash] \arrow[dd, dash, dashed] & \left(\gamma(k' \otimes_k R)\right)^{\otimes_{k'}(n+1)} \arrow[dd, dash, dashed] \arrow[dr, dash] & \\
k' \otimes_k \left(\gamma(R)^{\otimes_k(n+1)}\right) \arrow[ur, dash] \arrow[dr, dash] & & & \gamma\left((k' \otimes_k R)^{\otimes_{k'}(n+1)}\right) \arrow[dl, dash] \\
& k' \otimes_k \gamma\left(R^{\otimes_k(n+1)}\right) \arrow[r, dash] & \gamma\left(k' \otimes_k \left(R^{\otimes_k(n+1)}\right)\right) &
\end{tikzcd}$$

The two diagonal equivalences on the left and right arise from γ and $k' \otimes_k -$ being symmetric monoidal, and the two horizontal equivalences arise from the commutative diagram

$$\begin{tikzcd}
\mathsf{Ch}(k)^{\mathrm{cof}} \arrow[r, "k' \otimes_k -"] \arrow[d, "\gamma"'] & \mathsf{Ch}(k')^{\mathrm{cof}} \arrow[d, "\gamma"] \\
\mathcal{D}(k) \arrow[r, "k' \otimes_k -"'] & \mathcal{D}(k')
\end{tikzcd}$$

from Remark 4.3.2.2. This latter commutative square is actually a commutative square of symmetric monoidal functors, which is how we obtain the filler for the above diagram: The dashed equivalences (defined so as to make the left and right triangle commute) are precisely the equivalences exhibiting the compositions $k' \otimes_k \gamma(-)$ and $\gamma(k' \otimes_k -)$ as symmetric monoidal functors, and the filler for the square in the middle is the one exhibiting the homotopy between those two compositions being an homotopy of symmetric monoidal functors. $\diamondsuit$

Chapter 7.

Hochschild homology of polynomial algebras

In Definition 6.2.1.2 we defined a monoidal functor

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\colon \mathrm{Alg}\big(\mathcal{D}(k)\big) \to \mathcal{M}\mathrm{ixed}$$

that thus induces a functor

$$\mathrm{Alg}_{\mathbb{E}_2}\big(\mathcal{D}(k)\big) \simeq \mathrm{Alg}\Big(\mathrm{Alg}\big(\mathcal{D}(k)\big)\Big) \to \mathrm{Alg}(\mathcal{M}\mathrm{ixed})$$

that we will also denote by $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$.

An important collection of examples of commutative (so in particular $\mathbb{E}_2$-) algebras in $\mathcal{D}(k)$ is given by polynomial algebras, i. e. algebras of the form $k[X]$ for X a set[1], and the goal of this chapter is to describe $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of polynomial algebras as algebras in $\mathcal{M}\mathrm{ixed}$. Concretely, given a set X, we would like to obtain a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$, as an object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, i. e. an object A in $\mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})$ such that there is an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \simeq \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(A)$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. We would also like A to be as efficient (i. e. small) as possible.

By the results of Section 6.3.4 we already know that the standard Hochschild complex $\mathrm{C}(k[X])$ of a polynomial k-algebra $k[X]$, considered as either a commutative differential graded algebra, or a strict mixed complex, represents HH and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of $k[X]$, respectively. However, we have no comparison result available that compares $\mathrm{C}(k[X])$ and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as associative algebras of mixed complexes – while the standard Hochschild complex is a strict mixed complex as well as a differential graded algebra, it satisfies the Leibniz rule only up to homotopy, so we can not even consider it as a strict algebra in strict mixed complexes[2]! Even without this obstacle, $\mathrm{C}(k[X])$ would not be the kind of strict model we hope for, as it is not very efficient.

The first step on the road to finding a small strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ thus needs to be to define an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ that we later hope to prove is such a strict model. For R a commutative k-algebra we will thus in Section 7.1

[1] See Definition 7.0.0.1 for a definition.
[2] See Warning 6.3.2.13

review the definition of the strict mixed complex of de Rham forms on R, denoted by $\Omega^\bullet_{R/k}$, which has a very concise description. Indeed, as the underlying complex has no non-zero boundary operators, so it is not possible to find a "smaller" quasiisomorphic chain complex.

Our goal, which we will only be able to prove if $|X| \leq 2$, and which is formulated as Conjecture B, is then to produce an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \simeq \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^\bullet_{k[X]/k}\Big)$$

in Alg($\mathcal{M}$ixed), i. e. to show that $\Omega^\bullet_{k[X]/k}$ is a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object of Alg($\mathcal{M}$ixed).

In Section 7.2 we will begin comparing $\Omega^\bullet_{k[X]/k}$ with Hochschild homology of $k[X]$ by constructing a quasiisomorphism ϵ_X from $\Omega^\bullet_{k[X]/k}$ to the normalized standard Hochschild complex $\overline{\mathrm{C}}(k[X])$. This quasiisomorphism is multiplicative, so as we already know that $\mathrm{C}(k[X])$, and hence also $\overline{\mathrm{C}}(k[X])$, is a strict model for $\mathrm{HH}(k[X])$ as an object of $\mathrm{Alg}(\mathcal{D}(k))$, we can conclude that $\Omega^\bullet_{k[X]/k}$ is so as well.

To show that $\Omega^\bullet_{k[X]/k}$ is also a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object of $\mathcal{M}$ixed it would suffice to show that ϵ_X is even a morphism of strict mixed complexes. This is unfortunately not the case, but we can instead upgrade ϵ_X to a strongly homotopy linear quasiisomorphism[3], and will do so in Section 7.3.

The partial results regarding only the algebra and only the mixed structure from Sections 7.2 and 7.3 will then be used as input in Section 7.4, where we will show that $\Omega^\bullet_{k[X]/k}$ is even a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object of Alg($\mathcal{M}$ixed) as long as $|X| \leq 2$.

Suppose now that X is a set with $|X| \leq 2$ and f an element of $k[X]$. Denote the morphism of commutative k-algebras $k[t] \to k[X]$ that maps t to f by F. Now that we know that $\Omega^\bullet_{k[t]/k}$ represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t])$ and $\Omega^\bullet_{k[X]/k}$ represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ we can ask whether the induced morphism $\Omega^\bullet_{F/k}$ also represents the morphism $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)$ in Alg($\mathcal{M}$ixed). We are thus asking for a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{F/k}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[Y]\big) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^\bullet_{k[Y]/k}\Big)
\end{tikzcd}$$

in Alg($\mathcal{M}$ixed) such that the two horizontal morphisms are equivalences. We will formulate the claim that such a square exists for F as Conjecture C, and prove this conjecture for $|X| \leq 1$, as well as for $|X| = 2$ as long as 2 is invertible in k, in Section 7.5. We will also discuss Conjecture D, which is very closely related to Conjecture C and will be an essential ingredient in the results of later chapters.

[3] See Section 4.2.3 for this notion.

We end the introduction to this chapter by fixing some notation concerning polynomial algebras.

Definition 7.0.0.1. Let X be a set. Then $k[X]$ denotes the *polynomial k-algebra generated by X*, i. e. the free commutative k-algebra generated by X. Its underlying k-module is free, and a basis is given by elements of the form[4] $x^{\vec{i}}$ with $\vec{i}$ an element of $\mathbb{Z}_{\geq 0}^{\times X}$ such that all but finitely many components are zero. We also use notation such as $k[x_1, \ldots, x_n]$ for the polynomial k-algebra that is generated by n formal variables $x_1, \ldots, x_n$, and trust that this will not lead to confusion.

Note that as the underlying k-module of a polynomial k-algebra is free, a polynomial k-algebra is cofibrant when considered as a chain complex concentrated in degree 0.[5] ♢

7.1. The mixed complex of de Rham forms

Given a commutative k-algebra R, we denote by $\Omega^1_{R/k}$ the k-module of *Kähler differentials* – for a definition see [Lod98, 1.1.9 and 1.3.7 to 1.3.9]. One then defines [Lod98, 1.3.11] $\Omega^n_{R/k}$ for $n \geq 0$ to be the exterior product $\Lambda^n_R \Omega^1_{R/k}$. Equipping $\Omega^\bullet_{R/k}$ with the zero boundary operator we obtain a commutative differential graded algebra. $\Omega^1_{R/k}$ also comes with a derivation [Lod98, 1.3.8] $\mathrm{d}\colon \Omega^0_{R/k} = R \to \Omega^1_{R/k}$, and the unique extension of d to an operator of degree 1 on $\Omega^\bullet_{R/k}$ that satisfies $\mathrm{d} \circ \mathrm{d} = 0$ and the Leibniz rule makes $\Omega^\bullet_{R/k}$ into an object of $\mathrm{CAlg}(\mathsf{Mixed})$[6], called the *mixed complex of de Rham forms of R*. Elements of $\Omega^n_{R/k}$ are of the form $r_0 \,\mathrm{d}\, r_1 \cdots \mathrm{d}\, r_n$, with

$$\mathrm{d}(r_0 \,\mathrm{d}\, r_1 \cdots \mathrm{d}\, r_n) = \mathrm{d}\, r_0 \,\mathrm{d}\, r_1 \cdots \mathrm{d}\, r_n$$

and

$$(r_0 \,\mathrm{d}\, r_1 \cdots \mathrm{d}\, r_n) \cdot \big(r'_0 \,\mathrm{d}\, r'_1 \cdots \mathrm{d}\, r'_m\big) = r_0 r'_0 \,\mathrm{d}\, r_1 \cdots \mathrm{d}\, r_n \,\mathrm{d}\, r'_1 \cdots \mathrm{d}\, r'_m$$

describing the differential and multiplication [Lod98, 1.3.11 and 2.3.1]. This construction is functorial in morphisms of commutative k-algebras $f\colon R \to R'$ – there is a unique morphism in $\mathrm{CAlg}(\mathsf{Mixed})$ from $\Omega^\bullet_{R/k}$ to $\Omega^\bullet_{R'/k}$ that is given by f in degree 0.

For $R = k[X]$ for some set X, the $k[X]$-module $\Omega^1_{k[X]/k}$ is free with basis given by $\{\, \mathrm{d}\, x \mid x \in X \,\}$ – see [Lod98, 1.3.10 and 1.3.11]. It follows that we can identify $\Omega^\bullet_{k[X]/k}$ with $k[X] \otimes \Lambda_k(k \cdot \{\, \mathrm{d}\, x \mid x \in X \,\})$, where $k \cdot \{\, \mathrm{d}\, x \mid x \in X \,\}$ is the chain complex that is freely generated by $\{\, \mathrm{d}\, x \mid x \in X \,\}$, where we give the elements $\mathrm{d}\, x$ chain degree 1. In particular, $\Omega^\bullet_{k[X]/k}$ is levelwise free as a k-module, and hence cofibrant by [Hov99, 2.3.6]. We can thus make the following definition.

Definition 7.1.0.1. We denote by

$$\Omega^\bullet_{-/k}\colon \mathrm{CAlg}\big(\mathrm{LMod}_k(\mathsf{Ab})\big) \to \mathrm{CAlg}(\mathsf{Mixed})$$

[4] See Section 2.3 (32) for this notation.
[5] See [Hov99, 2.3.6]
[6] See Remark 4.2.1.12.

the functor sending a k-algebra R to the commutative algebra in strict mixed complexes $\Omega^\bullet_{R/k}$ discussed above. We also denote by[7]

$$\Omega^\bullet_{k[-]/k}\colon \mathsf{Set} \to \mathrm{CAlg}(\mathsf{Mixed}_{\mathrm{cof}})$$

the functor sending a set X to $\Omega^\bullet_{k[X]/k}$. ♢

Remark 7.1.0.2. $\Omega^\bullet_{-/k}$ is also functorial in k: For $\varphi\colon k \to k'$ a morphism of commutative rings and R a k-algebra, there is an evident isomorphism

$$k' \otimes_k \Omega^\bullet_{R/k} \cong \Omega^\bullet_{k'\otimes_k R/k'}, \qquad a \otimes (r_0\,\mathrm{d}\,r_1 \cdots \mathrm{d}\,r_n) \mapsto (a \otimes r_0)\,\mathrm{d}(1 \otimes r_1) \cdots \mathrm{d}(1 \otimes r_n)$$

in $\mathrm{CAlg}(\mathsf{Mixed}_{k'})$ that is natural in R and exhibits

$$\begin{tikzcd}
\mathrm{CAlg}\big(\mathrm{LMod}_k(\mathsf{Ab})\big) \arrow[r, "\Omega^\bullet_{-/k}"] \arrow[d, "k'\otimes_k -"'] & \mathrm{CAlg}(\mathsf{Mixed}_k) \arrow[d, "k'\otimes_k -"] \\
\mathrm{CAlg}\big(\mathrm{LMod}_{k'}(\mathsf{Ab})\big) \arrow[r, "\Omega^\bullet_{-/k'}"'] & \mathrm{CAlg}(\mathsf{Mixed}_{k'})
\end{tikzcd}$$

as a commutative diagram in Cat. ♢

7.2. De Rham forms as a strict model in $\mathrm{CAlg}(\mathsf{Ch}(k))$

The reason the mixed complex of de Rham forms is relevant for us is the close relationship with the (normalized) standard Hochschild complex that we will discuss in this section.

In Section 6.3.2.1 we discussed the bar resolution $\mathrm{C}^{\mathrm{Bar}}(A)$ of an associative algebra A and saw in Proposition 6.3.2.4 that the standard Hochschild complex of A is given by the relative tensor product $A \otimes_{A\otimes A^{\mathrm{op}}} \mathrm{C}^{\mathrm{Bar}}(A)$. In Section 7.2.1 we will, for a set X, construct a morphism $\widetilde{\epsilon}_X$ of left-$k[X] \otimes k[X]$-modules (in chain complexes) $\mathrm{C}^{\mathrm{sm}}(X) \to \mathrm{C}^{\mathrm{Bar}}(k[X])$. Tensoring with $k[X]$ over $k[X] \otimes k[X]$ we then obtain a morphism of chain complexes that we will be able to identify with a morphism $\Omega^\bullet_{k[X]/k} \to \mathrm{C}(k[X])$. In this manner we will obtain a natural transformation

$$\epsilon\colon \Omega^\bullet_{k[-]/k} \to \overline{\mathrm{C}}\big(k[-]\big)$$

of functors $\mathsf{Set} \to \mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$ that will turn out to be a pointwise quasiisomorphism, thereby providing a convenient multiplicative model $\Omega^\bullet_{k[X]/k}$ for $\mathrm{HH}(k[X])$. This will be discussed in Section 7.2.2.

While ϵ_X (for a set X) is a morphism of differential graded algebras, it is not a morphism of strict mixed complexes. However ϵ_X can be upgraded to a strongly homotopy linear morphism in the sense of Section 4.2.3. This will be shown in the next section, Section 7.3.

[7]See Definition 4.2.1.2 for a definition of $\mathsf{Mixed}_{\mathrm{cof}}$.

7.2.1. A smaller replacement for the bar complex

In this section we will first construct $\mathrm{C}^{\mathrm{sm}}(X)$ and $\widetilde{\epsilon}_X$ in Construction 7.2.1.1, before showing that they have good homotopical properties in Proposition 7.2.1.2.

Construction 7.2.1.1 ([Lod98, 3.2.2])**.** Let X be a set. We will construct a commutative triangle of left-$k[X] \otimes k[X]$-modules in $\mathsf{Ch}(k)$

$$\begin{tikzcd} \mathrm{C}^{\mathrm{sm}}(X) \arrow[rr, "\widetilde{\epsilon}_X"] \arrow[dr] & & \mathrm{C}^{\mathrm{Bar}}\big(k[X]\big) \arrow[dl] \\ & k[X] & \end{tikzcd} \tag{7.1}$$

where $\mathrm{C}^{\mathrm{Bar}}(k[X])$ refers to the bar resolution as constructed in Construction 6.3.2.1, and the right diagonal morphism is the one also defined in Construction 6.3.2.1. We will use notation from Section 2.3 (34).

Definition of $\mathrm{C}^{\mathrm{sm}}(X)$ *as a graded left-*$k[X] \otimes k[X]$*-module*: We define

$$\mathrm{C}^{\mathrm{sm}}(X)_n \coloneqq k[X] \otimes \Lambda^n(k \cdot X) \otimes k[X]$$

and the action of $k[X] \otimes k[X]$ as follows, with l', r', l, r elements of $k[X]$ and $x_1, \ldots, x_n$ elements of X.

$$\big(l' \otimes r'\big) \cdot (l \otimes x_1 \cdots x_n \otimes r) \coloneqq l'l \otimes x_1 \cdots x_n \otimes rr'$$

Note that if there exist $i \neq j$ with $x_i = x_j$, then the right hand side is also 0, so the action is well-defined[8].

Definition of the boundary operator on $\mathrm{C}^{\mathrm{sm}}(X)$: We make the following definition for l, r elements of $k[X]$ and $x_1, \ldots, x_n$ elements of X.

$$\begin{aligned} &\partial(l \otimes x_1 \cdots x_n \otimes r) \\ \coloneqq &\sum_{i=1}^{n} (-1)^{i-1}\big((lx_i \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes r) - (l \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_i r)\big) \end{aligned}$$

For well-definedness, assume that $1 \leq j < j' \leq n$ such that $x_j = x_{j'}$. We then have to check that the formula just given for $\partial(l \otimes x_1 \cdots x_n \otimes r)$ is zero. One can immediately see that the summands for $i \notin \{j, j'\}$ vanish, as the middle tensor factor $x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n$ then contains both x_j and $x_{j'}$ as factors. Thus we are left with the following sum.

$$\begin{aligned} &(-1)^{j-1}\big(lx_j \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_n \otimes r\big) \\ - &(-1)^{j-1}\big(l \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_n \otimes x_j r\big) \\ + &(-1)^{j'-1}\big(lx_{j'} \otimes x_1 \cdots x_{j'-1} \cdot x_{j'+1} \cdots x_n \otimes r\big) \end{aligned}$$

[8] See (29) in Section 2.3 for a definition of the exterior algebra $\Lambda(k \cdot X)$.

$$- (-1)^{j'-1}\big(l \otimes x_1 \cdots x_{j'-1} \cdot x_{j'+1} \cdots x_n \otimes x_{j'} r\big)$$

To see that this is zero, we will argue that the first and third terms cancel, the argument for the second and fourth term canceling is completely analogous. For this, we carry out the following calculation.

$$\begin{aligned} &(-1)^{j-1}\big(lx_j \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_n \otimes r\big) \\ &= (-1)^{j-1}\big(lx_j \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{j'-1} \cdot x_{j'} \cdot x_{j'+1} \cdots x_n \otimes r\big) \end{aligned}$$

Using that $x_{j'} = x_j$.

$$= (-1)^{j-1}\big(lx_{j'} \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{j'-1} \cdot x_j \cdot x_{j'+1} \cdots x_n \otimes r\big)$$

Now we move the factor x_j in the inner tensor factor to the spot between x_{j-1} and x_{j+1}. This involves moving past $j' - j - 1$ other factors, so incurs a sign $(-1)^{j'-j-1}$.

$$\begin{aligned} &= (-1)^{j-1}(-1)^{j'-j-1}\big(lx_{j'} \otimes x_1 \cdots x_{j'-1} \cdot x_{j'+1} \cdots x_n \otimes r\big) \\ &= -(-1)^{j'-1}\big(lx_{j'} \otimes x_1 \cdots x_{j'-1} \cdot x_{j'+1} \cdots x_n \otimes r\big) \end{aligned}$$

It is clear from the definition that ∂ is compatible with the left-$k[X] \otimes k[X]$-module structure.

∂ squares to zero on $\mathrm{C}^{\mathrm{sm}}(X)$: For l, r elements of $k[X]$ and $x_1, \ldots, x_n$ elements of X we obtain the following calculation[9], where we use $1_{j>i}$ as ad hoc notation for 0 if $j \not> i$ and 1 if $j > i$.

$$\begin{aligned} &\partial\big(\partial(l \otimes x_1 \cdots x_n \otimes r)\big) \\ &= \partial\left(\sum_{i=1}^{n} (-1)^{i-1}\big((lx_i \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes r)\big)\right) \\ &\quad - \partial\left(\sum_{i=1}^{n} (-1)^{i-1}\big((l \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_i r)\big)\right) \end{aligned}$$

The indices in the sums below range from 1 to n.

$$\begin{aligned} &= + \sum_{i \neq j} (-1)^{i-1}(-1)^{j-1_{j>i}-1}\big(lx_i x_j \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes r\big) \\ &\quad - \sum_{i \neq j} (-1)^{i-1}(-1)^{j-1_{j>i}-1}\big(lx_i \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_j r\big) \\ &\quad - \sum_{i \neq j} (-1)^{i-1}(-1)^{j-1_{j>i}-1}\big(lx_j \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_i r\big) \\ &\quad + \sum_{i \neq j} (-1)^{i-1}(-1)^{j-1_{j>i}-1}\big(l \otimes x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_j x_i r\big) \end{aligned}$$

The second and third line cancel by pairing the summand within the second line indexed by (i, j) with the summand within the third line indexed by (j, i), as the sign arising

[9] $x_1 \cdots x_{j-1} \cdot x_{j+1} \cdots x_{i-1} \cdot x_{i+1} \cdots x_n$ is to be as interpreted as the product from x_1 to x_n while omitting x_j and x_i, also when $j > i$.

from the $1_{j>i}$ expression will differ between the two terms. Furthermore, the first and fourth line each already vanish individually, which one sees by pairing the summand indexed by (i,j) with the one indexed by (j,i).

Definition of $\mathrm{C}^{\mathrm{sm}}(X) \to k[X]$ *as a morphism of graded* $k[X] \otimes k[X]$*-modules*: We define this morphism to be given by

$$(l \otimes x_1 \cdots x_n \otimes r) \mapsto \begin{cases} l \cdot r & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

for l, r elements of $k[X]$ and $x_1, \ldots, x_n$ elements of X. It is clear that this is well-defined and compatible with the $k[X] \otimes k[X]$-action.

Compatibility of $\mathrm{C}^{\mathrm{sm}}(X) \to k[X]$ *with* ∂: Let l and r be elements of $k[X]$ and x an element of X. We have to show that $\partial(l \otimes x \otimes r)$ is mapped to zero. But we have $\partial(l \otimes x \otimes r) = lx \otimes r - l \otimes xr$, which is mapped to $lxr - lxr = 0$.

Definition of $\widetilde{\epsilon}_X$ *as a morphism of graded* $k[X] \otimes k[X]$*-modules*: For l and r elements of $k[X]$ and $x_1, \ldots, x_n$ elements of X, we make the following definition.

$$\widetilde{\epsilon}_X(l \otimes x_1 \cdots x_n \otimes r) \coloneqq \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r$$

To see that this is well-defined on $k[X] \otimes \Lambda^n(k \cdot X) \otimes k[X]$, we need to verify that the formula on the right hand side is 0 if $x_i = x_j$ for some $1 \leq i < j \leq n$. But we can split up Σ_n as the union of left cosets of the subgroup $\{\mathrm{id}, (i\ j)\}$ in Σ_n, where $(i\ j)$ denotes the transposition that exchanges i and j, and thus carry out the following calculation.

$$\begin{aligned} &\sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\ = &\sum_{[\sigma] \in \Sigma_n/(i\ j)} \Big(\operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\ &\qquad + \operatorname{sgn}\big(\sigma \circ (i\ j)\big) l \otimes x_{(i\ j)\left(\sigma^{-1}(1)\right)} \otimes \cdots \otimes x_{(i\ j)\left(\sigma^{-1}(n)\right)} \otimes r \Big) \end{aligned}$$

As $x_i = x_j$, we can simplify the indices of x in the second summand. We also use that $\operatorname{sgn}\big((i\ j)\big) = -1$.

$$\begin{aligned} = &\sum_{[\sigma] \in \Sigma_n/(i\ j)} \big(\operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \\ &\qquad - \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \big) \\ = &\ 0 \end{aligned}$$

That the definition of $\widetilde{\epsilon}_X$ is compatible with the left-$k[X] \otimes k[X]$-module structures is clear.

Some comments on how to relate $\widetilde{\epsilon}_X$ *with actions of* Σ_n: We can define an action of the symmetric group Σ_n on $\mathrm{C}^{\mathrm{Bar}}(k[X])_n$ that is given by permuting the inner n tensor factors, i. e. we make the following definition for $y_0, \ldots, y_{n+1}$ elements of $k[X]$.

$$\sigma \cdot (y_0 \otimes y_1 \otimes \cdots \otimes y_n \otimes y_{n+1}) \coloneqq y_0 \otimes y_{\sigma^{-1}(1)} \otimes \cdots \otimes y_{\sigma^{-1}(n)} \otimes y_{n+1}$$

In particular we can then write $\widetilde{\epsilon}_X$ as follows, where $l, x_1, \ldots, x_n, r$ are elements of $k[X]$.

$$\widetilde{\epsilon}_X(l \otimes x_1 \cdots x_n \otimes r) = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)\big(\sigma \cdot (l \otimes x_1 \cdots x_n \otimes r)\big)$$

Finally, let us note that if S is a set with n elements and we write an element of $\mathrm{C}^{\mathrm{Bar}}(k[X])_n$ as $l \otimes y_{\varphi(1)} \otimes \cdots \otimes y_{\varphi(n)} \otimes r$ for $\varphi\colon \{1, \ldots, n\} \to S$ a bijection and $l, y_{\varphi(1)}, \ldots, y_{\varphi(n)}, r$ elements of $k[X]$, then the action of $\sigma \in \Sigma_n$ takes the following form.

$$\sigma \cdot \big(l \otimes y_{\varphi(1)} \otimes \cdots \otimes y_{\varphi(n)} \otimes r\big) = l \otimes y_{\varphi(\sigma^{-1}(1))} \otimes \cdots \otimes y_{\varphi(\sigma^{-1}(n))} \otimes r \qquad (*)$$

Compatibility of $\widetilde{\epsilon}_X$ with ∂: We carry out the following calculation, for l and r elements of $k[X]$ and $x_1, \ldots, x_n$ elements of X.

$$\begin{aligned}
&\partial\big(\widetilde{\epsilon}_X(l \otimes x_1 \cdots x_n \otimes r)\big)\\
&= \partial\left(\sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r\right)
\end{aligned}$$

We apply the formula for the boundary operator of $\mathrm{C}^{\mathrm{Bar}}\big(k[X]\big)$ as defined in Construction 6.3.2.1, writing the summands for $i = 0$ and $i = n$ as separate terms.

$$\begin{aligned}
&= \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) l x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r\\
&\quad + \sum_{i=1}^{n-1} (-1)^i \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i)} x_{\sigma^{-1}(i+1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r\\
&\quad + (-1)^n \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n-1)} \otimes x_{\sigma^{-1}(n)} r
\end{aligned}$$

We now split up the set Σ_n the sum in the second line is indexed over as the union of the right cosets of the subgroup generated by the transposition $(i\ i+1)$. Note that the right cosets have the form $\{\sigma, (i\ i+1)\sigma\}$.

$$\begin{aligned}
&= \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) l x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r\\
&\quad + \sum_{i=1}^{n-1} (-1)^i \sum_{[\sigma] \in (i\ i+1) \backslash \Sigma_n} \Big(\operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i)} x_{\sigma^{-1}(i+1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r\\
&\qquad\qquad + \operatorname{sgn}\big((i\ i+1) \circ \sigma\big) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i+1)} x_{\sigma^{-1}(i)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r\Big)\\
&\quad + (-1)^n \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n-1)} \otimes x_{\sigma^{-1}(n)} r
\end{aligned}$$

$$= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r$$
$$+ \sum_{i=1}^{n-1} (-1)^i \sum_{[\sigma]\in(i\; i+1)\backslash\Sigma_n} \Big(\operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i)} x_{\sigma^{-1}(i+1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r$$
$$- \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i+1)} x_{\sigma^{-1}(i)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r \Big)$$
$$+ (-1)^n \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n-1)} \otimes x_{\sigma^{-1}(n)} r$$

The middle summands now cancel, using that $x_{\sigma^{-1}(i)}$ and $x_{\sigma^{-1}(i+1)}$ commute in $k[X]$.

$$= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \otimes r$$
$$+ (-1)^n \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) l \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n-1)} \otimes x_{\sigma^{-1}(n)} r$$

Now let σ' be an element of Σ_n and assume that i is such that $\sigma'(i) = 1$. Then $\sigma = \sigma_{1\to n} \circ \sigma' \circ \sigma_{n\to i}$ fixes n, so that we can consider σ as an element of[10] Σ_{n-1}. The upshot is that if σ' maps i to 1, then we can write it uniquely as $\sigma' = \sigma_{n\to 1} \circ \sigma \circ \sigma_{i\to n}$ for σ an element of Σ_{n-1}. Analogously, if σ' maps i to n, then we can write it uniquely as $\sigma' = \sigma \circ \sigma_{i\to n}$ for σ an element of Σ_{n-1}.

Continuing the calculation from above, we can now rewrite the sums as follows.

$$= \sum_{i=1}^{n} \sum_{\sigma\in\Sigma_{n-1}} \Big(\operatorname{sgn}(\sigma_{n\to 1} \circ \sigma \circ \sigma_{i\to n}) \cdot$$
$$l x_{(\sigma_{n\to 1}\circ\sigma\circ\sigma_{i\to n})^{-1}(1)} \otimes x_{(\sigma_{n\to 1}\circ\sigma\circ\sigma_{i\to n})^{-1}(2)} \otimes \cdots \otimes x_{(\sigma_{n\to 1}\circ\sigma\circ\sigma_{i\to n})^{-1}(n)} \otimes r \Big)$$
$$+ (-1)^n \sum_{i=1}^{n} \sum_{\sigma\in\Sigma_{n-1}} \Big(\operatorname{sgn}(\sigma \circ \sigma_{i\to n}) \cdot$$
$$l \otimes x_{(\sigma\circ\sigma_{i\to n})^{-1}(1)} \otimes \cdots \otimes x_{(\sigma\circ\sigma_{i\to n})^{-1}(n-1)} \otimes x_{(\sigma\circ\sigma_{i\to n})^{-1}(n)} r \Big)$$

The sign of $\sigma_{j\to j'}$ is $(-1)^{j-j'}$, as one can see by writing $\sigma_{j\to j'}$ as the composition of transpositions $\big((j'+1)\; j'\big) \circ \big((j'+2)\; j'+1\big) \cdots \circ \big(j\; (j-1)\big)$ if $j > j'$, and similarly if $j' > j$.

$$= \sum_{i=1}^{n} \sum_{\sigma\in\Sigma_{n-1}} \Big((-1)^{n-1+i-n} \operatorname{sgn}(\sigma) \cdot$$
$$l x_{\sigma_{n\to i}\left(\sigma^{-1}\left(\sigma_{1\to n}(1)\right)\right)} \otimes x_{\sigma_{n\to i}\left(\sigma^{-1}\left(\sigma_{1\to n}(2)\right)\right)} \otimes \cdots \otimes x_{\sigma_{n\to i}\left(\sigma^{-1}\left(\sigma_{1\to n}(n)\right)\right)} \otimes r \Big)$$
$$+ (-1)^n \sum_{i=1}^{n} \sum_{\sigma\in\Sigma_{n-1}} \Big((-1)^{i-n} \operatorname{sgn}(\sigma) \cdot$$
$$l \otimes x_{\sigma_{n\to i}\left(\sigma^{-1}(1)\right)} \otimes \cdots \otimes x_{\sigma_{n\to i}\left(\sigma^{-1}(n-1)\right)} \otimes x_{\sigma_{n\to i}\left(\sigma^{-1}(n)\right)} r \Big)$$

[10] We consider Σ_{n-1} as a subset of Σ_n by extending with $n \mapsto n$.

$$= \sum_{i=1}^{n} \sum_{\sigma \in \Sigma_{n-1}} \Big((-1)^{i-1} \operatorname{sgn}(\sigma) l x_i \otimes x_{\sigma_{n\to i}(\sigma^{-1}(1))} \otimes \cdots \otimes x_{\sigma_{n\to i}(\sigma^{-1}(n-1))} \otimes r \Big)$$
$$- \sum_{i=1}^{n} \sum_{\sigma \in \Sigma_{n-1}} \Big((-1)^{i-1} \operatorname{sgn}(\sigma) l \otimes x_{\sigma_{n\to i}(\sigma^{-1}(1))} \otimes \cdots \otimes x_{\sigma_{n\to i}(\sigma^{-1}(n-1))} \otimes x_i r \Big)$$

We can now apply (∗).

$$= \sum_{i=1}^{n} \sum_{\sigma \in \Sigma_{n-1}} (-1)^{i-1} \operatorname{sgn}(\sigma) \Big(\sigma \cdot \big(l x_i \otimes x_{\sigma_{n\to i}(1)} \otimes \cdots \otimes x_{\sigma_{n\to i}(n-1)} \otimes r \big) \Big)$$
$$- \sum_{i=1}^{n} \sum_{\sigma \in \Sigma_{n-1}} (-1)^{i-1} \operatorname{sgn}(\sigma) \Big(\sigma \cdot \big(l \otimes x_{\sigma_{n\to i}(1)} \otimes \cdots \otimes x_{\sigma_{n\to i}(n-1)} \otimes x_i r \big) \Big)$$

We now evaluate $\sigma_{n\to i}$ in the indices.

$$= \sum_{i=1}^{n} \sum_{\sigma \in \Sigma_{n-1}} (-1)^{i-1} \operatorname{sgn}(\sigma) \big(\sigma \cdot (l x_i \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_n \otimes r) \big)$$
$$- \sum_{i=1}^{n} \sum_{\sigma \in \Sigma_{n-1}} (-1)^{i-1} \operatorname{sgn}(\sigma) \big(\sigma \cdot (l \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_n \otimes x_i r) \big)$$
$$= \sum_{\sigma \in \Sigma_{n-1}} \operatorname{sgn}(\sigma) \left(\sigma \cdot \left(\sum_{i=1}^{n} (-1)^{i-1} l x_i \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_n \otimes r \right) \right)$$
$$- \sum_{\sigma \in \Sigma_{n-1}} \operatorname{sgn}(\sigma) \left(\sigma \cdot \left(\sum_{i=1}^{n} (-1)^{i-1} l \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_n \otimes x_i r \right) \right)$$

We can now plug in the definition of the boundary operator on $\mathrm{C}^{\mathrm{sm}}(X)$.

$$= \sum_{\sigma \in \Sigma_{n-1}} \operatorname{sgn}(\sigma) \Big(\sigma \cdot \big(\partial (l \otimes x_1 \otimes \cdots \otimes x_n \otimes r) \big) \Big)$$

Finally, we can use the definition of $\widetilde{\epsilon}_X$.

$$= \widetilde{\epsilon}_X \big(\partial (l \otimes x_1 \otimes \cdots \otimes x_n \otimes r) \big)$$

Commutativity of diagram (7.1): Clear from the definitions. ♢

We next show that $\widetilde{\epsilon}_X$ is an equivalence between cofibrant replacements of $k[X]$ in $\mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k))$.

Proposition 7.2.1.2. *For X a set the following hold.*

(1) $\mathrm{C}^{\mathrm{sm}}(X)$ *as defined in Construction 7.2.1.1 is cofibrant as an object in the model category* $\mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k))$ *with respect to the model structure of Theorem 4.2.2.1 (where* $\mathsf{Ch}(k)$ *carries the model structure of Fact 4.1.3.1).*

(2) The morphism of chain complexes $\widetilde{\epsilon}_X \colon \mathrm{C}^{\mathrm{sm}}(X) \to \mathrm{C}^{\mathrm{Bar}}(X)$ *as defined in Construction 7.2.1.1 is a quasiisomorphism.* ♡

Proof. Proof of claim (1): The category of left-$k[X] \otimes k[X]$-modules in $\mathsf{Ch}(k)$ is isomorphic to $\mathsf{Ch}(k[X] \otimes k[X])$. We can equip $\mathsf{Ch}(k[X] \otimes k[X])$ with the projective model structure from Fact 4.1.3.1, and comparing weak equivalences and fibrations we then see that the isomorphism between $\mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k))$ and $\mathsf{Ch}(k[X] \otimes k[X])$ is even an isomorphism of model categories. As $\mathrm{C}^{\mathrm{sm}}(X)$ is concentrated in nonnegative degrees and is levelwise free as an $k[X] \otimes k[X]$-module we can then apply [Hov99, 2.3.6], which shows the claim.

Proof of claim (2): The proof of this claim follows the ideas of [Lod98, 3.2.2]. Considering only the underlying chain complexes, it follows directly from the definitions that morphisms in diagram (7.1) are natural in the set X. We thus obtain a commutative triangle

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{sm}}(-) \arrow[rr, "\widetilde{\epsilon}"] \arrow[dr, "p"'] & & \mathrm{C}^{\mathrm{Bar}}\big(k[-]\big) \arrow[dl] \\
& k[-] &
\end{tikzcd}$$

of natural transformations of functors $\mathsf{Set} \to \mathsf{Ch}(k)$. That the right diagonal morphism is a quasiisomorphism has been shown in Proposition 6.3.2.2, so it suffices to show that for any set X the left diagonal morphism $p_X\colon \mathrm{C}^{\mathrm{sm}}(X) \to k[X]$ is a quasiisomorphism.

Both $k[-]$ as well as $\Lambda^n(k\cdot -)$ preserve filtered colimits as functors $\mathsf{Set} \to \mathrm{LMod}_k(\mathsf{Ab})$[11]. Colimits of chain complexes are detected levelwise, the tensor product commutes with colimits in each variable separately, and if J is a filtered category and $n \geq 0$ an integer, then the diagonal functor $\mathsf{J} \to \mathsf{J}^n$ is cofinal [HTT, 5.3.1.22 and 4.1.1.8]. This implies that $\mathrm{C}^{\mathrm{sm}}(-)$ and $k[-]$ preserve filtered colimits as functors $\mathsf{Set} \to \mathsf{Ch}(k)$. Homology preserves filtered colimits as well [Wei94, 2.6.15], so quasiisomorphisms are closed under filtered colimits. As any set can be written as the filtered colimit of its finite subsets, this implies that it suffices to show that $p\colon \mathrm{C}^{\mathrm{sm}}(-) \to k[-]$ is a quasiisomorphism on finite sets.

[11] One can prove this by directly checking the universal property. We sketch this for $\Lambda^n(k\cdot -)$. So let J be a filtered category, $F\colon \mathsf{J} \to \mathsf{Set}$ a functor, Y a k-module, and $g_i\colon \Lambda^n(k\cdot F(i)) \to Y$ a morphism of k-modules for each object i of J such that $g_i \circ (\Lambda^n(k\cdot F(f))) = g_j$ for every morphism $f\colon j \to i$ in J. Then we have to check that there exists a unique morphism of k-modules $g\colon \Lambda^n(k\cdot(\operatorname{colim} F)) \to Y$ such that $g \circ (\Lambda^n(k\cdot \iota_i)) = g_i$ for every object i in J, where $\iota_i\colon F(i) \to \operatorname{colim} F$ is the morphism that exhibits $\operatorname{colim} F$ as a colimit. The k-module $\Lambda^n(k\cdot(\operatorname{colim} F))$ is free, with basis given by elements of the form $x_1 \cdots x_n$ with $x_1, \ldots, x_n$ elements of $\operatorname{colim} F$ such that $x_a \neq x_b$ for $a \neq b$. For such $x_1, \ldots, x_n$, there must be (as J is filtered) an object i of J and elements $x'_1, \ldots, x'_n$ of $F(i)$ such that $x_a = \iota_i(x'_a)$ for $1 \leq a \leq n$ (filteredness was used to find a single such i that works for all n elements at once). But then we must have $g(x_1 \cdots x_n) = (g \circ (\Lambda^n(k\cdot\iota_i)))(x'_1 \cdots x'_n) = g_i(x'_1 \cdots x'_n)$. This shows uniqueness. If i' is a different object of J and $x''_1, \ldots, x''_n$ elements of $F(i')$ such that $x_a = \iota_{i'}(x''_a)$ for $1 \leq a \leq n$, then, as J is filtered, there must exist morphisms $f\colon i \to j$ and $f'\colon i' \to j$ in J such that $F(f)(x'_a) = F(f')(x''_a)$ for $1 \leq a \leq n$. We thus obtain

$$\begin{aligned}
g_i\big(x'_1 \cdots x'_n\big) &= \big(g_j \circ (\Lambda^n(k\cdot F(f)))\big)\big(x'_1 \cdots x'_n\big) = g_j\big(F(f)(x'_1) \cdots F(f)(x'_n)\big) \\
&= g_j\big(F(f')(x''_1) \cdots F(f')(x''_n)\big) = \big(g_j \circ (\Lambda^n(k\cdot F(f')))\big)\big(x''_1 \cdots x''_n\big) = g_{i'}\big(x''_1 \cdots x''_n\big)
\end{aligned}$$

so that the above formula for $g(x_1 \cdots x_n)$ is independent of the choice of $x'_1, \ldots x'_n$, which implies that this defines a morphism g that is compatible with the g_i as required.

Now suppose that the set X is the disjoint union of Y and Y', with $\iota\colon Y \to X$ and $\iota'\colon Y' \to X$ the inclusions. We obtain a commutative diagram of chain complexes as follows, to be explained below.

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{sm}}(Y) \otimes \mathrm{C}^{\mathrm{sm}}(Y') \arrow[r] \arrow[d, "p_Y \otimes p_{Y'}"'] & \mathrm{C}^{\mathrm{sm}}(X) \arrow[d, "p_X"] \\
k[Y] \otimes k[Y'] \arrow[r] & k[X]
\end{tikzcd}$$

The top horizontal morphism is defined by k-linearly extending the assignment

$$\begin{gathered}
(l \otimes y_1 \cdots y_n \otimes r) \otimes (l' \otimes y'_1 \cdots y'_n \otimes r') \\
\mapsto k[\iota](l) \cdot k[\iota'](l') \otimes \iota(y_1) \cdots \iota(y_n) \cdot \iota'\big(y'_1\big) \cdots \iota'\big(y'_n\big) \otimes k[\iota](r) \cdot k[\iota'](r')
\end{gathered}$$

where l, r are elements of $k[Y]$, $y_1, \ldots y_n$ are elements of Y, l', r' are elements of $k[Y']$, and $y'_1, \ldots, y'_n$ are elements of Y'. It is immediate that this is well-defined, and checking compatibility with the boundary operator requires only unpacking the definitions and using that $k[X]$ is commutative. The bottom horizontal morphism is given by composing $k[\iota] \otimes k[\iota']$ with the multiplication $k[X] \otimes k[X] \to k[X]$.

Both the horizontal morphisms in the above diagram are isomorphisms, as one can easily see by considering the respective bases consisting of tensor products of monomials. To show that p_X is a quasiisomorphism, it thus suffices to show that $p_Y \otimes p_{Y'}$ is a quasiisomorphism.

Assume for the moment that p_Y and $p_{Y'}$ are quasiisomorphisms. As $k[Y]$ and $k[Y']$ are concentrated in degree 0, we can read off their homology and can thus conclude that $\mathrm{C}^{\mathrm{sm}}(Y)$, $\mathrm{C}^{\mathrm{sm}}(Y')$, $k[Y]$, and $k[Y']$ are all chain complexes that have free homology. The Künneth spectral sequences[12] that converge to the homology of the tensor products $\mathrm{C}^{\mathrm{sm}}(Y) \otimes \mathrm{C}^{\mathrm{sm}}(Y')$ and $k[Y] \otimes k[Y']$ thus collapse already on the second page, from which we can deduce that $p_Y \otimes p_{Y'}$ is also a quasiisomorphism.

It thus suffices to show that p_Y and $p_{Y'}$ are quasiisomorphisms in order to conclude that p_X is a quasiisomorphism as well, if X is the disjoint union of Y and Y'. As every finite set can be written as the disjoint union of sets that have exactly one element, we have thus reduced the claim to showing that $p_{\{x\}}$ is a quasiisomorphism.

We now show that $p_{\{x\}}$ is a chain homotopy equivalence. Note that the chain complex $\Lambda(k \cdot \{x\})$ is free with basis 1 in degree 0, free with basis x in degree 1, and zero in other degrees. We can define a section s of $p_{\{x\}}$ by $s(r) = 1 \otimes r$, so it suffices to construct a morphism of k-modules $h\colon k[x] \otimes k[x] \to k[x] \otimes k \cdot \{x\} \otimes k[x]$ that satisfies $\partial \circ h = \mathrm{id} - s \circ p_{\{x\}}$ on elements of degree 0 and $h \circ \partial = \mathrm{id}$ on elements of degree 1. For this we define h as follows on basis elements, where $n, m \geq 0$.

$$h(x^n \otimes x^m) \coloneqq \sum_{i=0}^{n-1} x^i \otimes x \otimes x^{n+m-i-1}$$

[12]See for example [Rot08, 10.90].

Then we obtain the following calculation for the first identity.

$$\begin{aligned}
&\partial\big(h(x^n \otimes x^m)\big)\\
&= \sum_{i=0}^{n-1} \partial\Big(x^i \otimes x \otimes x^{n+m-i-1}\Big)\\
&= \sum_{i=0}^{n-1} \Big(x^{i+1} \otimes x^{n+m-i-1} - x^i \otimes x^{n+m-i}\Big)\\
&= \sum_{i=1}^{n} x^i \otimes x^{n+m-i} - \sum_{i=0}^{n-1} x^i \otimes x^{n+m-i}\\
&= x^n \otimes x^m - 1 \otimes x^{n+m}\\
&= \big(\mathrm{id} - s \circ p_{\{x\}}\big)(x^n \otimes x^m)
\end{aligned}$$

The following calculation shows the second identity.

$$\begin{aligned}
&h\big(\partial(x^n \otimes x \otimes x^m)\big)\\
&= h\big(x^{n+1} \otimes x^m\big) - h\big(x^n \otimes x^{m+1}\big)\\
&= \sum_{i=0}^{n} x^i \otimes x \otimes x^{n+m-i} - \sum_{i=0}^{n-1} x^i \otimes x \otimes x^{n+m-i}\\
&= x^n \otimes x \otimes x^m\\
&= \mathrm{id}(x^n \otimes x \otimes x^m)
\end{aligned}$$

This proves the claim. □

7.2.2. A quasiisomorphism between de Rham forms and the standard Hochschild complex

In this section we define and discuss ϵ, a natural quasiisomorphism from $\Omega^\bullet_{k[-]/k}$ to $\overline{\mathrm{C}}(k[-])$.

Construction 7.2.2.1. For every set X we are going to construct a morphism of chain complexes

$$\epsilon_X \colon \Omega^\bullet_{k[X]/k} \to \overline{\mathrm{C}}(k[X])$$

where $\overline{\mathrm{C}}$ refers to the normalized standard Hochschild complex defined in Proposition 6.3.1.10.

So let X be a set. We define ϵ_X as a composition as follows, where we will explain the individual morphisms below.

$$\begin{tikzcd}
\Omega^{\bullet}_{k[X]/k} \arrow[rr, "\epsilon_X"] \arrow[d, "\epsilon'_X"', "\cong"] & & \overline{\mathrm{C}}\big(k[X]\big) \\
k[X] \otimes \Lambda(k \cdot X) \arrow[d, "\epsilon''_X"', "\cong"] & & \mathrm{C}\big(k[X]\big) \arrow[u] \\
k[X] \otimes_{k[X]\otimes k[X]} \mathrm{C}^{\mathrm{sm}}(X) \arrow[rr, "k[X]\otimes_{k[X]\otimes k[X]}\widetilde{\epsilon}_X"'] & & k[X] \otimes_{k[X]\otimes k[X]} \mathrm{C}^{\mathrm{Bar}}\big(k[X]\big) \arrow[u, "\cong", "\epsilon'''_X"']
\end{tikzcd}$$

In $k[X] \otimes \Lambda(k \cdot X)$ the elements of X in the exterior product are to have degree 1, and we make the resulting graded k-module into a chain complex by equipping it with the zero boundary operator. The isomorphism ϵ'_X is then the one suggested in Section 7.1, its inverse is defined by

$$l \otimes x_1 \cdots x_n \mapsto l \cdot \mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n$$

where l is an element of $k[X]$ and $x_1, \ldots, x_n$ are elements of X.

$\mathrm{C}^{\mathrm{sm}}(X)$ is as in Construction 7.2.1.1, so is given by $k[X] \otimes \Lambda(k \cdot X) \otimes k[X]$ as a graded k-module. We can thus define ϵ''_X as

$$l \otimes x_1 \cdots x_n \mapsto l \otimes (1 \otimes x_1 \cdots x_n \otimes 1)$$

where l is an element of $k[X]$ and $x_1, \ldots, x_n$ are elements of X, and it is clear that this is an isomorphism of graded k-modules. We still have to check that ϵ''_X is a morphism of chain complexes, i. e. is compatible with the boundary operators, which the following calculations shows it is.

$$\begin{aligned}
&\partial\big(l \otimes (1 \otimes x_1 \cdots x_n \otimes 1)\big) \\
&= \sum_{i=1}^{n} (-1)^{i-1} l \otimes (x_i \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes 1 - 1 \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes x_i) \\
&= \sum_{i=1}^{n} (-1)^{i-1} \big(x_i l \otimes (1 \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes 1) \\
&\qquad\qquad - l x_i \otimes (1 \otimes x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \otimes 1)\big) \\
&= \sum_{i=1}^{n} (-1)^{i-1} 0 = 0
\end{aligned}$$

$\widetilde{\epsilon}_X$ was defined in Construction 7.2.1.1, and the lower horizontal morphism is just the induced one. The isomorphism ϵ'''_X is to be the isomorphism from Proposition 6.3.2.4, given by

$$a \otimes (a_0 \otimes \cdots \otimes a_{n+1}) \mapsto (a_{n+1} \cdot a \cdot a_0) \otimes a_1 \otimes \cdots \otimes a_n$$

with $a, a_0, \ldots, a_n$ elements of $k[X]$. Finally, the morphism from the standard Hochschild complex to the normalized standard Hochschild complex is the quotient morphism from Proposition 6.3.1.10.

Going through all the definitions, ϵ_X is described by the following formula[13]

$$\begin{aligned}\epsilon_X(r \cdot \mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n) &= \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) r \otimes \overline{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes \overline{x_{\sigma^{-1}(n)}} \\ &= \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) \sigma \cdot (r \otimes \overline{x_1} \otimes \cdots \otimes \overline{x_n})\end{aligned}$$

where r is an element of $k[X]$ and $x_1, \ldots, x_n$ are elements of X. $\diamondsuit$

Proposition 7.2.2.2. *The following hold regarding the morphisms constructed in Construction 7.2.2.1.*

(1) Let X be a set, $x_1, \ldots, x_n$ elements of X, and r an element of $k[X]$. Then ϵ_X maps the element $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$ of $\Omega^n_{k[X]/k}$ to the element $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$ of $\overline{\mathrm{C}}_n(k[X])$.

(2) Let X be a set. Then ϵ_X is a morphism of commutative differential graded algebras, with respect to the commutative algebra structure on the normalized standard Hochschild complex from Proposition 6.3.2.11.

(3) The morphisms ϵ_X assemble to a natural transformation

$$\epsilon\colon \Omega^\bullet_{k[-]/k} \to \overline{\mathrm{C}}(k[-])$$

of functors $\mathsf{Set} \to \mathrm{CAlg}(\mathsf{Ch}(k))$.

(4) For every set X the chain complexes $\Omega^\bullet_{k[X]/k}$ and $\overline{\mathrm{C}}(k[X])$ are cofibrant, so the natural transformation $\epsilon\colon \Omega^\bullet_{k[-]/k} \to \overline{\mathrm{C}}(k[-])$ from claim (3) can be lifted to a natural transformation of functors $\mathsf{Set} \to \mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$.

(5) Let $\varphi\colon k \to k'$ be a morphism of commutative rings. Then the diagram

$$\begin{tikzcd} k' \otimes_k \Omega^\bullet_{k[-]/k} \arrow[r] \arrow[d, "\cong"'] & k' \otimes_k \overline{\mathrm{C}}\big(k[-]\big) \arrow[d, "\cong"] \\ \Omega^\bullet_{k'[-]/k'} \arrow[r] & \overline{\mathrm{C}}\big(k'[-]\big) \end{tikzcd} \tag{7.2}$$

of natural transformations of functors $\mathsf{Set} \to \mathrm{CAlg}(\mathsf{Ch}(k')^{\mathrm{cof}})$ commutes, where the horizontal functors are induced by ϵ, the left natural isomorphism is the one from Remark 7.1.0.2[14], and the right natural isomorphism is the one from Remark 6.3.1.11.

[13] For the action of Σ_n on $\overline{\mathrm{C}}(k[X])$, see Definition 6.3.2.9.

[14] Composed with the natural isomorphism $\Omega^\bullet_{k' \otimes_k k[-]/k} \cong \Omega^\bullet_{k'[-]/k}$ that is induced by the natural isomorphism $k' \otimes_k k[-] \cong k'[-]$ that is given by $l \otimes r \mapsto l \cdot \varphi[-](r)$.

(6) For every set X, the morphism ϵ_X is a quasiisomorphism. ♡

Proof. Proof of claim (1): If x is an element of X, then we can consider x as an element of $k[X]$ and thus of $\overline{\mathrm{C}}_0(k[X])$. By Proposition 6.3.1.10 we then have $\mathrm{d}\,x = 1 \otimes \overline{x}$ in $\overline{\mathrm{C}}_1(k[X])$, and using Proposition 6.3.2.10 we obtain that for $x_1, \ldots, x_n$ and r as in the claim the equation

$$r \cdot \mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n = \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) r \otimes \overline{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes \overline{x_{\sigma^{-1}(n)}}$$

holds in $\overline{\mathrm{C}}_n(k[X])$, which shows the claim, as the right hand side is the formula for $\epsilon_X(r \cdot \mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n)$ given in Construction 7.2.2.1.

Proof of claim (2): Follows immediately from claim (1).

Proof of claim (3): Let $f\colon X \to Y$ be a map of sets, and denote by $F = k[f]$ the induced morphism of commutative k-algebras $k[X] \to k[Y]$. We have to show that $\overline{\mathrm{C}}(F) \circ \epsilon_X = \epsilon_Y \circ \Omega^\bullet_{F/k}$. So let $x_1, \ldots, x_n$ be elements of X and r an element of $k[X]$. We first evaluate the left hand side on $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$. By (1), ϵ_X maps $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$ to $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$. As $\overline{\mathrm{C}}(F)$ is compatible with the strict mixed structure as well as multiplication, and given by F on degree 0 (see Propositions 6.3.1.10, 6.3.2.7 and 6.3.2.11) we obtain the following.

$$(\overline{\mathrm{C}}(F) \circ \epsilon_X)(r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n) = F(r)\,\mathrm{d}\,f(x_1) \cdots \mathrm{d}\,f(x_n)$$

We now evaluate $\epsilon_Y \circ \Omega^\bullet_{F/k}$ on $r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n$. The morphism $\Omega^\bullet_{F/k}$ maps this element to $F(r)\,\mathrm{d}\,f(x_1) \cdots \mathrm{d}\,f(x_n)$. It is crucial to note at this point that this description of this element is again of the form that allows us to apply (1), i. e. $f(x_i)$ is an element of the set Y, not merely an element of $k[Y]$, see also Warning 7.2.2.5. We can thus apply (1) to conclude that

$$(\epsilon_Y \circ \Omega^\bullet_{F/k})(r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n) = F(r)\,\mathrm{d}\,f(x_1) \cdots \mathrm{d}\,f(x_n)$$

which shows the claim.

Proof of claim (4): For $\Omega^\bullet_{k[X]/k}$ this is discussed before Definition 7.1.0.1. For $\overline{\mathrm{C}}(k[X])$, note that $k[X]$ and $\overline{(k[X])} = k[X]/(k \cdot 1)$ are free k-modules with bases $\left\{ x^{\vec{j}} \;\middle|\; \vec{j} \in \mathbb{Z}^{\times X} \right\}$ and $\left\{ x^{\vec{j}} \;\middle|\; \vec{j} \in \mathbb{Z}^{\times X}, \vec{j} \neq \vec{0} \right\}$, respectively, and thus $\overline{\mathrm{C}}(k[X])$ is cofibrant by Proposition 6.3.1.10 and [Hov99, 2.3.6].

Proof of claim (5): It suffices to check that the square commutes when evaluated at a set X, which can be checked by writing a generic element of the upper left chain complex as $r' \otimes (r\,\mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n)$ for $x_1, \ldots, x_n$ elements of X, r an element of $k[X]$, and r' an element of k', and verifying that the images in the lower right along the two compositions agree, by applying claim (1) in a manner similar to the proof of claim (3).

Proof of claim (6): ϵ_X is defined as the composite of five morphisms in Construction 7.2.2.1. Three of those were already remarked to be isomorphisms in Construction 7.2.2.1, and a fourth morphism is the quotient morphism $\mathrm{C}(k[X]) \to \overline{\mathrm{C}}(k[X])$, which was shown in Proposition 6.3.1.10 to be a quasiisomorphism. It thus remains to show that the fifth involved morphism, $k[X] \otimes_{k[X]\otimes k[X]} \widetilde{\epsilon}_X$, is a quasiisomorphism as well.

For this, we note as in the proof of claim (1) of Proposition 7.2.1.2 that the model categories $\mathrm{LMod}_{k[X]}(\mathsf{Ch}(k))$ and $\mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k))$ are isomorphic to $\mathsf{Ch}(k[X])$ and $\mathsf{Ch}(k[X] \otimes k[X])$, respectively. The functor

$$k[X] \otimes_{k[X]\otimes k[X]} - \colon \mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k)) \to \mathrm{LMod}_{k[X]}(\mathsf{Ch}(k))$$

can be identified with the extension of scalars functor along the multiplication morphism $k[X] \otimes k[X] \to k[X]$ and is thus by Fact 4.1.5.1 a left Quillen functor and hence preserves weak equivalences between cofibrant objects by [Hov99, 1.1.12]. But $\widetilde{\epsilon}_X$ is a quasiisomorphism by claim (2) of Proposition 7.2.1.2, $\mathrm{C}^{\mathrm{sm}}(X)$ is cofibrant as an object of $\mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k))$ by claim (1) of Proposition 7.2.1.2, and $\mathrm{C}^{\mathrm{Bar}}(k[X])$ is cofibrant as an object of $\mathrm{LMod}_{k[X]\otimes k[X]}(\mathsf{Ch}(k))$ by Proposition 6.3.2.3. □

As an immediate conclusion of Proposition 7.2.2.2 we obtain the following result showing that $\Omega^\bullet_{k[X]/k}$ is a strict multiplicative (but not mixed) model for $\mathrm{HH}(k[X])$.

Corollary 7.2.2.3. *Let X be a set. Then there is an equivalence*

$$\mathrm{HH}\big(k[X]\big) \simeq \mathrm{CAlg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big)$$

in $\mathrm{CAlg}(\mathcal{D}(k))$. Concretely, such an equivalence is given by the composition[15]

$$\mathrm{HH}\big(k[X]\big) \xleftarrow{\simeq} \mathrm{CAlg}(\gamma)\Big(\mathrm{C}\big(k[X]\big)\Big) \xrightarrow{\simeq} \mathrm{CAlg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[X]\big)\Big) \xleftarrow[\mathrm{CAlg}(\gamma)(\epsilon_X)]{\simeq} \mathrm{CAlg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big)$$

where the left equivalence is the one from Proposition 6.3.4.3, the middle one is induced by the quotient morphism from Propositions 6.3.1.10 and 6.3.2.11, and the right equivalence is induced from ϵ_X as constructed in Construction 7.2.2.1. ♡

Proof. Combine Propositions 6.3.4.3, 6.3.1.10 and 6.3.2.11 with Proposition 7.2.2.2 (2), (4), and (6). □

Proposition 7.2.2.4. *Let $\varphi\colon k \to k'$ be a morphism of commutative rings and X a set. Then there is a commutative square*

$$\begin{tikzcd}
k' \otimes_k \mathrm{HH}\big(k[X]\big) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & k' \otimes_k \mathrm{CAlg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[d, "\simeq"] \\
\mathrm{HH}\big(k' \otimes_k k[X]\big) \arrow[d, "\simeq"'] & \mathrm{CAlg}(\gamma)\Big(k' \otimes_k \Omega^\bullet_{k[X]/k}\Big) \arrow[d, "\simeq"] \\
\mathrm{HH}\big(k'[X]\big) \arrow[r, "\simeq"'] & \mathrm{CAlg}(\gamma)\Big(\Omega^\bullet_{k'[X]/k'}\Big)
\end{tikzcd} \tag{7.3}$$

[15] If we later refer to "the equivalence from Corollary 7.2.2.3" we mean this specific one.

in $\mathrm{CAlg}(\mathcal{D}(k'))$*, where the two horizontal equivalences are (induced from) those from Corollary 7.2.2.3, the top left vertical equivalence is the one from Remark 6.2.1.6, the bottom left vertical equivalence is induced from the isomorphism* $k' \otimes_k k[X] \cong k'[X]$ *that is given by including both tensor factors in* $k'[X]$ *and then multiplying, the top right vertical equivalence is the one from Remark 4.4.1.3, and the bottom right equivalence is induced by the isomorphism that is given by applying the unit in the first tensor factor and* $\Omega^\bullet_{\iota_k[X]/k}$ *in the second, and then multiplying.* ♡

Proof. Consider the following diagram in $\mathrm{CAlg}(\mathcal{D}(k'))$ that will be explained below.

$$\begin{tikzcd}
k' \otimes_k \mathrm{HH}\big(k[X]\big) \arrow[r, "\simeq"] \arrow[d, dash, "\simeq"] & \mathrm{HH}\big(k' \otimes_k k[X]\big) \arrow[r, "\simeq"] & \mathrm{HH}\big(k'[X]\big) \arrow[d, dash, "\simeq"] \\
k' \otimes_k \mathrm{CAlg}(\gamma)\Big(\mathrm{C}\big(k[X]\big)\Big) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathrm{CAlg}(\gamma)\Big(k' \otimes_k \mathrm{C}\big(k[X]\big)\Big) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathrm{CAlg}(\gamma)\Big(\mathrm{C}\big(k'[X]\big)\Big) \arrow[d, "\simeq"] \\
k' \otimes_k \mathrm{CAlg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[X]\big)\Big) \arrow[r, "\simeq"] & \mathrm{CAlg}(\gamma)\Big(k' \otimes_k \overline{\mathrm{C}}\big(k[X]\big)\Big) \arrow[r, "\simeq"] & \mathrm{CAlg}(\gamma)\Big(\overline{\mathrm{C}}\big(k'[X]\big)\Big) \\
k' \otimes_k \mathrm{CAlg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[r, "\simeq"] \arrow[u, "k'\otimes_k \mathrm{CAlg}(\gamma)(\epsilon_X)", "\simeq"'] & \mathrm{CAlg}(\gamma)\Big(k' \otimes_k \Omega^\bullet_{k'[X]/k}\Big) \arrow[r, "\simeq"] \arrow[u, "\mathrm{CAlg}(\gamma)\big(k'\otimes_k \epsilon_X\big)", "\simeq"'] & \mathrm{CAlg}(\gamma)\Big(\Omega^\bullet_{k'[X]/k}\Big) \arrow[u, "\mathrm{CAlg}(\gamma)(\epsilon_X)", "\simeq"']
\end{tikzcd}$$

The big outer rectangle is exactly given by the transpose of diagram (without a filler so far) (7.3), after replacing the horizontal equivalences by their definition in Corollary 7.2.2.3. The middle vertical morphisms are all induced by the quotient morphism from the standard Hochschild complex to the normalized standard Hochschild complex, see Propositions 6.3.1.10 and 6.3.2.11. The two middle left horizontal equivalences are the ones from Remark 4.4.1.3, the middle right horizontal equivalences are the ones from Remarks 6.3.1.7 and 6.3.1.11, combined with the equivalence $k' \otimes_k k[X] \cong k'[X]$ that was already mentioned in the statement.

It now suffices to give a filler for all the small squares and rectangles in the above diagram. The top rectangle has a filler by Remark 6.3.4.4 and minor considerations regarding the isomorphism $k' \otimes_k k[X] \cong k'[X]$ using naturality of the equivalence Proposition 6.3.4.3. The middle left and bottom left squares have fillers by naturality of the equivalences from Remark 4.4.1.3. The middle right square has a filler by Remark 6.3.1.11. The bottom right square has a filler by Proposition 7.2.2.2 (5). □

Warning 7.2.2.5. Let X be a nonempty set. Then ϵ_X is not *not* strictly compatible with the strict mixed structures on domain and codomain. Indeed, if x is an element of X, then we have

$$\mathrm{d}\Big(\epsilon_X\big(x^2\big)\Big) = \mathrm{d}\big(x^2\big) = 1 \otimes \overline{x^2}$$

which is not equal (though homologous) to the following.

$$\epsilon_X\Big(\mathrm{d}\big(x^2\big)\Big) = \epsilon_X(2x\,\mathrm{d}\,x) = 2x \otimes \overline{x}$$

In Section 7.3 we will however see that ϵ can be upgraded to a strongly homotopy linear morphism. ♢

Warning 7.2.2.6. A previous version of this text claimed that ϵ as defined in Construction 7.2.2.1 can even be considered as a natural transformation $\Omega^\bullet_{-/k} \to \overline{\mathrm{C}}(-)$ of functors from the full subcategory of the category of k-algebras spanned by the polynomial algebras, to $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}})$, a claim that fed into the eventual proof of the main result Theorem A.

That claim is however incorrect, as was pointed out by Thomas Nikolaus. Indeed, if we consider the morphism of commutative rings $\varphi\colon \mathbb{Z}[x] \to \mathbb{Z}[y]$ that maps x to y^2, then the diagram

$$\begin{tikzcd}
\Omega^\bullet_{\mathbb{Z}[x]/\mathbb{Z}} \arrow[r, "\epsilon_{\{x\}}"] \arrow[d, "\Omega^\bullet_{\varphi/\mathbb{Z}}"'] & \overline{\mathrm{C}}(\mathbb{Z}[x]) \arrow[d, "\overline{\mathrm{C}}(\varphi)"] \\
\Omega^\bullet_{\mathbb{Z}[y]/\mathbb{Z}} \arrow[r, "\epsilon_{\{y\}}"'] & \overline{\mathrm{C}}(\mathbb{Z}[y])
\end{tikzcd}$$

does not commute, as one can check using the element $\mathrm{d}\,x$ of the top left; The composition along the top right maps this element to $1 \otimes \overline{y^2}$ in the bottom right, whereas the composition along the bottom left maps this element to $2y \otimes \overline{y}$. This phenomenon is closely related to ϵ failing to preserve the differential, see Warning 7.2.2.5. ♢

7.3. De Rham forms as a strict model in Mixed

Let X be a set. As a conclusion to Section 7.2 we showed in Corollary 7.2.2.3 that $\Omega^\bullet_{k[X]/k}$ is a strict model for $\mathrm{HH}(k[X])$ as an object in $\mathrm{CAlg}(\mathcal{D}(k))$. In this section we show that it is also a model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object in $\mathcal{M}\mathrm{ixed}$.

To do so we show that ϵ_X can be upgraded to a strongly homotopy linear morphism in the sense of Section 4.2.3. We will define the data necessary for this, i. e. morphisms $\epsilon_X^{(l)}$ for $l \geq 0$ (where $\epsilon_X^{(0)} = \epsilon_X$), in Section 7.3.1, and the rest of the section will then be devoted to proving that this makes ϵ_X into a strongly homotopy linear morphism.

As $\Omega^\bullet_{k[X]/k}$ has zero boundary operator, this amounts to

$$\partial \circ \epsilon_X^{(l)} = \epsilon_X^{(l-1)} \circ \mathrm{d} - \mathrm{d} \circ \epsilon_X^{(l-1)} \tag{7.4}$$

holding for $l \geq 1$. We will be able to use the partial Leibniz rule for d on the normalized standard Hochschild complex that we proved in Proposition 6.3.2.14 to reduce to only needing to show the above identity for elements of degree 0. This will make up the bulk of this section.

A general pattern that will occur many times in this verification will be that we are given a sum of two sums, each of which are indexed over somewhat complicated indexing sets. We then produce a bijection between those two sets and show that the summands that correspond along this bijection agree, perhaps up to sign. The strategy to show that (7.4) holds will thus be to write both sides as sums over some indexing set, then to subdivide the respective indexing sets sufficiently to be able to pairwise match up the subsets; some will match up on the same side of (7.4) and cancel, others from one side will match with the other side. As the indexing sets we consider will often involve permutations, we will make heavy use of notation and definitions from Section 2.3 (34).

We now give a short overview over the main steps of the proof.

In Section 7.3.2 we will begin by writing the left hand side of (7.4) as a sum indexed by a set I. We then write I as a disjoint union of various subsets, some of which have "cancel" in their notation, and show that the sums over those subsets vanish.

In Section 7.3.3 we begin by considering $\epsilon_X^{(l-1)} \circ \mathrm{d}$, and immediately subdivide the resulting summands into two types. We will also match up the summands of the first type with sums over some subsets of I, i.e. with summands from the left hand side of (7.4). In Section 7.3.4 we will then turn towards the summands of the second type, and rewrite them as a sum over a new indexing set I^{d} that is better suited for later simplifications. In Section 7.3.5 we consider $\mathrm{d} \circ \epsilon_X^{(l-1)}$ and write this as a sum over a indexing set I^1. We then sum up the progress made so far in showing (7.4) in Section 7.3.6.

While I^{d} and I^1 are defined using similar notions, this does not hold for I, so in Section 7.3.7 we replace the remaining subsets of I (those over which the sums have not been matched up yet) by sets $I^{\partial}_{\mathrm{even}}$ and $I^{\partial}_{\mathrm{odd}}$ that are defined in a way similar to I^{d} and I^1.

In Section 7.3.8 we then write $I^{\partial}_{\mathrm{even}}$, $I^{\partial}_{\mathrm{odd}}$, I^{d}, and I^1 as disjoint unions of various subsets. In Section 7.3.9 we show how the sums over some of the subsets of I^{d} cancel with each other, and in Section 7.3.10 we show how the remaining sums match up with each other.

Finally, we put everything together in Section 7.3.11 to prove that $\epsilon_X^{(\bullet)}$ indeed upgrades ϵ_X to a strongly homotopy linear morphism.

7.3.1. Definition of the higher homotopies

Construction 7.3.1.1. Let X be a totally ordered set. We will construct morphisms of $\mathbb{Z}$-graded k-modules

$$\epsilon_X^{(l)} \colon \Omega^{\bullet}_{k[X]/k} \to \overline{\mathrm{C}}\big(k[X]\big)$$

of degree $2l$ for every $l \geq 0$, such that $\epsilon_X^{(0)} = \epsilon_X$, where ϵ_X is as defined in Construction 7.2.2.1.

The construction and later verifications that we will need to do to show that $\epsilon_X^{(\bullet)}$ forms a strongly homotopy linear morphism are somewhat involved, so we begin by introducing some auxiliary notation and definitions.

First let $l \geq 1$ be an integer. Then we let E_l be the following subset of the symmetric group Σ_{2l}[16], where we consider σ to be extended by $\sigma(0) = 0$.

$$E_l \coloneqq \Big\{\sigma \in \Sigma_{2l} \;\Big|\; \forall\; 0 \leq i \leq l-1\colon \sigma \text{ cyclically preserves the ordering of } \{2i, 2i+1, 2i+2\}\Big\}$$

Note that as $\sigma(0)$ was defined to be 0 the condition in particular implies that $\sigma(1) < \sigma(2)$.

Next, if $l, m \geq 0$ are integers, then we first define a set $C(l, m)$ as follows.

$$C(l,m) \coloneqq \left\{\, (c_1, \ldots, c_{l+1}) \in \{1, \ldots, m+1\}^{l+1} \;\middle|\; c_{l+1} = m+1 \text{ and } c_i + 1 \leq c_{i+1} - 1 \text{ for } 1 \leq i \leq l \,\right\}$$

Let $l, m \geq 0$ be integers, $y_1, \ldots, y_m$ elements of $k[X]$, and $(c_1, \ldots, c_{l+1})$ an element of $C(l, m)$. Then we define an element $T((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1}))$ in $\overline{\mathrm{C}}_{2l}(k[X])$ as follows.

$$T\big((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1})\big) \coloneqq \prod_{j=1}^{c_1 - 1} y_j \otimes \overline{y_{c_1}} \otimes \overline{\prod_{j=c_1+1}^{c_2-1} y_j} \otimes \cdots \otimes \overline{y_{c_l}} \otimes \overline{\prod_{j=c_l+1}^{c_{l+1}-1} y_j}$$

Note that as $c_{l+1} - 1 = m + 1 - 1 = m$, the last tensor factor does not contain undefined factors. The condition $c_i + 1 \leq c_{i+1} - 1$ in the definition of $C(l, m)$ is made precisely to ensure that the products $\prod_{j=c_i+1}^{c_{i+1}-1} y_j$ are not 1 and thus that $T((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1}))$ is not zero. We will furthermore use the notation $T\big((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1})\big)_i$, where $0 \leq i \leq 2l$, for the i-th tensor factor of $T\big((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1})\big)$.

We can now define $\epsilon_X^{(l)}$ on degree 0, where we can prescribe the value on monomials in X and then extend k-linearly. Every monomial in X can be written uniquely as $\prod_{j=1}^{m} y_j$ where $m \geq 0$, each y_j is an element of X, and such that $j < j'$ implies $y_j < y_{j'}$. For example if $X = \{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$, then the monomial $x_1^2 x_2 x_3^3$ would be written as the product $x_1 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_3 \cdot x_3$. On elements of this form we define $\epsilon_X^{(l)}$ as

$$\epsilon_X^{(l)}\left(\prod_{j=1}^{m} y_j\right) = \sum_{\sigma \in E_l} \operatorname{sgn}(\sigma) \cdot \sigma \cdot \left(\sum_{\substack{(c_1,\ldots,c_{l+1}) \\ \in C(l,m)}} T\big((y_1, \ldots, y_m), (c_1, \ldots, c_{l+1})\big)\right)$$

Note that in the case $l = 0$ the set E_l consists only of the identity, $C(l, m)$ only of the 1-tuple $(m+1)$, and that $T((y_1, \ldots, y_m), (m+1)) = \prod_{j=1}^{m} y_j$. The above definition of $\epsilon_X^{(0)}$ thus recovers the definition of ϵ_X from Construction 7.2.2.1 on elements of degree 0.

To define $\epsilon_X^{(l)}$ in degrees other than 0, we set

$$\epsilon_X^{(l)}(f \,\mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n) \coloneqq \epsilon_X^{(l)}(f) \cdot \epsilon_X(\mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n)$$

for f an element of $k[X]$ and $x_1, \ldots, x_n$ elements of X, and extend k-linearly. Note that Proposition 7.2.2.2 (2) implies that $\epsilon_X^{(0)} = \epsilon_X$. ♢

[16] The symmetric group Σ_{2l} is the group of bijections of the set $\{1, \ldots, 2l\}$.

7.3.2. Simplification of the boundary

We begin the verification that

$$\partial \circ \epsilon_X^{(l)} = \epsilon_X^{(l-1)} \circ \mathrm{d} - \mathrm{d} \circ \epsilon_X^{(l-1)}$$

holds for $l \geq 1$ by subdividing the left side, and showing that some parts cancel directly.

Definition 7.3.2.1. Let X be a set. Then we define for integers $0 \leq i \leq n$ a morphism of k-modules

$$\partial_i \colon \overline{\mathrm{C}}_n(k[X]) \to \overline{\mathrm{C}}_{n-1}(k[X])$$

as the k-linear extension of

$$\partial_i \colon x^{\vec{v_0}} \otimes \overline{x^{\vec{v_1}}} \otimes \cdots \otimes \overline{x^{\vec{v_n}}} \mapsto x^{\vec{v_0}} \otimes \cdots \otimes \overline{x^{\vec{v_{i-1}}}} \otimes \overline{x^{\vec{v_i}+\vec{v_{i+1}}}} \otimes \overline{x^{\vec{v_{i+2}}}} \otimes \cdots \otimes \overline{x^{\vec{v_n}}}$$

for $0 \leq i \leq n-1$ and

$$\partial_n \colon x^{\vec{v_0}} \otimes \overline{x^{\vec{v_1}}} \otimes \cdots \otimes \overline{x^{\vec{v_n}}} \mapsto x^{\vec{v_n}+\vec{v_0}} \otimes \overline{x^{\vec{v_1}}} \otimes \cdots \otimes \overline{x^{\vec{v_{n-1}}}}$$

for $i = n$, with $\vec{v_0}, \ldots, \vec{v_n}$ elements of $\mathbb{Z}_{\geq 0}^X$ (with all but finitely many components zero) such that $\vec{v_1}, \ldots, \vec{v_n}$ are non-zero. ♢

Remark 7.3.2.2. Let X be a totally ordered set. Then it follows directly from the definition of the boundary operator on the normalized standard Hochschild complex of $k[X]$ in Propositions 6.3.1.9 and 6.3.1.10 that for $n \geq 1$

$$\partial \colon \overline{\mathrm{C}}_n(k[X]) \to \overline{\mathrm{C}}_{n-1}(k[X])$$

is given by the following sum.

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i$$

This implies in particular the following formula, where $l \geq 1$, and $y_1, \ldots, y_m$ and other notation is as in Construction 7.3.1.1.

$$\partial\left(\epsilon_X^{(l)}\left(\prod_{j=1}^{m} y_j\right)\right) = \sum_{\substack{0 \leq i \leq 2l,\\ \sigma \in E_l\\ \vec{c} \in C(l,m)}} (-1)^i \cdot \mathrm{sgn}(\sigma) \cdot \partial_i\Big(\sigma \cdot T\big((y_1, \ldots, y_m), \vec{c}\big)\Big)$$

♢

Definition 7.3.2.3. In this definition we use notation from Construction 7.3.1.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. We will define several subsets of the set

$$I := \{0, \ldots, 2l\} \times E_l \times C(l, m)$$

that by Remark 7.3.2.2 is the indexing set of a sum we can express $\partial(\epsilon_X^{(l)}(\prod_{j=1}^{m} y_j))$ as.

For $1 \leq i \leq 2l-1$ we define the following set.

$$I_i^{\text{cancel}} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = i \text{ and for all } 0 \leq p \leq l-1 \text{ it holds that } \{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \nsubseteq \{2p, 2p+1, 2p+2\} \Big\}$$

For $i = 0$ and $i = 2l$ we make the following definitions.

$$I_0^{\text{cancel}} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = 0 \text{ and } \sigma^{-1}(1) \neq 1 \Big\}$$

$$I_{2l}^{\text{cancel}} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = 2l \text{ and } \sigma^{-1}(2l) \neq 2 \Big\}$$

The above three subsets of I cover the large part of I where $\sigma^{-1}(i)$ and $\sigma^{-1}(i+1)$ do not take certain special values. We now define a number of additional subsets to deal with the remaining elements. We begin with the case in which i is neither 0 nor $2l$, and where $2p+1$ is involved. So we make the following definitions for $1 \leq i \leq 2l-1$ and $1 \leq p \leq l-1$.

$$I_{i,2p,2p+1}^{\text{cancel}} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 2p \text{ and } \sigma^{-1}(i+1) = 2p+1 \text{ and } c_{p+1} + 1 < c_{p+2} - 1 \Big\}$$

$$I_{i,2p+1,2p+2}^{\text{cancel}} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 2p+1 \text{ and } \sigma^{-1}(i+1) = 2p+2 \text{ and } c_p + 1 < c_{p+1} - 1 \Big\}$$

While $p = 0$ would be impossible in the definition of $I_{i,2p,2p+1}^{\text{cancel}}$, it is possible for $I_{i,2p+1,2p+2}^{\text{cancel}}$, though we need a slightly different definition, as there is no c_0. So we make the following definition for $1 \leq i \leq 2l-1$.

$$I_{i,1,2}^{\text{cancel}} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 1 \text{ and } \sigma^{-1}(i+1) = 2 \text{ and } 0 < c_1 - 1 \Big\}$$

Now we consider the case where $2p+1$ is not involved. We make the following definition for $1 \leq i \leq 2l-1$ and $1 \leq p \leq l-1$.

$$I_{i,2p+2,2p} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 2p+2 \text{ and } \sigma^{-1}(i+1) = 2p \Big\}$$

We next consider the cases $i = 0$ and $i = 2l$.[17]

$$I_{0,0,1}^{\text{cancel}} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = 0 \text{ and } \sigma^{-1}(1) = 1 \text{ and } c_1 + 1 < c_2 - 1 \Big\}$$

$$I_{0,0,1} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = 0 \text{ and } \sigma^{-1}(1) = 1 \text{ and } c_1 + 1 = c_2 - 1 \Big\}$$

$$I_{2l,2,0} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = 2l \text{ and } \sigma^{-1}(2l) = 2 \Big\}$$

We now need to cover the left over complement. So we make the following definition for $1 \leq i \leq 2l-1$ and $1 \leq p \leq l-1$.

$$I_{i,2p,2p+1} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\Big|\; i' = i \text{ and } \sigma^{-1}(i) = 2p \text{ and } \sigma^{-1}(i+1) = 2p+1 \text{ and } c_{p+1} + 1 = c_{p+2} - 1 \Big\}$$

$$I_{i,2p+1,2p+2} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\big|\; i' = i \text{ and } \sigma^{-1}(i) = 2p+1 \text{ and } \sigma^{-1}(i+1) = 2p+2 \\ \text{and } c_p + 1 = c_{p+1} - 1 \Big\}$$

Finally, we define the following for $1 \leq i \leq 2l - 1$.

$$I_{i,1,2} \coloneqq \Big\{ \big(i', \sigma, \vec{c}\big) \in I \;\big|\; i' = i \text{ and } \sigma^{-1}(i) = 1 \text{ and } \sigma^{-1}(i+1) = 2 \\ \text{and } c_1 = 1 \Big\}$$

Still with l, m, and $y_1, \ldots, y_m$ as above, we also introduce the following shorthand notation. For $(i, \sigma, \vec{c})$ an element of I we define

$$B\big((i, \sigma, \vec{c})\big) \coloneqq (-1)^i \cdot \operatorname{sgn}(\sigma) \cdot \partial_i \Big(\sigma \cdot T\big((y_1, \ldots, y_m), \vec{c}\big) \Big)$$

so that we with Remark 7.3.2.2 have the following concise formula for the boundary of $\epsilon_X^{(l)}(\prod_{j=1}^m y_j)$.

$$\partial \left(\epsilon_X^{(l)} \left(\prod_{j=1}^{m} y_j \right) \right) = \sum_{v \in I} B(v)$$

♢

Proposition 7.3.2.4. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then I is the disjoint union of the following subsets.*

I_i^{cancel}	*for* $0 \leq i \leq 2l$
$I_{i,2p,2p+1}^{\text{cancel}}$	*for* $1 \leq i \leq 2l-1$ *and* $1 \leq p \leq l-1$
$I_{i,2p+1,2p+2}^{\text{cancel}}$	*for* $1 \leq i \leq 2l-1$ *and* $0 \leq p \leq l-1$
$I_{i,2p+2,2p}$	*for* $1 \leq i \leq 2l-1$ *and* $1 \leq p \leq l-1$
$I_{i,2p,2p+1}$	*for* $1 \leq i \leq 2l-1$ *and* $1 \leq p \leq l-1$
$I_{i,2p+1,2p+2}$	*for* $1 \leq i \leq 2l-1$ *and* $0 \leq p \leq l-1$
$I_{0,0,1}^{\text{cancel}}$	
$I_{0,0,1}$	
$I_{2l,2,0}$	

♡

Proof. We provide a proof here, but even the very diligent reader that otherwise reads all proofs might prefer to go through the case distinctions for themselves rather than

[17] Note that $l \geq 1$ implies that c_2 is well-defined.

reading the proof. The only arguments appearing apart from nested case distinctions is to look into the definitions of E_l and $C(l,m)$ to see how they exclude certain values, e. g. $\sigma^{-1}(i)$ can not be 0 if $i > 0$ or $\sigma(2p) = \sigma(2p+1)+1$ is not possible.

In all listed subsets there is a unique integer occurring as the first component of the elements. We can thus consider the possible values for the first component separately.

We begin with the value 0. So let $(0, \sigma, \vec{c})$ be an element of I. We have to show that $(0, \sigma, \vec{c})$ is an element of exactly one of the subsets I_0^{cancel}, $I_{0,0,1}^{\text{cancel}}$, and $I_{0,0,1}$. If $\sigma^{-1}(1) \neq 1$, then the element lies in I_0^{cancel} but not in the other two subsets. If instead $\sigma^{-1}(1) = 1$, then the element lies in $I_{0,0,1}^{\text{cancel}}$ if and only if $c_1 + 1 < c_2 - 1$ and in $I_{0,0,1}$ if and only if $c_1 + 1 = c_2 - 1$. As $c_1 + 1 \leq c_2 - 1$ by the definition of $C(l,m)$, this covers all cases.

We next consider elements for which the first component is $2l$. So let $(2l, \sigma, \vec{c})$ be an element of I. We have to show that $(0, \sigma, \vec{c})$ is an element of exactly one of the subsets I_{2l}^{cancel} and $I_{2l,2,0}$. But the element is in I_{2l}^{cancel} if and only if $\sigma^{-1}(2l) \neq 2$, and in $I_{2l,2,0}$ otherwise.

Now let $1 \leq i \leq 2l-1$ and $(i, \sigma, \vec{c})$ an element of I. We have to show that this element lies in precisely one of the following subsets of I.

$$\begin{array}{rl}
I_i^{\text{cancel}} & \\
I_{i,2p,2p+1}^{\text{cancel}} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p+1,2p+2}^{\text{cancel}} & \text{for } 0 \leq p \leq l-1 \\
I_{i,2p+2,2p} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p,2p+1} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p+1,2p+2} & \text{for } 0 \leq p \leq l-1
\end{array}$$

We first note that $(i, \sigma, \vec{c})$ is an element of I_i^{cancel} if and only if the condition is satisfied that for all $0 \leq p \leq l-1$ it holds that $\{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \nsubseteq \{2p, 2p+1, 2p+2\}$. It thus remains to show that $(i, \sigma, \vec{c})$ is an element of one of the other subsets listed above if and only if there exists a $0 \leq p \leq l-1$ such that $\{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \subseteq \{2p, 2p+1, 2p+2\}$. It follows directly from the definitions that if $(i, \sigma, \vec{c})$ is an element of one of those subsets, then there exists such a $0 \leq p \leq l-1$.

We thus assume that $0 \leq p \leq l-1$ is such that $\{\sigma^{-1}(i), \sigma^{-1}(i+1)\} \subseteq \{2p, 2p+1, 2p+2\}$, and what we need to show is that $(i, \sigma, \vec{c})$ is an element of exactly one of the subsets of I listed below.

$$\begin{array}{rl}
I_{i,2p+2,2p} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p,2p+1}^{\text{cancel}} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p,2p+1} & \text{for } 1 \leq p \leq l-1 \\
I_{i,2p+1,2p+2}^{\text{cancel}} & \text{for } 0 \leq p \leq l-1 \\
I_{i,2p+1,2p+2} & \text{for } 0 \leq p \leq l-1
\end{array}$$

By definition of E_l it must hold either that

$$\sigma(2p) < \sigma(2p+1) < \sigma(2p+2)$$

or

$$\sigma(2p+2) < \sigma(2p) \qquad < \sigma(2p+1)$$

or

$$\sigma(2p+1) < \sigma(2p+2) < \sigma(2p)$$

which implies that it is not possible to have one of the following three equalities.

$$\begin{aligned} \sigma(2p+1) &= \sigma(2p+2)+1 \\ \sigma(2p+2) &= \sigma(2p)+1 \\ \sigma(2p) &= \sigma(2p+1)+1 \end{aligned}$$

This means that we must be in precisely one of the following three cases.

(a) $\sigma^{-1}(i) = 2p+2$ and $\sigma^{-1}(i+1) = 2p$.

(b) $\sigma^{-1}(i) = 2p$ and $\sigma^{-1}(i+1) = 2p+1$.

(c) $\sigma^{-1}(i) = 2p+1$ and $\sigma^{-1}(i+1) = 2p+2$.

We now go through these cases individually.

In case (a), we first note that $(i, \sigma, \vec{c})$ can only possibly be an element of a subset of the first type listed above. Furthermore, note that p can not be 0, because $\sigma(0) = 0 \neq i+1$. Thus we must have $1 \leq p \leq l-1$, and so $(i, \sigma, \vec{c})$ is indeed an element of $I_{i,2p+2,2p}$.

In case (b), the element $(i, \sigma, \vec{c})$ can only possibly be an element of the second or third type of subset listed above, i. e. $I^{\mathrm{cancel}}_{i,2q,2q+1}$ and $I_{i,2q,2q+1}$ for $1 \leq q \leq l-1$. Again p can not be 0, as $\sigma(0) = 0 \neq i$. By definition of $C(l,m)$ we must have $c_{p+1}+1 \leq c_{p+2}-1$, so we have either $c_{p+1}+1 < c_{p+2}-1$ or $c_{p+1}+1 = c_{p+2}-1$. The element $(i, \sigma, \vec{c})$ is an element of $I^{\mathrm{cancel}}_{i,2p,2p+1}$ precisely in the first case and of $I_{i,2p,2p+1}$ precisely in the second case.

Finally, in the case (c), the element $(i, \sigma, \vec{c})$ can only possibly be an element of the fourth or fifth type of subset listed above, i. e. $I^{\mathrm{cancel}}_{i,2q+1,2q+2}$ and $I_{i,2q+1,2q+2}$ for $0 \leq q \leq l-1$. If $p > 0$, then the argument is analogous to the case (b), but it remains to show that if $p = 0$, then $(i, \sigma, \vec{c})$ is an element of precisely one of $I^{\mathrm{cancel}}_{i,1,2}$ and $I_{i,1,2}$. It is an element of the first precisely if $c_1 > 1$ and of the second precisely if $c_1 = 1$. As $c_1 \geq 1$ by the definition of $C(l,m)$, this finishes the proof. □

Proposition 7.3.2.5. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds for every $1 \leq i \leq 2l-1$.*

$$\sum_{v \in I_i^{\mathrm{cancel}}} B(v) = 0$$

♡

Proof. Let $(i, \sigma, \vec{c})$ be an element of I_i^{cancel}. Then we claim that $(i, (i\ i+1) \circ \sigma, \vec{c})$ is also an element of I_i^{cancel}. For this we need to show that $(i\ i+1) \circ \sigma$ is again an element of E_l, and that for all $0 \le p \le l-1$ the following holds.

$$\left\{\ \sigma^{-1}\big((i\ i+1)^{-1}(i)\big), \sigma^{-1}\big((i\ i+1)^{-1}(i+1)\big)\ \right\} \nsubseteq \{2p, 2p+1, 2p+2\}$$

This latter condition follows directly from $(i, \sigma, \vec{c})$ being an element of I_i^{cancel} given the following short calculation.

$$\begin{aligned} &\left\{\ \sigma^{-1}\big((i\ i+1)^{-1}(i)\big), \sigma^{-1}\big((i\ i+1)^{-1}(i+1)\big)\ \right\} \\ = &\left\{\ \sigma^{-1}(i+1), \sigma^{-1}(i)\ \right\} = \left\{\ \sigma^{-1}(i), \sigma^{-1}(i+1)\ \right\} \end{aligned}$$

We still have to show that $(i\ i+1) \circ \sigma$ is an element of E_l. So let $0 \le p \le l-1$. Then there is a condition on the ordering of the three integers obtained by applying $(i\ i+1) \circ \sigma$ to $2p$, $2p+1$, and $2p+2$. Applying σ to those three elements, the condition is satisfied as σ is in E_l. As postcomposing with $(i\ i+1)$ only swaps i and $i+1$, the condition will thus also be satisfied for $(i\ i+1) \circ \sigma$ as long as at most one of i and $i+1$ occurs as a value of $2p$, $2p+1$, and $2p+2$ under σ. But this is ensured by the condition that

$$\left\{\ \sigma^{-1}(i), \sigma^{-1}(i+1)\ \right\} \nsubseteq \{2p, 2p+1, 2p+2\}$$

that holds due to $(i', \sigma, \vec{c})$ being an element of I_i^{cancel}.

Now let S be a subset of Σ_{2l} containing exactly one representative of each right coset of $\{\text{id}, (i\ i+1)\}$. We then obtain

$$\sum_{v \in I_i^{\text{cancel}}} B(v) = \sum_{\substack{(i,\sigma,\vec{c}) \in I_i^{\text{cancel}} \\ \text{such that} \\ \sigma \in S}} \Big(B\big((i, \sigma, \vec{c})\big) + B\Big(\big(i, (i\ i+1) \circ \sigma, \vec{c}\big)\Big) \Big)$$

so that it suffices to show that if $(i, \sigma, \vec{c})$ is an element of I_i^{cancel}, then the following holds.

$$B((i, \sigma, \vec{c})) + B((i, (i\ i+1) \circ \sigma, \vec{c})) = 0$$

But as ∂_i multiplies together the i-th and $i+1$-th tensor factor we have

$$\partial_i\Big((i\ i+1) \cdot \Big(\sigma \cdot T\big((y_1, \ldots, y_m), \vec{c}\big)\Big)\Big) = \partial_i\Big(\sigma \cdot T\big((y_1, \ldots, y_m), \vec{c}\big)\Big)$$

which together with $\operatorname{sgn}((i\ i+1)) = -1$ finishes the proof. □

Proposition 7.3.2.6. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \ge 1$ and $m \ge 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I_0^{\text{cancel}}} B(v) + \sum_{v \in I_{2l}^{\text{cancel}}} B(v) = 0$$

♡

Proof. We prove this by constructing a bijection

$$\varphi\colon I_0^{\text{cancel}} \to I_{2l}^{\text{cancel}}$$

such that for every element v of I_0^{cancel} we have $B(\varphi(v)) = -B(v)$.

We define φ as follows.

$$\varphi\colon (0, \sigma, \vec{c}) \mapsto (2l, \sigma_{1\to 2l} \circ \sigma, \vec{c})$$

We also directly define the candidate inverse map as follows.

$$\psi\colon (2l, \sigma, \vec{c}) \mapsto (0, \sigma_{2l\to 1} \circ \sigma, \vec{c})$$

It is clear that φ and ψ will be mutually inverse bijections as long as both are well-defined.

Before showing well-definedness we begin with a small observation. Let $(0, \sigma, \vec{c})$ be an element of I_0^{cancel}. Then the definition of I_0^{cancel} rules out that $\sigma^{-1}(1) = 1$, and we claim that the requirement that σ is an element of E_l also rules out $\sigma^{-1}(1) = 2$. Indeed, if we had $\sigma(2) = 1$, then, as $\sigma(0) = 0$, we would have $\sigma(0) < \sigma(2)$, which due to $\sigma \in E_l$ requires that $\sigma(1)$ is an integer bigger than $\sigma(0)$ and smaller than $\sigma(2)$, which would be impossible. In a completely analogous way one can see that if $(2l, \sigma, \vec{c})$ is an element of I_{2l}^{cancel}, then $\sigma^{-1}(2l)$ can be neither 1 nor 2.

Now we turn to showing that φ is well-defined. So let $(0, \sigma, \vec{c})$ be an element of I_0^{cancel}. We have to show that $(2l, \sigma_{1\to 2l} \circ \sigma, \vec{c})$ is an element of I_{2l}^{cancel}.

We first show that $\sigma_{1\to 2l} \circ \sigma$ is an element of E_l. So let $0 \leq p \leq l-1$. As $\sigma_{1\to 2l}$ preserves the ordering of the subset $\{2, \dots 2l\}$ it is immediate that $\sigma_{1\to 2l} \circ \sigma$ cyclically preserves the ordering of $\{2p, 2p+1, 2p+2\}$ as long as none of the three values $\sigma(2p)$, $\sigma(2p+1)$, and $\sigma(2p+2)$ is 1. So assume that $0 \leq p \leq l-1$ is such that one of these three values is 1. Our previous observation rules out that this can happen when $p = 0$, so we may assume that $1 \leq p \leq l-1$, which implies that $2p$, $2p+1$, and $2p+2$ are all at least 1 and hence their images under σ will also be at least 1, which implies that the one that is 1 will be the minimum, and σ being in E_l will then imply which of the other two values must be bigger. We now consider the three possible cases separately. So assume first that $\sigma(2p) = 1$. We then obtain that

$$\sigma(2p) < \sigma(2p+1) < \sigma(2p+2)$$

which implies the following.

$$(\sigma_{1\to 2l} \circ \sigma)(2p+1) < (\sigma_{1\to 2l} \circ \sigma)(2p+2) < (\sigma_{1\to 2l} \circ \sigma)(2p)$$

Next, assume that $\sigma(2p+1) = 1$. In this case we must have

$$\sigma(2p+1) < \sigma(2p+2) < \sigma(2p)$$

which implies the following.

$$(\sigma_{1\to 2l} \circ \sigma)(2p+2) < (\sigma_{1\to 2l} \circ \sigma)(2p) < (\sigma_{1\to 2l} \circ \sigma)(2p+1)$$

Finally, assume that $\sigma(2p+2) = 1$. Then we must have

$$\sigma(2p+2) < \sigma(2p) < \sigma(2p+1)$$

which implies the following.

$$(\sigma_{1\to 2l} \circ \sigma)(2p) < (\sigma_{1\to 2l} \circ \sigma)(2p+1) < (\sigma_{1\to 2l} \circ \sigma)(2p+2)$$

This shows that $\sigma_{1\to 2l} \circ \sigma$ is an element of E_l. To show that φ is well-defined we still need to show that

$$(\sigma_{1\to 2l} \circ \sigma)^{-1}(2l) = \sigma^{-1}\big(\sigma_{1\to 2l}^{-1}(2l)\big) = \sigma^{-1}(1)$$

is not 2. But this has been shown in the observation we made above.

We have now shown that φ is well-defined. That ψ is well-defined can be shown in a completely analogous way.

It remains to show that for every element v of I_0^{cancel} we have $B(\varphi(v)) = -B(v)$. So let $(0, \sigma, \vec{c})$ be an element of I_0^{cancel}. Then we have the following calculation.

$$\begin{aligned}
&B\Big(\varphi\big((0,\sigma,\vec{c})\big)\Big)\\
&= B\big((2l, \sigma_{1\to 2l}\circ\sigma, \vec{c})\big)\\
&= (-1)^{2l}\cdot \operatorname{sgn}(\sigma_{1\to 2l}\circ\sigma)\cdot \partial_{2l}\bigg(\sigma_{1\to 2l}\cdot\Big(\sigma\cdot T\big((y_1,\ldots,y_m),\vec{c}\big)\Big)\bigg)\\
&= \operatorname{sgn}(\sigma_{1\to 2l})\cdot\operatorname{sgn}(\sigma)\cdot\partial_0\Big(\sigma\cdot T\big((y_1,\ldots,y_m),\vec{c}\big)\Big)\\
&= (-1)\cdot\operatorname{sgn}(\sigma)\cdot\partial_0\Big(\sigma\cdot T\big((y_1,\ldots,y_m),\vec{c}\big)\Big)\\
&= -B\big((0,\sigma,\vec{c})\big)
\end{aligned}$$

□

Proposition 7.3.2.7. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Let $1 \leq p \leq l-1$ be an integer. Then the following holds.*

$$\sum_{\substack{1\leq i\leq 2l-1\\ v\in I_{i,2p,2p+1}^{\mathrm{cancel}}}} B(v) + \sum_{\substack{1\leq i\leq 2l-1\\ v\in I_{i,2p+1,2p+2}^{\mathrm{cancel}}}} B(v) = 0$$

♡

Proof. We use the following notation.

$$\begin{aligned}
J &:= \Big\{\ (i,v)\in\{1,\ldots,2l-1\}\times I\ \Big|\ v\in I_{i,2p,2p+1}^{\mathrm{cancel}}\ \Big\}\\
J' &:= \Big\{\ (i,v)\in\{1,\ldots,2l-1\}\times I\ \Big|\ v\in I_{i,2p+1,2p+2}^{\mathrm{cancel}}\ \Big\}
\end{aligned}$$

To prove this proposition it then suffices to construct maps

$$\varphi\colon J\to J' \qquad \text{and} \qquad \psi\colon J'\to J$$

that are mutually inverse bijections such that for every element (i, v) of J we have $B(w) = -B(v)$ if w is the second component of $\varphi((i, v))$.

So let $(i, (i, \sigma, \vec{c}))$ be an element of J. By definition of $I^{\mathrm{cancel}}_{i,2p,2p+1}$ we have $\sigma(2p) = i$ and $\sigma(2p+1) = i+1$ so that $\sigma(2p) < \sigma(2p+1)$. The definition of E_l then implies that we are in one of the following two cases.

(a) $\sigma(2p) < \sigma(2p+1) < \sigma(2p+2)$

(b) $\sigma(2p+2) < \sigma(2p) < \sigma(2p+1)$

If we are in case (a) we let $\tau = \sigma_{i+1 \to \sigma(2p+2)-1}$[18], and if we are instead in case (b) we let $\tau = \sigma_{i+1 \to \sigma(2p+2)}$. In both cases we define φ as follows.

$$\varphi\Big(\big(i, (i, \sigma, \vec{c})\big)\Big) := \Big(\tau(i+1), \big(\tau(i+1), \tau \circ \sigma, \vec{c} + \vec{e_{p+1}}\big)\Big)$$

We will show later that φ is actually well-defined, but will first define ψ. So let $(i, (i, \sigma, \vec{c}))$ be an element of J'. By definition of $I^{\mathrm{cancel}}_{i,2p+1,2p+2}$ we have $\sigma(2p+1) = i$ and $\sigma(2p+2) = i+1$ so that $\sigma(2p+1) < \sigma(2p+2)$. The definition of E_l then implies that we are in one of the following two cases.

(a) $\sigma(2p) < \sigma(2p+1) < \sigma(2p+2)$

(b) $\sigma(2p+1) < \sigma(2p+2) < \sigma(2p)$

If we are in case (a) we let $\tau' = \sigma_{i \to \sigma(2p)+1}$[19] and if we are instead in case (b) we let $\tau' = \sigma_{i \to \sigma(2p)}$. In both cases we define ψ as follows.

$$\psi\Big(\big(i, (i, \sigma, \vec{c})\big)\Big) := \Big(\tau'(i) - 1, \big(\tau'(i) - 1, \tau' \circ \sigma, \vec{c} - \vec{e_{p+1}}\big)\Big)$$

We next show that φ is well-defined. So let $(i, (i, \sigma, \vec{c}))$ be an element of J. We first show that $1 \le \tau(i+1) \le 2l - 1$. That $1 \le \tau(i+1)$ is clear. In case (a) we have that $\tau(i+1)$ is by definition strictly smaller than $\sigma(2p+2)$, which can be at most $2l$, and in case (b) we can use that $\sigma(2p+2)$ is strictly smaller than $\sigma(2p)$ by virtue of us being in case (b), and $\sigma(2p)$ is at most $2l$. This show that $\tau(i+1) \le 2l - 1$.

Next we need to show that $\tau \circ \sigma$ is an element of E_l. As τ preserves the ordering of the complement of $\{\sigma(2p+1)\}$ it immediately follows from σ cyclically preserving the ordering of $\{2q, 2q+1, 2q+2\}$ that $\tau \circ \sigma$ does so as well, as long as $0 \le q \le l - 1$ with $q \ne p$. But if we are in case (a) then we have

$$(\tau \circ \sigma)(2p) < (\tau \circ \sigma)(2p+1) < (\tau \circ \sigma)(2p+2)$$

and in case (b) we have

$$(\tau \circ \sigma)(2p+1) < (\tau \circ \sigma)(2p+2) < (\tau \circ \sigma)(2p)$$

[18] Note that $\sigma(2p+1) < \sigma(2p+2)$ implies $\sigma(2p+2) - 1 \ge \sigma(2p+1) \ge 1$, so τ is well-defined.
[19] Note that $\sigma(2p) < \sigma(2p+1)$ implies $\sigma(2p) + 1 \le \sigma(2p+1) \le 2l$, so τ' is well-defined.

so that $\tau \circ \sigma$ cyclically preserves the ordering of $\{2p, 2p+1, 2p+2\}$ as well.

To finish showing that $(\tau(i+1), \tau \circ \sigma, \vec{c} + \vec{e_{p+1}})$ is an element of I we need to show that $\vec{c'} = \vec{c} + \vec{e_{p+1}}$ is an element of $C(l, m)$. Most of the (in)equalities that need to be satisfied for this are inherited from $\vec{c}$, as $\vec{c'}$ has all components except the $p+1$-th component in common with $\vec{c}$, so we are left to show that $c_p + 1 \leq \big(c_{p+1} + 1\big) - 1$ and $\big(c_{p+1} + 1\big) + 1 \leq c_{p+2} - 1$. The former follows directly from $c_p + 1 \leq c_{p+1} - 1$, and the latter follows from $c_{p+1} + 1 < c_{p+2} - 1$, which is part of the definition of $I^{\text{cancel}}_{i,2p,2p+1}$.

We have now shown that $(\tau(i+1), \tau \circ \sigma, \vec{c} + \vec{e_{p+1}})$ is an element of I, and we need to show that it is even an element of $I^{\text{cancel}}_{\tau(p+1),2p+1,2p+2}$. The condition on $\tau \circ \sigma$ holds as

$$\tau\big(\sigma(2p+1)\big) = \tau(i+1)$$

and τ is defined exactly so that $\tau(i+1) + 1 = \tau(\sigma(2p+2))$. The condition on $\vec{c} + \vec{e_{p+1}}$ requires that

$$\big(c_p\big) + 1 < \big(c_{p+1} + 1\big) - 1$$

which holds as $c_p + 1 \leq c_{p+1} - 1$ due to $\vec{c}$ being in $C(l, m)$.

This finishes the proof that φ is well-defined. That ψ is well-defined can be shown completely analogously.

We next show that $\psi \circ \varphi = \mathrm{id}$. So let $(i, (i, \sigma, \vec{c}))$ be an element of J and τ as in the definition of φ so that the following holds.

$$\varphi\Big(\big(i, (i, \sigma, \vec{c})\big)\Big) \coloneqq \Big(\tau(i+1), \big(\tau(i+1), \tau \circ \sigma, \vec{c} + \vec{e_{p+1}}\big)\Big)$$

Then let τ' be as in the definition of ψ such that we have the following.

$$\psi\bigg(\varphi\Big(\big(i, (i, \sigma, \vec{c})\big)\Big)\bigg) \coloneqq \bigg(\tau'\big(\tau(i+1)\big) - 1, \Big(\tau'\big(\tau(i+1)\big) - 1, \tau' \circ \tau \circ \sigma, \vec{c} + \vec{e_{p+1}} - \vec{e_{p+1}}\Big)\bigg)$$

Inspecting this it is clear that it suffices to show that $\tau' \circ \tau$ is the identity. Note that τ maps $i+1$ to some element but preserves the ordering of the complement, whereas τ' preserves the ordering of the complement of $\{\tau(i+1)\}$. The composition thus also preserves the ordering of the complement of $\{i+1\}$, so that it suffices to show that $\tau' \circ \tau$ maps $i+1$ to $i+1$.

For this we distinguish between the two cases. Let us first assume case (a). Then τ maps $i+1$ to $\sigma(2p+2) - 1$. In showing that φ is well-defined we already saw that $\varphi((i, (i, \sigma, \vec{c})))$ will be as in case (a) for ψ. Thus τ' is defined by mapping $\tau(i+1)$ to $(\tau \circ \sigma)(2p) + 1$. As $\sigma(2p)$ is smaller than both $\sigma(2p+1)$ and $\sigma(2p+2)$, we have $(\tau \circ \sigma)(2p) = \sigma(2p)$ so that we obtain the following calculation, where the second equality comes from the definition of $I^{\text{cancel}}_{i,2p,2p+1}$.

$$(\tau \circ \sigma)(2p) + 1 = \sigma(2p) + 1 = i + 1$$

Let us now assume case (b). Then τ maps $i+1$ to $\sigma(2p+2)$. In showing that φ is well-defined we already saw that $\varphi((i, (i, \sigma, \vec{c})))$ will be as in case (b) for ψ. Thus τ' is defined by mapping $\tau(i+1)$ to $(\tau \circ \sigma)(2p)$. As $\sigma(2p)$ is smaller than $\sigma(2p+1)$ but bigger than $\sigma(2p+2)$, we have $\tau(\sigma(2p)) = \sigma(2p)+1$ so that we obtain the following calculation, where the second equality comes from the definition of $I^{\text{cancel}}_{i,2p,2p+1}$.

$$(\tau \circ \sigma)(2p) = \sigma(2p) + 1 = i + 1$$

We have now shown that $\psi \circ \varphi = \mathrm{id}$. That $\varphi \circ \psi = \mathrm{id}$ can be proven in an analogous way.

It remains to show that for every element $(i, (i, \sigma, \vec{c}))$ of J

$$B(w) = -B\big((i, \sigma, \vec{c})\big)$$

holds if w is the second component of $\varphi((i, (i, \sigma, \vec{c})))$. Let τ again be like in the definition of $\varphi((i, (i, \sigma, \vec{c})))$, so that $\varphi((i, (i, \sigma, \vec{c})))$ is given by $(\tau(i+1), (\tau(i+1), \tau \circ \sigma, \vec{c} + \vec{e_{p+1}}))$. We can then carry out the following calculation.

$$\begin{aligned}
&B\big((\tau(i+1), \tau \circ \sigma, \vec{c} + \vec{e_{p+1}})\big)\\
&= (-1)^{\tau(i+1)} \cdot \mathrm{sgn}(\tau \circ \sigma) \cdot \partial_{\tau(i+1)}\Big((\tau \circ \sigma) \cdot T\big((y_1, \ldots, y_m), \vec{c} + \vec{e_{p+1}}\big)\Big)\\
&= (-1)^{\tau(i+1)} \cdot \mathrm{sgn}(\tau) \cdot \mathrm{sgn}(\sigma) \cdot \partial_{\tau(i+1)}\Big((\tau \circ \sigma) \cdot T\big((y_1, \ldots, y_m), \vec{c} + \vec{e_{p+1}}\big)\Big)\\
&= (-1)^{\tau(i+1)} \cdot (-1)^{\tau(i+1)-(i+1)} \cdot \mathrm{sgn}(\sigma) \cdot \partial_{\tau(i+1)}\Big((\tau \circ \sigma) \cdot T\big((y_1, \ldots, y_m), \vec{c} + \vec{e_{p+1}}\big)\Big)\\
&= -(-1)^{i} \cdot \mathrm{sgn}(\sigma)\partial_{\tau(i+1)}\Big((\tau \circ \sigma) \cdot T\big((y_1, \ldots, y_m), \vec{c} + \vec{e_{p+1}}\big)\Big)
\end{aligned}$$

It now remains to show that

$$\partial_{\tau(i+1)}\Big((\tau \circ \sigma) \cdot T\big((y_1, \ldots, y_m), \vec{c} + \vec{e_{p+1}}\big)\Big) = \partial_i\Big(\sigma \cdot T\big((y_1, \ldots, y_m), \vec{c}\big)\Big) \qquad (*)$$

On the left hand side we start with $T\big((y_1, \ldots, y_m), \vec{c} + \vec{e_{p+1}}\big)$, permute the tensor factors with $\tau \circ \sigma$, and then multiply the $\tau(i+1)$-th and $\tau(i+1)+1$-th tensor factor together. Note that $(\tau \circ \sigma)^{-1}(\tau(i+1)) = \sigma^{-1}(i+1) = 2p+1$, and in both cases we distinguished one can furthermore check that $\tau^{-1}(\tau(i+1)+1) = \sigma(2p+2)$. As τ preserves the ordering of the complement of $\{i+1\}$, we can thus describe the process of obtaining the left hand side of $(*)$ from $T\big((y_1, \ldots, y_m), \vec{c} + \vec{e_{p+1}}\big)$ also as follows: First we permute the tensor factors using σ, then we remove the $\sigma(2p+1) = i+1$-th tensor factor and replace the $\sigma(2p+2)$-th tensor factor by its product with the $\sigma(2p+1)$-th tensor factor.

The $\sigma(2p+2)$-th tensor factor is given by

$$T\big((y_1, \ldots, y_m), \vec{c} + \vec{e_{p+1}}\big)_{2p+2} = \prod_{j=c'_{p+1}+1}^{c'_{p+2}-1} y_j$$

where we define $\vec{c'} = \vec{c} + \vec{e_{p+1}}$ for ease of notation, and the $\sigma(2p+1)$-th tensor factor is given by

$$T\big((y_1, \ldots, y_m), \vec{c} + \vec{e_{p+1}}\big)_{2p+1} = y_{c'_{p+1}}$$

so that, using that $c'_{p+1} = c_{p+1} + 1$ and that the other components of c' equal those of c, we obtain that the product is

$$y_{c_{p+1}+1} \cdot \left(\prod_{j=c_{p+1}+2}^{c_{p+2}-1} y_j \right) = \prod_{j=c_{p+1}+1}^{c_{p+2}-1} y_j$$

which is exactly the $2p+2$-th tensor factor of $T\big((y_1, \ldots, y_m), \vec{c}\big)$. As the tensor factors of $T\Big((y_1, \ldots, y_m), \vec{c'}\Big)$ and $T\big((y_1, \ldots, y_m), \vec{c}\big)$ are equal except the $2p$-th, $2p+1$-th, and $2p+2$-th, we can thus describe the process of obtaining the left hand side of $(*)$ from $T\big((y_1, \ldots, y_m), \vec{c}\big)$ as follows (note that the second argument of T is now $\vec{c}$, not $\vec{c'}$): First we permute the tensor factors using σ, then we remove the $\sigma(2p+1)$-th tensor factor and replace the $\sigma(2p)$-th tensor factor by the $2p$-th tensor factor of $T\Big((y_1, \ldots, y_m), \vec{c'}\Big)$.

We have

$$\begin{aligned}
T\Big((y_1, \ldots, y_m), \vec{c'}\Big)_{2p} &= \prod_{j=c_p+1}^{(c_{p+1}+1)-1} y_j \\
&= \left(\prod_{j=c_p+1}^{c_{p+1}-1} y_j \right) \cdot y_{c_{p+1}} \\
&= T\big((y_1, \ldots, y_m), \vec{c}\big)_{2p} \cdot T\big((y_1, \ldots, y_m), \vec{c}\big)_{2p+1}
\end{aligned}$$

so that we can also describe the process of obtaining the left hand side of $(*)$ from $T\big((y_1, \ldots, y_m), \vec{c}\big)$ as follows: First we permute the tensor factors using σ, then we remove the $\sigma(2p+1)$-th tensor factor and replace the $\sigma(2p)$-th tensor factor by the product of the $\sigma(2p)$-th tensor factor with the $\sigma(2p+1)$-th tensor factor. But this is exactly the definition of the right hand side, as $\sigma(2p) = i$. □

Proposition 7.3.2.8. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I^{\text{cancel}}_{0,0,1}} B(v) + \sum_{\substack{1 \leq i \leq 2l-1 \\ v \in I^{\text{cancel}}_{i,1,2}}} B(v) = 0$$

♡

Proof. This proposition and proof is very similar to Proposition 7.3.2.7, but a easier, as we are always in case (a).[20] We will thus refer to the proof of Proposition 7.3.2.7 for

[20]The reason this is a separate proposition is the fact that the condition that c_1 needs to satisfy for $\vec{c} \in C(l, m)$ is not precisely of the same form as for c_i with $i > 1$, which makes the definitions a little different, and that $\sigma(2p)$ is always 0 if $p = 0$. Those differences don't add any complications to the proof and instead make it simpler however.

more details of the proof. We use the following notation.

$$J := \left\{ (i, v) \in \{1, \ldots, 2l-1\} \times I \;\middle|\; v \in I^{\text{cancel}}_{i,1,2} \right\}$$

To prove this proposition it then suffices to construct maps

$$\varphi\colon I^{\text{cancel}}_{0,0,1} \to J \qquad \text{and} \qquad \psi\colon J \to I^{\text{cancel}}_{0,0,1}$$

that are mutually inverse bijections such that for every element v of J the identity $B(w) = -B(v)$ holds if w is the second component of $\varphi(v)$.

We begin by defining φ, which we do as follows.[21]

$$\varphi\big((0, \sigma, \vec{c})\big) = \Big(\sigma(2) - 1, \big(\sigma(2) - 1, \sigma_{1\to\sigma(2)-1} \circ \sigma, \vec{c} + \vec{e_1}\big)\Big)$$

Now let $(i, (i, \sigma, \vec{c}))$ be an element of J. Then we define ψ as follows.

$$\psi\Big(\big(i, (i, \sigma, \vec{c})\big)\Big) = (0, \sigma_{i\to 1} \circ \sigma, \vec{c} - \vec{e_1})$$

We next show that φ is well-defined. So let $(0, \sigma, \vec{c})$ be an element of $I^{\text{cancel}}_{0,0,1}$. Then $1 \le \sigma(2) - 1 \le 2l - 1$ and $\sigma_{1\to\sigma(2)-1} \circ \sigma \in E_l$ can be shown exactly as in the proof of Proposition 7.3.2.7. To see that $\vec{c} + \vec{e_1}$ is an element of $C(l, m)$ we need to show that $(c_1 + 1) + 1 \le c_2 - 1$, which follows from the condition $c_1 + 1 < c_2 - 1$ that is part of the definition of $I^{\text{cancel}}_{0,0,1}$. To see that $(\sigma(2) - 1, \sigma_{1\to\sigma(2)-1} \circ \sigma, \vec{c} + \vec{e_1})$ is even an element of $I^{\text{cancel}}_{i,1,2}$ we need to show a condition on the values of 1 and 2 under $\sigma_{1\to\sigma(2)-1} \circ \sigma$, which can be done exactly as in Proposition 7.3.2.7, and that $0 < (c_1 + 1) - 1$, which follows from $c_1 \ge 1$.

The proof that ψ is well-defined is very similar. That φ and ψ are mutually inverse can be shown just as in Proposition 7.3.2.7 (though the proof is easier, as only one case needs to be considered). Finally, that $B(w) = -B(v)$ for every element v of J with w the second component of $\varphi(v)$ can also be shown in exactly the same way as in the proof of Proposition 7.3.2.7. □

We sum up the progress made in this section with the following proposition.

Proposition 7.3.2.9. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \ge 1$ and $m \ge 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Let $1 \le p \le l-1$ be an integer. Then the following holds.*

$$\partial\Big(\epsilon^{(l)}_X(y_1 \cdots y_m)\Big) = \sum_{\substack{1\le i\le 2l-1,\\ 1\le q\le l-1,\\ v\in I_{i,2q+2,2q}}} B(v) + \sum_{\substack{1\le i\le 2l-1,\\ 1\le q\le l-1,\\ v\in I_{i,2q,2q+1}}} B(v) + \sum_{\substack{1\le i\le 2l-1,\\ 0\le q\le l-1,\\ v\in I_{i,2q+1,2q+2}}} B(v) + \sum_{v\in I_{0,0,1}} B(v) + \sum_{v\in I_{2l,2,0}} B(v)$$

♡

[21] As $\sigma(1) = 1$ by definition of $I^{\text{cancel}}_{0,0,1}$ we must have $\sigma(2) \ge 2$, so $\sigma(2) - 1 \ge 1$.

Proof. This follows by combining the previous results as follows. We start by applying Remark 7.3.2.2 and Definition 7.3.2.3.

$$\begin{aligned}
&\partial\Big(\epsilon_X^{(l)}(y_1\cdots y_m)\Big)\\
&=\sum_{v\in I} B(v)
\end{aligned}$$

Now we apply the decomposition of I from Proposition 7.3.2.4.

$$\begin{aligned}
&= \sum_{\substack{1\le i\le 2l-1,\\ v\in I_i^{\mathrm{cancel}}}} B(v)\\
&\quad + \sum_{v\in I_0^{\mathrm{cancel}}} B(v) + \sum_{v\in I_{2l}^{\mathrm{cancel}}} B(v)\\
&\quad + \sum_{1\le q\le l-1}\left(\sum_{\substack{1\le i\le 2l-1,\\ v\in I_{i,2q,2q+1}^{\mathrm{cancel}}}} B(v) + \sum_{\substack{1\le i\le 2l-1,\\ v\in I_{i,2q+1,2q+2}^{\mathrm{cancel}}}} B(v)\right)\\
&\quad + \sum_{\substack{1\le i\le 2l-1,\\ v\in I_{i,1,2}^{\mathrm{cancel}}}} B(v) + \sum_{v\in I_{0,0,1}^{\mathrm{cancel}}} B(v)\\
&\quad + \sum_{\substack{1\le i\le 2l-1,\\ 1\le q\le l-1,\\ v\in I_{i,2q+2,2q}}} B(v) + \sum_{\substack{1\le i\le 2l-1,\\ 1\le q\le l-1,\\ v\in I_{i,2q,2q+1}}} B(v) + \sum_{\substack{1\le i\le 2l-1,\\ 0\le q\le l-1,\\ v\in I_{i,2q+1,2q+2}}} B(v)\\
&\quad + \sum_{v\in I_{0,0,1}} B(v) + \sum_{v\in I_{2l,2,0}} B(v)
\end{aligned}$$

The first line is zero by Proposition 7.3.2.5, the second line by Proposition 7.3.2.6, the third line by Proposition 7.3.2.7, and the fourth line by Proposition 7.3.2.8, which shows the claim. □

7.3.3. Identification of summands of $\epsilon_X^{(l-1)}\circ \mathrm{d}$ of a first type

We now begin looking into the term $\epsilon_X^{(l-1)}\big(\mathrm{d}(y_1\cdots y_m)\big)$. We can write this as a sum of terms of two types, and one one type can immediately be identified with summands from $\partial(\epsilon_X^{(l)}(y_1\cdots y_m))$.

Remark 7.3.3.1. In this remark we use notation from Construction 7.3.1.1. Let X be a totally ordered set, $l\ge 1$ and $m\ge 0$ integers, and $y_1,\ldots,y_m$ as in Construction 7.3.1.1.

We consider $\epsilon_X^{(l-1)}\big(\mathrm{d}(y_1\cdots y_m)\big)$. Unpacking the definition, we obtain the following.

$$\begin{aligned}
&\epsilon_X^{(l-1)}\big(\mathrm{d}(y_1\cdots y_m)\big)\\
&=\epsilon_X^{(l-1)}\left(\sum_{s=1}^{m} y_1\cdots y_{s-1}\cdot y_{s+1}\cdots y_m\cdot \mathrm{d}\,y_s\right)
\end{aligned}$$

$$= \sum_{s=1}^{m} \epsilon_X^{(l-1)}(y_1 \cdots y_{s-1} \cdot y_{s+1} \cdots y_m) \cdot (1 \otimes \overline{y_s})$$
$$= \sum_{\substack{1 \leq s \leq m \\ \sigma \in E_{l-1} \\ \vec{c} \in C(l-1, m-1)}} \Big(\operatorname{sgn}(\sigma) \cdot \sigma \cdot T\big((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c}\big)\Big) \cdot (1 \otimes \overline{y_s})$$

We can distinguish two types of summands: Those in which y_{s-1} and y_{s+1} appear together in a tensor factor, and those in which they don't. The former happens precisely if there exists an integer $1 \leq p \leq l-1$ with $c_p < s-1$ and $c_{p+1} > s$[22], or if $c_1 > s$. Note that these possibilities exclude each other, i. e. if we count $c_1 > s$ as being the condition for $p = 0$, then if there exists a $0 \leq p \leq l-1$ satisfying the condition, then it is unique. $\diamondsuit$

We begin by identifying the summands of $\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m))$ in which y_{s-1} and y_{s+1} occur in the same tensor factor.

Proposition 7.3.3.2. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{\substack{1 \leq i \leq 2l-1, \\ 1 \leq p \leq l-1}} \sum_{v \in I_{i,2p+2,2p}} B(v)$$
$$= \sum_{\substack{1 \leq s \leq m, \\ \sigma \in E_{l-1}, \\ \vec{c} \in C(l-1, m-1), \\ 1 \leq p \leq l-1 \\ \text{such that} \\ c_p < s-1 < s < c_{p+1}}} \Big(\operatorname{sgn}(\sigma) \cdot \sigma \cdot T\big((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c}\big)\Big) \cdot (1 \otimes \overline{y_s})$$

$\heartsuit$

Proof. We first evaluate the product occurring in the summands on the right hand side of the equation, which by Propositions 6.3.2.10 and 6.3.2.11 yields the following for s, σ, $\vec{c}$, and p as in the sum in the statement.

$$\Big(\operatorname{sgn}(\sigma) \cdot \sigma \cdot T\big((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c}\big)\Big) \cdot (1 \otimes \overline{y_s})$$
$$= \sum_{1 \leq t \leq 2l-1} \operatorname{sgn}\Big(\sigma_{2l-1 \to t} \circ \big(\sigma \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot$$
$$\Big(\sigma_{2l-1 \to t} \circ \big(\sigma \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot \Big(T\big((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c}\big) \otimes \overline{y_s}\Big)$$

We now make the following definitions.

$$J := \left\{ (i, p, v) \in \{1, \ldots, 2l-1\} \times \{1, \ldots, l-1\} \times I \;\middle|\; v \in I_{i,2p+2,2p} \right\}$$

[22] Note that we "jump over" y_s, so y_{s+1} has index s rather than $s+1$.

$$J' := \Big\{ \Big(s, \sigma'', \vec{c'}, p, t\Big) \in \{1, \dots, m\} \times E_{l-1} \times C(l-1, m-1) \times \{1, \dots, l-1\} \times \{1, \dots, 2l-1\} \;\Big|\; c'_p < s-1 < s < c'_{p+1} \Big\}$$

Furthermore, for $(s, \sigma'', \vec{c'}, p, \tau)$ an element of J' we will use the following notation.

$$B'\Big(\Big(s, \sigma'', \vec{c'}, p, t\Big)\Big) := \operatorname{sgn}\Big(\sigma_{2l-1\to t} \circ \big(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot \Big(\sigma_{2l-1\to t} \circ \big(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot \Big(T\Big((y_1, \dots, y_{s-1}, y_{s+1}, \dots, y_m), \vec{c'}\Big) \otimes \overline{y_s}\Big)$$

It thus suffices to construct a bijection of sets

$$\Phi\colon J \to J'$$

such that for each element (i, p, v) of J it holds that $B'(\Phi((i, p, v))) = B(v)$.

So let $(i, p, (i, \sigma, \vec{c}))$ be an element of J. Let $s := c_{p+1}$. As $1 \le p \le l-1$ we have $2 \le p+1 \le l$, so that c_{p+1} is defined and satisfies $1 \le c_{p+1} \le (m+1) - 2 < m$. Next we define σ' as follows.

$$\sigma' = \sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2}$$

Note that $\sigma'(2l) = 2l$ so that we can consider σ' as an element of Σ_{2l-1}. We let $t := \sigma'(2p+1)$. Note that $1 \le t \le 2l-1$ and that t is $\sigma(2p+1)$ if $\sigma(2p+1) < \sigma(2p+2)$ and $\sigma(2p+1) - 1$ otherwise. We can now define another permutation σ'' to be the following composition.

$$\begin{aligned}\sigma'' &:= \sigma_{t\to 2l-1} \circ \sigma' \circ \sigma_{2l-1\to 2p+1}\\ &= \sigma_{t\to 2l-1} \circ \sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2} \circ \sigma_{2l-1\to 2p+1}\end{aligned}$$

With this definition σ'' is an element of Σ_{2l-1} that satisfies $\sigma''(2l-1) = 2l-1$, so that we can consider σ'' as an element of $\Sigma_{2(l-1)}$.

We claim that σ'' is an element of E_{l-1}. So let $0 \le a \le l-2$. We have to show that σ'' cyclically preserves the ordering of $\{2a, 2a+1, 2a+2\}$. We note first that as $a \le l-2$ implies $2a+2 \le 2l-2$, so the image of $\{2a, 2a+1, 2a+2\}$ under $\sigma \circ \sigma_{2l\to 2p+2} \circ \sigma_{2l-1\to 2p+1}$ will have image in the complement of $\{\sigma(2p+2), \sigma^{-1}_{\sigma(2p+2)\to 2l}(t)\}$, so that $\sigma_{t\to 2l-1} \circ \sigma_{\sigma(2p+2)\to 2l}$ is order-preserving on this image. This means that it suffices to show $0 \le a \le l-2$ that $\sigma \circ \sigma_{2l\to 2p+2} \circ \sigma_{2l-1\to 2p+1}$ cyclically preserves the ordering of $\{2a, 2a+1, 2a+2\}$.

We first consider the case of $a < p$. Then the claim follows as $\sigma_{2l-1\to 2p+1}$ and $\sigma_{2l\to 2p+2}$ are the identity on $\{2a, 2a+1, 2a+2\}$, and σ cyclically preserves the ordering of $\{2a, 2a+1, 2a+2\}$. We next consider the case of $a > p$. In this case both $\sigma_{2l-1\to 2p+1}$ is given by addition with 1 on $\{2a, 2a+1, 2a+2\}$, and $\sigma_{2l\to 2p+2}$ is given by addition with 1 on $\{2a+1, 2a+2, 2a+3\}$. The claim thus follows from σ cyclically preserving the ordering of $\{2(a+1), 2(a+1)+1, 2(a+1)+2\}$. It remains to consider the case $a = p$. In this case $2p$, $2p+1$, and $2p+2$ are mapped by $\sigma_{2l-1\to 2p+1}$ to $2p$, $2p+2$, and

$2p+3$, which are mapped by $\sigma_{2l\to 2p+2}$ to $2p$, $2p+3$, and $2p+4$, which are mapped by σ to $\sigma(2p)$, $\sigma(2p+3)$, and $\sigma(2p+4)$, respectively. So we have to show that σ cyclically preserves the ordering of $\{2p, 2p+3, 2p+4\}$. But by assumption σ is an element of $I_{i,2p+2,2p}$, which implies that $\sigma(2p) = \sigma(2p+2)+1$. This means that σ cyclically preserves the ordering of $\{2p, 2p+3, 2p+4\}$ if and only if σ cyclically preserves the ordering of $\{2p+2, 2p+3, 2p+4\}$, which is the case, as σ is an element of E_l.

We define $\vec{c'} \in \{1, \ldots, m\}^l$ as follows.

$$c'_a := \begin{cases} c_a & \text{for } a \leq p \\ c_{a+1} - 1 & \text{for } a > p \end{cases} \qquad \text{for } 1 \leq a \leq l$$

Note that as $c_a \leq m+1$ for $1 \leq a \leq l+1$ we obtain $c'_a \leq m$ for $1 \leq a \leq l$. Furthermore, as $p \geq 1$, and $1 \leq c_1 < c_2 < \ldots < c_{l+1}$ we also obtain that $c'_a \geq 1$ for $1 \leq a \leq l$, so that $\vec{c'}$ is indeed an element of $\{1, \ldots, m\}^l$. We claim that $\vec{c'}$ is in fact an element of $C(l-1, m-1)$. For this we first note that as $p \leq l-1$ we have $c'_l = c_{l+1} - 1 = m+1-1 = m$, which handles one of the conditions. That $c'_a + 1 \leq c'_{a+1} - 1$ for $1 \leq a \leq l-1$ follows directly from the corresponding property for $\vec{c}$ as long as $a \neq p$. For $a = p$ we have

$$c'_p + 1 = c_p + 1 \leq c_{p+1} - 1 \leq c_{p+2} - 3 = c'_{p+1} - 2 \leq c'_{p+1} - 1$$

which finishes the proof that $\vec{c'}$ is an element of $C(l-1, m-1)$.

We can now define Φ as follows.

$$\Phi\Big(\big(i, p, (i, \sigma, \vec{c})\big)\Big) := \Big(s, \sigma'', \vec{c'}, p, t\Big) = \Big(c_{p+1}, \sigma'', \vec{c'}, p, \sigma'(2p+1)\Big)$$

To show that Φ is well-defined it remains to show that it holds in the above situation that

$$c'_p < s - 1 < s < c'_{p+1}$$

but unpacking the definitions, this become the following.

$$c_p < c_{p+1} - 1 < c_{p+1} < c_{p+2} - 1$$

which holds as $\vec{c}$ is an element of $C(l, m)$.

We next show that for each element (i, p, v) of J it holds that $B'(\Phi((i, p, v))) = B(v)$. We continue using the notation we introduced up to now for this. We first check that the signs of the two terms agree. For this we have the following calculation.

$$\begin{aligned}
&\operatorname{sgn}\Big(\sigma_{2l-1\to t} \circ \big(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \\
&= \operatorname{sgn}(\sigma_{2l-1\to t}) \cdot \operatorname{sgn}(\sigma'') \\
&= (-1)^{2l-1-t} \cdot \operatorname{sgn}\big(\sigma_{t\to 2l-1} \circ \sigma' \circ \sigma_{2l-1\to 2p+1}\big) \\
&= (-1)^{2l-1-t} \cdot (-1)^{2l-1-t} \cdot \operatorname{sgn}(\sigma') \cdot (-1)^{2l-1-2p-1} \\
&= \operatorname{sgn}(\sigma') \\
&= \operatorname{sgn}\big(\sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2}\big)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{2l-\sigma(2p+2)} \cdot \operatorname{sgn}(\sigma) \cdot (-1)^{2p+2-2l} \\
&= (-1)^{\sigma(2p+2)} \cdot \operatorname{sgn}(\sigma) \\
&= (-1)^{i} \cdot \operatorname{sgn}(\sigma)
\end{aligned}
$$

To complete the proof of $B'(\Phi((i,p,v))) = B(v)$ it remains to show the following.

$$
\begin{aligned}
&\Big(\sigma_{2l-1\to t} \circ \big(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot \Big(T\Big((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c'}\Big) \otimes \overline{y_s}\Big) \\
&= \partial_i\Big(\sigma \cdot T\big((y_1, \ldots, y_m), \vec{c}\big)\Big)
\end{aligned}
$$

We begin by considering $T((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c'})$, in the following calculation, where we let $y'_1 = y_1, \ldots, y'_{s-1} = y_{s-1}, y'_s = y_{s+1}, \ldots, y'_{m-1} = y_m$.

$$
\begin{aligned}
&T\Big((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c'}\Big) \\
&= \prod_{j=1}^{c'_1-1} y'_j \otimes \overline{y'_{c'_1}} \otimes \overline{\prod_{j=c'_1+1}^{c'_2-1} y'_j} \otimes \cdots \otimes \overline{y'_{c'_{l-1}}} \otimes \overline{\prod_{j=c'_{l-1}+1}^{c'_l-1} y'_j} \\
&= \prod_{j=1}^{c_1-1} y'_j \otimes \overline{y'_{c_1}} \otimes \overline{\prod_{j=c_1+1}^{c_2-1} y'_j} \otimes \cdots \otimes \overline{\prod_{j=c_p+1}^{c_{p+2}-1-1} y'_j} \otimes \overline{y'_{c_{p+2}-1}} \otimes \overline{\prod_{j=c_{p+2}-1+1}^{c_{p+3}-1-1} y'_j} \otimes \overline{y'_{c_{p+3}-1}} \otimes \cdots
\end{aligned}
$$

Using that $s = c_{p+1}$.

$$
= \prod_{j=1}^{c_1-1} y_j \otimes \overline{y_{c_1}} \otimes \overline{\prod_{j=c_1+1}^{c_2-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_p+1}^{c_{p+1}-1} y_j \cdot \prod_{j=c_{p+1}+1}^{c_{p+2}-1} y_j} \otimes \overline{y_{c_{p+2}}} \otimes \overline{\prod_{j=c_{p+2}+1}^{c_{p+3}-1} y_j} \otimes \cdots
$$

Abbreviating $T((y_1, \ldots, y_m), \vec{c})$ as $T = T_0 \otimes \ldots T_{2l}$, we obtain the following.

$$
\begin{aligned}
&\Big(\sigma_{2l-1\to t} \circ \big(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot \Big(T\Big((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c'}\Big) \otimes \overline{y_s}\Big) \\
&= \Big(\sigma_{2l-1\to t} \circ \big(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \\
&\quad \cdot \Big(T_0 \otimes \cdots \otimes T_{2p-1} \otimes \big(T_{2p} \cdot T_{2p+2}\big) \otimes T_{2p+3} \otimes \cdots \otimes T_{2l} \otimes T_{2p+1}\Big) \\
&= \big(\sigma_{2l-1\to t} \circ \sigma_{t\to 2l-1} \circ \sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2} \circ \sigma_{2l-1\to 2p+1}\big)|_{\{1,\ldots,2l-1\}} \\
&\quad \cdot \Big(T_0 \otimes \cdots \otimes T_{2p-1} \otimes \big(T_{2p} \cdot T_{2p+2}\big) \otimes T_{2p+3} \otimes \cdots \otimes T_{2l} \otimes T_{2p+1}\Big) \\
&= \big(\sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2} \circ \sigma_{2l-1\to 2p+1}\big)|_{\{1,\ldots,2l-1\}} \\
&\quad \cdot \Big(T_0 \otimes \cdots \otimes T_{2p-1} \otimes \big(T_{2p} \cdot T_{2p+2}\big) \otimes T_{2p+3} \otimes \cdots \otimes T_{2l} \otimes T_{2p+1}\Big) \\
&= \big(\sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2}\big)|_{\{1,\ldots,2l-1\}} \\
&\quad \cdot \Big(T_0 \otimes \cdots \otimes T_{2p-1} \otimes \big(T_{2p} \cdot T_{2p+2}\big) \otimes T_{2p+1} \otimes T_{2p+3} \otimes T_{2p+4} \otimes \cdots \otimes T_{2l}\Big)
\end{aligned}
$$

Recall that $\sigma(2p+2) = i$ and $\sigma(2p) = i+1$. We now have to distinguish several cases. We start with $1 \leq j \leq 2p-1$ such that $\sigma(j) < i$. Then the permutation $\sigma' = \sigma_{\sigma(2p+2)\to 2l} \circ \sigma \circ \sigma_{2l\to 2p+2}$ maps j to $\sigma(j)$, as both $\sigma_{\sigma(2p+2)\to 2l}$ as well as $\sigma_{2l\to 2p+2}$ act as the identity on the relevant elements. Thus the $\sigma(j)$-th tensor factor in the result is given by T_j. If instead $2p+2 < j \leq 2l$ and $\sigma(j) < i$, then σ' maps $j-1$ to $\sigma(j)$. As the $j-1$-th tensor factor of the unpermuted tensor product is given by T_j, we can again conclude that the $\sigma(j)$-th tensor factor of the result is given by T_j. If $j = 2p$ or $j = 2p+2$ then we can not have $\sigma(j) < i$. If $\sigma(2p+1) < i$, then we get that $\sigma'(2p+1) = \sigma(2p+1)$. The upshot is that the 0-th to $(i-1)$-th tensor factors of the result will be given by $T_0 \otimes T_{\sigma^{-1}(1)} \otimes \cdots \otimes T_{\sigma^{-1}(i-1)}$.

We have

$$\sigma'(2p) = \sigma_{i\to 2l}(\sigma(2p)) = \sigma_{i\to 2l}(i+1) = i$$

so that we can conclude that the i-th tensor factor is given by $T_{2p} \cdot T_{2p+2} = T_{2p+2} \cdot T_{2p}$.

Now let $1 \leq j \leq 2p-1$ with $\sigma(j) > i$. Then $\sigma'(j) = \sigma(j) - 1$, so the $(\sigma(j)-1)$-th tensor factor of the result is given by T_j. If instead $2p+2 < j \leq 2l$ and $\sigma(j) > i$, then $\sigma'(j-1) = \sigma(j) - 1$, so that we can again conclude that the $(\sigma(j)-1)$-th tensor factor of the result is given by T_j. Finally, if $\sigma(2p+1) > i$, then $\sigma'(2p+1) = \sigma(2p+1) - 1$ as well. As $\sigma(\{2p, 2p+2\}) = \{i, i+1\}$, the image of $\{1, \ldots, 2p-1, 2p+3, \ldots, 2l\}$ under σ contains $\{i+2, \ldots, 2l\}$. The upshot is that the $(i+1)$-th through $2l-1$-th tensor factors of the product are given by $T_{\sigma^{-1}(i+2)} \otimes \cdots \otimes T_{\sigma^{-1}(2l)}$.

Thus we obtain

$$\begin{aligned}
&\Big(\sigma_{2l-1\to t} \circ \big(\sigma'' \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot \Big(T\big((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c'}\big) \otimes \overline{y_s}\Big) \\
&= T_0 \otimes T_{\sigma^{-1}(1)} \otimes \cdots T_{\sigma^{-1}(i-1)} \otimes T_{\sigma^{-1}(i)} \cdot T_{\sigma^{-1}(i+1)} \otimes T_{\sigma^{-1}(i+2)} \otimes \cdots \otimes T_{\sigma^{-1}(2l)} \\
&= \partial_i\Big(\sigma \cdot T\big((y_1, \ldots, y_m), \vec{c}\big)\Big)
\end{aligned}$$

To finish the proof of this proposition it remains to show that Φ is a bijection. For this we construct an inverse Ψ. So let $(s, \sigma'', \vec{c'}, p, t)$ be an element of J'. Then we define

$$\sigma' \coloneqq \sigma_{2l-1\to t} \circ \sigma'' \circ \sigma_{2p+1\to 2l-1}$$

as an element of Σ_{2l-1}. We then define $i \coloneqq \sigma'(2p)$ and define σ as follows, as an element of Σ_{2l}.

$$\sigma \coloneqq \sigma_{2l\to i} \circ \sigma' \circ \sigma_{2p+2\to 2l}$$

Note that as σ' is an element of Σ_{2l-1} we have that $1 \leq i \leq 2l-1$.

We also claim that σ is an element of E_l. So let $0 \leq a \leq l-1$. We have to show that σ cyclically preserves the ordering of $\{2a, 2a+1, 2a+2\}$. For this we distinguish four cases. If $a < p$, then $2a$, $2a+1$, and $2a+2$ are mapped to $2a$, $2a+1$, and $2a+2$ by $\sigma_{2p+2\to 2l}$ and $\sigma_{2p+1\to 2l-1}$. The permutation σ'' cyclically preserves the ordering of $\{2a, 2a+1, 2a+2\}$, and as $a < p \leq l-1$, the image under σ'' lies in $\{1, \ldots, 2l-2\}$, so that $\sigma_{2l\to i}$ and $\sigma_{2l-1\to t}$ preserve the ordering.

Next we consider the case $a = p$. In this case we have the following.

$$\begin{aligned}\sigma(2p) &= \big(\sigma_{2l\to i}\circ\sigma'\circ\sigma_{2p+2\to 2l}\big)(2p) = \big(\sigma_{2l\to i}\circ\sigma'\big)(2p) = \sigma_{2l\to i}(i) = i+1\\ \sigma(2p+2) &= \big(\sigma_{2l\to i}\circ\sigma'\circ\sigma_{2p+2\to 2l}\big)(2p+2) = \big(\sigma_{2l\to i}\circ\sigma'\big)(2l) = \sigma_{2l\to i}(2l) = i\end{aligned}$$

Which shows that σ cyclically preserves the ordering of $\{2p, 2p+1, 2p+2\}$ (it does not matter where $2p+1$ is mapped to).

We now consider the case $a = p+1$.

$$\begin{aligned}\sigma(2p+3) &= \big(\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''\circ\sigma_{2p+1\to 2l-1}\circ\sigma_{2p+2\to 2l}\big)(2p+3)\\ &= \big(\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''\circ\sigma_{2p+1\to 2l-1}\big)(2p+2)\\ &= \big(\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''\big)(2p+1)\\ \sigma(2p+3) &= \big(\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''\circ\sigma_{2p+1\to 2l-1}\circ\sigma_{2p+2\to 2l}\big)(2p+4)\\ &= \big(\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''\circ\sigma_{2p+1\to 2l-1}\big)(2p+3)\\ &= \big(\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''\big)(2p+2)\end{aligned}$$

What we thus need to show is that the three distinct integers i, $(\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma'')(2p+1)$, and $(\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma'')(2p+2)$ are cyclically ordered. We now note that

$$\begin{aligned}\big(\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''\big)(2p) &= \big(\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''\circ\sigma_{2p+1\to 2l-1}\big)(2p)\\ &= \big(\sigma_{2l\to i}\circ\sigma'\big)(2p)\\ &= \sigma_{2l\to i}(i)\\ &= i+1\end{aligned}$$

so as $2p+1 \neq 2p$ and $2p+2 \neq 2p$, we can replace i by $i+1$ and instead show that $\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''$ cyclically preserves the ordering of $\{2p, 2p+1, 2p+2\}$. Note that $a \leq l-1$ and we are looking at the case where $a = p+1$, which implies that $p \leq l-2$ (even though p in general can be $l-1$ as well), which implies that the set $\{2p, 2p+1, 2p+2\}$ is mapped by σ'' to the complement of $\{2l-1, 2l\}$, so that $\sigma_{2l\to i}\circ\sigma_{2l-1\to t}$ is order preserving on this image. That $\sigma_{2l\to i}\circ\sigma_{2l-1\to t}\circ\sigma''$ cyclically preserves the order of $\{2p, 2p+1, 2p+2\}$ thus follows from σ'' doing so.

Finally, we consider the case $p+1 < a \leq l-1$. Then $2a$, $2a+1$, and $2a+2$ are mapped by $\sigma_{2p+1\to 2l-1}\circ\sigma_{2p+2\to 2l}$ to $2(a-1)$, $2(a-1)+1$, and $2(a-1)+2$. As $a \leq l-1$ we have $a-1 \leq l-2$, so that σ'' maps these elements into the complement of $\{2l-1, 2l\}$, on which $\sigma_{2l\to i}\circ\sigma_{2l-1\to t}$ is order preserving. The claim thus follows from σ'' cyclically preserving the order of $\{2(a-1), 2(a-1)+1, 2(a-1)+2\}$.

To define Ψ we still need to define $\vec{c}$, which we do as follows.

$$c_a \coloneqq \begin{cases} c'_a & \text{for } 1 \leq a \leq p\\ s & \text{for } a = p+1\\ c'_{a-1}+1 & \text{for } p+2 \leq a \leq l+1\end{cases} \qquad \text{for } 1 \leq a \leq l+1$$

We first note that as $1 \leq s \leq m$ and $1 \leq c'_a \leq m$ for all $1 \leq a \leq l$, we have that $\vec{c}$ is an element of $\{1, \ldots, m+1\}^{l+1}$. We next need to show that $\vec{c}$ is an element of $C(l, m)$. For this we first note that $p+1 \leq l-1+1 = l$, so $c_{l+1} = c'_l + 1 = m+1$. Furthermore, that $c_a + 1 \leq c_{a+1} - 1$ for $1 \leq a \leq l$ follows directly from $\vec{c'}$ being in $C(l-1, m-1)$ as long as $a < p$ or $a \geq p+2$, so that it only remains to consider the cases $a = p$ and $a = p+1$. But we have

$$c_p = c'_p, \qquad c_{p+1} = s, \qquad c_{p+2} = c'_{p+1} + 1$$

so that the required property follows from

$$c'_p < s - 1 < s < c'_{p+1}$$

which holds as $(s, \sigma'', \vec{c'}, p, t)$ is an element of J'.

We have now defined i, σ, and $\vec{c}$ and shown that $(i, \sigma, \vec{c})$ is an element of I. In the course of doing so we also already showed that $\sigma(2p) = i+1$ and $\sigma(2p+2) = i$, so that $(i, \sigma, \vec{c})$ is even an element of $I_{i,2p+2,2p}$. We can thus define Ψ as follows.

$$\Psi\Big(\big(s, \sigma'', \vec{c'}, p, t\big)\Big) := \big(i, p, (i, \sigma, \vec{c})\big)$$

It remains to show that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the respective identity maps. So let $(i, p, (i, \sigma, \vec{c}))$ be an element of J, and let s, σ', σ'', $\vec{c'}$, and t be as in the definition of $\Phi((i, p, (i, \sigma, \vec{c})))$. Then recall that σ' and σ'' were defined (in the definition of Φ) as follows.

$$\begin{aligned}\sigma' &= \sigma_{i \to 2l} \circ \sigma \circ \sigma_{2l \to 2p+2} \\ \sigma'' &= \sigma_{t \to 2l-1} \circ \sigma' \circ \sigma_{2l-1 \to 2p+1}\end{aligned}$$

We first note that then

$$\sigma_{2l-1 \to t} \circ \sigma'' \circ \sigma_{2p+1 \to 2l-1} = \sigma'$$

so that the σ' defined from σ'' in the definition of $\Psi((s, \sigma'', \vec{c'}, p, t))$ recovers the σ' used in the definition of $\Phi((i, p, (i, \sigma, \vec{c})))$. Next we have

$$\begin{aligned}\sigma'(2p) &= \big(\sigma_{i \to 2l} \circ \sigma \circ \sigma_{2l \to 2p+2}\big)(2p) \\ &= (\sigma_{i \to 2l} \circ \sigma)(2p) \\ &= (\sigma_{i \to 2l})(i+1) \\ &= i\end{aligned}$$

so that we in the definition of $\Psi((s, \sigma'', \vec{c'}, p, t))$ also recover the correct i. It then also follows immediately from the definition that the correct σ is recovered as well. Let $\underline{c}$ be what was called $\vec{c}$ in the definition of $\Psi((s, \sigma'', \vec{c'}, p, t))$. Then we have for $1 \leq a \leq p$ that

$$\underline{c}_a = c'_a = c_a$$

while for $p+2 \leq a \leq l+1$ we have

$$\underline{c}_a = c'_{a-1} + 1 = c_{a-1+1} - 1 + 1 = c_a$$

and finally, we have the following.

$$\underline{c}_{p+1} = s = c_{p+1}$$

This shows that $\Psi \circ \Phi$ is the identity.

Now let $(s, \sigma'', \vec{c'}, p, t)$ be an element of J', and let σ', σ, i, and $\vec{c}$ be as in the definition of $\Psi((s, \sigma'', \vec{c'}, p, t))$. Let $\Phi(i, p, (i, \sigma, \vec{c})) = (\underline{s}, \underline{\sigma''}, \underline{\vec{c'}}, \underline{p}, \underline{t})$. Then we directly obtain $\underline{s} = c_{p+1} = s$ and $\underline{p} = p$. It then follows from the definition that the σ' constructed in the definition of $\Phi(i, p, (i, \sigma, \vec{c}))$ recovers the σ' constructed in the definition of $\Psi((s, \sigma'', \vec{c'}, p, t))$. We then obtain that

$$\begin{aligned}
\underline{t} &= \sigma'(2p+1) \\
&= \big(\sigma_{2l-1 \to t} \circ \sigma'' \circ \sigma_{2p+1 \to 2l-1}\big)(2p+1) \\
&= t
\end{aligned}$$

from which we can then also conclude that $\underline{\sigma''} = \sigma''$. It remains to show that $\underline{\vec{c'}} = \vec{c'}$. If $1 \leq a \leq p$ then we have

$$\underline{c'}_a = c_a = c'_a$$

and if instead $p < a \leq l$, then we have

$$\underline{c'}_a = c_{a+1} - 1 = c'_{a+1-1} + 1 - 1 = c'_a$$

which finishes the proof. $\square$

The next proposition is exactly like Proposition 7.3.3.2, just for $p = 0$.

Proposition 7.3.3.3. *In this proposition we use notation from Construction 7.3.1.1 and Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\begin{aligned}
&\sum_{v \in I_{2l,2,0}} B(v) \\
&= \sum_{\substack{1 \leq s \leq m, \\ \sigma \in E_{l-1}, \\ \vec{c} \in C(l-1, m-1) \\ \text{such that} \\ s < c_1}} \Big(\operatorname{sgn}(\sigma) \cdot \sigma \cdot T\big((y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_m), \vec{c}\big)\Big) \cdot (1 \otimes \overline{y_s})
\end{aligned}$$

♡

Proof. The proof is very similar to the proof of Proposition 7.3.3.2, but has some differences that require some minor changes. For example there is only $I_{2l,2,0}$ rather than $I_{i,2,0}$ for various values of i, which is related to the relevant permutations σ being forced to map 0 to 0. We will point out how the main steps differ in the case at hand to the case considered in Proposition 7.3.3.2, but avoid details, for which Proposition 7.3.3.2 should be consulted.

The proof of Proposition 7.3.3.2 begins with an unpacking of the product occurring on the right hand side, which applies in the same way in our case. We then define

$$\begin{aligned} J' \coloneqq \Big\{ \Big(s, \sigma'', \vec{c'}, t\Big) \in \\ &\{1, \ldots, m\} \times E_{l-1} \times C(l-1, m-1) \times \{1, \ldots, 2l-1\} \\ &\Big|\ s < c'_1 \Big\} \end{aligned}$$

and for an element $(s, \sigma'', \vec{c'}, t)$ of J' we define $B'((s, \sigma'', \vec{c'}, t))$ in exactly the same way as in the proof of Proposition 7.3.3.2 (note the definition of B' there does not depend on p). It thus suffices to construct a bijection of sets

$$\Phi\colon I_{2l,2,0} \to J'$$

such that for each element v of $I_{2l,2,0}$ it holds that $B'(\Phi(v)) = B(v)$.

For the construction of Φ, let $(i, \sigma, \vec{c})$ be an element of $I_{2l,2,0}$. Then we define s, σ'', $\vec{c'}$, and t in exactly the same way as in Proposition 7.3.3.2. The verification of the required property of $\vec{c'}$ differs slightly, we have to show that $s < c'_1$ which amounts to $c_1 < c_2 - 1$, which is satisfied as $\vec{c}$ is an element of $C(l, m)$.

The proof of Proposition 7.3.3.2 continues with a verification of $B'(\Phi(v)) = B(v)$, which can be done in essentially the same way, only requiring very minor modification, and less cases.

The construction of Ψ requires some modifications from the way it was done in Proposition 7.3.3.2. To start with we do not have p given as part of the input, and instead set $p = 0$. The definition of i, which is defined as $\sigma'(2p) = 0$ in Proposition 7.3.3.2, needs to be changed to $i \coloneqq 2l$. The definition of σ', σ, and $\vec{c}$, using these values for p and i, is then exactly as in Proposition 7.3.3.2. The verification that σ is in E_l needs to be modified when checking the cases $a = p$ and $a = p + 1$. In the case $a = p = 0$ we have $\sigma(0) = 0$ and $\sigma(2) = 2l$, so σ cyclically preserves the ordering of $\{0, 1, 2\}$ as $1 \leq \sigma(2) < 2l$. For the case $a = p+1 = 1$ one arrives as in the proof of Proposition 7.3.3.2 to showing that $2l$, $(\sigma_{2l-1\to t} \circ \sigma'')(1)$, and $(\sigma_{2l-1\to t} \circ \sigma'')(2)$ are cyclically ordered, which is the case if and only if 0, $(\sigma_{2l-1\to t} \circ \sigma'')(1)$, and $(\sigma_{2l-1\to t} \circ \sigma'')(2)$ are cyclically ordered. One now uses that $(\sigma_{2l-1\to t} \circ \sigma'')(0) = 0$ and proceeds as in the proof of Proposition 7.3.3.2. The remaining verification steps in the construction of Ψ are exactly as in the proof of Proposition 7.3.3.2.

The verification of $\Psi \circ \Phi = \mathrm{id}$ is the same as in the proof of Proposition 7.3.3.2 except for the argument showing that i is correctly recovered, which instead in our case is a tautology. The situation for the verification for $\Phi \circ \Psi = \mathrm{id}$ is analogous. □

7.3.4. Reindexing of summands of $\epsilon_X^{(l-1)} \circ \mathrm{d}$ of a second type

We have now shown how the summands of $\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m))$ in which y_{s-1} and y_{s+1} occur together as factors of a single tensor factor match up with summands of

$\partial(\epsilon_X^{(l)}(y_1 \cdots y_m))$. We now consider those summands in which y_{s-1} and y_{s+1} do *not* occur together as factors of a single tensor factor. For this it will be helpful to introduce some further notation, and while doing so we will also immediately introduce relevant analogous definitions that will be used in the next sections for $\mathrm{d}(\epsilon_X^{(l-1)}(y_1 \cdots y_m))$ and the remaining summands from $\partial(\epsilon_X^{(l)}(y_1 \cdots y_m))$.

Definition 7.3.4.1. Let $n \geq 1$ be an integer and σ an element of Σ_n. Let us for the moment denote by $P(\sigma)$ the following set.

$$P(\sigma) \coloneqq \left\{ p \in \{1, \ldots, n-1\} \;\middle|\; \sigma \text{ cyclically preserves the ordering of } \{p-1, p, p+1\} \right\}$$

Then we make the following definitions

$$e_{\mathrm{even}}(\sigma) \coloneqq \max\Big(\left\{ p \in \{1, \ldots, n-1\} \;\middle|\; p \notin P(\sigma) \text{ and } 2 \mid p \right\}\Big)$$
$$e_{\mathrm{odd}}(\sigma) \coloneqq \min\Big(\left\{ p \in \{1, \ldots, n-1\} \;\middle|\; p \notin P(\sigma) \text{ and } 2 \nmid p \right\}\Big)$$

where we set $e_{\mathrm{even}}(\sigma) = -\infty$ if the set over which the maximum is taken is empty, and $e_{\mathrm{odd}}(\sigma) = \infty$ if the set over which the minimum is taken is empty.

Now let $n, m \geq 0$ be integers. Then we define a set $C^{\mathrm{full}}(n, m)$ as follows.

$$C^{\mathrm{full}}(n, m) \coloneqq \Big\{ (c_1, \ldots, c_{n+1}) \in \{1, \ldots, m+1\}^{n+1} \;\Big|\; c_1 < c_2 < \ldots < c_n < c_{n+1} \text{ and } c_{n+1} = m+1 \Big\}$$

Now let X be a totally ordered set, $n \geq 1$ and $m \geq 0$ integers, $y_1, \ldots, y_m$ as in Construction 7.3.1.1, and $\vec{c}$ an element of $C^{\mathrm{full}}(n, m)$. Then we define an element $T^{\mathrm{full}}((y_1, \ldots, y_m), \vec{c})$ in $\overline{\mathrm{C}}_n(k[X])$ as follows.

$$T^{\mathrm{full}}\big((y_1, \ldots, y_m), \vec{c}\big) \coloneqq \prod_{j=1}^{c_1 - 1} y_j \otimes \overline{\prod_{j=c_1}^{c_2-1} y_j} \otimes \overline{\prod_{j=c_2}^{c_3-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_n}^{c_{n+1}-1} y_j}$$

Finally, we also make the following definition for $n, m \geq 0$ and $\vec{c}$ an element of $C^{\mathrm{full}}(n, m)$.

$$e_{\mathrm{even}}(\vec{c}) \coloneqq \max\Big(\left\{ p \in \{1, \ldots, n\} \;\middle|\; c_p + 1 < c_{p+1} \text{ and } 2 \mid p \right\} \cup \left\{ p \in \{0\} \;\middle|\; 1 < c_1 \right\}\Big)$$
$$e_{\mathrm{odd}}(\vec{c}) \coloneqq \min\Big(\left\{ p \in \{1, \ldots, n\} \;\middle|\; c_p + 1 < c_{p+1} \text{ and } 2 \nmid p \right\}\Big)$$

Again, if the set over which we take the maximum is empty we set $e_{\mathrm{even}}(\vec{c}) = -\infty$ and if the set over which we take the minimum we set $e_{\mathrm{odd}}(\vec{c}) = \infty$. ♢

Definition 7.3.4.2. In this definition we use notation from Construction 7.3.1.1 and continue on with similar definition as Definition 7.3.2.3. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. The following set

I^{d} will act as an indexing set for the summands of $\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m))$ that were not yet considered, while the set I^1 will be used for $\mathrm{d}(\epsilon_X^{(l-1)}(y_1 \cdots y_m))$.

$$\begin{aligned} I^{\mathrm{d}} \coloneqq \Big\{ &(\sigma, \vec{c}, p) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \times \{1, \ldots, 2l-1\} \\ &\Big| \; e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c}) \text{ and } e_{\mathrm{even}}(\sigma) - 2 < p < e_{\mathrm{odd}}(\sigma) + 2 \\ &\quad \text{and } \sigma \text{ cycl. pres. the ord. of } \{p-2, p-1, p+1\} \text{ if } 2 \mid p \text{ and } p \leq 2l-2 \\ &\quad \text{and } \sigma \text{ cycl. pres. the ord. of } \{p-1, p+1, p+2\} \text{ if } 2 \nmid p \text{ and } p \leq 2l-3 \Big\} \\ I^1 \coloneqq \Big\{ &(\sigma, \vec{c}) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \;\Big|\; e_{\mathrm{even}}(\sigma) = -\infty \text{ and } e_{\mathrm{even}}(\vec{c}) = -\infty \Big\} \end{aligned}$$

One should think of I^{d} as something like $E_l \times C(l, m)$, but where we have an extra component p that we "jump over" in the properties that E_l and $C(l, m)$ need to satisfy. We also define some new indexing sets that we will use to reindex sums appearing in $\partial(\epsilon_X^{(l)})(y_1 \cdots y_m)$.

$$\begin{aligned} I^{\partial}_{\mathrm{even}} \coloneqq \Big\{ &(\sigma, \vec{c}) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \\ &\Big| \; e_{\mathrm{even}}(\vec{c}) \neq -\infty \text{ and } e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{even}}(\vec{c}) \\ &\quad \text{and } e_{\mathrm{odd}}(\vec{c}) \geq e_{\mathrm{even}}(\vec{c}) + 3 \text{ and } e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{even}}(\vec{c}) + 1 \Big\} \\ I^{\partial}_{\mathrm{odd}} \coloneqq \Big\{ &(\sigma, \vec{c}) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \\ &\Big| \; e_{\mathrm{odd}}(\vec{c}) \neq \infty \text{ and } e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{odd}}(\vec{c}) \\ &\quad \text{and } e_{\mathrm{even}}(\vec{c}) \leq e_{\mathrm{odd}}(\vec{c}) - 3 \text{ and } e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{odd}}(\vec{c}) - 1 \Big\} \end{aligned}$$

We also define B'' and B' as follows for $(\sigma', \vec{c'})$ an element of $\Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m)$ and $(\sigma, \vec{c}, p)$ an element of I^{d}

$$\begin{aligned} B''\big((\sigma, \vec{c})\big) &\coloneqq \mathrm{sgn}(\sigma) \cdot \sigma \cdot T^{\mathrm{full}}\big((y_1, \ldots, y_m), \vec{c}\big) \\ B'\big((\sigma, \vec{c}, p)\big) &\coloneqq (-1)^{p+1} \cdot B''\big((\sigma, \vec{c})\big) \end{aligned} \qquad \diamondsuit$$

Proposition 7.3.4.3. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\begin{aligned} \epsilon_X^{(l-1)}\big(\mathrm{d}(y_1 \cdots y_m)\big) = &\sum_{\substack{1 \leq i \leq 2l-1, \\ 1 \leq p \leq l-1}} \sum_{v \in I_{i,2p+2,2p}} B(v) + \sum_{v \in I_{2l,2,0}} B(v) \\ &+ \sum_{v \in I^{\mathrm{d}}} B'(v) \end{aligned}$$

$\heartsuit$

Proof. Define a set J' as follows.

$$J' := \Big\{(s,\sigma,\vec{c}) \in \{1,\dots,m\} \times E_{l-1} \times C(l-1,m-1) \\ \Big| \text{ there is no } 1 \le p \le l-1 \text{ such that } c_p < s-1 < s < c_{p+1}, \\ \text{and } c_1 \not> s\Big\}$$

Then Remark 7.3.3.1 together with Propositions 7.3.3.2 and 7.3.3.3 imply

$$\begin{aligned} &\epsilon_X^{(l-1)}\big(\mathrm{d}(y_1 \cdots y_m)\big) \\ &= \sum_{\substack{1 \le i \le 2l-1,\\ 1 \le p \le l-1}} \sum_{v \in I_{i,2p+2,2p}} B(v) + \sum_{v \in I_{2l,2,0}} B(v) \\ &\quad + \sum_{(s,\sigma,\vec{c}) \in J'} \Big(\mathrm{sgn}(\sigma) \cdot \sigma \cdot T\big((y_1,\dots,y_{s-1},y_{s+1},\dots,y_m),\vec{c}\big)\Big) \cdot (1 \otimes \overline{y_s}) \end{aligned}$$

so that it suffices to show the following.

$$\sum_{v \in I^{\mathrm{d}}} B'(v) = \sum_{(s,\sigma,\vec{c}) \in J'} \Big(\mathrm{sgn}(\sigma) \cdot \sigma \cdot T\big((y_1,\dots,y_{s-1},y_{s+1},\dots,y_m),\vec{c}\big)\Big) \cdot (1 \otimes \overline{y_s})$$

As in the proof of Proposition 7.3.3.2, we begin by evaluating the product occurring in the summands on the right hand side of the equation, which by Propositions 6.3.2.10 and 6.3.2.11 yields the following for $(s,\sigma,\vec{c})$ an element of J'.

$$\begin{aligned} &\Big(\mathrm{sgn}(\sigma) \cdot \sigma \cdot T\big((y_1,\dots,y_{s-1},y_{s+1},\dots,y_m),\vec{c}\big)\Big) \cdot (1 \otimes \overline{y_s}) \\ &= \sum_{1 \le t \le 2l-1} \mathrm{sgn}\Big(\sigma_{2l-1 \to t} \circ \big(\sigma \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot \\ &\qquad \Big(\sigma_{2l-1 \to t} \circ \big(\sigma \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot \Big(T\big((y_1,\dots,y_{i-1},y_{i+1},\dots,y_m),\vec{c}\big) \otimes \overline{y_s}\Big) \end{aligned}$$

Defining a set J as follows

$$J := \Big\{(s,t,\sigma,\vec{c}) \in \{1,\dots,m\} \times \{1,\dots,2l-1\} \times E_{l-1} \times C(l-1,m-1) \\ \Big| \text{ there is no } 1 \le q \le l-1 \text{ such that } c_q < s-1 < s < c_{q+1}, \\ \text{and } c_1 \not> s\Big\}$$

it then suffices to show that there exists a bijection

$$\Phi\colon J \to I^{\mathrm{d}}$$

such that the following holds for all elements $(s,t,\sigma,\vec{c})$ of J.

$$\begin{aligned} &B'\Big(\Phi\big((s,t,\sigma,\vec{c})\big)\Big) \\ &= \mathrm{sgn}\Big(\sigma_{2l-1 \to t} \circ \big(\sigma \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \\ &\quad \cdot \Big(\sigma_{2l-1 \to t} \circ \big(\sigma \amalg \mathrm{id}_{\{2l-1\}}\big)\Big) \cdot \Big(T\big((y_1,\dots,y_{i-1},y_{i+1},\dots,y_m),\vec{c}\big) \otimes \overline{y_s}\Big) \end{aligned}$$

So to define Φ, let $(s, t, \sigma, \vec{c})$ be an element of J. As $c_1 \not> s$ we must have $c_1 \leq s$. We also have $s \leq m = c_l$. As $c_1 < c_2 < \cdots < c_l$ there must thus either exists a $1 \leq q \leq l$ with $c_q = s$ or with $c_q < s < c_{q+1}$. But as we ruled out $c_q < s-1 < s < c_{q+1}$, the latter implies $c_q = s-1$. The upshot is that there is a $1 \leq q \leq l$ with either $c_q = s$ or $c_q = s-1$. If $c_q = s$ then set $p \coloneqq 2q-1$. If instead $c_q = s-1$, then set $p \coloneqq 2q$. We then define

$$\sigma' \coloneqq \sigma_{2l-1\to t} \circ \sigma \circ \sigma_{p\to 2l-1}$$

as an element of Σ_{2l-1} and $\vec{c'}$ as follows.

$$c'_a \coloneqq \begin{cases} c_{a/2} + 1 & \text{if } 2 \mid a \\ c_{(a+1)/2} & \text{if } a \leq p \text{ and } 2 \nmid a \\ c_{(a-1)/2} + 2 & \text{if } a > p \text{ and } 2 \nmid a \end{cases} \qquad \text{for } 1 \leq a \leq 2l$$

We want to define Φ by setting

$$\Phi\big((s, t, \sigma, \vec{c})\big) \coloneqq \left(\sigma', \vec{c'}, p\right)$$

and for this we need to check various things to ensure that this is well-defined.

To begin with, we have $1 \leq q \leq l$ and defined p as either $2q$ or $2q-1$. We can thus conclude that $1 \leq p \leq 2l$, and are left to exclude that $p = 2l$ can occur. This could only occur if we had $c_l = s-1$, which can not happen, as $c_l = m$ and $s-1 < m$. Thus $1 \leq p \leq 2l-1$.

We next show that $e_{\text{even}}(\sigma') - 2 < p < e_{\text{odd}}(\sigma') + 2$. We begin with the left inequality. To show that $e_{\text{even}}(\sigma') < p+2$ we need to show that if $p+2 \leq a \leq 2l-2$ and a is even, then σ' cyclically preserves the ordering of $\{a-1, a, a+1\}$. Unpacking the definition of σ' this amounts to σ cyclically preserving the ordering of $\{a-2, a-1, a\}$, which it does as $a-1$ is odd, $1 \leq a-1 \leq 2l-3$[23], and σ is an element of E_{l-1}. Similarly, to show that $e_{\text{odd}}(\sigma') > p-2$, we need to show that if $1 \leq a \leq p-2$ and a is odd, then σ' cyclically preserves $\{a-1, a, a+1\}$, which unpacking the definition of σ amounts to σ cyclically preserving the ordering of $\{a-1, a, a+1\}$, which it does as it is an element of E_{l-1}. Similarly we can show the extra condition on σ around p, where this time the elements are "split up" by $\sigma_{p\to 2l-1}$. If $p \leq 2l-2$ is even, then σ' cyclically preserving the ordering of $\{p-2, p-1, p+1\}$ amounts to σ cyclically preserving the ordering of $\{p-2, p-1, p\}$, which it does as $1 \leq p-1 \leq 2l-3$ is odd[24] and σ is an element of E_{l-1}. Similarly, if $p \leq 2l-3$ is odd, then σ' cyclically preserving the ordering of $\{p-1, p+1, p+2\}$ amounts to σ cyclically preserving the ordering of $\{p-1, p, p+1\}$, which it does as $1 \leq p \leq 2l-3$ is odd.

We now show that $\vec{c'}$ is an element of $C^{\text{full}}(2l-1, m)$. For this we first need to show that c'_a is a well-defined element of $\{1, \ldots, m+1\}$ for $1 \leq a \leq 2l$. If $1 \leq a \leq 2l$ is even, then $1 \leq a/2 \leq l$, so $1 \leq c_{a/2} \leq m$ is well-defined, implying that $1 \leq c'_{a/2} \leq m+1$. If a is odd and $1 \leq a \leq p \leq 2l-1$, then $2 \leq a+1 \leq 2l$, so $1 \leq (a+1)/2 \leq l$ and

[23] $1 \leq a-1$ is implied by $p+2 \leq a$.
[24] $1 \leq p-1$, as $p = 1$ conflicts with the assumption that p is even.

$1 \leq c_{(a+1)/2} \leq m$ is well-defined. If instead a is odd with $2 \leq p+1 \leq a \leq 2l$, then $1 \leq a-1 \leq 2l-1$. As $a-1$ is even this implies that $1 \leq (a-1)/2 \leq l-1$ so that $c_{(a-1)/2}$ is well-defined and $1 \leq c_{(a-1)/2} \leq m$. As $(a-1)/2 \leq l-1$ we furthermore have that $c_{(a-1)/2} \leq c_l - 2 = m-2$, so that $1 \leq c_{(a-1)/2} + 2 \leq m$. So far we showed that $\vec{c'}$ is an element of $\{1, \ldots, m+1\}^{2l}$, so we still need to verify the (in)equalities the components need to satisfy. It follows immediately from the definition that $c'_{2l} = c_l + 1 = m+1$. It remains to show that $c'_1 < \cdots < c'_{2l}$. So let $1 \leq a \leq 2l$ be even. Assume that $2 \leq a$. Then we need to show that $c'_{a-1} < c'_a$. Depending on whether $a - 1 \leq p$ or not this amounts to either $c_{a/2} < c_{a/2} + 1$, which clearly true, or $c_{(a/2)-1} + 2 < c_{a/2} + 1$, which is true as $\vec{c}$ is an element of $C(l-1, m-1)$. Now assume that $a \leq 2l-2$. Then we have to show that $c'_a < c'_{a+1}$. Again we have two cases and this amounts to either $c_{a/2} + 1 < c_{(a/2)+1}$, which is true as $\vec{c}$ is an element of $C(l-1, m-1)$, or to $c_{a/2} + 1 < c_{a/2} + 2$, which is trivially true.

To show that Φ is well-defined it only remains to show that $e_{\mathrm{even}}(\vec{c'}) < p < e_{\mathrm{odd}}(\vec{c'})$. We begin with $e_{\mathrm{even}}(\vec{c'}) < p$. So let $p \leq a \leq 2l$ be even. Then we have to show that $c'_a + 1 = c'_{a+1}$. But unpacking the definition of $\vec{c'}$ we have $c'_a = c_{a/2} + 1$ and $c'_{a+1} = c_{a/2} + 2$, so this holds. For $p < e_{\mathrm{odd}}(\vec{c'})$ let $1 \leq a \leq p$ be odd. Then we have to show that $c'_a + 1 = c'_{a+1}$. This time we have by definition $c'_a = c_{(a+1)/2}$, and $c'_{a+1} = c_{(a+1)/2} + 1$. This finishes the proof that Φ is well-defined.

Now let $(s, t, \sigma, \vec{c})$ be an element of J, and $\Phi((s, t, \sigma, \vec{c})) = (\sigma', \vec{c'}, p)$. We want to verify the identity for $B'(\Phi((s, t, \sigma, \vec{c})))$. We begin with the following calculation.

$$
\begin{aligned}
& B'\Big(\Phi\big((s,t,\sigma,\vec{c})\big)\Big) \\
&= (-1)^{p+1} \cdot \operatorname{sgn}(\sigma') \cdot \sigma' \cdot T^{\mathrm{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big) \\
&= (-1)^{p+1} \cdot \operatorname{sgn}\big(\sigma_{2l-1\to t} \circ \sigma \circ \sigma_{p\to 2l-1}\big) \cdot \big(\sigma_{2l-1\to t} \circ \sigma \circ \sigma_{p\to 2l-1}\big) \cdot \\
&\quad \left(\prod_{j=1}^{c'_1-1} y_j \otimes \overline{\prod_{j=c'_1}^{c'_2-1} y_j} \otimes \overline{\prod_{j=c'_2}^{c'_3-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \right) \\
&= (-1)^{p+1} \cdot (-1)^{p-(2l-1)} \operatorname{sgn}(\sigma_{2l-1\to t} \circ \sigma) \cdot (\sigma_{2l-1\to t} \circ \sigma) \cdot \\
&\quad \left(\prod_{j=1}^{c'_1-1} y_j \otimes \overline{\prod_{j=c'_1}^{c'_2-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{p-1}}^{c'_p-1} y_j} \otimes \overline{\prod_{j=c'_{p+1}}^{c'_{p+2}-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \otimes \overline{\prod_{j=c'_p}^{c'_{p+1}-1} y_j} \right) \\
&= \operatorname{sgn}(\sigma_{2l-1\to t} \circ \sigma) \cdot (\sigma_{2l-1\to t} \circ \sigma) \cdot \\
&\quad \left(\prod_{j=1}^{c'_1-1} y_j \otimes \overline{\prod_{j=c'_1}^{c'_2-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{p-1}}^{c'_p-1} y_j} \otimes \overline{\prod_{j=c'_{p+1}}^{c'_{p+2}-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \otimes \overline{\prod_{j=c'_p}^{c'_{p+1}-1} y_j} \right)
\end{aligned}
$$

Let $y'_1 = y_1, \ldots, y'_{s-1} = y_{s-1}$, and $y'_s = y_{s+1}, \ldots, y'_{m-1} = y_m$. It then suffices to show the following.

$$\prod_{j=1}^{c'_1-1} y_j \otimes \overline{\prod_{j=c'_1}^{c'_2-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{p-1}}^{c'_p-1} y_j} \otimes \overline{\prod_{j=c'_{p+1}}^{c'_{p+2}-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \otimes \overline{\prod_{j=c'_p}^{c'_{p+1}-1} y_j}$$
$$= T\Big(\big(y'_1, \ldots, y'_{m-1}\big), \vec{c}\Big) \otimes \overline{y_s}$$

For this we distinguish two cases according to the parity of p. If p is odd, then we obtain the following by unpacking the definition of $\vec{c'}$ and p.

$$\prod_{j=1}^{c'_1-1} y_j \otimes \overline{\prod_{j=c'_1}^{c'_2-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{p-1}}^{c'_p-1} y_j} \otimes \overline{\prod_{j=c'_{p+1}}^{c'_{p+2}-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \otimes \overline{\prod_{j=c'_p}^{c'_{p+1}-1} y_j}$$
$$= \prod_{j=1}^{c_1-1} y_j \otimes \overline{\prod_{j=c_1}^{c_1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_{q-1}+1}^{c_q-1} y_j} \otimes \overline{\prod_{j=c_q+1}^{c_q+1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+2}^{c_l} y_j} \otimes \overline{\prod_{j=c_q}^{c_q} y_j}$$
$$= \prod_{j=1}^{c_1-1} y_j \otimes \overline{y_{c_1}} \otimes \cdots \otimes \overline{\prod_{j=c_{q-1}+1}^{c_q-1} y_j} \otimes \overline{y_{c_q+1}} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+2}^{c_l} y_j} \otimes \overline{y_{c_q}}$$
$$= \prod_{j=1}^{c_1-1} y'_j \otimes \overline{y'_{c_1}} \otimes \cdots \otimes \overline{\prod_{j=c_{q-1}+1}^{c_q-1} y'_j} \otimes \overline{y'_{c_q}} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+1}^{c_l-1} y'_j} \otimes \overline{y_s}$$
$$= T\Big(\big(y'_1, \ldots, y'_{m-1}\big), \vec{c}\Big) \otimes \overline{y_s}$$

If p is instead even, one obtains the following instead. There is only a slight difference in the middle.

$$\prod_{j=1}^{c'_1-1} y_j \otimes \overline{\prod_{j=c'_1}^{c'_2-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{p-1}}^{c'_p-1} y_j} \otimes \overline{\prod_{j=c'_{p+1}}^{c'_{p+2}-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j} \otimes \overline{\prod_{j=c'_p}^{c'_{p+1}-1} y_j}$$
$$= \prod_{j=1}^{c_1-1} y_j \otimes \overline{\prod_{j=c_1}^{c_1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_q}^{c_q} y_j} \otimes \overline{\prod_{j=c_q+2}^{c_q+1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+2}^{c_l} y_j} \otimes \overline{\prod_{j=c_q}^{c_q} y_j}$$
$$= \prod_{j=1}^{c_1-1} y_j \otimes \overline{y_{c_1}} \otimes \cdots \otimes \overline{y_{c_q}} \otimes \overline{\prod_{j=c_q+2}^{c_{q+1}} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+2}^{c_l} y_j} \otimes \overline{y_{c_q}}$$
$$= \prod_{j=1}^{c_1-1} y'_j \otimes \overline{y'_{c_1}} \otimes \cdots \otimes \overline{y'_{c_q}} \otimes \overline{\prod_{j=c_q+1}^{c_{q+1}-1} y'_j} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+1}^{c_l-1} y'_j} \otimes \overline{y_s}$$
$$= T\Big(\big(y'_1, \ldots, y'_{m-1}\big), \vec{c}\Big) \otimes \overline{y_s}$$

To finish the proof of this proposition it remains to show that Φ is a bijection, for which we construct an inverse Ψ. So let $(\sigma', \vec{c'}, p)$ be an element of I^{d}. Then we define s,

t, σ, and $\vec{c}$ as follows.

$$\begin{aligned} s &:= c'_p \\ t &:= \sigma'(p) \\ \sigma &:= \sigma_{t\to 2l-1} \circ \sigma' \circ \sigma_{2l-1\to p} \\ c_a &:= c'_{2a} - 1 \qquad \text{for } 1 \leq a \leq l \end{aligned}$$

We want to define Ψ as

$$\Psi\Big((\sigma', \vec{c'}, p)\Big) := (s, t, \sigma, \vec{c})$$

for which we need to check various things to ensure that this is well-defined.

We first note that $\vec{c'}$ is an element of $C^{\mathrm{full}}(2l-1, m)$ and $1 \leq p \leq 2l-1$, so c'_p is defined and satisfies $1 \leq c_p < c_{2l} = m+1$, so $1 \leq s \leq m$. Next, σ' is an element of Σ_{2l-1}, so $1 \leq t \leq 2l-1$ is also well-defined.

We next need to show that σ is an element of E_{l-1}. For this we first note that it follows from the definition of t and σ that σ is an element of Σ_{2l-2}. So now let $1 \leq a \leq 2l-3$ be an odd integer. We have to show that σ cyclically preserves the ordering of $\{a-1, a, a+1\}$. As $a \leq 2l-3$ we have $a+1 < 2l-1$, so this amounts to showing that σ' cyclically preserves the ordering of $\{\sigma_{2l-1\to p}(a-1), \sigma_{2l-1\to p}(a), \sigma_{2l-1\to p}(a+1)\}$. For this we need to distinguish four cases. First consider the case $a < p-1$. Then we have to show that σ' cyclically preserves the ordering of $\{a-1, a, a+1\}$, which it does, as a is odd and $a \leq p-2 < e_{\mathrm{odd}}(\sigma')$. Next consider the case $a > p$. Then we have to show that σ' cyclically preserves the ordering of $\{a, a+1, a+2\}$, which it does, as $a+1$ is even and $e_{\mathrm{even}}(\sigma') < p+2 \leq a+1$. The cases $a = p-1$ and $a = p$ remain. So assume $a = p-1$. Then we have to show that σ' cyclically preserves the ordering of $\{p-2, p-1, p+1\}$. Now $a \leq 2l-3$ being odd implies that $p \leq 2l-2$ is even, so this is part of the condition for $(\sigma', \vec{c'}, p)$ being an element of I^{d}. Similarly, if we assume $a = p$, then we have to show that σ' cyclically preserves the ordering of $\{p-1, p+1, p+2\}$, which it does as $p = a \leq 2l-3$ is even.

We now turn to showing that $\vec{c}$ is an element of $C(l-1, m-1)$. If $1 \leq a \leq l$, then $2 \leq 2a \leq 2l$, so c'_{2a} is defined and satisfies $1 \leq c'_1 < c'_{2a} \leq m+1$, so that c_a is well-defined and satisfies $1 \leq c_a \leq m$. We also obtain $c_l = c'_{2l} - 1 = m+1-1 = m$. So now let $1 \leq a \leq l-1$. Then we have to show that $c_a + 1 \leq c_{a+1} - 1$. This amounts to showing that $c'_{2a} \leq c'_{2a+2} - 2$. But this follows from $c'_{2a} < c'_{2a+1} < c'_{2a+2}$.

To finish the proof that Ψ is well-defined it remains to show that $c_1 \not> s$ and that there is no $1 \leq q \leq l-1$ such that $c_q < s-1 < s < c_{q+1}$. Applying the definitions of s and $\vec{c}$, this means we have to show that $c'_2 - 1 \not> c'_p$ and that there is no $1 \leq q \leq l-1$ such that $c'_{2q} - 1 < c'_p - 1 < c'_p < c'_{2q+2} - 1$. Let us first tackle the first claim. Assume that $c'_p < c'_2 - 1$, so $c'_p + 1 < c'_2$. As $c'_1 < c'_2 < c'_3 < \cdots$ this implies that $p = 1$. As $p < e_{\mathrm{odd}}(\vec{c'})$ and $p = 1$ is odd, this means that $c'_1 + 1 = c'_2$, which contradicts $c'_p + 1 < c'_2$. Next, assume $1 \leq q \leq l-1$ such that $c'_{2q} - 1 < c'_p - 1 < c'_p < c'_{2q+2} - 1$. Again as $c'_1 < c'_2 < \cdots$ we obtain that we must have $2q < p < 2q+2$, so $p = q+1$. As $e_{\mathrm{even}}(\vec{c'}) < p < e_{\mathrm{odd}}(\vec{c'})$ we can then conclude that $c'_p + 1 = c'_{p+1}$, which contradicts the assumption that $c'_p < c'_{2q+2} - 1$. This finishes the proof that Ψ is well-defined.

It remains to show that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the respective identities. So let $(s, t, \sigma, \vec{c})$ be an element of J and $\Phi((s, t, \sigma, \vec{c})) = (\sigma', \vec{c'}, p)$. Let $\Psi((\sigma', \vec{c'}, p)) = (\underline{s}, \underline{t}, \underline{\sigma}, \underline{\vec{c}})$. It follows directly from the definitions that $\underline{t} = \sigma'(p) = t$, from which we can then conclude $\underline{\sigma} = \sigma$ as well. It is also immediate from the definitions that $\underline{\vec{c}} = \vec{c}$. To show that $\underline{s} = s$, one needs to distinguish by the parity of p, and then this also follows directly by unpacking the definitions.

Now let $(\sigma', \vec{c'}, p)$ be an element of I^{d} and let $\Psi((\sigma', \vec{c'}, p)) = (s, t, \sigma, \vec{c})$, as well as $\Phi((s, t, \sigma, \vec{c})) = (\underline{\sigma'}, \underline{\vec{c'}}, \underline{p})$. We again need to distinguish by the parity of p. If p is odd, then $p < e_{\mathrm{odd}}(\vec{c'})$ implies that $c'_{p+1} - 1 = c'_p$. From this we obtain $c_{(p+1)/2} = c'_{p+1} - 1 = c'_p = s$. Thus we obtain $\underline{p} = 2((p+1)/2) - 1 = p$. If instead p is even, then we directly obtain $c_{p/2} = c'_p - 1 = s - 1$, so that $\underline{p} = 2(p/2) = p$. As $\underline{p} = p$ it then follows from the definition that $\underline{\sigma'} = \sigma'$. For $\underline{\vec{c'}}$ we obtain the following for $1 \leq a \leq 2l$.

$$\underline{c'}_a := \begin{cases} c'_a & \text{if } 2 \mid a \\ c'_{a+1} - 1 & \text{if } a \leq p \text{ and } 2 \nmid a \\ c'_{a-1} + 1 & \text{if } a > p \text{ and } 2 \nmid a \end{cases}$$

So let $a \leq p$ be odd. Then $a < e_{\mathrm{odd}}(\vec{c'})$, so that $c'_a = c'_{a+1} - 1$. Now let $a > p$ be odd. Then $a - 1 \geq p > e_{\mathrm{even}}(\vec{c'})$ is even, so $c'_{a-1} + 1 = c'_a$. This shows that $\underline{\vec{c'}} = \vec{c'}$ and thus finishes the proof that $\Phi \circ \Psi = \mathrm{id}$ and thus the proof of this proposition. □

7.3.5. A first look at $\mathrm{d} \circ \epsilon_X^{(l-1)}$

We now turn to $\mathrm{d}(\epsilon_X^{(l-1)}(y_1 \cdots y_m))$ and write it as a sum over I^1.

Proposition 7.3.5.1. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big) = \sum_{v \in I^1} B''(v)$$

♡

Proof. We begin by evaluating the left hand side using the definition of $\epsilon_X^{(l-1)}$ from Construction 7.3.1.1 and of the differential on the normalized standard Hochschild complex in Proposition 6.3.1.10.

$$\begin{aligned} &\mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big) \\ &= \mathrm{d}\left(\sum_{\substack{\sigma \in E_{l-1}, \\ \vec{c} \in C(l-1,m)}} \mathrm{sgn}(\sigma) \cdot \sigma \cdot T\big((y_1, \ldots, y_m), \vec{c}\big) \right) \end{aligned}$$

$$= \sum_{0 \le t \le 2l-2} \sigma^t_{\mathrm{cyc},2l-1} \cdot \left(1 \otimes \left(\sum_{\substack{\sigma \in E_{l-1},\\ \vec{c} \in C(l-1,m)}} \operatorname{sgn}(\sigma) \cdot \sigma \cdot T\big((y_1, \ldots, y_m), \vec{c}\big) \right) \right)$$

Note that $\operatorname{sgn}(\sigma_{\mathrm{cyc},2l-1}) = (-1)^{(2l-1)-1} = 1$.

$$= \sum_{0 \le t \le 2l-2} \sum_{\substack{\sigma \in E_{l-1},\\ \vec{c} \in C(l-1,m)}} \operatorname{sgn}\Big(\sigma^t_{\mathrm{cyc},2l-1} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot$$
$$\Big(\sigma^t_{\mathrm{cyc},2l-1} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot \Big(1 \otimes T\big((y_1, \ldots, y_m), \vec{c}\big)\Big)$$

Finally, we note that if we had $c_1 = 1$, then the first tensor factor of $T((y_1, \ldots, y_m), \vec{c})$ would be 1, making $1 \otimes T((y_1, \ldots, y_m), \vec{c}) = 0$. We can thus remove those summands.

$$= \sum_{0 \le t \le 2l-2} \sum_{\substack{\sigma \in E_{l-1},\\ \vec{c} \in C(l-1,m)\\ \text{such that}\\ c_1 > 1}} \operatorname{sgn}\Big(\sigma^t_{\mathrm{cyc},2l-1} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot$$
$$\Big(\sigma^t_{\mathrm{cyc},2l-1} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot \Big(1 \otimes T\big((y_1, \ldots, y_m), \vec{c}\big)\Big)$$

This leads us to defining a set J as follows.

$$J := \Big\{ (t, \sigma, \vec{c}) \in \{0, \ldots, 2l-2\} \times E_{l-1} \times C(l-1, m) \;\Big|\; c_1 > 1 \Big\}$$

It then suffices to construct a bijection

$$\Phi \colon J \to I^1$$

such that for every element $(t, \sigma, \vec{c})$ of J the following holds.

$$B''\Big(\Phi\big((t, \sigma, \vec{c})\big)\Big) = \operatorname{sgn}\Big(\sigma^t_{\mathrm{cyc},2l-1} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot$$
$$\Big(\sigma^t_{\mathrm{cyc},2l-1} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot \Big(1 \otimes T\big((y_1, \ldots, y_m), \vec{c}\big)\Big)$$

So let $(t, \sigma, \vec{c})$ be an element of J. Then we make the following definitions.

$$\sigma' := \sigma^t_{\mathrm{cyc},2l-1} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)$$
$$c'_a := \begin{cases} 1 & \text{if } a = 1 \\ c_{a/2} & \text{if } a \text{ is even} \\ c_{(a-1)/2} + 1 & \text{if } 1 < a \text{ is odd} \end{cases} \qquad \text{for } 1 \le a \le 2l$$
$$\Phi\big((t, \sigma, \vec{c})\big) := \Big(\sigma', \vec{c'}\Big)$$

We need to show that $(\sigma', \vec{c'})$ defined like this is a well-defined element of I^1. For this note first that as σ is an element of Σ_{2l-2} the permutation σ' is indeed an element of

Σ_{2l-1}. We also need $e_{\mathrm{even}}(\sigma') = -\infty$. So let $2 \leq a \leq 2l-2$ be even. Then we have to show that σ' cyclically preserves the ordering of $\{a-1, a, a+1\}$. This amounts to σ cyclically preserving the ordering of $\{a-2, a-1, a\}$, which it does as σ is an element of E_{l-1} and $1 \leq a-1 \leq 2l-3$ is odd. Next we need to show that $\vec{c'}$ is a well-defined element of $C^{\mathrm{full}}(2l-1, m)$. If $2 \leq a \leq 2l$ is even, then $1 \leq a/2 \leq l$, so c'_a is well defined and satisfies $1 \leq c'_a \leq m+1$. If $3 \leq a \leq 2l-1$ is odd, then $1 \leq (a-1)/2 \leq l-1$ so that $c_{(a-1)/2}$ is defined and satisfies $1 \leq c_{(a-1)/2} < c_l = m+1$, which implies that $1 \leq c'_a \leq m+1$. Thus $\vec{c'}$ is an element of $\{1, \ldots, m+1\}^{2l}$. We also have $c'_{2l} = c_l = m+1$. It remains to show that $c'_1 < \cdots < c'_{2l}$. This amounts to $1 < c_1 < c_1 + 1 < c_2 < \cdots < c_l$, which holds as $c_1 > 1$ by assumption on $(t, \sigma, \vec{c})$, and as $c_a + 1 \leq c_{a+1} - 1$ for $1 \leq a \leq l-1$ as $\vec{c}$ is an element of $C(l-1, m)$. To show that $(\sigma', \vec{c'})$ is an element of I^1 it still remains to show that $e_{\mathrm{even}}(\vec{c'}) = -\infty$, which amounts to showing that $c'_1 = 1$ and that $c'_{a+1} = c'_a + 1$ for $2 \leq a \leq 2l-2$ even, both of which is the case directly from the definition of $\vec{c'}$.

We now verify the identity that needs to be satisfied for $B''(\Phi((t, \sigma, \vec{c})))$.

$$\begin{aligned}
&B''\Big(\Phi\big((t,\sigma,\vec{c})\big)\Big)\\
&= B''\Big(\big(\sigma', \vec{c'}\big)\Big)\\
&= \operatorname{sgn}\Big(\sigma^t_{\mathrm{cyc},2l-1} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot \\
&\qquad \Big(\sigma^t_{\mathrm{cyc},2l-1} \circ \big(\mathrm{id}_{\{1\}} \amalg \sigma\big)\Big) \cdot T^{\mathrm{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big)
\end{aligned}$$

Verification of the identity that is needed for $B''(\Phi((t, \sigma, \vec{c})))$ is now completed by the following calculation.

$$\begin{aligned}
&T^{\mathrm{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big)\\
&= \prod_{j=1}^{c'_1-1} y_j \otimes \overline{\prod_{j=c'_1}^{c'_2-1} y_j} \otimes \overline{\prod_{j=c'_2}^{c'_3-1} y_j} \otimes \overline{\prod_{j=c'_3}^{c'_4-1} y_j} \otimes \overline{\prod_{j=c'_4}^{c'_5-1} y_j} \otimes \cdots \otimes \overline{\prod_{j=c'_{2l-1}}^{c'_{2l}-1} y_j}\\
&= \prod_{j=1}^{0} y_j \otimes \overline{\prod_{j=1}^{c_1-1} y_j} \otimes \overline{\prod_{j=c_1}^{c_1} y_j} \otimes \overline{\prod_{j=c_1+1}^{c_2-1} y_j} \otimes \overline{\prod_{j=c_2}^{c_2} y_j} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+1}^{c_l-1} y_j}\\
&= 1 \otimes \overline{\prod_{j=1}^{c_1-1} y_j} \otimes \overline{y_{c_1}} \otimes \overline{\prod_{j=c_1+1}^{c_2-1} y_j} \otimes \overline{y_{c_2}} \otimes \cdots \otimes \overline{\prod_{j=c_{l-1}+1}^{c_l-1} y_j}\\
&= 1 \otimes T\big((y_1, \ldots, y_m), \vec{c}\big)
\end{aligned}$$

It remains to show that Φ is a bijection. As usual we construct an inverse Ψ. So let $(\sigma', \vec{c'})$ be an element of I^1. Then we define $\Psi((\sigma', \vec{c'}))$ as follows.

$$\begin{aligned}
t &:= \sigma'(1) - 1\\
\sigma &:= r_{\{2,\ldots,2l-1\}}\Big(\sigma^{-t}_{\mathrm{cyc},2l-1} \circ \sigma'\Big)
\end{aligned}$$

$$
\begin{aligned}
c_a &:= c'_{2a} \qquad \text{for } 1 \leq a \leq l \\
\Psi\left(\left(\sigma', \vec{c'}\right)\right) &:= (t, \sigma, \vec{c})
\end{aligned}
$$

Again we have to check some things to verify that this is well-defined. First, as σ' is an element of Σ_{2l-1}, the value of t satisfies indeed $0 \leq t \leq 2l-2$, and the above definition of σ is an element of Σ_{2l-1}. We need to show that σ is even an element of E_{l-1}. So let $1 \leq a \leq 2l-3$ be an odd integer. We have to show that σ cyclically preserves the ordering of $\{a-1, a, a+1\}$. But as $\sigma_{\mathrm{cyc},2l-1}^{-t}$ cyclically preserves the ordering of any set, the restriction means that what we have to show amounts to showing that σ' cyclically preserves the ordering of $\{a, a+1, a+2\}$, which it does as $2 \leq a+1 \leq 2l-2$ is even and $e_{\mathrm{even}}(\sigma) = -\infty$. We also need to show that $\vec{c}$ is an element of $C(l-1, m)$ satisfying $c_1 > 1$. For this we note that for $1 \leq a \leq l$ we have $2 \leq 2a \leq 2l$. Thus $1 \leq c'_1 < c'_{2a} \leq m+1$, from which it follows that $\vec{c}$ is an element of $\{1, \ldots, m+1\}^l$ with $c_1 > 1$. Directly from the definition we have $c_l = c'_{2l} = m+1$, and if $a < l$, then we have $c_a = c'_{2a} < c'_{2a+1} < c'_{2a+2} = c_{a+1}$, from which $c_a + 1 \leq c_{a+1} - 1$ follows. This shows that Ψ is well-defined.

To finish the proof of this propositions we are left to show that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the respective identity maps. So let $(t, \sigma, \vec{c})$ be an element of J, and set $\Phi((t, \sigma, \vec{c})) = (\sigma', \vec{c'})$ and $\Psi((\sigma', \vec{c'})) = (\underline{t}, \underline{\sigma}, \underline{\vec{c}})$. Then the following calculations show that $\Psi \circ \Phi$ is the identity.

$$
\begin{aligned}
\underline{t} &= \sigma'(1) - 1 = \sigma_{\mathrm{cyc},2l-1}^{t}(1) - 1 = 1 + t - 1 = t \\
\underline{\sigma} &= r_{\{2,\ldots,2l-1\}}\left(\sigma_{\mathrm{cyc},2l-1}^{-t} \circ \sigma_{\mathrm{cyc},2l-1}^{t} \circ \left(\mathrm{id}_{\{1\}} \amalg \sigma\right)\right) = r_{\{2,\ldots,2l-1\}}\left(\mathrm{id}_{\{1\}} \amalg \sigma\right) = \sigma \\
\underline{c}_a &= c'_{2a} = c_a \qquad \text{for } 1 \leq a \leq l
\end{aligned}
$$

Now let $(\sigma', \vec{c'})$ be an element of I^1. Let $\Psi((\sigma', \vec{c'})) = (t, \sigma, \vec{c})$ and $\Phi((t, \sigma, \vec{c})) = (\underline{\sigma'}, \underline{\vec{c'}})$. We begin by the following calculation.

$$
\left(\sigma_{\mathrm{cyc},2l-1}^{-t} \circ \sigma'\right)(1) = \sigma_{\mathrm{cyc},2l-1}^{-\left(\sigma'(1)-1\right)}\left(\sigma'(1)\right) = \sigma'(1) - \left(\sigma'(1) - 1\right) = 1
$$

This implies the following calculation showing $\underline{\sigma'} = \sigma'$.

$$
\begin{aligned}
\underline{\sigma'} &= \sigma_{\mathrm{cyc},2l-1}^{t} \circ \left(\mathrm{id}_{\{1\}} \amalg \sigma\right) \\
&= \sigma_{\mathrm{cyc},2l-1}^{t} \circ \left(\mathrm{id}_{\{1\}} \amalg r_{\{2,\ldots,2l-1\}}\left(\sigma_{\mathrm{cyc},2l-1}^{-t} \circ \sigma'\right)\right) \\
&= \sigma_{\mathrm{cyc},2l-1}^{t} \circ \sigma_{\mathrm{cyc},2l-1}^{-t} \circ \sigma' \\
&= \sigma'
\end{aligned}
$$

It remains to show that $\underline{\vec{c'}} = \vec{c'}$. So let $1 \leq a \leq 2l$. Then we have the following calculation.

$$
\underline{c'}_a = \begin{cases} 1 & \text{if } a = 1 \\ c_{a/2} & \text{if } a \text{ is even} \\ c_{(a-1)/2} + 1 & \text{if } 1 < a \text{ is odd} \end{cases}
$$

$$= \begin{cases} 1 & \text{if } a = 1 \\ c'_a & \text{if } a \text{ is even} \\ c'_{a-1} + 1 & \text{if } 1 < a \text{ is odd} \end{cases}$$

As $e_{\mathrm{even}}(\vec{c'}) = -\infty$ by definition of I^1 we have $c'_1 = 1$. Furthermore, if $3 \leq a \leq 2l - 1$ is odd, then $2 \leq a - 1 \leq 2l - 2$ is even, so $c'_{a-1+1} = c'_{a-1} + 1$ for the same reason. This finishes the proof of $\Phi \circ \Psi = \mathrm{id}$. □

7.3.6. Progress so far

We can sum up progress so far as in the following proposition. Our goal is to show that the left hand side of the equation is zero.

Proposition 7.3.6.1. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\begin{aligned} &\partial\Big(\epsilon_X^{(l)}(y_1 \cdots y_m)\Big) - \epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m)) + \mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big) \\ = &\sum_{\substack{1 \leq i \leq 2l-1, \\ 1 \leq q \leq l-1, \\ v \in I_{i,2q,2q+1}}} B(v) + \sum_{\substack{1 \leq i \leq 2l-1, \\ 0 \leq q \leq l-1, \\ v \in I_{i,2q+1,2q+2}}} B(v) + \sum_{v \in I_{0,0,1}} B(v) \\ &- \sum_{v \in I^{\mathrm{d}}} B'(v) + \sum_{v \in I^1} B''(v) \end{aligned} \tag{$\heartsuit$}$$

Proof. By combining Proposition 7.3.2.9 (first two lines, for $\partial(\epsilon_X^{(l)}(y_1 \cdots y_m))$), Proposition 7.3.4.3 (third line, for $-\epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m))$) Proposition 7.3.5.1 (fourth line, for $\mathrm{d}(\epsilon_X^{(l-1)}(y_1 \cdots y_m))$) we obtain the following.

$$\begin{aligned} &\partial\Big(\epsilon_X^{(l)}(y_1 \cdots y_m)\Big) - \epsilon_X^{(l-1)}(\mathrm{d}(y_1 \cdots y_m)) + \mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big) \\ = &\sum_{\substack{1 \leq i \leq 2l-1, \\ 1 \leq q \leq l-1, \\ v \in I_{i,2q+2,2q}}} B(v) + \sum_{\substack{1 \leq i \leq 2l-1, \\ 1 \leq q \leq l-1, \\ v \in I_{i,2q,2q+1}}} B(v) + \sum_{\substack{1 \leq i \leq 2l-1, \\ 0 \leq q \leq l-1, \\ v \in I_{i,2q+1,2q+2}}} B(v) \\ &+ \sum_{v \in I_{0,0,1}} B(v) + \sum_{v \in I_{2l,2,0}} B(v) \\ &- \sum_{\substack{1 \leq i \leq 2l-1, \\ 1 \leq p \leq l-1, \\ v \in I_{i,2p+2,2p}}} B(v) - \sum_{v \in I_{2l,2,0}} B(v) - \sum_{v \in I^{\mathrm{d}}} B'(v) \\ &+ \sum_{v \in I^1} B''(v) \end{aligned}$$

Now some summands cancel and the result follows. □

7.3.7. Reindexing remaining summands from the boundary

We want to show that the left hand side of the equation in Proposition 7.3.6.1 is zero, doing so via the right hand side. Of the terms there, the last two terms are written as sums of summands that are obtained by applying T^{full} to an element of $C^{\mathrm{full}}(2l-1, m)$ and then permuting and perhaps adding a sign. The other terms are however given differently, so in this section we reindex those sums to bring them into a similar form.

Proposition 7.3.7.1. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{\substack{1\leq i\leq 2l-1,\\ 1\leq q\leq l-1,\\ v\in I_{i,2q,2q+1}}} B(v) + \sum_{v\in I_{0,0,1}} B(v) = \sum_{v\in I^{\partial}_{\mathrm{even}}} B''(v)$$

♡

Proof. Define the subset J of I as follows.[25]

$$J \coloneqq I_{0,0,1} \cup \bigcup_{\substack{1\leq i\leq 2l-1,\\ 1\leq q\leq l-1}} I_{i,2q,2q+1}$$

It then suffices to produce a bijection

$$\Phi\colon J \to I^{\partial}_{\mathrm{even}}$$

such that the following holds for every element v of J.

$$B''\big(\Phi(v)\big) = B(v)$$

So let $(i, \sigma, \vec{c})$ be an element of J. Then we make the following definitions.

$$\begin{aligned}
q &\coloneqq \sigma^{-1}(i)/2\\
\sigma' &\coloneqq \sigma_{i+1\to 2l} \circ \sigma \circ \sigma_{2l\to 2q+1}\\
c'_a &\coloneqq \begin{cases} c_{(a+1)/2} & \text{if } 2 \nmid a \text{ and } a \leq 2q\\ c_{a/2}+1 & \text{if } 2 \mid a \text{ and } a \leq 2q\\ c_{(a+1)/2}+1 & \text{if } 2 \nmid a \text{ and } a \geq 2q+1\\ c_{a/2+1} & \text{if } 2 \mid a \text{ and } a \geq 2q+1 \end{cases} \qquad \text{for } 1 \leq a \leq 2l\\
\Phi\big((i,\sigma,\vec{c})\big) &\coloneqq \Big(\sigma', \vec{c'}\Big)
\end{aligned}$$

[25] The definition of $I_{0,0,1}$ is really the same one as for $I_{i,2q,2q+1}$ if we set $i = 0$ and $q = 0$, so we mostly do not need to treat this as a separate case. The only difference is that $I_{i,0,1}$ is empty unless $i = 0$, as $\sigma(0) = 0$ for every element σ of Σ_{2l}.

There are various things that we need to check to verify that this is well-defined. First, by assumption on elements of J we have that $\sigma^{-1}(i+1) = \sigma^{-1}(i)+1$, so as $0 \leq i \leq 2l-1$ implies $1 \leq \sigma^{-1}(i+1) \leq 2l$ we can conclude that we must have $0 \leq \sigma^{-1}(i) \leq 2l-1$. Furthermore, the definition of J implies that $\sigma^{-1}(i)$ is even, so q is a well-defined integer satisfying $0 \leq q \leq l-1$. This makes σ' into a well-defined element of Σ_{2l}. Furthermore, we have

$$\begin{aligned}
\sigma'(2l) &= \sigma_{i+1\to 2l}\big(\sigma(2q+1)\big) = \sigma_{i+1\to 2l}\Big(\sigma\big(\sigma^{-1}(i)+1\big)\Big) \\
&= \sigma_{i+1\to 2l}\Big(\sigma\big(\sigma^{-1}(i)\big)+1\Big) = \sigma_{i+1\to 2l}(i+1) \\
&= 2l
\end{aligned}$$

so that we can even consider σ' as an element of Σ_{2l-1}.

We next show that $\vec{c'}$ is a well-defined element of $C^{\mathrm{full}}(2l-1, m)$. Using that $\vec{c}$ is an element of $C(l,m)$ one easily sees that in all four cases c'_a is a well-defined integer satisfying $1 \leq c'_a \leq m+1$[26]. We also have $c'_{2l} = c_{l+1} = m+1$. It remains to show that $c'_a < c'_{a+1}$ for $1 \leq a \leq 2l-1$. If $a \leq 2q-1$ is odd or $a \geq 2q+2$ even then is immediate. If $a \leq 2q-2$ is even, then $c'_a = c_{a/2}+1$ and $c'_{a+1} = c_{a/2+1}$, so $c'_a < c'_{a+1}$ follows from $c_{a/2}+1 \leq c_{a/2+1}-1$. If $a \geq 2q+1$ is odd, then $c'_a = c_{(a+1)/2}+1$ and $c'_{a+1} = c_{(a+1)/2+1}$, so that $c'_a < c'_{a+1}$ follows analogously. It remains to consider $a = 2q$. In this case $c'_{2q} = c_q+1$ and $c'_{2q+1} = c_{q+1}+1$, so $c'_{2q} < c'_{2q+1}$ as $c_q < c_{q+1}$. This completes the proof that $\vec{c'}$ is a well-defined element of $C^{\mathrm{full}}(2l-1, m)$.

We now verify the conditions required for $(\sigma', \vec{c'})$ to be an element of $I^{\partial}_{\mathrm{even}}$. Concretely we make the following claims.

$$\begin{aligned}
e_{\mathrm{even}}\Big(\vec{c'}\Big) &= 2q \\
e_{\mathrm{even}}\big(\sigma'\big) &\leq 2q \\
e_{\mathrm{odd}}\Big(\vec{c'}\Big) &\geq 2q+3 \\
e_{\mathrm{odd}}\big(\sigma'\big) &\geq 2q+1
\end{aligned}$$

To show that $e_{\mathrm{even}}(\vec{c'}) = 2q$, we first note that $c'_{2q} = c_q+1$ and $c'_{2q+1} = c_{q+1}+1$. As $\vec{c}$ is an element of $C(l,m)$, we have $c_q+1 < c_{q+1}$, which implies that $c'_{2q}+1 < c'_{2q+1}$, so $e_{\mathrm{even}}(\vec{c'}) \geq 2q$. Now let $2q+2 \leq a \leq 2l-2$ be even. Then $c'_a+1 = c_{a/2+1}+1 = c'_{a+1}$, which shows that $e_{\mathrm{even}}(\vec{c'}) = 2q$. Next let $1 \leq a \leq 2q-1$ be odd. Then $c'_a+1 = c_{(a+1)/2}+1 = c'_{a+1}$, so $e_{\mathrm{odd}}(\vec{c'}) \geq 2q+1$. Furthermore, we have $c'_{2q+1} = c_{q+1}+1$ and $c'_{2q+2} = c_{q+2}$. By definition of J it holds that $c_{q+1}+1 = c_{q+2}-1$, which then implies $c'_{2q+1}+1 = c'_{2q+2}$. Thus we even get $e_{\mathrm{odd}}(\vec{c'}) \geq 2q+3$. We next show that $e_{\mathrm{even}}(\sigma') \leq 2q$. So let $2q+2 \leq a \leq 2l-2$ be even. Then we have to show that σ' cyclically preserves the ordering of $\{a-1, a, a+1\}$, which amounts to σ cyclically preserving the ordering of $\{a, a+1, a+2\}$, which is the

[26]To exclude that we get $m+2$ in the two cases in which 1 is added to a component of $\vec{c}$, note that in those cases the index is at most l, and $c_l < c_{l+1} = m+1$.

case as $a+1$ is odd and satisfies $1 \leq a+1 \leq 2l-1$. To show that $e_{\mathrm{odd}}(\sigma') \geq 2q+1$ we let $1 \leq a \leq 2q-1$ be odd, and have to show that σ' cyclically preserves the ordering of $\{a-1, a, a+1\}$, which it does as σ does. This completes the proof that Φ is well-defined.

Keeping the notation used so far, we now show that $B''(\Phi((i,\sigma,\vec{c}))) = B((i,\sigma,\vec{c}))$. We first consider the signs.

$$\begin{aligned}
&\operatorname{sgn}\big(\sigma'\big)\\
&= \operatorname{sgn}\big(\sigma_{i+1\to 2l} \circ \sigma \circ \sigma_{2l \to 2q+1}\big)\\
&= (-1)^{2l-(i+1)} \cdot \operatorname{sgn}(\sigma) \cdot (-1)^{2q+1-2l}\\
&= (-1)^{i} \cdot \operatorname{sgn}(\sigma)
\end{aligned}$$

It thus remains to show the following.

$$\partial_i\Big(\sigma \cdot T\big((y_1,\ldots,y_m),\vec{c}\big)\Big) = \sigma' \cdot T^{\mathrm{full}}\Big((y_1,\ldots,y_m),\vec{c'}\Big)$$

For this let us write $T = T((y_1,\ldots,y_m),\vec{c})$ and T_a for the a-th tensor factor of T. Then we obtain the following for the a-th tensor factor of $T^{\mathrm{full}}((y_1,\ldots,y_m),\vec{c'})$, with $0 \leq a \leq 2l-1$.

$$\begin{aligned}
&T^{\mathrm{full}}\Big((y_1,\ldots,y_m),\vec{c'}\Big)_a\\
&= \begin{cases}
\prod_{j=1}^{c'_1-1} y_j & \text{if } a = 0\\
\overline{\prod_{j=c'_a}^{c'_{a+1}-1} y_j} & \text{if } a > 0
\end{cases}\\
&= \begin{cases}
\prod_{j=1}^{c_1} y_j & \text{if } a = 0 = q\\
\prod_{j=1}^{c_1-1} y_j & \text{if } a = 0 < q\\
\overline{\prod_{j=c_{(a+1)/2}}^{c_{(a+1)/2}} y_j} & \text{if } 0 < a \leq 2q-1 \text{ is odd}\\
\overline{\prod_{j=c_{a/2}+1}^{c_{a/2+1}-1} y_j} & \text{if } 0 < a \leq 2q-1 \text{ is even}\\
\overline{\prod_{j=c_q+1}^{c_{q+1}} y_j} & \text{if } 0 < a = 2q\\
\overline{\prod_{j=c_{(a+1)/2+1}}^{c_{(a+1)/2+1}-1} y_j} & \text{if } a \geq 2q+1 \text{ is odd}\\
\overline{\prod_{j=c_{a/2+1}}^{c_{a/2+1}} y_j} & \text{if } a \geq 2q+1 \text{ is even}
\end{cases}\\
&= \begin{cases}
\prod_{j=1}^{c_1} y_j & \text{if } a = 0 = q\\
\prod_{j=1}^{c_1-1} y_j & \text{if } a = 0 < q\\
\overline{y_{c_{(a+1)/2}}} & \text{if } 0 < a \leq 2q-1 \text{ is odd}\\
\overline{\prod_{j=c_{a/2}+1}^{c_{a/2+1}-1} y_j} & \text{if } 0 < a \leq 2q-1 \text{ is even}\\
\overline{\prod_{j=c_q+1}^{c_{q+1}} y_j} & \text{if } 0 < a = 2q\\
\overline{\prod_{j=c_{(a+1)/2+1}}^{c_{(a+1)/2+1}-1} y_j} & \text{if } a \geq 2q+1 \text{ is odd}\\
\overline{y_{c_{a/2+1}}} & \text{if } a \geq 2q+1 \text{ is even}
\end{cases}
\end{aligned}$$

$$= \begin{cases} T_0 \cdot T_1 & \text{if } a = 0 = q \\ T_a & \text{if } a = 0 < q \\ T_a & \text{if } 0 < a \leq 2q-1 \text{ is odd} \\ T_a & \text{if } 0 < a \leq 2q-1 \text{ is even} \\ T_{2q} \cdot T_{2q+1} & \text{if } 0 < a = 2q \\ T_{a+1} & \text{if } a \geq 2q+1 \text{ is odd} \\ T_{a+1} & \text{if } a \geq 2q+1 \text{ is even} \end{cases}$$

$$= \begin{cases} T_a & \text{if } a \leq 2q-1 \\ T_{2q} \cdot T_{2q+1} & \text{if } a = 2q \\ T_{a+1} & \text{if } a \geq 2q+1 \end{cases}$$

Note that the inverse of σ' is given by

$$\sigma'^{-1} = \sigma_{2q+1\to 2l} \circ \sigma^{-1} \circ \sigma_{2l\to i+1}$$

so that we have the following values for $0 \leq a \leq 2l-1$ (note that the cases below are exhaustive, as $2q+1$ can not occur due to $a \neq 2l$).

$$\sigma'^{-1}(a) = \begin{cases} \sigma^{-1}(a) & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \leq 2q \\ \sigma^{-1}(a) - 1 & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \geq 2q+2 \\ \sigma^{-1}(a+1) & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \leq 2q \\ \sigma^{-1}(a+1) - 1 & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \geq 2q+2 \end{cases}$$

The upshot is that the a-th tensor factor of $\sigma' \cdot T^{\text{full}}((y_1, \ldots, y_m), \vec{c'})$ is given by

$$\begin{cases} T^{\text{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big)_{\sigma^{-1}(a)} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \leq 2q \\ T^{\text{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big)_{\sigma^{-1}(a)-1} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \geq 2q+2 \\ T^{\text{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big)_{\sigma^{-1}(a+1)} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \leq 2q \\ T^{\text{full}}\Big((y_1, \ldots, y_m), \vec{c'}\Big)_{\sigma^{-1}(a+1)-1} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \geq 2q+2 \end{cases}$$

$$= \begin{cases} T_{\sigma^{-1}(a)} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \leq 2q-1 \\ T_{2q} \cdot T_{2q+1} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) = 2q \\ T_{\sigma^{-1}(a)-1+1} & \text{if } a \leq i \text{ and } \sigma^{-1}(a) \geq 2q+2 \\ T_{\sigma^{-1}(a+1)} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \leq 2q-1 \\ T_{2q} \cdot T_{2q+1} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) = 2q \\ T_{\sigma^{-1}(a+1)-1+1} & \text{if } a \geq i+1 \text{ and } \sigma^{-1}(a+1) \geq 2q+2 \end{cases}$$

Note that $\sigma(2q) = i$ and $\sigma(2q+1) = i+1$.

$$= \begin{cases} T_{\sigma^{-1}(a)} & \text{if } a \le i \text{ and } \sigma^{-1}(a) \le 2q-1 \\ T_{\sigma^{-1}(i)} \cdot T_{\sigma^{-1}(i+1)} & \text{if } a = i \\ T_{\sigma^{-1}(a)} & \text{if } a \le i \text{ and } \sigma^{-1}(a) \ge 2q+2 \\ T_{\sigma^{-1}(a+1)} & \text{if } a \ge i+1 \text{ and } \sigma^{-1}(a+1) \le 2q-1 \\ T_{\sigma^{-1}(a+1)} & \text{if } a \ge i+1 \text{ and } \sigma^{-1}(a+1) \ge 2q+2 \end{cases}$$

$$= \begin{cases} T_{\sigma^{-1}(a)} & \text{if } a \le i \\ T_{\sigma^{-1}(i)} \cdot T_{\sigma^{-1}(i+1)} & \text{if } a = i \\ T_{\sigma^{-1}(a+1)} & \text{if } a \ge i+1 \end{cases}$$

$$= \left(\partial_i\left(\sigma \cdot T\big((y_1, \ldots, y_m), \vec{c}\big)\right)\right)_a$$

This finishes the proof that $B''(\Phi((i, \sigma, \vec{c}))) = B((i, \sigma, \vec{c}))$.

We still have to show that Φ is a bijection. For this we construct an inverse Ψ. So let $(\sigma', \vec{c'})$ be an element of $I^{\partial}_{\mathrm{even}}$. Then we make the following definitions.

$$\begin{aligned} q &:= e_{\mathrm{even}}\left(\vec{c'}\right)/2 \\ i &:= \sigma'(2q) \\ \sigma &:= \sigma_{2l \to i+1} \circ \sigma' \circ \sigma_{2q+1 \to 2l} \\ c_a &:= \begin{cases} c'_{2a-1} & \text{if } a \le q \\ c'_{2a-1} - 1 & \text{if } q+1 \le a \le l \\ m+1 & \text{if } a = l+1 \end{cases} \qquad \text{for } 1 \le a \le l+1 \\ \Psi\left((\sigma', \vec{c'})\right) &:= (i, \sigma, \vec{c}) \end{aligned}$$

As usual various checks are needed to show that this is indeed well-defined. To begin with $e_{\mathrm{even}}\left(\vec{c'}\right) \neq -\infty$ by definition of $I^{\partial}_{\mathrm{even}}$, so $0 \le e_{\mathrm{even}}\left(\vec{c'}\right) \le 2l-2$, implying that q is a well-defined integer satisfying $0 \le q \le l-1$. This makes i a well defined integer satisfying $0 \le i \le 2l-1$. We note here that $i = 0$ if and only if $q = 0$.

We next show that σ is an element of E_l. So let $1 \le a \le 2l-1$ be odd. We have to show that σ cyclically preserves the ordering of $\{a-1, a, a+1\}$. If $a \le 2q-1$, then this amounts to showing that σ' cyclically preserves $\{a-1, a, a+1\}$, which it does as $e_{\mathrm{odd}}(\sigma') \ge e_{\mathrm{even}}(\vec{c'}) + 1 = 2q+1$. If instead $a \ge 2q+3$, then this amounts to showing that σ' cyclically preserves $\{a-2, a-1, a\}$, which it does as $a-1$ is even, satisfies $a-1 \ge 2q+2$, and $e_{\mathrm{even}}(\sigma') \le e_{\mathrm{even}}(\vec{c'}) = 2q$. The case $a = 2q+1$ remains. For this we just evaluate σ at $2q$ and $2q+1$ as follows

$$\sigma(2q) = i \qquad \sigma(2q+1) = i+1$$

which already shows the claim, no matter what $\sigma(2q+2)$ may be. It also handles the condition on σ required for $(i, \sigma, \vec{c})$ to be an element of $I_{i,2q,2q+1}$.

Now we show that $\vec{c}$ is an element of $C(l,m)$. We have $c_{l+1} = m+1$ by definition, and for $1 \leq a \leq l$ we have $1 \leq 2a-1 \leq 2l-1$ so that c'_{2a-1} is a well-defined integer. If furthermore $a \leq q$, then, as $q \leq l-1$, we have the following chain of inequalities.

$$1 \leq c'_{2a-1} \leq c'_{2l-3} \leq c'_{2l} - 3 = m-2$$

If instead $q+1 \leq a$ as well as $2 \leq a$, then we have the following chain of inequalities.

$$1 \leq c'_2 \leq c'_3 - 1 \leq c'_{2a-1} - 1 \leq c'_{2l-1} - 1 \leq c'_{2l} - 2 = m-1$$

Finally, if $a = 1$ and $q = 0$, then $e_{\mathrm{even}}(\vec{c'}) = 0$, which implies that $1 \leq c'_1 - 1$, while $c'_{2a-1} - 1 \leq m-1$ as in the previous case. We have thus shown so far that $c_{l+1} = m+1$ while $1 \leq c_a \leq m-1$ for $1 \leq a \leq l$. So let $1 \leq a \leq l-1$. We still have to show that $c_a + 1 \leq c_{a+1} - 1$. If $a \leq q-1$ or $a \geq q+1$ this follows from $c'_{2a-1} < c'_{2a} < c'_{2a+1}$. The case $a = q$ remains, where we have $c_q = c'_{2q-1}$ and $c_{q+1} = c'_{2q+1} - 1$. But as $e_{\mathrm{even}}(\vec{c'}) = 2q$, we obtain the last inequality in the following chain $c'_{2q-1} < c'_{2q} < c'_{2q} + 1 < c'_{2q+1}$, which shows the claim. Using that $c'_{2q+1} = c'_{2q+2} - 1 = c'_{2q+3} - 2$ due to $e_{\mathrm{odd}}(\vec{c'}) \geq 2q+3$ and $e_{\mathrm{even}}(\vec{c'}) = 2q$ we obtain the short calculation

$$c_{q+1} + 1 = c'_{2q+1} - 1 + 1 = c'_{2q+1} = c'_{2q+2} - 1 = c'_{2q+3} - 2 = c_{q+2} - 1$$

which finishes the proof that Ψ is well-defined as a map to J.

It remains to show that Ψ is an inverse map to Φ. So let $(i, \sigma, \vec{c})$ be an element of J, and set $\Phi((i,\sigma,\vec{c})) = (\sigma', \vec{c'})$ and $q = \sigma^{-1}(i)/2$ as in the definition of Φ. Set furthermore $\Psi((\sigma', \vec{c'})) = (\underline{i}, \underline{\sigma}, \underline{\vec{c}})$ and $\underline{q} = e_{\mathrm{even}}(\vec{c'})/2$ as in the definition of Ψ. In the definition of Φ it was shown that $e_{\mathrm{even}}(\vec{c'}) = 2q$, so that $\underline{q} = q$, and unpacking the definition we then have $\underline{i} = \sigma'(2q) = i$. It then follows immediately that also $\underline{\sigma} = \sigma$, and the following calculation shows that $\underline{\vec{c}} = \vec{c}$, where $1 \leq a \leq l$.

$$\begin{aligned}
\underline{c}_a &= \begin{cases} c'_{2a-1} & \text{if } a \leq q \\ c'_{2a-1} - 1 & \text{if } q+1 \leq a \leq l \end{cases} \\
&= \begin{cases} c_a & \text{if } a \leq q \\ c_a + 1 - 1 & \text{if } q+1 \leq a \leq l \end{cases} \\
&= c_a
\end{aligned}$$

This shows that $\Psi \circ \Phi = \mathrm{id}$.

Now let $(\sigma', \vec{c'})$ be an element of $I^{\partial}_{\mathrm{even}}$. Set $\Psi((\sigma', \vec{c'})) = (i, \sigma, \vec{c})$ and let $q = e_{\mathrm{even}}(\vec{c'})/2$ be as in the definition of Ψ. Let furthermore $\Phi((i,\sigma,\vec{c})) = (\underline{\sigma'}, \underline{\vec{c'}})$ and $\underline{q} = \sigma^{-1}(i)/2$ as in the definition of Φ. Then we have

$$\sigma(2q) = \big(\sigma_{2l\to i+1} \circ \sigma' \circ \sigma_{2q+1\to 2l}\big)(2q) = \sigma_{2l\to i+1}\big(\sigma'(2q)\big) = \sigma_{2l\to i+1}(i) = i$$

so that $\underline{q} = q$. It then follows that $\underline{\sigma'} = \sigma'$. It remains to show that $\underline{\vec{c'}} = \vec{c'}$. So let $1 \leq a \leq 2l$. Then this is shown by the following calculation.

$$\begin{aligned}
\underline{c'}_a &= \begin{cases} c_{(a+1)/2} & \text{if } 2 \nmid a \text{ and } a \leq 2q \\ c_{a/2} + 1 & \text{if } 2 \mid a \text{ and } a \leq 2q \\ c_{(a+1)/2} + 1 & \text{if } 2 \nmid a \text{ and } a \geq 2q+1 \\ c_{a/2+1} & \text{if } 2 \mid a \text{ and } a \geq 2q+1 \end{cases} \\
&= \begin{cases} c'_a & \text{if } 2 \nmid a \text{ and } a \leq 2q \\ c'_{a-1} + 1 & \text{if } 2 \mid a \text{ and } a \leq 2q \\ c'_a - 1 + 1 & \text{if } 2 \nmid a \text{ and } a \geq 2q+1 \\ c'_{a+1} - 1 & \text{if } 2 \mid a \text{ and } 2l > a \geq 2q+1 \\ m+1 & \text{if } a = 2l \end{cases}
\end{aligned}$$

Using that $e_{\mathrm{odd}}(\vec{c'}) \geq 2q+3$ we obtain $c'_{a-1} + 1 = c'_a$ in the second case, and using $e_{\mathrm{even}}(\vec{c'}) = 2q$ we obtain $c'_a = c'_{a+1} - 1$ in the fourth case.

$$\begin{aligned}
&= \begin{cases} c'_a & \text{if } 2 \nmid a \text{ and } a \leq 2q \\ c'_a & \text{if } 2 \mid a \text{ and } a \leq 2q \\ c'_a & \text{if } 2 \nmid a \text{ and } a \geq 2q+1 \\ c'_a & \text{if } 2 \mid a \text{ and } 2l > a \geq 2q+1 \\ m+1 & \text{if } a = 2l \end{cases} \\
&= c'_a
\end{aligned}$$

□

Proposition 7.3.7.2. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \dots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{\substack{1 \leq i \leq 2l-1, \\ 0 \leq q \leq l-1, \\ v \in I_{i,2q+1,2q+2}}} B(v) = - \sum_{v \in I^{\partial}_{\mathrm{odd}}} B''(v)$$

♡

Proof. The proof is completely analogous to the proof of Proposition 7.3.7.1, so we omit the details. The formulas used to define Φ in this case are

$$\begin{aligned}
q &:= \big(\sigma^{-1}(i) - 1\big)/2 \\
\sigma' &:= \sigma_{i \to 2l} \circ \sigma \circ \sigma_{2l \to 2q+1} \\
c'_a &:= \begin{cases} c_{(a+1)/2} & \text{if } 2 \nmid a \text{ and } a \leq 2q+1 \\ c_{a/2} + 1 & \text{if } 2 \mid a \text{ and } a \leq 2q+1 \\ c_{(a+1)/2} + 1 & \text{if } 2 \nmid a \text{ and } a \geq 2q+2 \\ c_{a/2+1} & \text{if } 2 \mid a \text{ and } a \geq 2q+2 \end{cases} \qquad \text{for } 1 \leq a \leq 2l
\end{aligned}$$

$$\Phi\Big((i, \sigma, \vec{c})\Big) := \Big(\sigma', \vec{c'}\Big)$$

and in this case $e_{\mathrm{odd}}(\vec{c'}) = 2q + 1$.

The special assumption on $\vec{c}$ from the definition of J has in this case, in contrast to the proof of Proposition 7.3.7.1, a different form depending on whether $q = 0$ or not, as there is no c_0. Where this property was used in the proof of Proposition 7.3.7.1 was to show that $e_{\mathrm{odd}}(\vec{c'}) \neq 2q + 1$. In our case here this property is needed to show that $e_{\mathrm{even}}(\vec{c'}) \neq 2q$, and the distinction between the cases $q = 0$ and $q \neq 0$ corresponds to the analogous distinction in the definition of e_{even}.

That the definition of σ' involves i instead of $i + 1$ introduces an extra minus sign in $\mathrm{sgn}(\sigma')$, which explains the minus sign in the result.

The formulas used to define Ψ are as follows.

$$\begin{aligned}
q &:= \left(e_{\mathrm{even}}\left(\vec{c'}\right) - 1\right)/2 \\
i &:= \sigma'(2q + 1) \\
\sigma &:= \sigma_{2l \to i} \circ \sigma' \circ \sigma_{2q+1 \to 2l} \\
c_a &:= \begin{cases} c'_{2a-1} & \text{if } a \leq q + 1 \\ c'_{2a-1} - 1 & \text{if } q + 2 \leq a \leq l \\ m + 1 & \text{if } a = l + 1 \end{cases} \qquad \text{for } 1 \leq a \leq l + 1 \\
\Psi\left((\sigma', \vec{c'})\right) &:= (i, \sigma, \vec{c})
\end{aligned}$$

Again the proof that this is well-defined is analogous to the proof of Proposition 7.3.7.1 except the special treatment of $q = 0$ as discussed above. □

We sum up our current progress.

Proposition 7.3.7.3. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\begin{aligned}
&\partial\Big(\epsilon_X^{(l)}(y_1 \cdots y_m)\Big) - \epsilon_X^{(l-1)}\big(\mathrm{d}(y_1 \cdots y_m)\big) + \mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big) \\
&= \sum_{v \in I^{\partial}_{\mathrm{even}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd}}} B''(v) + \sum_{v \in I^{1}} B''(v) - \sum_{v \in I^{\mathrm{d}}} B'(v)
\end{aligned} \qquad \heartsuit$$

Proof. Combine Propositions 7.3.6.1, 7.3.7.1 and 7.3.7.2. □

7.3.8. Subdivisions of the remaining indexing sets

To continue we need to subdivide I^{d}, $I^{\partial}_{\mathrm{even}}$, $I^{\partial}_{\mathrm{odd}}$, and I^{1} into a disjoint unions of subsets, which we do in this section.

Definition 7.3.8.1. In this definition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1 and 7.3.4.2. Let $l \geq 1$ and $m \geq 0$ be integers. We define the following subsets of I^{d}.

$$\begin{aligned}
I^{\mathrm{d,cancel}}_{>} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}} \;\Big|\; e_{\mathrm{even}}(\sigma) > e_{\mathrm{odd}}(\sigma)\Big\}\\
I^{\mathrm{d}}_{<} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}} \;\Big|\; e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)\Big\}\\
I^{\mathrm{d}}_{<,\mathrm{top}} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<} \;\Big|\; \text{if } e_{\mathrm{odd}}(\sigma) \neq \infty \text{ then } p = e_{\mathrm{odd}}(\sigma), \text{ else } p = 2l-1\Big\}\\
I^{\mathrm{d,cancel}}_{<,\mathrm{top}} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<,\mathrm{top}} \;\Big|\; e_{\mathrm{even}}(\sigma) \neq -\infty \text{ and } e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma)\Big\}\\
I^{\mathrm{d}}_{<,\mathrm{top},\partial} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<,\mathrm{top}} \;\Big|\; e_{\mathrm{even}}(\vec{c}) \neq -\infty \text{ and } e_{\mathrm{even}}(\vec{c}) \geq e_{\mathrm{even}}(\sigma)\Big\}\\
I^{\mathrm{d}}_{<,\mathrm{top},1} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<,\mathrm{top}} \;\Big|\; e_{\mathrm{even}}(\vec{c}) = -\infty \text{ and } e_{\mathrm{even}}(\sigma) = -\infty\Big\}\\
I^{\mathrm{d}}_{<,\mathrm{bottom}} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<} \;\Big|\; p = e_{\mathrm{even}}(\sigma)\Big\}\\
I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<,\mathrm{bottom}} \;\Big|\; \text{if } e_{\mathrm{odd}}(\sigma) = \infty \text{ then } e_{\mathrm{odd}}(\vec{c}) = \infty,\\
&\qquad\qquad \text{else } e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})\Big\}\\
I^{\mathrm{d}}_{<,\mathrm{bottom},\partial} &\coloneqq \Big\{(\sigma, \vec{c}, p) \in I^{\mathrm{d}}_{<,\mathrm{bottom}} \;\Big|\; e_{\mathrm{odd}}(\vec{c}) \neq \infty \text{ and } e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{odd}}(\vec{c})\Big\}
\end{aligned}$$

The following subset is to be defined for $1 \leq p \leq 2l-2$.

$$I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p} \coloneqq \Big\{\big(\sigma, \vec{c}, p'\big) \in I^{\mathrm{d}}_{<} \;\Big|\; p' = p \text{ and } e_{\mathrm{even}}(\sigma) < p < e_{\mathrm{odd}}(\sigma)\Big\}$$

We also define the following subsets of $\Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m)$.

$$\begin{aligned}
I^{\partial}_{\mathrm{even,d}} &\coloneqq \Big\{(\sigma, \vec{c}) \in I^{\partial}_{\mathrm{even}} \;\Big|\; \text{if } e_{\mathrm{odd}}(\sigma) = \infty \text{ then } e_{\mathrm{odd}}(\vec{c}) = \infty,\\
&\qquad\qquad \text{else } e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})\Big\}\\
I^{\partial}_{\mathrm{odd,d}} &\coloneqq \Big\{(\sigma, \vec{c}) \in I^{\partial}_{\mathrm{odd}} \;\Big|\; e_{\mathrm{even}}(\sigma) \neq -\infty \text{ and } e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma)\Big\}\\
I^{\partial}_{\mathrm{odd-even}} &\coloneqq \Big\{(\sigma, \vec{c}) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \;\Big|\; e_{\mathrm{even}}(\vec{c}) \neq -\infty \text{ and } e_{\mathrm{odd}}(\vec{c}) \neq \infty \text{ and}\\
&\qquad\qquad e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{even}}(\vec{c}) \leq e_{\mathrm{odd}}(\vec{c}) - 3 \leq e_{\mathrm{odd}}(\sigma) - 3\Big\}\\
I^{\partial}_{\mathrm{odd},1} &\coloneqq \Big\{(\sigma, \vec{c}) \in \Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m) \;\Big|\; e_{\mathrm{even}}(\vec{c}) = -\infty \text{ and } e_{\mathrm{odd}}(\vec{c}) \neq \infty \text{ and}\\
&\qquad\qquad e_{\mathrm{even}}(\sigma) = -\infty \text{ and } e_{\mathrm{odd}}(\vec{c}) \leq e_{\mathrm{odd}}(\sigma)\Big\}\\
I^{1}_{\mathrm{d}} &\coloneqq \Big\{(\sigma, \vec{c}) \in I^{1} \;\Big|\; \text{if } e_{\mathrm{odd}}(\sigma) = \infty \text{ then } e_{\mathrm{odd}}(\vec{c}) = \infty, \text{else } e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})\Big\} \qquad \diamondsuit
\end{aligned}$$

Proposition 7.3.8.2. *In this definition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let $l \geq 1$ and $m \geq 0$ be integers. Then the set I^{d} is the disjoint union of the following subsets.*

- $I^{\mathrm{d,cancel}}_{>}$
- $I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p}$ *for* $1 \leq p \leq 2l - 2$
- $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$
- $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$
- $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$
- $I^{\mathrm{d}}_{<,\mathrm{top},1}$
- $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$ ♡

Proof. As $e_{\mathrm{even}}(\sigma) = e_{\mathrm{odd}}(\sigma)$ is never possible for parity reasons, we must always either have $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$ or $e_{\mathrm{even}}(\sigma) > e_{\mathrm{odd}}(\sigma)$, showing that I^{d} is the disjoint union of $I^{\mathrm{d,cancel}}_{>}$ and $I^{\mathrm{d}}_{<}$.

Now assume that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d}}_{<}$. We will show that then

$$e_{\mathrm{even}}(\sigma) \leq p \leq e_{\mathrm{odd}}(\sigma)$$

which implies that $I^{\mathrm{d}}_{<}$ is the disjoint union of the subsets $I^{\mathrm{d,cancel}}_{<,\mathrm{mid},q}$ for $1 \leq q \leq 2l - 2$, $I^{\mathrm{d}}_{<,\mathrm{top}}$, and $I^{\mathrm{d}}_{<,\mathrm{bottom}}$. By definition of I^{d} we must have

$$e_{\mathrm{even}}(\sigma) - 1 \leq p \leq e_{\mathrm{odd}}(\sigma) + 1$$

so that we only must rule out that $p = e_{\mathrm{even}}(\sigma) - 1$ and $p = e_{\mathrm{odd}}(\sigma) + 1$. For this, note that by definition of $e_{\mathrm{even}}(\sigma)$ the permutation σ does *not* cyclically preserve the ordering of $\{e_{\mathrm{even}}(\sigma) - 1, e_{\mathrm{even}}(\sigma), e_{\mathrm{even}}(\sigma) + 1\}$, which means that

$$e_{\mathrm{even}}(\sigma) - 1, \qquad e_{\mathrm{even}}(\sigma) + 1, \qquad e_{\mathrm{even}}(\sigma)$$

will be cyclically ordered. As $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$ by definition of $I^{\mathrm{d}}_{<}$, we also know that

$$e_{\mathrm{even}}(\sigma) - 2, \qquad e_{\mathrm{even}}(\sigma) - 1, \qquad e_{\mathrm{even}}(\sigma)$$

is cyclically ordered. Combining both we obtain that

$$e_{\mathrm{even}}(\sigma) - 2, \qquad e_{\mathrm{even}}(\sigma) - 1, \qquad e_{\mathrm{even}}(\sigma) + 1, \qquad e_{\mathrm{even}}(\sigma)$$

is cyclically ordered. But this means that

$$e_{\mathrm{even}}(\sigma) - 2, \qquad e_{\mathrm{even}}(\sigma), \qquad e_{\mathrm{even}}(\sigma) + 1$$

is *not* cyclically ordered, which rules out $p = e_{\mathrm{even}}(\sigma) - 1$. Analogously one can rule out $p = e_{\mathrm{odd}}(\sigma) + 1$.

We have now shown that I^{d} is the disjoint union of the following subsets.

- $I^{\mathrm{d,cancel}}_{>}$

- $I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p}$ for $1 \leq p \leq 2l-2$
- $I^{\mathrm{d}}_{<,\mathrm{top}}$
- $I^{\mathrm{d}}_{<,\mathrm{bottom}}$

It thus remains to show the following two claims. Firstly that $I^{\mathrm{d}}_{<,\mathrm{top}}$ is a disjoint union of the following subsets.

- $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$
- $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$
- $I^{\mathrm{d}}_{<,\mathrm{top},1}$

And secondly that $I^{\mathrm{d}}_{<,\mathrm{bottom}}$ is a disjoint union of the following subsets.

- $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$
- $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$

For the first claim we begin by noting that clearly the three subsets are pairwise disjoint. So now let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{top}}$. First assume that $e_{\mathrm{even}}(\sigma) \neq -\infty$. If $e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma)$ then $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$. If instead $e_{\mathrm{even}}(\vec{c}) \geq e_{\mathrm{even}}(\sigma)$ then it follows from $e_{\mathrm{even}}(\sigma) \neq -\infty$ that also $e_{\mathrm{even}}(\vec{c}) \neq -\infty$ and $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$. Next assume that $e_{\mathrm{even}}(\sigma) = -\infty$. If also $e_{\mathrm{even}}(\vec{c}) = -\infty$, then $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d}}_{<,\mathrm{top},1}$, and otherwise it will be an element of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$.

For the second claim we can again note immediately that the two subsets are disjoint. So now let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{bottom}}$ and assume it is not an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$. If $e_{\mathrm{odd}}(\sigma) = \infty$, then this means $e_{\mathrm{odd}}(\vec{c}) \neq \infty$, and this implies that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. If instead $e_{\mathrm{odd}}(\sigma) \neq \infty$, then this implies $e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{odd}}(\vec{c})$, so in particular $e_{\mathrm{odd}}(\vec{c}) \neq \infty$, and thus $(\sigma, \vec{c}, p)$ is again an element of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. □

Proposition 7.3.8.3. *In this definition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let $l \geq 1$ and $m \geq 0$ be integers. Then the set $I^{\partial}_{\mathrm{odd}}$ is the disjoint union of the following subsets.*

- $I^{\partial}_{\mathrm{odd,d}}$
- $I^{\partial}_{\mathrm{odd-even}}$
- $I^{\partial}_{\mathrm{odd},1}$

Furthermore the set $I^{\partial}_{\mathrm{even}}$ is the disjoint union of the following subsets.

- $I^{\partial}_{\mathrm{even,d}}$
- $I^{\partial}_{\mathrm{odd-even}}$ ♡

Proof. While $I^{\partial}_{\mathrm{odd,d}}$ was defined as a subset of $I^{\partial}_{\mathrm{odd}}$ and $I^{\partial}_{\mathrm{even,d}}$ as a subset of $I^{\partial}_{\mathrm{even}}$, the other two relevant sets have only be defined as a subset of $\Sigma_{2l-1} \times C^{\mathrm{full}}(2l-1, m)$. However it follows easily from the definition that $\vec{c}$ and σ have the necessary properties for the required subset inclusions.

We first discuss $I^{\partial}_{\mathrm{odd}}$. So let $(\sigma, \vec{c})$ be an element of $I^{\partial}_{\mathrm{odd}}$. If $e_{\mathrm{even}}(\vec{c}) = -\infty$ as well as $e_{\mathrm{even}}(\sigma) = -\infty$, then $(\sigma, \vec{c})$ could (out of the three subsets in question) only possibly be an element of $I^{\partial}_{\mathrm{odd},1}$, and indeed it is, as the other two required properties are part of the definition of $I^{\partial}_{\mathrm{odd}}$. If instead $e_{\mathrm{even}}(\vec{c}) = -\infty$ and $e_{\mathrm{even}}(\sigma) > -\infty$, then $(\sigma, \vec{c})$ is an element of (only) $I^{\partial}_{\mathrm{odd,d}}$. If we have $e_{\mathrm{even}}(\vec{c}) \neq -\infty$, and $e_{\mathrm{even}}(\sigma) > e_{\mathrm{even}}(\vec{c})$, then $(\sigma, \vec{c})$ is also only element of $I^{\partial}_{\mathrm{odd,d}}$. The last case is when $e_{\mathrm{even}}(\vec{c}) \neq -\infty$, and $e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{even}}(\vec{c})$, in which case $(\sigma, \vec{c})$ is an element of precisely $I^{\partial}_{\mathrm{odd-even}}$, with the remaining inequalities arising from the definition of $I^{\partial}_{\mathrm{odd}}$.

We now discuss $I^{\partial}_{\mathrm{even}}$. It is easy to see that elements of $I^{\partial}_{\mathrm{even,d}}$ are not elements of $I^{\partial}_{\mathrm{odd-even}}$, so the two subsets are disjoint. Now let $(\sigma, \vec{c})$ be an element of $I^{\partial}_{\mathrm{even}}$ that is not in $I^{\partial}_{\mathrm{even,d}}$. If $e_{\mathrm{odd}}(\sigma) = \infty$ this means that $e_{\mathrm{odd}}(\vec{c}) \neq \infty$, and then $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{odd-even}}$, with the other inequalities being part of the definition of $I^{\partial}_{\mathrm{even}}$. If instead $e_{\mathrm{odd}}(\sigma) \neq \infty$, then $e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{odd}}(\vec{c})$, which implies $e_{\mathrm{odd}}(\vec{c}) \neq \infty$, and combined with the properties arising from the definition of $I^{\partial}_{\mathrm{even}}$ this again shows that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{odd-even}}$. □

Proposition 7.3.8.4. *In this definition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let $l \geq 1$ and $m \geq 0$ be integers. Then the set I^1 is the disjoint union of the following subsets.*

- I^1_{d}
- $I^{\partial}_{\mathrm{odd},1}$ ♡

Proof. While I^1_{d} was defined as a subset of I^1, this is not the case for $I^{\partial}_{\mathrm{odd},1}$, but that it is a subset is clear from the definition. It is also straightforward that the two subsets are disjoint. Now let $(\sigma, \vec{c})$ be an element of I^1. Assume $e_{\mathrm{odd}}(\sigma) = \infty$. Then either $e_{\mathrm{odd}}(\vec{c}) = \infty$, in which case $(\sigma, \vec{c})$ is an element of I^1_{d}, or $e_{\mathrm{odd}}(\vec{c}) \leq e_{\mathrm{odd}}(\sigma)$, in which case $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{odd},1}$. Now assume $e_{\mathrm{odd}}(\sigma) \neq \infty$. Then either $e_{\mathrm{odd}}(\vec{c}) > e_{\mathrm{odd}}(\sigma)$, in which case $(\sigma, \vec{c})$ is an element of I^1_{d}, or $e_{\mathrm{odd}}(\vec{c}) \leq e_{\mathrm{odd}}(\sigma)$, which implies $e_{\mathrm{odd}}(\vec{c}) \neq \infty$, so that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{odd},1}$. □

7.3.9. Canceling of some summands of $\epsilon_X^{(l-1)} \circ \mathrm{d}$

Several of the subsets we defined for I^{d} are such that the relevant sums over them cancel (which we indicated by naming them $I^{\mathrm{d,cancel}}$ with some subscript). This is what we show in this subsection.

Proposition 7.3.9.1. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and*

$m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.

$$\sum_{v \in I_{>}^{\mathrm{d,cancel}}} B'(v) = 0 \qquad \heartsuit$$

Proof. Let $(\sigma, \vec{c}, p)$ be an element of $I_{>}^{\mathrm{d,cancel}}$. Then

$$e_{\mathrm{odd}}(\sigma) \leq e_{\mathrm{even}}(\sigma) - 1 \leq p \leq e_{\mathrm{odd}}(\sigma) + 1 \leq e_{\mathrm{even}}(\sigma)$$

holds, where the middle inequality is from the definition of I^{d} and the other two are from the definition of $I_{>}^{\mathrm{d,cancel}}$. This implies that[27]

$$e_{\mathrm{odd}}(\sigma) + 1 = e_{\mathrm{even}}(\sigma)$$

and either $p = e_{\mathrm{odd}}(\sigma)$ or $p = e_{\mathrm{odd}}(\sigma) + 1$.

It thus suffices to show that the map

$$\Phi\colon \left\{ (\sigma, \vec{c}, p) \in I_{>}^{\mathrm{d,cancel}} \;\middle|\; p = e_{\mathrm{odd}}(\sigma) \right\} \to \left\{ (\sigma, \vec{c}, p) \in I_{>}^{\mathrm{d,cancel}} \;\middle|\; p = e_{\mathrm{odd}}(\sigma) + 1 \right\}$$
$$(\sigma, \vec{c}, p) \mapsto (\sigma, \vec{c}, p+1)$$

is a well-defined bijection and that for every element $(\sigma, \vec{c}, p)$ of $I_{>}^{\mathrm{d,cancel}}$ with $p = e_{\mathrm{odd}}(\sigma)$ it holds that $B'((\sigma, \vec{c}, p+1)) = -B'((\sigma, \vec{c}, p))$. This property of B' is obvious from the definition, so it only remains to show that Φ is a well-defined bijection.

So let $(\sigma, \vec{c}, p)$ be an element of $I_{>}^{\mathrm{d,cancel}}$ with $p = e_{\mathrm{odd}}(\sigma)$. Note that this implies that p is odd with $p \leq 2l - 3$. Thus $1 \leq p + 1 \leq 2l - 2$. We have to show that $(\sigma, \vec{c}, p+1)$ is again an element of I^{d}. It follows from

$$e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c})$$

that also

$$e_{\mathrm{even}}(\vec{c}) < p + 1 < e_{\mathrm{odd}}(\vec{c})$$

for parity reasons. The discussion at the start of this proof shows that

$$e_{\mathrm{even}}(\sigma) - 1 \leq p + 1 \leq e_{\mathrm{odd}}(\sigma) + 1$$

holds as well. It thus remains to show that

$$\sigma\big(e_{\mathrm{odd}}(\sigma) - 1\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma) + 2\big)$$

is cyclically ordered. But as $(\sigma, \vec{c}, p)$ is an element of I^{d} we know that

$$\sigma\big(e_{\mathrm{odd}}(\sigma) - 1\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma) + 1\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma) + 2\big)$$

is cyclically ordered, and the definition of $e_{\mathrm{even}}(\sigma) = e_{\mathrm{odd}}(\sigma) + 1$ implies that

$$\sigma\big(e_{\mathrm{odd}}(\sigma)\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma) + 2\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma) + 1\big)$$

[27] $e_{\mathrm{even}}(\sigma) - 1 \leq e_{\mathrm{odd}}(\sigma) + 1 \leq e_{\mathrm{even}}(\sigma)$ but for parity reasons $e_{\mathrm{odd}}(\sigma) + 1 = e_{\mathrm{even}}(\sigma) - 1$ is not possible.

is cyclically ordered. Rotating the first of these two we can phrase this as the following two lines each being cyclically ordered

$$\begin{array}{ccc} \sigma\big(e_{\mathrm{odd}}(\sigma)+2\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)-1\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)+1\big) \\ \sigma\big(e_{\mathrm{odd}}(\sigma)\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)+2\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)+1\big) \end{array}$$

which combines to

$$\sigma\big(e_{\mathrm{odd}}(\sigma)\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)+2\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)-1\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)+1\big)$$

being cyclically ordered, from which the claim follows, so Φ is well-defined.

To show that Φ is a bijection, we let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d,cancel}}_{>}$ with $p = e_{\mathrm{odd}}(\sigma)+1$. We have to show that $(\sigma, \vec{c}, p-1)$ is again an element of I^{d}. The first two properties for this are shown completely analogously to the argument above. It thus remains to show that

$$\sigma\big(e_{\mathrm{odd}}(\sigma)-1\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)+1\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)+2\big)$$

is cyclically ordered. Similarly to the argument above one finds that the following two lines are each being cyclically ordered, the first arising from $(\sigma, \vec{c}, p)$ being an element of I^{d}, the second from the definition of e_{odd}.

$$\begin{array}{ccc} \sigma\big(e_{\mathrm{odd}}(\sigma)-1\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)+2\big) \\ \sigma\big(e_{\mathrm{odd}}(\sigma)-1\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)+1\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)\big) \end{array}$$

which combines to

$$\sigma\big(e_{\mathrm{odd}}(\sigma)-1\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)+1\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)+2\big)$$

being cyclically ordered, from which the claim follows. □

Proposition 7.3.9.2. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Let $1 \leq p \leq 2l-3$ be odd. Then the following holds.*

$$\sum_{v \in I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p}} B'(v) + \sum_{v \in I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p+1}} B'(v) = 0 \qquad \heartsuit$$

Proof. Let σ be an element of Σ_{2l-1} and $\vec{c}$ an element of $C^{\mathrm{full}}(2l-1, m)$. It suffices to show that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p}$ if and only if $(\sigma, \vec{c}, p+1)$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p+1}$, and that in this case it holds that $B'((\sigma, \vec{c}, p+1)) = -B'((\sigma, \vec{c}, p))$. This latter property is clear from definition.

Purely for parity reasons we immediately have that

$$e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c}) \qquad \text{and} \qquad e_{\mathrm{even}}(\sigma) < p < e_{\mathrm{odd}}(\sigma)$$

if and only if

$$e_{\mathrm{even}}(\vec{c}) < p+1 < e_{\mathrm{odd}}(\vec{c}) \qquad \text{and} \qquad e_{\mathrm{even}}(\sigma) < p+1 < e_{\mathrm{odd}}(\sigma)$$

It thus remains to show that σ cyclically preserves the ordering of $\{p-1, p+1, p+2\}$ if and only if σ cyclically preserves the ordering of $\{p-1, p, p+2\}$. So assume first that σ cyclically preserves the ordering of $\{p-1, p+1, p+2\}$. As $p < e_{\mathrm{odd}}(\sigma)$ is odd, we know that σ cyclically preserves the ordering of $\{p-1, p, p+1\}$, which combined with the assumption yields the claim. For the other direction we combine the assumption with $p+1 > e_{\mathrm{even}}(\sigma)$ being even, which means that σ cyclically preserves the ordering of $\{p, p+1, p+2\}$. □

Proposition 7.3.9.3. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I^{\mathrm{d,cancel}}_{<,\mathrm{top}}} B'(v) + \sum_{v \in I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}} B'(v) = 0 \qquad \heartsuit$$

Proof. It suffices to show that

$$\begin{aligned} \Phi\colon I^{\mathrm{d,cancel}}_{<,\mathrm{top}} &\to I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}} \\ (\sigma, \vec{c}, p) &\mapsto \big(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma)\big) \end{aligned}$$

is a well-defined bijection that satisfies $B'(\Phi(v)) = -B'(v)$ for every element v of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$.

So let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$. We first handle the property for B'. We have

$$\begin{aligned} B'\Big(\big(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma)\big)\Big) &= (-1)^{e_{\mathrm{even}}(\sigma)+1} B''\big((\sigma, \vec{c})\big) = -B''\big((\sigma, \vec{c})\big) \\ &= -(-1)^{p+1} B''\big((\sigma, \vec{c})\big) = -B'\big((\sigma, \vec{c}, p)\big) \end{aligned}$$

where we used that p is odd.

Next we need to show that $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$. First we show that this is an element of I^{d}. For this we first show the following inequality.

$$e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\vec{c})$$

The inequality on the left holds by definition of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$. By definition of $I^{\mathrm{d}}_{<}$ we have $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$, which together with $e_{\mathrm{odd}}(\sigma) \leq e_{\mathrm{odd}}(\vec{c})$ due to what p is (and $(\sigma, \vec{c}, p)$ being an element of I^{d}) implies the inequality on the right. Next we show the following inequality.

$$e_{\mathrm{even}}(\sigma) - 2 < e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma) + 2$$

The left inequality is clear, and the right inequality follows from $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$, which holds by definition of $I^{\mathrm{d}}_{<}$. To finish showing that $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of I^{d} it remains to show that

$$\sigma\big(e_{\mathrm{even}}(\sigma) - 2\big), \qquad \sigma\big(e_{\mathrm{even}}(\sigma) - 1\big), \qquad \sigma\big(e_{\mathrm{even}}(\sigma) + 1\big)$$

is cyclically ordered For this we use that the following two lines are cyclically ordered, where the first one arises from the definition of $e_{\mathrm{even}}(\sigma)$, and the second from $e_{\mathrm{even}}(\sigma) - 1 < e_{\mathrm{odd}}(\sigma)$ being odd.

$$\begin{array}{lll} \sigma\big(e_{\mathrm{even}}(\sigma)-1\big), & \sigma\big(e_{\mathrm{even}}(\sigma)+1\big), & \sigma\big(e_{\mathrm{even}}(\sigma)\big) \\ \sigma\big(e_{\mathrm{even}}(\sigma)-2\big), & \sigma\big(e_{\mathrm{even}}(\sigma)-1\big), & \sigma\big(e_{\mathrm{even}}(\sigma)\big) \end{array}$$

Combining these two we obtain that

$$\sigma\big(e_{\mathrm{even}}(\sigma)-2\big), \qquad \sigma\big(e_{\mathrm{even}}(\sigma)-1\big), \qquad \sigma\big(e_{\mathrm{even}}(\sigma)+1\big), \qquad \sigma\big(e_{\mathrm{even}}(\sigma)\big)$$

is cyclically ordered, from which the claim follows. We have now shown that $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of I^{d}. That $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is then an element of $I^{\mathrm{d}}_{<,\mathrm{bottom}}$ is clear. To show that it is even an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$, we have to show that either $e_{\mathrm{odd}}(\sigma) = e_{\mathrm{odd}}(\vec{c}) = \infty$ or $e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})$. But this follows from what p must be from the definition of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$ together with the inequalities p must satisfy in the definition of I^{d}.

So far we have shown that Φ is a well-defined map, and it is clearly an injection, as σ and $\vec{c}$ already determine the value of p if $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$. It remains to show that Φ is surjective. So let $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ be an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$. If $e_{\mathrm{odd}}(\sigma) = \infty$ set $p = 2l-1$, otherwise let $p = e_{\mathrm{odd}}(\sigma)$. Then we have to show that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d,cancel}}_{<,\mathrm{top}}$. From $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ being an element of I^{d} we can immediately conclude that $e_{\mathrm{even}}(\sigma) \neq -\infty$ and that $e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma)$. It thus only remains to show that $(\sigma, \vec{c}, p)$ is an element of I^{d}. For this we first show the following inequalities.

$$e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c})$$

That $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of I^{d} implies that $e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma)$, which together with $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$ from the definition of $I^{\mathrm{d}}_{<}$ implies the left inequality. The right inequality follows instead from the definition of $I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}$. We next show the following inequalities.

$$e_{\mathrm{even}}(\sigma) - 2 < p < e_{\mathrm{odd}}(\sigma) + 2$$

Here the left inequality follows from $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$ from the definition of $I^{\mathrm{d}}_{<}$, and the right inequality is clear. It remains to show that σ cyclically preserves the ordering of $\{p-1, p+1, p+2\}$, as long as $p \leq 2l-3$. So assume that $p \leq 2l-3$, which implies that we are in the case in which $p = e_{\mathrm{odd}}(\sigma) \neq \infty$. Then we use that the following two lines are cyclically ordered, where the first one arises from the definition of $e_{\mathrm{odd}}(\sigma)$, and the second from $e_{\mathrm{odd}}(\sigma) + 1 > e_{\mathrm{even}}(\sigma)$ being odd.

$$\begin{array}{lll} \sigma\big(e_{\mathrm{odd}}(\sigma)-1\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)+1\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)\big) \\ \sigma\big(e_{\mathrm{odd}}(\sigma)\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)+1\big), & \sigma\big(e_{\mathrm{odd}}(\sigma)+2\big) \end{array}$$

Combining these two we obtain that

$$\sigma\big(e_{\mathrm{odd}}(\sigma)\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)-1\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)+1\big), \qquad \sigma\big(e_{\mathrm{odd}}(\sigma)+2\big)$$

from which the claim follows. $\square$

7.3.10. Matching up of the remaining summands

In this section we show how the sums over various subsets of I^{d}, I^{1}, $I^{\partial}_{\mathrm{even}}$, and $I^{\partial}_{\mathrm{odd}}$ match up.

Proposition 7.3.10.1. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},\partial}} B'(v) = \sum_{v \in I^{\partial}_{\mathrm{even},\mathrm{d}}} B''(v) \qquad \heartsuit$$

Proof. Let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$. Then p is odd, so

$$B'\big((\sigma, \vec{c}, p)\big) = B''\big((\sigma, \vec{c})\big)$$

so that it suffices to show that

$$\Phi\colon I^{\mathrm{d}}_{<,\mathrm{top},\partial} \to I^{\partial}_{\mathrm{even},\mathrm{d}}$$
$$\Phi\colon (\sigma, \vec{c}, p) \mapsto (\sigma, \vec{c})$$

is a well-defined bijection.

So let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$. We first show that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{even}}$. For this we need that $e_{\mathrm{even}}(\vec{c}) \neq \infty$ and $e_{\mathrm{even}}(\sigma) \geq e_{\mathrm{even}}(\vec{c})$, both properties that are part of the definition of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$, and we need that $e_{\mathrm{odd}}(\vec{c}) \geq e_{\mathrm{even}}(\vec{c}) + 3$, which follows from the condition $e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c})$ from the definition of I^{d} together with the parities, and finally we need that $e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{even}}(\vec{c}) + 1$, which follows from left part of the inequalities just used together with the definition of p in $I^{\mathrm{d}}_{<,\mathrm{top}}$. So now we have shown that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{even}}$. The properties that σ needs to satisfy for $(\sigma, \vec{c})$ to even be an element of $I^{\partial}_{\mathrm{even},\mathrm{d}}$ follow from what p is by the definition of $I^{\mathrm{d}}_{<,\mathrm{top}}$ and that $p < e_{\mathrm{odd}}(\vec{c})$ by the definition of I^{d}. This shows that Φ is well-defined. As p is uniquely determined by σ and $\vec{c}$ in the definition of $I^{\mathrm{d}}_{<,\mathrm{top}}$, we can also conclude that Φ is injective.

It remains to show that Φ is surjective. So let $(\sigma, \vec{c})$ be an element of $I^{\partial}_{\mathrm{even},\mathrm{d}}$. If $e_{\mathrm{odd}}(\sigma) = \infty$ set $p = 2l - 1$, otherwise let $p = e_{\mathrm{odd}}(\sigma)$. Then we have to show that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$. We can first note that the two inequalities in the definition of $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$ also occur in the definition of $I^{\partial}_{\mathrm{even}}$, so that it suffices to show that $(\sigma, \vec{c}, p)$ is an element of $I^{\mathrm{d}}_{<}$. By the definition of $I^{\partial}_{\mathrm{even}}$ we have

$$e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{odd}}(\sigma)$$

so that is only remains to show that $(\sigma, \vec{c}, p)$ is an element of I^{d}. For this we note that

$$e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c})$$

follows from the definition of $I^{\partial}_{\mathrm{even}}$ for the left inequality and from the definition of $I^{\partial}_{\mathrm{even},\mathrm{d}}$ for the right inequality. Next we consider the following inequalities.

$$e_{\mathrm{even}}(\sigma) - 2 < p < e_{\mathrm{odd}}(\sigma) + 2$$

The left inequality follows from $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$, which we already showed above, and the right inequality is clear. Finally, we have to show that σ cyclically preserves the ordering of $\{p-1, p+1, p+2\}$ as long as $p \leq 2l-3$, which implies that $p = e_{\mathrm{odd}}(\sigma) \neq \infty$. The argument for this is identical to the argument used at the end of the proof of Proposition 7.3.9.3. □

Proposition 7.3.10.2. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \dots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}} B'(v) = -\sum_{v \in I^{\partial}_{\mathrm{odd,d}}} B''(v) \qquad \heartsuit$$

Proof. Let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. Then p is even, so

$$B'\big((\sigma, \vec{c}, p)\big) = -B''\big((\sigma, \vec{c})\big)$$

so that it suffices to show that

$$\Phi\colon I^{\mathrm{d}}_{<,\mathrm{bottom},\partial} \to I^{\partial}_{\mathrm{odd,d}}$$
$$\Phi\colon (\sigma, \vec{c}, p) \mapsto (\sigma, \vec{c})$$

is a well-defined bijection.

So let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. We first show that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{odd}}$. For this we need that $e_{\mathrm{odd}}(\vec{c}) \neq \infty$ and $e_{\mathrm{odd}}(\sigma) \geq e_{\mathrm{odd}}(\vec{c})$, both properties that are part of the definition of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. We also need that $e_{\mathrm{even}}(\vec{c}) \leq e_{\mathrm{odd}}(\vec{c}) - 3$, which follows from the condition $e_{\mathrm{even}}(\vec{c}) < p < e_{\mathrm{odd}}(\vec{c})$ from the definition of I^{d} together with parities. Finally, we need that $e_{\mathrm{even}}(\sigma) \leq e_{\mathrm{odd}}(\vec{c}) - 1$, which follows from $p < e_{\mathrm{odd}}(\vec{c})$ together with $p = e_{\mathrm{even}}(\sigma)$ from the definition of $I^{\mathrm{d}}_{<,\mathrm{bottom}}$. This finishes the proof that $(\sigma, \vec{c})$ is an element of $I^{\partial}_{\mathrm{odd}}$. The properties that $(\sigma, \vec{c})$ needs to satisfy to also be an element of the subset $I^{\partial}_{\mathrm{odd,d}}$ follow from $p = e_{\mathrm{even}}(\sigma)$ and the definition of I^{d}. This shows that Φ is well-defined. As p is uniquely determined by σ and $\vec{c}$ in the definition of $I^{\mathrm{d}}_{<,\mathrm{bottom}}$ we can also conclude that Φ is injective.

It remains to show that Φ is surjective. So let $(\sigma, \vec{c})$ be an element of $I^{\partial}_{\mathrm{odd,d}}$. We have to show that $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$. We first note that the two inequalities in the definition of $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$ also occur in the definition of $I^{\partial}_{\mathrm{odd}}$, so that it suffices to show that $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of $I^{\mathrm{d}}_{<}$. By the definition of $I^{\partial}_{\mathrm{odd}}$ we have

$$e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\vec{c}) \leq e_{\mathrm{odd}}(\sigma)$$

so that is only remains to show that $(\sigma, \vec{c}, e_{\mathrm{even}}(\sigma))$ is an element of I^{d}. For this we note that

$$e_{\mathrm{even}}(\vec{c}) < e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\vec{c})$$

follows from the definition of $I^{\partial}_{\mathrm{odd,d}}$ for the left inequality and from the definition of $I^{\partial}_{\mathrm{odd}}$ for the right inequality. Next we consider the following inequalities.

$$e_{\mathrm{even}}(\sigma) - 2 < e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma) + 2$$

The left inequality is clear and the right inequality follows from $e_{\mathrm{even}}(\sigma) < e_{\mathrm{odd}}(\sigma)$, which we already showed above. Finally, we have to show that σ cyclically preserves the ordering of $\{e_{\mathrm{even}}(\sigma)-2, e_{\mathrm{even}}(\sigma)-1, e_{\mathrm{even}}(\sigma)+1\}$ The argument for this is identical to the argument used at the middle of the proof of Proposition 7.3.9.3, where it is shown that the map Φ used there is well-defined. □

Proposition 7.3.10.3. *In this proposition we use notation from Construction 7.3.1.1 and Definitions 7.3.4.1, 7.3.4.2 and 7.3.8.1. Let X be a totally ordered set, $l \geq 1$ and $m \geq 0$ integers, and $y_1, \ldots, y_m$ as in Construction 7.3.1.1. Then the following holds.*

$$\sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},1}} B'(v) = \sum_{v \in I^1_{\mathrm{d}}} B''(v) \qquad \heartsuit$$

Proof. Let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{top},1}$. Then p is odd, so

$$B'\big((\sigma, \vec{c}, p)\big) = B''\big((\sigma, \vec{c})\big)$$

so that it suffices to show that

$$\begin{aligned} \Phi\colon I^{\mathrm{d}}_{<,\mathrm{top},1} &\to I^1_{\mathrm{d}} \\ \Phi\colon (\sigma, \vec{c}, p) &\mapsto (\sigma, \vec{c}) \end{aligned}$$

is a well-defined bijection.

So let $(\sigma, \vec{c}, p)$ be an element of $I^{\mathrm{d}}_{<,\mathrm{top},1}$. That $(\sigma, \vec{c})$ is an element of I^1 then follows directly from the definition of $I^{\mathrm{d}}_{<,\mathrm{top},1}$. Suppose now that $e_{\mathrm{odd}}(\sigma) = \infty$. Then we must have $p = 2l-1$ by the definition of $I^{\mathrm{d}}_{<,\mathrm{top}}$, which by the definition of I^{d} implies that $e_{\mathrm{odd}}(\vec{c}) > 2l-1$ so that we can conclude that $e_{\mathrm{odd}}(\vec{c}) = \infty$ as well. If instead $e_{\mathrm{odd}}(\sigma) \neq \infty$ Then we must have $p = e_{\mathrm{odd}}(\sigma)$ by the definition of $I^{\mathrm{d}}_{<,\mathrm{top}}$, which by the definition of I^{d} implies that $e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})$. This finishes the proof that Φ is well-defined. As p is uniquely determined by σ and $\vec{c}$ in the definition of $I^{\mathrm{d}}_{<,\mathrm{top}}$ we also obtain that Φ is injective.

It remains to show that Φ is surjective. So let $(\sigma, \vec{c})$ be an element of I^1_{d}. Assume first that $e_{\mathrm{odd}}(\sigma) = \infty$. Then the definition of I^1_{d} implies that $e_{\mathrm{odd}}(\vec{c}) = \infty$ as well, and by the definition of I^1 we have that $e_{\mathrm{even}}(\vec{c}) = \infty = e_{\mathrm{even}}(\sigma)$. This directly implies all the properties needed for $(\sigma, \vec{c}, 2l-1)$ to be an element of $I^{\mathrm{d}}_{<,\mathrm{top},1}$. Assume now that $e_{\mathrm{odd}}(\sigma) \neq \infty$. Then the definition of I^1_{d} implies that $e_{\mathrm{odd}}(\sigma) < e_{\mathrm{odd}}(\vec{c})$. This time all properties needed for $(\sigma, \vec{c}, e_{\mathrm{odd}}(\sigma))$ to be an element of $I^{\mathrm{d}}_{<,\mathrm{top},1}$ are directly implies except that σ must cyclically preserve the ordering of $\{e_{\mathrm{odd}}(\sigma)-1, e_{\mathrm{odd}}(\sigma)+1, e_{\mathrm{odd}}(\sigma)+2\}$, which follows with the same argument used at the end of the proof of Proposition 7.3.9.3. □

7.3.11. Conclusion

We can now put everything together to show that $\epsilon_X^{(\bullet)}$ forms a strongly homotopy linear morphism. As an intermediate step we first show that the identity required for this holds on elements of degree 0.

Proposition 7.3.11.1. *Let X be a totally ordered set and $l \geq 1$ an integer. Then*

$$\partial \circ \epsilon_X^{(l)} = \epsilon_X^{(l-1)} \circ \mathrm{d} - \mathrm{d} \circ \epsilon_X^{(l-1)}$$

holds on elements of $\Omega^0_{k[X]/k}$, where $\epsilon_X^{(\bullet)}$ defined as in Construction 7.3.1.1. ♡

Proof. The equation we have to show is k-linear on both sides, so it suffices to show it for a set of generators. So let $m \geq 0$ be an integer and $y_1, \ldots, y_m$ be as in Construction 7.3.1.1. It suffices to show that

$$\partial\Big(\epsilon_X^{(l)}(y_1 \cdots y_m)\Big) - \epsilon_X^{(l-1)}\big(\mathrm{d}(y_1 \cdots y_m)\big) + \mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big) = 0$$

This is done by combining various previous results as follows.

$$\partial\Big(\epsilon_X^{(l)}(y_1 \cdots y_m)\Big) - \epsilon_X^{(l-1)}\big(\mathrm{d}(y_1 \cdots y_m)\big) + \mathrm{d}\Big(\epsilon_X^{(l-1)}(y_1 \cdots y_m)\Big)$$

Applying Proposition 7.3.7.3.

$$= \sum_{v \in I^{\partial}_{\mathrm{even}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd}}} B''(v) + \sum_{v \in I^{1}} B''(v) - \sum_{v \in I^{\mathrm{d}}} B'(v)$$

Applying Proposition 7.3.8.3 for $I^{\partial}_{\mathrm{even}}$ (first line) and $I^{\partial}_{\mathrm{odd}}$ (second line), Proposition 7.3.8.4 for I^1 (third line), and Proposition 7.3.8.2 for I^{d} (rest).

$$\begin{aligned}
= &\sum_{v \in I^{\partial}_{\mathrm{even,d}}} B''(v) + \sum_{v \in I^{\partial}_{\mathrm{odd-even}}} B''(v) \\
&- \sum_{v \in I^{\partial}_{\mathrm{odd,d}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd-even}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd,1}}} B''(v) \\
&+ \sum_{v \in I^{1}_{\mathrm{d}}} B''(v) + \sum_{v \in I^{\partial}_{\mathrm{odd,1}}} B''(v) \\
&- \sum_{v \in I^{\mathrm{d,cancel}}_{>}} B'(v) - \sum_{\substack{1 \leq p \leq 2l-q \\ v \in I^{\mathrm{d,cancel}}_{<,\mathrm{mid},p}}} B'(v) \\
&- \left(\sum_{v \in I^{\mathrm{d,cancel}}_{<,\mathrm{top}}} B'(v) + \sum_{v \in I^{\mathrm{d,cancel}}_{<,\mathrm{bottom}}} B'(v) \right) \\
&- \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},\partial}} B'(v) - \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},1}} B'(v) - \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}} B'(v)
\end{aligned}$$

The terms involving $I^{\partial}_{\mathrm{odd-even}}$ in the first and second line cancel. Similarly, the terms involving $I^{\partial}_{\mathrm{odd,1}}$ in the second and third line cancel. Furthermore the terms in the fourth and fifth line are zero by Propositions 7.3.9.1, 7.3.9.2 and 7.3.9.3.

$$\begin{aligned}
= &\sum_{v \in I^{\partial}_{\mathrm{even,d}}} B''(v) - \sum_{v \in I^{\partial}_{\mathrm{odd,d}}} B''(v) + \sum_{v \in I^{1}_{\mathrm{d}}} B''(v) \\
&- \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},\partial}} B'(v) - \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{top},1}} B'(v) - \sum_{v \in I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}} B'(v)
\end{aligned}$$

Applying Proposition 7.3.10.1 for the term involving $I^{\mathrm{d}}_{<,\mathrm{top},\partial}$, Proposition 7.3.10.3 for the term involving $I^{\mathrm{d}}_{<,\mathrm{top},1}$, and Proposition 7.3.10.2 for the term involving $I^{\mathrm{d}}_{<,\mathrm{bottom},\partial}$.

$$\begin{aligned}
&= \sum_{v\in I^{\partial}_{\mathrm{even,d}}} B''(v) - \sum_{v\in I^{\partial}_{\mathrm{odd,d}}} B''(v) + \sum_{v\in I^{1}_{\mathrm{d}}} B''(v)\\
&\quad - \sum_{v\in I^{\partial}_{\mathrm{even,d}}} B''(v) - \sum_{v\in I^{1}_{\mathrm{d}}} B''(v) + \sum_{v\in I^{\partial}_{\mathrm{odd,d}}} B''(v)\\
&= 0
\end{aligned}$$

□

Proposition 7.3.11.2. *Let X be a totally ordered set. Then the quasiisomorphism of chain complexes*

$$\epsilon_X \colon \Omega^{\bullet}_{k[X]/k} \to \overline{\mathrm{C}}\big(k[X]\big)$$

from Construction 7.2.2.1 and Proposition 7.2.2.2 can be upgraded to a strongly homotopy linear quasiisomorphism by equipping it with $\epsilon_X^{(\bullet)}$ as defined in Construction 7.3.1.1. ♡

Proof. By Definition 4.2.3.1 we have to show that

$$\partial \circ \epsilon_X^{(l)} = \epsilon_X^{(l-1)} \circ \mathrm{d} - \mathrm{d} \circ \epsilon_X^{(l-1)}$$

holds for $l > 0$[28]. As both sides of the above equation are k-linear it suffices to show this on a set of generators of $\Omega^{\bullet}_{k[X]/k}$. So let f be an element of $k[X]$ and $y_1, \ldots, y_n$ elements of X. Then the following calculation shows that the above identity is satisfied on the element $f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m$.

$$\begin{aligned}
&\left(\partial \circ \epsilon_X^{(l)}\right)(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\\
&= \partial\left(\epsilon_X^{(l)}(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\right)
\end{aligned}$$

Applying the definition of $\epsilon_X^{(l)}$ from Construction 7.3.1.1.

$$= \partial\left(\epsilon_X^{(l)}(f) \cdot \epsilon_X(\mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\right)$$

Applying Proposition 7.2.2.2 (1).

$$= \partial\left(\epsilon_X^{(l)}(f) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m\right)$$

Applying the Leibniz rule for ∂, and using that $\partial(\mathrm{d}\, x) = 0$ in $\overline{\mathrm{C}}(k[X])$ for every element x of X, which can be seen either by direct calculation or by using that $\partial(\mathrm{d}\, x) = -\,\mathrm{d}(\partial x) = 0$ for degree reasons.

$$= \partial\left(\epsilon_X^{(l)}(f)\right) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m$$

Applying Proposition 7.3.11.1.

$$\begin{aligned}
&= \left(\epsilon_X^{(l-1)}\big(\mathrm{d}(f)\big) - \mathrm{d}\left(\epsilon_X^{(l-1)}(f)\right)\right) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m\\
&= \epsilon_X^{(l-1)}\big(\mathrm{d}(f)\big) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m - \mathrm{d}\left(\epsilon_X^{(l-1)}(f)\right) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m
\end{aligned}$$

[28]The case $l = 0$ is equivalent to ϵ_X being a morphism of chain complexes, which we already know.

Using Proposition 7.2.2.2 (1) for the first summand and Proposition 6.3.2.14 for the second summand.

$$= \epsilon_X^{(l-1)}\big(\mathrm{d}(f)\big) \cdot \epsilon_X(\mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m) - \mathrm{d}\Big(\epsilon_X^{(l-1)}(f) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m\Big)$$

Also using Proposition 7.2.2.2 (1) for the second summand.

$$= \epsilon_X^{(l-1)}\big(\mathrm{d}(f)\big) \cdot \epsilon_X(\mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m) - \mathrm{d}\Big(\epsilon_X^{(l-1)}(f) \cdot \epsilon_X(\mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\Big)$$

Using the definition of $\epsilon_X^{(l-1)}$ from Construction 7.3.1.1.

$$= \epsilon_X^{(l-1)}\big(\mathrm{d}(f) \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m\big) - \mathrm{d}\Big(\epsilon_X^{(l-1)}(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\Big)$$

Using the Leibniz rule for d in $\Omega^\bullet_{k[X]/k}$ (and that $\mathrm{d} \circ \mathrm{d} = 0$).

$$= \epsilon_X^{(l-1)}\big(\mathrm{d}(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\big) - \mathrm{d}\Big(\epsilon_X^{(l-1)}(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)\Big)$$

$$= \Big(\epsilon_X^{(l-1)} \circ \mathrm{d} - \mathrm{d} \circ \epsilon_X^{(l-1)}\Big)(f \cdot \mathrm{d}\, y_1 \cdots \mathrm{d}\, y_m)$$

This shows that ϵ_X can be upgraded to a strongly homotopy linear quasiisomorphism using $\epsilon_X^{(\bullet)}$ constructed in Construction 7.3.1.1. □

As the end result of this section we can now use Proposition 7.3.11.2 to obtain an equivalence between $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ and $\gamma_{\mathsf{Mixed}}(\Omega^\bullet_{k[X]/k})$ in $\mathcal{M}$ixed, showing that $\Omega^\bullet_{k[X]/k}$ is a strict mixed model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$.

Construction 7.3.11.3. Let X be a totally ordered set. The strongly homotopy linear quasiisomorphism ϵ_X from Proposition 7.3.11.2 induces by Proposition 7.2.2.2 (4) and Construction 4.4.4.1 a morphism

$$\gamma_{\mathsf{Mixed}}\Big(\Omega^\bullet_{k[X]/k}\Big) \to \gamma_{\mathsf{Mixed}}\Big(\overline{\mathrm{C}}\big(k[X]\big)\Big)$$

in $\mathcal{M}$ixed, which is even an equivalence by Remark 4.4.4.2. Composing this equivalence with the equivalences from Proposition 6.3.4.1 and Proposition 6.3.1.10 yields an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \simeq \gamma_{\mathsf{Mixed}}\Big(\Omega^\bullet_{k[X]/k}\Big)$$

in the ∞-category $\mathcal{M}$ixed. ♢

7.4. De Rham forms as a strict model in $\mathrm{Alg}(\mathsf{Mixed})$

In Sections 7.2 and 7.3 we showed that $\Omega^\bullet_{k[X]/k}$, which is an object in $\mathrm{CAlg}(\mathsf{Mixed}_{\mathrm{cof}})$, is a model for both $\mathrm{HH}(k[X])$ considered as an object in $\mathrm{CAlg}(\mathcal{D}(k))$, by forgetting the strict mixed structure, and of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object in $\mathcal{M}$ixed, by forgetting the algebra structure. An improved version of the latter result would be to show that $\Omega^\bullet_{k[X]/k}$ is also a model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. While it seems reasonable to expect this to hold, we will unfortunately not be able to show this in general, so we first formulate this as the following conjecture.

Conjecture B. *Let X be a set. Then there exists an equivalence*

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\Big(k[X]\Big) \simeq \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[X]/k}\Big)$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$.

We will often refer to the existence of such an equivalence for a specific set X as "Conjecture B holds for X". ♧

While we will not be able to show Conjecture B in general, we will be able to show that it holds for sets X with $|X| \leq 2$, and this is the goal of this section.

Let us now give an overview of the strategy to prove Conjecture B for $|X| \leq 2$. The very rough idea is to lift $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ to some cofibrant strict model in Alg(Mixed), use the previous results to obtain two equivalences from this model to $\Omega^{\bullet}_{k[X]/k}$, one respecting the strict mixed structure and one respecting the algebra structure, and finally use this to construct an equivalence between $\Omega^{\bullet}_{k[X]/k}$ and our generic lift that respects both.

To implement this plan we begin in Section 7.4.1 by lifting $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ to a cofibrant object $\widetilde{\mathrm{C}}''(X)$ of Alg(Mixed).

As the underlying differential graded algebra of $\widetilde{\mathrm{C}}''(X)$ is also cofibrant, we could then already lift the equivalence from Corollary 7.2.2.3 to a multiplicative quasiisomorphism as follows.

$$\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}''(X)\Big) \to \Omega^{\bullet}_{k[X]/k}$$

However, we can not carry out the same argument to obtain such a quasiisomorphism that is compatible with the strict mixed structure from the equivalence from Construction 7.3.11.3, as the underlying strict mixed complex $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}''(X))$ of $\widetilde{\mathrm{C}}''(X)$ need not be cofibrant. This problem is related to the fact that the monoidal unit k of Mixed is not cofibrant as a strict mixed complex. To deal with this issue we will thus not actually use $\widetilde{\mathrm{C}}''(X)$, but replace it along a quasiisomorphism

$$\widetilde{\mathrm{C}}(X) \to \widetilde{\mathrm{C}}''(X)$$

in Alg(Mixed) by $\widetilde{\mathrm{C}}(X)$, which is also cofibrant and constructed so as to satisfy some specific properties that we will need. In particular, $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$ will be given by a coproduct $k \oplus \widetilde{\mathrm{C}}'(X)$, with the inclusion of the first summand given by the unit morphism, and such that $\widetilde{\mathrm{C}}'(X)$ is cofibrant as a strict mixed complex. The construction of $\widetilde{\mathrm{C}}(X)$ will be carried out in Section 7.4.2.

Now we can lift the equivalence from Corollary 7.2.2.3 to a quasiisomorphism

$$\Phi'_X\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}(X)\Big) \to \Omega^{\bullet}_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$, and the equivalence from Construction 7.3.11.3 to a quasiisomorphism

$$k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X) \to \Omega^{\bullet}_{k[X]/k}$$

in Mixed, and we only need to verify that the restriction to k^{cof} factors over k to obtain a quasiisomorphism

$$\Psi_X\colon\ \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right) \to \Omega^\bullet_{k[X]/k}$$

in Mixed as desired. This will be done in Section 7.4.3.

So now let us get back to what we actually want to show, that $\widetilde{\mathrm{C}}(X)$ is equivalent to $\Omega^\bullet_{k[X]/k}$ in $\mathrm{Alg}(\mathsf{Mixed})$. As $\widetilde{\mathrm{C}}(X)$ is cofibrant such an equivalence could be realized by a quasiisomorphism

$$\widetilde{\mathrm{C}}(X) \to \Omega^\bullet_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Mixed})$. However, we know little about the elements of $\widetilde{\mathrm{C}}(X)$, apart from those that must exist by virtue of the quasiisomorphisms discussed above, so it would be easier to construct morphisms *into* rather than *out of* $\widetilde{\mathrm{C}}(X)$. As $\Omega^\bullet_{k[X]/k}$ is not cofibrant as an object in $\mathrm{Alg}(\mathsf{Mixed})$, we can not hope for there to be an actual strict morphism

$$\Omega^\bullet_{k[X]/k} \to \widetilde{\mathrm{C}}(X)$$

in $\mathrm{Alg}(\mathsf{Mixed})$, so instead we will attempt to construct a morphism Ξ_X from a cofibrant replacement of $\Omega^\bullet_{k[X]/k}$ to $\widetilde{\mathrm{C}}(X)$.

To be able to actually construct Ξ_X will require good control over the (low-degree) generators of said cofibrant replacement, so we construct a specific cofibrant replacement $\Omega'^\bullet_{k[X]/k}$ of $\Omega^\bullet_{k[X]/k}$ in Section 7.4.5.

The set X will occur as free generators of $\Omega'^\bullet_{k[X]/k}$ in degree 0, so the construction of Ξ_X will begin by defining $\Xi_X(x)$ to be such that $(\Phi'_X \circ \Xi_X)(x) = x$ for elements x in X. As Φ'_X is a quasiisomorphism it suffices to check that $\Phi'_X \circ \Xi_X$ is a quasiisomorphism to conclude that Ξ_X is one. The information mentioned so far would suffice to show that Ξ_X induces an isomorphism on H_0, but to handle the other homology groups we also need control over where $\Phi'_X \circ \Xi_X$ maps $\mathrm{d}\,x$ for x an element in X.

Thus we need to study how Φ'_X interacts with d. In Section 7.4.4 we will begin with the one variable case $\Phi_{\{t\}}$. We will not quite be able to show that the $\Phi'_{\{t\}}$ is compatible with d, but we find that this holds up to sign. By postcomposing with an automorphism that tweaks signs we can thus define new morphisms Φ_X to replace the usage of Φ'_X such that $\Phi_{\{t\}}$ is compatible with d.

To deduce from this that Φ_X is also compatible with d on elements of degree 0, as long as $|X| \leq 2$, we need a naturality statement for Φ. We show the required statement in Section 7.4.7, after we showed a similar naturality statement for ϵ in Section 7.4.6. The reason we only show this naturality statement for ϵ in Section 7.4.6 rather than earlier is that the proof uses the cofibrant resolution of $\Omega^\bullet_{k[t]/k}$ that was constructed in Section 7.4.5. After having handled the required naturality of Φ we can then show that Φ_X is compatible with d on degree 0 elements in Section 7.4.8.

Finally, in Section 7.4.9 we will put everything together and actually construct the quasiisomorphism

$$\Xi_X\colon \Omega'^\bullet_{k[X]/k} \to \widetilde{\mathrm{C}}(X)$$

that is a morphism in $\mathrm{Alg}(\mathsf{Mixed})$, and thereby prove Conjecture B for $|X| \leq 2$. To do so it will be very relevant to use the comparison morphisms Φ_X as well as Ψ_X; to begin with we need to prescribe the images of the generators X as we mentioned before, which we do by lifting elements along Φ_X, and in later steps there will be obstructions in the form of cycles that need to be boundaries, which we can verify by checking that the homology class represented by the cycle maps to zero along one of the two comparison morphisms.

7.4.1. A first cofibrant model

In this section we lift $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ to a first cofibrant model $\widetilde{\mathrm{C}}''(X)$ in $\mathrm{Alg}(\mathsf{Mixed})$. We actually need slightly more and lift not only $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$, but the morphism $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \to \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ that is induced by the unit morphism. We need this relative version in order to carry out the identification of the restriction to k that is needed for the strict mixed comparison morphism, as was explained in the introduction to Section 7.4.

Proposition 7.4.1.1. *Let X be a set. Then there exists a morphism*

$$\widetilde{\iota}''\colon \widetilde{\mathrm{C}}''(\emptyset) \to \widetilde{\mathrm{C}}''(X)$$

in $\mathrm{Alg}(\mathsf{Mixed})$*, such that $\widetilde{\mathrm{C}}''(\emptyset)$ and $\widetilde{\mathrm{C}}''(X)$ are cofibrant, together with a commutative square*

$$\begin{tikzcd} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\iota_{k[X]})"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(\emptyset)\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\widetilde{\iota}'')"] \\ \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(X)\right) \end{tikzcd} \tag{7.5}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$*, where the left morphism is induced by the unit morphism $\iota_{k[X]}\colon k \to k[X]$ and the horizontal morphisms are equivalences.* ♡

Proof. By Propositions 4.4.1.7 and 4.4.2.3 the ∞-category $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ is the underlying ∞-category of the combinatorial model category $\mathrm{Alg}(\mathsf{Mixed})$, where $\mathrm{Alg}(\mathsf{Mixed})$ carries the model structure from Proposition 4.2.2.9. As $[1]$ is a small category[29], we can apply [HA, 1.3.4.25] to lift functors $[1] \to \mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ to functors $[1] \to \mathrm{Alg}(\mathsf{Mixed})$ that are cofibrant with respect to the projective model structure.

Let us for the moment denote the functor $[1] \to \mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ that is encoded by the morphism $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\iota_{k[X]})$ by θ. Applying [HA, 1.3.4.25] to θ we thus obtain a functor

$$\Theta\colon [1] \to \mathrm{Alg}(\mathsf{Mixed})$$

[29]By $[1]$ we mean the 1-category with two objects 0 and 1, and a unique non-identity morphism $0 \to 1$.

that is cofibrant with respect to the projective model structure on the functor category $\mathrm{Fun}([1], \mathrm{Alg}(\mathsf{Mixed}))$, and that lifts θ in the sense that there is a commutative diagram as follows.

$$\begin{tikzcd}
{[1]} \arrow[r, "\Theta"] \arrow[dr, "\theta"'] & \mathrm{Alg}(\mathsf{Mixed}) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})"] \\
& \mathrm{Alg}(\mathcal{M}\mathrm{ixed})
\end{tikzcd}$$

The functor Θ corresponds to a morphism in $\mathrm{Alg}(\mathsf{Mixed})$ that we are going to denote by

$$\widetilde{\iota}'' \colon \widetilde{\mathrm{C}}''(\emptyset) \to \widetilde{\mathrm{C}}''(X)$$

so that the commutative triangle above corresponds exactly to the commuting square (7.5).

It remains to show that $\widetilde{\mathrm{C}}''(\emptyset)$ and $\widetilde{\mathrm{C}}''(X)$ are cofibrant objects. As Θ is cofibrant with respect to the projective model structure, it is also cofibrant with respect to the injective model structure by [HTT, A.2.8.5], which by definition[30] means that it is pointwise cofibrant. □

We can directly improve Proposition 7.4.1.1 by showing that we can replace $\widetilde{\mathrm{C}}''(\emptyset)$ by k, which we do in the following proposition.

Proposition 7.4.1.2. *Let X be a set. Then there exists a cofibrant object $\widetilde{\mathrm{C}}''(X)$ in $\mathrm{Alg}(\mathsf{Mixed})$ so that there is a commutative square*

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq", dash] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}''(X)}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, "\simeq"', dash] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(X)\right)
\end{tikzcd} \tag{7.6}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, where the left morphism is induced by the unit morphism $\iota_{k[X]} \colon k \to k[X]$, the right morphism is induced by the unit morphism $\iota_{\widetilde{\mathrm{C}}''(X)} \colon k \to \widetilde{\mathrm{C}}''(X)$, and the horizontal morphisms are equivalences. ♡

Proof. Let

$$\widetilde{\iota}'' \colon \widetilde{\mathrm{C}}''(\emptyset) \to \widetilde{\mathrm{C}}''(X)$$

[30] See [HTT, A.2.8.1 and A.2.8.2].

be as in Proposition 7.4.1.1. Then $\widetilde{\mathrm{C}}''(X)$ is cofibrant and the diagram

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\iota_{k[X]})"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(\emptyset)\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\tilde{\iota}'')"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[l, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}''(\emptyset)}\right)"'] \arrow[dl, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}''(X)}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(X)\right) &
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ commutes, where $\iota_{\widetilde{\mathrm{C}}''(\emptyset)}\colon k \to \widetilde{\mathrm{C}}''(\emptyset)$ is the unit morphism and the square is the one supplied by Proposition 7.4.1.1. It thus suffices to show that $\iota_{\widetilde{\mathrm{C}}''(\emptyset)}\colon k \to \widetilde{\mathrm{C}}''(\emptyset)$ is a quasiisomorphism.

As quasiisomorphisms are detected on underlying morphisms of chain complexes, we can forget about the strict mixed structure and only consider the unit morphism of the differential graded algebra $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}''(\emptyset))$. There is a composite equivalence

$$\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\left(\widetilde{\mathrm{C}}''(\emptyset)\right)\right) \simeq \mathrm{HH}(k) \simeq \mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k/k}\right) \simeq \mathrm{Alg}(\gamma)(k)$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where the first equivalence is obtained by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ to the equivalence at the top left in the diagram above combined with compatibility of $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1, the second equivalence is the one from Corollary 7.2.2.3, and the third equivalence arises from the isomorphism $\Omega^{\bullet}_{k/k} \cong k$.

As initial object k is cofibrant in $\mathrm{Alg}(\mathsf{Ch}(k))$, so as every object in $\mathrm{Alg}(\mathsf{Ch}(k))$ is fibrant, the above equivalence in $\mathrm{Alg}(\mathcal{D}(k))$ can be lifted to a quasiisomorphism

$$k \to \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\left(\widetilde{\mathrm{C}}''(\emptyset)\right)$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$. But as k is the initial object in this category, this morphism must be exactly $\iota_{\widetilde{\mathrm{C}}''(\emptyset)}$, which has thus been proven to be a quasiisomorphism. □

7.4.2. An improved cofibrant model

$\widetilde{\mathrm{C}}''(X)$ as in Proposition 7.4.1.1 is a cofibrant model in $\mathrm{Alg}(\mathsf{Mixed})$ for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$, but apart from that we know nothing about $\widetilde{\mathrm{C}}''(X)$. In this section we will use $\widetilde{\mathrm{C}}''(X)$ to construct a new cofibrant model $\widetilde{\mathrm{C}}(X)$ over which we will have more control.

Before we state the result of this section we begin with some notation and a remark on pushouts of certain free algebras in strict mixed complexes.

Notation 7.4.2.1. In this section we are often going to use free associative algebras in strict mixed complexes that are generated by strict mixed complexes that are themselves free. To simplify notation, we thus define

$$\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})} := \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}_{\mathsf{Mixed}} \circ \mathrm{Free}^{\mathsf{Mixed}}$$

where $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}_{\mathsf{Mixed}}$ and $\mathrm{Free}^{\mathsf{Mixed}}$ are as in Notation 4.2.2.10. ♢

Remark 7.4.2.2. Let X be an object in $\mathrm{Alg}(\mathsf{Mixed})$, let E be a $\mathbb{Z}$-graded set, and let $i'\colon E \to X$ be a map of $\mathbb{Z}$-graded sets. Assume that the image of i' consists only of cycles in X. Define B' to be the chain complex whose underlying graded k-module is $k \cdot E$ (i. e. the free one on E), equipped with the zero boundary operator. We also define a $\mathbb{Z}$-graded k-module $\underline{B'} := B' \oplus B'[1]$. Then $\underline{B'}$ has two generators corresponding to every element e of E; the one in the left summand is in the same degree as e, and we will also denote this generator by e, and the one in the right summand has degree one higher than e, and we will denote this generator by $\underline{e}$. We can upgrade $\underline{B'}$ to a chain complex by defining $\partial(\underline{e}) = e$ and $\partial(e) = 0$ for every element e of E. There is an obvious morphism of chain complexes $j'\colon B' \to \underline{B'}$ that maps e to e.

We will consider the pushout diagram

$$\begin{tikzcd}
\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(B') \arrow[r, "\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(j')"] \arrow[d, "i"'] & \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(\underline{B'}) \arrow[d, "\underline{i}"] \\
X \arrow[r, "\iota"'] & \underline{X}
\end{tikzcd} \tag{$*$}$$

in $\mathrm{Alg}(\mathsf{Mixed})$, where i is the morphism that is determined by the morphism of chain complexes $B' \to X$ that is given by mapping e to $i'(e)$ for every element e of E (this is a morphism of chain complexes by the assumption that $i'(e)$ is a cycle).

Let Y be a chain complex. Then the underlying graded k-algebra of $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(Y)$ is given by the free graded k-algebra generated by the graded k-module $\mathrm{D} \otimes Y \cong Y \oplus Y[1]$. This follows from Proposition 4.2.2.11 and the analogous statement proven with Proposition E.7.2.2 (2) in the same manner by using that the forgetful functor from $\mathsf{Ch}(k)$ to the category of $\mathbb{Z}$-graded k-modules is symmetric monoidal and preserves colimits.

As the forgetful functor from $\mathrm{Alg}(\mathsf{Mixed})$ to $\mathrm{Alg}(\mathsf{Ch}(k))$ preserves colimits by Proposition 4.2.2.12 and the forgetful functor from $\mathrm{Alg}(\mathsf{Ch}(k))$ to the category of $\mathbb{Z}$-graded k-algebras does so as well by Proposition E.7.3.1, we then obtain that diagram ($*$) is on underlying graded k-algebras given by a pushout[31]

$$\begin{tikzcd}
\mathrm{Free}(k \cdot E \oplus k \cdot \mathrm{d}\,E) \arrow[r] \arrow[d, "i"'] & \mathrm{Free}(k \cdot E \oplus k \cdot \mathrm{d}\,E) \amalg \mathrm{Free}(k \cdot \underline{E} \oplus k \cdot \mathrm{d}\,\underline{E}) \arrow[d, "\underline{i}"] \\
X \arrow[r, "\iota"'] & \underline{X}
\end{tikzcd}$$

[31] We denote by $\mathrm{d}\,E$ a $\mathbb{Z}$-graded set that consists of an element that we denote by $\mathrm{d}\,e$ of degree one higher than e for each element e of E. We use a similar convention for $\underline{E}$.

where Free is ad hoc notation for the free associative $\mathbb{Z}$-graded k-algebra on a $\mathbb{Z}$-graded k-module[32], $\amalg$ refers to the coproduct in the category of $\mathbb{Z}$-graded k-algebras, i. e. the free product, and the top morphism is the inclusion of the first summand. From this it follows that the underlying graded k-algebra of $\underline{X}$ is given by the coproduct (in graded k-algebras) of X and the free graded k-algebra on elements $\underline{e}$ and $\mathrm{d}\,\underline{e}$ for $e \in E$. $\diamondsuit$

Proposition 7.4.2.3. *Let Y be an object in $\mathrm{Alg}(\mathsf{Mixed})$ and Y' a sub-$\mathbb{Z}$-graded-k-module of $\mathrm{H}_*(Y)$ such that $\mathrm{H}_*(Y)$ is the direct sum of Y' with a copy of k generated by the homology class $[1]$ that is represented by the multiplicative unit 1 of Y[33]. Assume furthermore that the homology of Y is concentrated in non-negative degrees.*

Then there exists a quasiisomorphism

$$\Theta\colon X \to Y$$

in $\mathrm{Alg}(\mathsf{Mixed})$ such that X is cofibrant, concentrated in nonnegative degrees, and satisfies the following additional property. There must exist a sub-strict-mixed-complex X' of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$ that is cofibrant as an object of Mixed such that the morphism of strict mixed complexes

$$k \oplus X' \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$$

that is induced by the unit $k \to X$ and the inclusion $X' \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$ is an isomorphism. Furthermore, the restriction of $\mathrm{H}_*(\Theta)$ to $\mathrm{H}_*(X')$ must corestrict to an isomorphism $\mathrm{H}_*(X') \xrightarrow{\cong} Y'$. $\heartsuit$

Proof. We will inductively construct a diagram in $\mathrm{Alg}(\mathsf{Mixed})$ as indicated below, satisfying properties (a), (b), (c), (d), (e), (f) and (g) that will be explained below.

$$\begin{tikzcd}
 & & k & & \\
X_{-1} \arrow[r, "\iota_{-1}^{0}"] \arrow[urr, bend left, "p_{-1}"] \arrow[drr, bend right, "\Theta_{-1}"'] & X_0 \arrow[r, "\iota_0^1"] \arrow[ur, "p_0"] \arrow[dr, "\Theta_0"'] & X_1 \arrow[r, "\iota_1^2"] \arrow[u, "p_1"] \arrow[d, "\Theta_1"] & X_2 \arrow[r] \arrow[ul, bend right, "p_2"'] \arrow[dl, bend left, "\Theta_2"] & \cdots \\
 & & Y & &
\end{tikzcd} \qquad (*)$$

Beyond the notation indicated in the diagram, we will denote the morphism from X_n to X_m for $-1 \leq n \leq m$ by $\iota_n^m := \iota_{m-1}^m \circ \cdots \circ \iota_n^{n+1}$. All morphisms ι_n^m are going to be levelwise injective, so if x is an element of X_n, we will also just write x for the element $\iota_n^m(x)$ of X_m. Finally, we define $K_n := \mathrm{Ker}(p_n)$ for $n \geq -1$. Note that as p_n is a morphism of chain complexes K_n will be closed under ∂.

Now we can formulate the properties that (7.4.2.3) needs to satisfy.

[32]We also use that Free preserves coproducts to rewrite the top right object as a coproduct.

[33]This element is a cycle and satisfies $\mathrm{d}(1) = 0$ due to the Leibniz rule that is satisfied by both ∂ as well as d.

(a) $X_{-1} = k$.

(b) X_n is concentrated in non-negative degrees for all $n \geq -1$.

(c) $\mathrm{H}_*(\Theta_n)$ is an isomorphism for $* < n$ if $n \geq -1$ and surjective for all $*$ if $n \geq 0$.

(d) $\mathrm{H}_*(\Theta_n)$ maps $\mathrm{H}_*(K_n)$ into Y' for all $n \geq -1$.

(e) Let $n \geq -1$. Then there is a $\mathbb{Z}$-graded set E_n and a morphism of $\mathbb{Z}$-graded sets $i'_n\colon E_n \to X_n$ satisfying the following properties. Let e be an element of E_n. Then the image $i'_n(e)$ in X_n must be a cycle as well as lie in K_n. We denote by $B'_n \coloneqq k \cdot E_n$ the chain complex with zero boundary operator whose underlying $\mathbb{Z}$-graded k-module is freely generated by E_n. We furthermore denote by $\underline{B}'_n$ the $\mathbb{Z}$-graded k-module that is given by $(k \cdot E_n) \oplus (k \cdot E_n)[1]$. If e is an element of E_n, then we will also use e to refer to e as en element of the left summand, and $\underline{e}$ to refer to e as an element of the right summand. Note that $\underline{e}$ has degree 1 higher than e. We can then make $\underline{B}'_n$ into a chain complex by defining $\partial(\underline{e}) = e$ and $\partial(e) = 0$ for every element e of E_n. There is a morphism of chain complexes $j'_n\colon B'_n \to \underline{B}'_n$ that maps e to e. Now we can finally formulate the property that E_n needs to satisfy. We require that there is a pushout diagram

$$\begin{tikzcd}
\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(B'_n) \arrow[r, "\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}\left(j'_n\right)"] \arrow[d, "i_n"'] & \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}\left(\underline{B}'_n\right) \arrow[d, "\underline{i}_n"] \\
X_n \arrow[r, "\iota_n^{n+1}"'] & X_{n+1}
\end{tikzcd} \tag{$**$}$$

in $\mathrm{Alg}(\mathsf{Mixed})$, where i_n is the morphism that is determined by the morphism of chain complexes $B'_n \to X_n$ that is given by mapping e to $i'_n(e)$ for e an element of E_n (this is a morphism of chain complexes by the assumption that every element of E_n be a cycle in X_n).

(f) ι_n^{n+1} is a cofibration in $\mathrm{Alg}(\mathsf{Mixed})$ for $n \geq -1$.

(g) $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_n^{n+1})$ is a cofibration in Mixed for $n \geq -1$.

Before we construct diagram ($*$) with these properties, let us first explain how to deduce the claim from it. We define

$$X \coloneqq \operatorname*{colim}_{n \geq -1} X_n$$

with the colimit taken in $\mathrm{Alg}(\mathsf{Mixed})$, and let $p\colon X \to k$ and $\Theta\colon X \to Y$ be the morphisms induced by p_n and Θ_n. We furthermore define

$$X' \coloneqq \mathrm{Ker}\Big(\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(p)\Big)$$

which is a sub-strict-mixed-complex of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$ as $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(p)$ is a morphism of strict mixed complexes. It remains to check the properties that X and Θ need to satisfy. Before we go through the individual claims, let us first note that the forgetful functors from $\mathrm{Alg}(\mathsf{Mixed})$ to $\mathrm{Alg}(\mathsf{Ch}(k))$, Mixed, as well as $\mathsf{Ch}(k)$ all detect filtered colimits by Proposition 4.2.2.12, so in particular every element of X already occurs in X_n for some $n \geq -1$. That X is concentrated in nonnegative degrees then follows directly from (b).

We continue by showing that Θ is a quasiisomorphism. It follows immediately from (c) that $\mathrm{H}_m(\Theta)$ is surjective for any integer m. Now assume that m is an integer and z is a cycle of chain degree m in X such that $\Theta(z)$ is a boundary. There must be an $n \geq -1$ such that z is an element of X_n, and we may assume that $n > m$. Then (c) implies that $\mathrm{H}_m(\Theta_n)$ is an isomorphism, so z must be a boundary in X_n and hence in X. Thus Θ is a quasiisomorphism.

Next we need to show that X is a cofibrant object in $\mathrm{Alg}(\mathsf{Mixed})$. This means that the morphism from the initial object k must be a cofibration. By (a) we can identify this morphism with the inclusion $X_{-1} \to X$, which is a transfinite composition of

$$X_{-1} \xrightarrow{\iota_{-1}^{0}} X_0 \xrightarrow{\iota_0^1} X_1 \longrightarrow \cdots$$

so that the claim follows from each ι_n^{n+1} being a cofibration in $\mathrm{Alg}(\mathsf{Mixed})$ by (f), as cofibrations are closed under transfinite compositions.

We now turn towards the properties X' needs to satisfy. As p is a morphism in $\mathrm{Alg}(\mathsf{Mixed})$, it must be compatible with the respective unit morphisms, so that the composition of the unit morphism $k \to X$ with p must be the identity. The splitting lemma now implies that the morphism of strict mixed complexes $k \oplus X' \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$ that is induced by the unit $k \to X$ and the inclusion $X' \to X$ is an isomorphism. Let m be an integer. Using the just mentioned isomorphism and the one from the statement of the proposition we obtain a composition

$$\mathrm{H}_m(k) \oplus \mathrm{H}_n(X') \xrightarrow{\cong} \mathrm{H}_m(X) \xrightarrow{\mathrm{H}_m(\Theta)} \mathrm{H}_m(Y) \xrightarrow{\cong} \mathrm{H}_m(k \cdot \{[1]\}) \oplus Y'$$

that we can write as a 2×2 matrix (thinking of the direct sums as column vectors), and showing that the restriction of $\mathrm{H}_*(\Theta)$ to $\mathrm{H}_*(X')$ corestricts to an isomorphism $\mathrm{H}_*(X') \xrightarrow{\cong} Y'$ means showing the the component $\mathrm{H}_n(X') \to \mathrm{H}_m(k \cdot \{[1]\})$ is zero and the component $\mathrm{H}_n(X') \to Y'$ is an isomorphism. (d) implies that the restriction of $\mathrm{H}_m(\Theta)$ to $\mathrm{H}_m(X')$ factors over Y', which handles the former. As Θ is a morphism in $\mathrm{Alg}(\mathsf{Mixed})$ we also know that the composition of Θ with the unit morphism $k \to X$ is given by the unit morphism $k \to Y$, which shows that matrix is of the form

$$\begin{bmatrix} \cong & 0 \\ 0 & ? \end{bmatrix}$$

Combining this with the fact that $\mathrm{H}_m(\Theta)$ is an isomorphism as we already showed above we can conclude that the component $\mathrm{H}_n(X') \to Y'$ (indicated with a question mark above) must be an isomorphism as well.

It remains to show that X' is a cofibrant strict mixed complex. Using that the forgetful functor $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ from $\mathrm{Alg}(\mathsf{Mixed})$ to Mixed preserves transfinite compositions we can show, using the same argument as when we showed that X was cofibrant in $\mathrm{Alg}(\mathsf{Mixed})$, only this time using (g) instead of (f), that the unit morphism $k \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X)$ is a cofibration in Mixed. We can identify this unit morphism with the inclusion of the first summand $k \to k \oplus X'$. This means that the top horizontal morphism in the pushout diagram

$$\begin{tikzcd} k \arrow[r, "\mathrm{id}_k \times 0"] \arrow[d] & k \oplus X' \arrow[d, "0 \amalg \mathrm{id}_{X'}"] \\ 0 \arrow[r] & X' \end{tikzcd} \tag{$***$}$$

in Mixed is a cofibration, and hence so is the bottom horizontal morphism, i. e. X' is cofibrant as an object of Mixed.

We have shown that constructing diagram $(*)$ satisfying properties (a), (b), (c), (d), (e), (f) and (g) will imply the statement of the proposition, so we now turn towards actually constructing this diagram. This has two main parts. We will inductively construct X_n together with ι_{n-1}^n, p_n and Θ_n satisfying (a), (b), (c), (d) and (e), and separately show that this implies that (f) and (g) hold as well.

We first get this latter part out of the way. So assume that we are given a diagram $(*)$ satisfying properties (a), (b), (c), (d) and (e). Then the morphisms j_n' defined in (e) for $n \geq -1$ are cofibrations of chain complexes as they are coproducts of generating cofibrations, see [Hov99, 2.3.3 and 2.3.11][34]. The functor $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ is a left Quillen functor by Theorem 4.2.2.1, so the morphisms $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(j_n')$ in $\mathrm{Alg}(\mathsf{Mixed})$ are cofibrations as well, and hence so are the morphisms ι_n^{n+1} by the pushout diagram that is part of (e). This proves (f).

Showing (g) requires a more detailed analysis of the underlying objects of pushouts in associative algebras. Luckily, Schwede and Shipley already did most of the work for us in the proof of [SS00, 6.2], and the following argument assumes that the reader has familiarized themselves with the proof of [SS00, 6.2]. We prove (g) by induction, letting $n \geq -1$, assuming that $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_{-1}^0), \ldots, \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_{n-1}^n)$ are cofibrations in Mixed, and proving that then also $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_n^{n+1})$ is a cofibration in Mixed. By (e) the morphism ι_n^{n+1} is given by a pushout

$$\begin{tikzcd} \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}\left(\mathrm{Free}^{\mathsf{Mixed}}(B_n')\right) \arrow[r, "\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}\left(\mathrm{Free}^{\mathsf{Mixed}}(j_n')\right)"] \arrow[d, "i_n"'] & \mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}\left(\mathrm{Free}^{\mathsf{Mixed}}(\underline{B}_n')\right) \arrow[d, "\underline{i}_n"] \\ X_n \arrow[r, "\iota_n^{n+1}"'] & X_{n+1} \end{tikzcd}$$

in $\mathrm{Alg}(\mathsf{Mixed})$. This is also the situation considered in the proof of [SS00, 6.2], with their functor T being given by $\mathrm{Free}_{\mathsf{Mixed}}^{\mathrm{Alg}(\mathsf{Mixed})}$, and the the pushout diagram above then

[34] The relevant generating cofibrations are denoted by $S^{m-1} \to D^m$ in [Hov99, 2.3.3].

corresponding to the pushout diagram

$$\begin{tikzcd} T(K) \arrow[r] \arrow[d] & T(L) \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

that is considered at the start of the proof of [SS00, 6.2]. The proof then shows (using their notation for the intermediate steps, but ours for the end points) that $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_n^{n+1})$ is a transfinite composition of a sequence

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n) = P_0 \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_m \longrightarrow \cdots$$

in Mixed. As cofibrations are closed under transfinite compositions, it thus suffices to show that the morphism $P_{m-1} \to P_m$ is a cofibration for every $m \geq 1$. This morphism is defined as a pushout

$$\begin{tikzcd} Q_m \arrow[r] \arrow[d] & \left(\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n) \otimes \mathrm{Free}^{\mathsf{Mixed}}\left(\underline{B}'_n\right)\right)^{\otimes m} \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n) \arrow[d] \\ P_{m-1} \arrow[r] & P_m \end{tikzcd}$$

in Mixed, so that it suffices to show that the morphism

$$Q_m \to \left(\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n) \otimes \mathrm{Free}^{\mathsf{Mixed}}\left(\underline{B}'_n\right)\right)^{\otimes m} \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)$$

is a cofibration in Mixed. This morphism is in turn isomorphic to a morphism

$$\overline{Q}_m \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)} \to \mathrm{Free}^{\mathsf{Mixed}}\left(\underline{B}'_n\right)^{\otimes m} \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)}$$

that is given as a tensor product of a morphism $\overline{Q}_m \to \mathrm{Free}^{\mathsf{Mixed}}\left(\underline{B}'_n\right)^{\otimes m}$ and the identity of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)}$. Now $\overline{Q}_m$ is the colimit of a punctured hypercube built up from $\mathrm{Free}^{\mathsf{Mixed}}(j'_n)$. As j'_n is a cofibration of chain complexes[35] and $\mathrm{Free}^{\mathsf{Mixed}}$ is a left Quillen functor by Theorem 4.2.2.1, $\mathrm{Free}^{\mathsf{Mixed}}(j'_n)$ is a cofibration in Mixed. Just like in the proof of [SS00, 6.2] one can now conclude by iterated application of the pushout-product that the morphism $\overline{Q}_m \to \mathrm{Free}^{\mathsf{Mixed}}\left(\underline{B}'_n\right)^{\otimes m}$ is a cofibration in Mixed.

Where we have to deviate from the proof of [SS00, 6.2] is in how we conclude from this that the morphism

$$\overline{Q}_m \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)} \to \mathrm{Free}^{\mathsf{Mixed}}\left(\underline{B}'_n\right)^{\otimes m} \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)}$$

[35]This was shown above when we proved (f).

is a cofibration as well. While $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)$ is assumed to be cofibrant in the context of [SS00, 6.2], $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)$ actually *not* cofibrant in our situation. However, with arguments completely analogous to the proof that the statement of the proposition follows from the existence of a diagram $(*)$ satisfying properties (a), (b), (c), (d), (e), (f) and (g), we can see that $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)$ is given by the direct sum of the sub-strict-mixed-complex K_n and the image of unit morphism $k \to X_n$. That unit morphism can furthermore be identified with the morphism $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\iota_{-1}^n)$, which is a cofibration in Mixed by the induction assumption. Using a pushout diagram analogous to $(***)$ we can then conclude that K_n is cofibrant as an object of Mixed. Let us now return to showing that

$$\overline{Q}_m \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)} \to \mathrm{Free}^{\mathsf{Mixed}}\big(\underline{B}'_n\big)^{\otimes m} \otimes \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)}$$

is a cofibration. The tensor product $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(X_n)^{\otimes(m+1)} \cong (k \oplus K_n)^{\otimes(m+1)}$ is isomorphic to a direct sum of terms of the form $K_n^{\otimes i} \otimes k^{\otimes(m+1-i)} \cong K_n^{\otimes i}$. As cofibrations are closed under coproducts, it thus suffices to show that

$$\overline{Q}_m \otimes K_n^{\otimes i} \to \mathrm{Free}^{\mathsf{Mixed}}\big(\underline{B}'_n\big)^{\otimes m} \otimes K_n^{\otimes i}$$

is a cofibration in Mixed for any $i \geq 0$. Here we need to distinguish two cases. If $i > 0$, then $K_n^{\otimes i}$ is cofibrant in Mixed as K_n is cofibrant as just shown, and combining this with $\overline{Q}_m \to \mathrm{Free}^{\mathsf{Mixed}}\big(\underline{B}'_n\big)^{\otimes m}$ being a cofibration and the pushout-product axiom we obtain that the morphism above is indeed a cofibration. If instead $i = 0$, then $K_n^{\otimes i} \cong k$. This is not cofibrant as a strict mixed complex, but as it is the monoidal unit, we obtain that the above morphism in question is isomorphic to $\overline{Q}_m \to \mathrm{Free}^{\mathsf{Mixed}}\big(\underline{B}'_n\big)^{\otimes m}$ and hence nevertheless a cofibration.

We have now shown that given a diagram $(*)$ satisfying properties (a), (b), (c), (d) and (e) also properties (f) and (g) hold. So now it remains to actually construct a diagram $(*)$ satisfying properties (a), (b), (c), (d) and (e), which we do inductively.

We begin by setting $X_{-1} := k$, $p_{-1} := \mathrm{id}_k$, and $\Theta_{-1} \colon k \to Y$ the unit morphism of Y. Then (a) is handled, and (b) clearly holds for $n = -1$. As Y was assumed to have homology concentrated in non-negative degrees, and k has the same property we also have (c) for $n = -1$. Finally, $K_{-1} = 0$, so (d) is clear for $n = -1$.

Now let Z be the graded subset of Y that is given by cycles that represent a non-zero homology class in Y'. We let E_{-1} be $Z[-1]$, i. e. the $\mathbb{Z}$-graded set in which the elements of Z are all given a degree that has been lowered by 1, and define $i'_n \colon E_{-1} \to X_{-1} = k$ as the map that maps every element to 0. As the element 0 in every degree of k is an element of K_{-1} as well as a cycle we can now define X_0 via the pushout diagram $(**)$, so that (e) is satisfied for $n = -1$. We also need to define p_0 and Θ_0, which we do using the universal property of the pushout, which ultimately amounts to prescribing a cycle of the appropriate degree in k and Y to the elements $\underline{e}$ of $\underline{B}'_{-1}$ for each element e of E_{-1}. For p_0 we simply let $\underline{e}$ map to 0. For Θ_0 we note that an element e of E_{-1} corresponds to a cycle z in Y, and the degrees of $\underline{e}$ and z agree. We can thus define Θ_0 by mapping $\underline{e}$ to the corresponding cycle z.

We now need to show that (b), (c) and (d) hold for $n = 0$. By assumption Y has homology concentrated in non-negative degrees, so by construction of E_{-1} every element

e of E_{-1} is of degree bigger or equal to -1, which means that the corresponding elements $\underline{e}$ are all of non-negative degrees. Applying Remark 7.4.2.2 we can thus conclude that X_0 is concentrated in non-negative degrees, which shows (b) for $n = 0$. By construction of E_{-1} and Θ_0 it is clear that Y' is contained in the image of $\mathrm{H}_*(\Theta_0)$. As 1 must also be in the image by virtue of Θ_0 being multiplicative, we can conclude from the assumption that $\mathrm{H}_*(Y) \cong k \cdot \{[1]\} \oplus Y'$ that $\mathrm{H}_*(\Theta_0)$ is surjective. As both X_0 and Y have homology that is concentrated in non-negative degrees it is also clear that $\mathrm{H}_*(\Theta_0)$ is an isomorphism for $* < 0$. Thus (c) follows for $n = 0$. Finally, it is clear from the definitions and Remark 7.4.2.2 that a basis for K_0 is given by non-empty words in the multiplicative generators $\underline{e}$ and $\mathrm{d}\,\underline{e}$ of X_0 for e elements of E_{-1}. As Θ_0 maps every element of the form $\underline{e}$ to a cycle that represents a homology class in Y', the same is true for elements of the form $\mathrm{d}\,\underline{e}$, as Y' is closed under d for degree reasons[36]. Multiplicativity of Θ_0 now implies that $\mathrm{H}_*(\Theta_0)$ maps $\mathrm{H}_*(K_0)$ into Y', showing (d) for $n = 0$.

We now define the remainder of diagram $(*)$ by induction. So we assume that $m > 0$ such that $X_{-1}, \ldots, X_{m-1}$ as well as $p_{-1}, \ldots, p_{m-1}$ and $\Theta_{-1}, \ldots, \Theta_{m-1}$ have already been defined in such a way that (e) holds for $n = -1, \ldots, m-2$ and (b), (c) and (d) hold for $n = -1, \ldots, m-1$. We then define X_m, p_m, and Θ_m in such a way that (e) holds for $n = m-1$ and (b), (c) and (d) hold for $n = m$.

Let $L \coloneqq \mathrm{Ker}(\mathrm{H}_{m-1}(\Theta_{m-1}))$. We want to define E_{m-1} as a $\mathbb{Z}$-graded subset of K_{m-1} whose elements are cycles representing nonzero homology classes in L, and which contains at least one such cycle for each nonzero homology class in L. Note that E_{m-1} will then be concentrated in degree $m-1$. We have to show that this is in fact possible, i. e. that every homology class in L is represented by a cycle that lies in K_{m-1}. Note that, as we already mentioned before, X_{m-1} decomposes as a direct sum of $k \cdot \{1\}$ and K_{m-1}. If $m > 1$, then this immediately implies the claim, as $k \cdot \{1\}$ is then concentrated in degree $0 < m-1$ so that *every* cycle of degree $m-1$ in X_{m-1} will be in K_{m-1}. If instead $m = 1$, then a cycle representing a homology class in L is given by a sum $a \cdot 1 + l$, with a an element of k and l a cycle in K_0 of degree 0. That Θ_0 is an algebra morphism as well as (d) for $n = m-1$ imply that

$$\mathrm{H}_0(\Theta_0)\big([a \cdot 1] + [l]\big) = a \cdot [1] + \mathrm{H}_0(\Theta_0)\big([l]\big)$$

with $\mathrm{H}_0(\Theta_0)([l])$ an element of Y'. The assumption that $\mathrm{H}_*(Y)$ is the direct sum of $k \cdot \{[1]\}$ and Y' then implies that we must have $a = 0$. Thus a $\mathbb{Z}$-graded subset E_{m-1} of K_{m-1} of the form described above exists.

We let $i'_{m-1} \colon E_{m-1} \to X_{m-1}$ be the inclusion map and define ι^m_{m-1} and X_m via the pushout diagram $(**)$, so that (e) is satisfied for $n = m-1$. We next define p_m and Θ_m using the universal property of the pushout. We define p_m by extending p_{m-1} by mapping $\underline{e}$ to 0 for every element e of E_{m-1}, which is compatible as $p_{m-1} \circ i_{m-1}$ maps every element of E_{m-1} to 0 as E_{m-1} is a subset of K_{m-1}.

We also define Θ_m as follows. Let e be an element of E_{m-1}. By definition $i'_{m-1}(e)$ is a cycle that represents a homology class that is in the kernel of $\mathrm{H}_{m-1}(\Theta_{m-1})$. There thus

[36] Y' is concentrated in nonnegative degrees, so the images of d applied to elements of Y' lie in degrees greater or equal to 1, and in those degrees Y' is equal to $\mathrm{H}_*(Y)$, as k is concentrated in degree 0.

exists an element in degree m of Y whose boundary is $\Theta_{m-1}(i'_{m-1}(e))$, and we can thus define Θ_m as an extension of Θ_{m-1} by mapping $\underline{e}$ to one such element. It now remains to show that with these definitions (b), (c) and (d) hold for $n = m$.

Combining that E_{m-1} is concentrated in degree $m-1 \geq 0$ with Remark 7.4.2.2 we obtain that the underlying $\mathbb{Z}$-graded k-algebra of X_m is multiplicatively generated by X_{m-1} and elements of the form $\underline{e}$ of degree m and $\mathrm{d}\,\underline{e}$ of degree $m+1$ for $e \in E_{m-1}$. Combining this with (b) for $n = m-1$ we obtain (b) for $n = m$.

This also implies that ι^m_{m-1} is an isomorphism in degrees less than or equal to $m-1$[37], and thus an isomorphism in homology in degrees less than or equal to $m-2$. Combining this with (c) for $n = m-1$ we obtain that $\mathrm{H}_*(\Theta_m)$ is an isomorphism for $* < m-1$. That $\mathrm{H}_*(\Theta_m)$ is surjective for all $*$ follows directly from $\mathrm{H}_*(\Theta_{m-1})$ being surjective for all $*$ by (c) for $n = m-1$. To show (c) for $n = m$ it thus remains to show that $\mathrm{H}_{m-1}(\Theta_m)$ is injective. As we noted that ι^m_{m-1} is an isomorphism in degrees less than or equal to $m-1$, any homology class in the kernel of $\mathrm{H}_{m-1}(\Theta_m)$ must already lie in the kernel of $\mathrm{H}_{m-1}(\Theta_{m-1})$ and hence in L. But the construction of X_m then directly implies that that homology class is zero in $\mathrm{H}_{m-1}(X_m)$. This shows (c) for $n = m$.

Finally, ι^m_{m-1} being an isomorphism in degrees less than or equal to $m-1$ implies that the restriction and corestriction of ι^m_{m-1} to a morphism of chain complexes $K_{m-1} \to K_m$ is also an isomorphism in those degrees. As $m-1 \geq 0$ this implies that the image of the restriction of $\mathrm{H}_0(\Theta_m)$ to $\mathrm{H}_0(K_m)$ is contained in the image of the restriction of $\mathrm{H}_0(\Theta_{m-1})$ to $\mathrm{H}_0(K_{m-1})$, which together with (d) for $n = m-1$ shows that $\mathrm{H}_0(\Theta_m)$ maps $\mathrm{H}_0(K_m)$ into Y'. As Y' is equal to $\mathrm{H}_*(Y)$ in degrees $* \neq 0$, this shows (d) for $n = m$. □

We can now apply Proposition 7.4.2.3 to improve the cofibrant model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ from Proposition 7.4.1.2.

Proposition 7.4.2.4. *Let X be a set. Then there exists a cofibrant object $\widetilde{\mathrm{C}}(X)$ in $\mathrm{Alg}(\mathsf{Mixed})$ that is concentrated in nonnegative degrees satisfying the following properties.*

Firstly, there has to a be a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\iota_{k[X]})"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\iota_{\widetilde{\mathrm{C}}(X)})"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(X)\right)
\end{tikzcd} \tag{7.7}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, *where the left morphism is induced by the unit morphism $\iota_{k[X]}\colon k \to k[X]$, the right morphism is induced by the unit morphism $\iota_{\widetilde{\mathrm{C}}(X)}\colon k \to \widetilde{\mathrm{C}}(X)$, and the horizontal morphisms are equivalences.*

[37] This is one reason why (b) is part of the properties that we need to require of diagram (∗) even if we did not need this property to conclude the statement of the proposition; without assuming it in the induction each new multiplicative generator also causes new elements of potentially arbitrary low degree.

Secondly, there must exists a sub-strict-mixed-complex $\widetilde{\mathrm{C}}'(X)$ *of* $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$ *that is cofibrant as an object of* Mixed *and such that the morphism of strict mixed complexes*

$$k \oplus \widetilde{\mathrm{C}}'(X) \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right)$$

that is induced by the unit $k \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$ *and inclusion* $\widetilde{\mathrm{C}}'(X) \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$ *is an isomorphism.* ♡

Proof. Let $\widetilde{\mathrm{C}}''(X)$ be as in Proposition 7.4.1.2. Then there is a composite equivalence

$$\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\left(\widetilde{\mathrm{C}}''(X)\right)\right) \simeq \mathrm{HH}\big(k[X]\big) \simeq \mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k[X]/k}\right)$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where the first equivalence is obtained by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ to the equivalence at the bottom of diagram (7.6) supplied by Proposition 7.4.1.2 combined with compatibility of $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1, and the second equivalence is the one from Corollary 7.2.2.3. This implies that there is an isomorphism of $\mathbb{Z}$-graded k-algebras as follows.

$$\mathrm{H}_*\left(\widetilde{\mathrm{C}}''(X)\right) \cong \mathrm{H}_*\left(\Omega^{\bullet}_{k[X]/k}\right) \cong \Omega^{\bullet}_{k[X]/k}$$

As $\Omega^{\bullet}_{k[X]/k}$ is concentrated in nonnegative degrees and can be written as a direct sum of a copy of k generated by the multiplicative unit 1 and some complement we can transfer this sum decomposition to the homology of $\widetilde{\mathrm{C}}''(X)$ and use it to apply Proposition 7.4.2.3. This yields a quasiisomorphism

$$\Theta\colon \widetilde{\mathrm{C}}(X) \to \widetilde{\mathrm{C}}''(X)$$

in $\mathrm{Alg}(\mathsf{Mixed})$ such that $\widetilde{\mathrm{C}}(X)$ is cofibrant, concentrated in nonnegative degrees, and such that there exists a cofibrant sub-strict-mixed-complex $\widetilde{\mathrm{C}}'(X)$ of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$ such that the morphism of strict mixed complexes

$$k \oplus \widetilde{\mathrm{C}}'(X) \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}(X))$$

that is induced by the unit and inclusion is an isomorphism. This already shows the second property that $\widetilde{\mathrm{C}}(X)$ needs to satisfy.

It remains to show the existence of a commutative square (7.7) in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. This is obtained as the outer square of the commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\iota_{k[X]})"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}''(X)}\right)"'] \arrow[dr, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}(X)}\right)"] & \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}''(X)\right) & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(X)\right) \arrow[l, "\simeq", "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Theta)"']
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, with the left commutative square being the one supplied by Proposition 7.4.1.2 and the right triangle commuting because k is initial in $\mathrm{Alg}(\mathsf{Mixed})$. □

As it will later be relevant to keep using the same equivalences as in diagram (7.7) of Proposition 7.4.2.4, we now fix $\widetilde{\mathrm{C}}$ once and for all.

Construction 7.4.2.5. Let X be set. Then we define $\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)$ to be a cofibrant object of $\mathrm{Alg}(\mathsf{Mixed}_{\mathbb{Z}})$ satisfying the conditions of Proposition 7.4.2.4. Together with $\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)$ we fix once and for all a commutative square

$$\begin{tikzcd} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{\mathbb{Z}[X]}\right)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\mathbb{Z}) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)}\right)"] \\ \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\mathbb{Z}[X]\right) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\right) \end{tikzcd} \tag{7.8}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed}_{\mathbb{Z}})$ and a cofibrant sub-strict-mixed-complex $\widetilde{\mathrm{C}}'_{\mathbb{Z}}(X)$ of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X))$ as supplied by Proposition 7.4.2.4.

For other commutative rings k we then define

$$\widetilde{\mathrm{C}}_k(X) := k \otimes_{\mathbb{Z}} \widetilde{\mathrm{C}}_{\mathbb{Z}}(X)$$

which is a cofibrant object of $\mathrm{Alg}(\mathsf{Mixed}_k)$ by Proposition 4.2.2.13. It also follows directly from $\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)$ being concentrated in nonnegative degrees that the same holds true for $\widetilde{\mathrm{C}}_k(X)$. Applying $k \otimes_{\mathbb{Z}} -$ to the inclusion of $\widetilde{\mathrm{C}}'_{\mathbb{Z}}(X)$ into $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X))$ we obtain an injection into a strict mixed complex that we can identify with $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}_k(X))$. We define $\widetilde{\mathrm{C}}'_k(X)$ to be the image of that injection, as a sub-strict-mixed-complex of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}_k(X))$. It then follows immediately from the analogous property for $\widetilde{\mathrm{C}}'_{\mathbb{Z}}$ that the morphism of strict mixed complexes

$$k \oplus \widetilde{\mathrm{C}}'_k(X) \to \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}_k(X)\right)$$

that is induced by the unit and inclusion is then an isomorphism. Furthermore, as the functor

$$k \otimes_{\mathbb{Z}} - \colon \mathsf{Mixed}_{\mathbb{Z}} \to \mathsf{Mixed}_k$$

preserves cofibrations by Proposition 4.2.2.3 we can also conclude that $\widetilde{\mathrm{C}}'_k(X)$ is cofibrant as an object of Mixed_k.

We also obtain the following diagram in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed}_k)$

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[rrr, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"] \arrow[d, dash, "\simeq"'] &&& \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \arrow[d, dash, "\simeq"] \\
k \otimes_{\mathbb{Z}} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}) \arrow[rrr, "k\otimes_{\mathbb{Z}}\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{\mathbb{Z}[X]}\right)"] \arrow[d, dash, "\simeq"'] &&& k \otimes_{\mathbb{Z}} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[X]\big) \arrow[d, dash, "\simeq"] \\
k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\mathbb{Z}) \arrow[rrr, "k\otimes_{\mathbb{Z}}\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)}\right)"] \arrow[d, dash, "\simeq"'] &&& k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\right) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[rrr, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}_k(X)}\right)"] &&& \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}_k(X)\right)
\end{tikzcd}$$

where the top square arises from compatibility of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ with extension of scalars as in Remark 6.2.1.6 (plus using the obvious isomorphisms $k \otimes_{\mathbb{Z}} \mathbb{Z} \cong k$ and $k \otimes_{\mathbb{Z}} \mathbb{Z}[X] \cong k[X]$ that are given by including both tensor factors into the codomain and then multiplying), the middle square is obtained by applying $k \otimes_{\mathbb{Z}} -$ to the transpose of diagram (7.8), and the bottom square arises from compatibility of $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ with extension of scalars by Remark 4.4.1.3 (together again with the isomorphism $k \otimes_Z \mathbb{Z} \cong k$). Transposing the outer commutative rectangle we obtain a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(k) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\iota_{\widetilde{\mathrm{C}}_k(X)}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}_k(X)\right)
\end{tikzcd} \tag{7.9}$$

which we fix once and for all. With the chosen diagram (7.9) and sub-strict-mixed-complex $\widetilde{\mathrm{C}}'_k(X)$ of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}(\widetilde{\mathrm{C}}_k(X))$ we have thus provided the data that shows that $\widetilde{\mathrm{C}}_k(X)$ as we defined it here satisfies the conclusion of Proposition 7.4.2.4.

If the base ring is clear from context we will as usual omit it from the notation and just write e. g. $\widetilde{\mathrm{C}}(X)$ instead of $\widetilde{\mathrm{C}}_k(X)$.

Now let X and Y be two sets and $F\colon k[X] \to k[Y]$ a morphism of commutative k-algebras. Then the composite morphism

$$\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(X)\right) \xrightarrow{\simeq} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \xrightarrow{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[Y]\big) \xrightarrow{\simeq} \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(Y)\right)$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, where the first and third equivalences are the ones from (7.9), can be lifted[38] to a morphism $\widetilde{\mathrm{C}}(F)$ in $\mathrm{Alg}(\mathsf{Mixed})$, which we chose once and for all. $\widetilde{\mathrm{C}}(F)$ comes

[38] As $\widetilde{\mathrm{C}}(X)$ is cofibrant and $\widetilde{\mathrm{C}}(Y)$ fibrant in $\mathrm{Alg}(\mathsf{Mixed})$.

together with a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(X)\Big) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\widetilde{\mathrm{C}}(F))"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[Y]\big) \arrow[r, dash, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(Y)\Big)
\end{tikzcd} \tag{7.10}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, where the horizontal equivalences are those from (7.9). ♢

7.4.3. Comparing the algebra and mixed structure separately

Construction 7.4.2.5 provides a reasonably nice strict model $\widetilde{\mathrm{C}}(X)$ for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X])$ as an algebra in mixed complexes. In this section we will construct comparison morphisms from the underlying differential graded algebra and strict mixed complex of $\widetilde{\mathrm{C}}(X)$ to $\Omega^{\bullet}_{k[X]/k}$.

Construction 7.4.3.1. Let X be a set. We will construct a quasiisomorphism

$$\Phi'_{k,X}\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \to \Omega^{\bullet}_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$. If the base ring is clear from context we will also write Φ'_X, and even Φ' if the set X is clear as well.

As in Construction 7.4.2.5 we first construct $\Phi'_{\mathbb{Z},X}$ and then extend scalars for $\Phi'_{k,X}$. There is a composite equivalence

$$\mathrm{Alg}(\gamma)\Bigg(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\Big)\Bigg) \simeq \mathrm{HH}\big(\mathbb{Z}[X]\big) \simeq \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{\mathbb{Z}[X]/\mathbb{Z}}\Big)$$

in $\mathrm{Alg}(\mathcal{D}(\mathbb{Z}))$, where the first equivalence is obtained by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ to the equivalence at the bottom of diagram (7.8) in Construction 7.4.2.5 combined with compatibility of $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1, and the second equivalence is the one from Corollary 7.2.2.3. By Proposition 4.2.2.12 $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ preserves cofibrant objects, so $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X))$ is cofibrant as an object in $\mathrm{Alg}(\mathsf{Ch}(\mathbb{Z}))$. As $\Omega^{\bullet}_{\mathbb{Z}[X]/\mathbb{Z}}$ is fibrant (like every object), we can thus lift the above equivalence in $\mathrm{Alg}(\mathcal{D}(\mathbb{Z}))$ to a quasiisomorphism $\Phi'_{\mathbb{Z},X}$ (see [Hov99, 1.2.10 (ii)] and Proposition A.1.0.1) as claimed.

We now define

$$\Phi'_{k,X}\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \to \Omega^{\bullet}_{k[X]/k}$$

as the composition

$$\begin{aligned}
\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) = \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(k \otimes_{\mathbb{Z}} \widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\Big) &\xrightarrow{\cong} k \otimes_{\mathbb{Z}} \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\Big) \\
&\xrightarrow{k\otimes_{\mathbb{Z}}\Phi'_{\mathbb{Z},X}} k \otimes_{\mathbb{Z}} \Omega^{\bullet}_{\mathbb{Z}[X]/\mathbb{Z}} \xrightarrow{\cong} \Omega^{\bullet}_{k[X]/k}
\end{aligned}$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$, where the first equality is by definition, the isomorphism on the first line the one from compatibility of $\mathrm{ev}_{\mathfrak{m}}$ with extension of scalars as in Remark 4.2.1.3, and the isomorphism in the second line is given by applying the unit in the first tensor factor and $\Omega^{\bullet}_{\iota_k[X]/k}$ in the second, and then multiplying. To see that $\Phi'_{k,X}$ is indeed a quasiisomorphism we only need to argue that $k \otimes_{\mathbb{Z}} \Phi'_{\mathbb{Z},X}$ is a quasiisomorphism. Note that the underlying morphism of chain complexes can be identified with $k \otimes_{\mathbb{Z}} \mathrm{ev}_{\mathfrak{a}}(\Phi'_{\mathbb{Z},X})$, and the functor

$$k \otimes_{\mathbb{Z}} - \colon \mathsf{Ch}(\mathbb{Z}) \to \mathsf{Ch}(k)$$

is a left Quillen functor by Fact 4.1.5.1 and so preserves weak equivalences between cofibrant objects. By Proposition 4.2.2.12 $\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)$ has cofibrant underlying chain complex, and by the discussion surrounding Definition 7.1.0.1 $\Omega^{\bullet}_{\mathbb{Z}[X]/\mathbb{Z}}$ has cofibrant underlying chain complex as well, so as $\Phi'_{\mathbb{Z},X}$ is a quasiisomorphism we obtain that $\Phi'_{k,X}$ is one as well. ♢

Proposition 7.4.3.2. *Let X be a set. Then there is a commutative triangle*

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\widetilde{\mathrm{C}}_k(X)\big)\Big) \arrow[rr, "\mathrm{Alg}(\gamma)(\Phi'_{k,X})"] \arrow[dr, dash, "\simeq"'] & & \mathrm{Alg}(\gamma)\big(\Omega^{\bullet}_{k[X]/k}\big) \arrow[dl, dash, "\simeq"] \\
& \mathrm{HH}\big(k[X]\big) &
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where the left diagonal equivalence is obtained by applying the forgetful functor $\mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}$ to the equivalence at the bottom of diagram (7.9) in Construction 7.4.2.5 combined with compatibility of $\mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1, and the right diagonal equivalence is the one from Corollary 7.2.2.3. ♡

Proof. We drop the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ from the notation in this proof to improve readability. Consider the following diagram in $\mathrm{Alg}(\mathcal{D}(k))$ that will be explained below.

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\left(\widetilde{\mathrm{C}}_k(X)\right) \arrow[r, "\mathrm{Alg}(\gamma)\left(\Phi'_{k,X}\right)"] \arrow[d, dash, "\mathrm{id}"'] & \mathrm{Alg}(\gamma)\left(\Omega^\bullet_{k[X]/k}\right) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma)\left(k \otimes_{\mathbb{Z}} \widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\right) \arrow[r, "\mathrm{Alg}(\gamma)\left(k\otimes_{\mathbb{Z}}\Phi'_{\mathbb{Z},X}\right)"] \arrow[d, dash, "\simeq"'] & \mathrm{Alg}(\gamma)\left(k \otimes_{\mathbb{Z}} \Omega^\bullet_{\mathbb{Z}[X]/\mathbb{Z}}\right) \arrow[d, dash, "\simeq"] \\
k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma)\left(\widetilde{\mathrm{C}}_{\mathbb{Z}}(X)\right) \arrow[r, "k\otimes_{\mathbb{Z}}\mathrm{Alg}(\gamma)\left(\Phi'_{\mathbb{Z},X}\right)"] \arrow[d, dash, "\simeq"'] & k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma)\left(\Omega^\bullet_{\mathbb{Z}[X]/\mathbb{Z}}\right) \arrow[d, dash, "\mathrm{id}"] \\
k \otimes_{\mathbb{Z}} \mathrm{HH}\left(\mathbb{Z}[X]\right) \arrow[r, "\simeq"] \arrow[d, dash, "\simeq"'] & k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma)\left(\Omega^\bullet_{\mathbb{Z}[X]/\mathbb{Z}}\right) \arrow[d, dash, "\simeq"] \\
\mathrm{HH}\left(k \otimes_{\mathbb{Z}} \mathbb{Z}[X]\right) \arrow[d, dash, "\simeq"'] & \mathrm{Alg}(\gamma)\left(k \otimes_{\mathbb{Z}} \Omega^\bullet_{\mathbb{Z}[X]/\mathbb{Z}}\right) \arrow[d, dash, "\simeq"] \\
\mathrm{HH}\left(k[X]\right) \arrow[r, "\simeq"] & \mathrm{Alg}(\gamma)\left(\Omega^\bullet_{k[X]/k}\right)
\end{tikzcd}$$

The first square from the top is built from the composition $\Phi'_{k,X}$ is defined as in Construction 7.4.3.1. The second square is the naturality square for the equivalence in Remark 4.4.1.3. The third square is obtained from the definition of $\Phi'_{\mathbb{Z},X}$ by applying $k \otimes_{\mathbb{Z}} \mathrm{Alg}(\gamma)(-)$, the left equivalence is obtained by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ to the equivalence at the bottom of diagram (7.8) in Construction 7.4.2.5 combined with compatibility of $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1 and at the end tensoring with k, and the bottom equivalence is obtained by tensoring the equivalence from Corollary 7.2.2.3 (for base ring $\mathbb{Z}$) with k. Finally, the bottom rectangle is the one from Proposition 7.2.2.4, so that in particular the bottom equivalence of the full rectangle is the one from Corollary 7.2.2.3.

Now note that on the right the top two equivalences are the same as the bottom two equivalences, so the composition of the right column is the identity. The bottom equivalence is exactly the one occurring as the right diagonal equivalence in the statement. Finally, the composition on the left is exactly the definition of the equivalence at the bottom of diagram (7.9) in Construction 7.4.2.5. □

Proposition 7.4.3.3. *Let X be a totally ordered set. Then there exists a quasiisomorphism*

$$\Psi\colon\ \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right) \to \Omega^\bullet_{k[X]/k}$$

in Mixed. ♡

Proof. Some parts of this proof will be analogous to Construction 7.4.3.1, but we need some additional arguments as $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right)$ is not a cofibrant object of Mixed. Proposition 7.4.2.4 and Construction 7.4.2.5 isolate this problem to the non-cofibrancy of the summand k. So let $j\colon k^{\mathrm{cof}} \to k$ be a cofibrant replacement of k in Mixed. It then follows from Construction 7.4.2.5 that $k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)$ is a cofibrant strict mixed complex and that the composition

$$k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X) \xrightarrow{j \oplus \mathrm{id}} k \oplus \widetilde{\mathrm{C}}'(X) \xrightarrow{\cong} \mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right) \tag{$*$}$$

is a quasiisomorphism, where the second morphism is induced by the unit and inclusion. There is a composite equivalence

$$\gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)\right) \simeq \gamma_{\mathsf{Mixed}}\left(\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\left(\widetilde{\mathrm{C}}(X)\right)\right) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \simeq \gamma_{\mathsf{Mixed}}\left(\Omega^{\bullet}_{k[X]/k}\right) \tag{$**$}$$

in $\mathcal{M}$ixed, where the first equivalences arises from the composite quasiisomorphism ($*$), the second equivalence is obtained by applying the forgetful functor $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ to the equivalence at the bottom of diagram (7.9) in Construction 7.4.2.5 combined with compatibility of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1, and the third equivalence is the one from Construction 7.3.11.3.

Using that $k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)$ is a cofibrant object of Mixed and that every object, so in particular $\Omega^{\bullet}_{k[X]/k}$, is fibrant, we can now lift the composite equivalence from ($**$) to a quasiisomorphism

$$\Psi'\colon k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X) \to \Omega^{\bullet}_{k[X]/k}$$

in Mixed.

In the following we will use the notation i_1 and i_2 for the inclusions of the first and second summands of the sums $k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)$ and $k \oplus \widetilde{\mathrm{C}}'(X)$, with the context making clear which of the two sums we are including into. We now claim the following.

Claim 1: There exist morphisms

$$\Psi''\colon k^{\mathrm{cof}} \to k \qquad \text{and} \qquad \Psi'''\colon k \to \Omega^{\bullet}_{k[X]/k}$$

in Mixed such that Ψ'' is a quasiisomorphism and such that there exists a commutative square

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}}\right) \arrow[r, "\gamma_{\mathsf{Mixed}}(i_1)"] \arrow[d, "\gamma_{\mathsf{Mixed}}(\Psi'')"'] & \gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)\right) \arrow[d, "\gamma_{\mathsf{Mixed}}(\Psi')"] \\
\gamma_{\mathsf{Mixed}}(k) \arrow[r, "\gamma_{\mathsf{Mixed}}(\Psi''')"'] & \gamma_{\mathsf{Mixed}}\left(\Omega^{\bullet}_{k[X]/k}\right)
\end{tikzcd} \tag{$***$}$$

in $\mathcal{M}$ixed.

Before showing the claim we discuss how the claim implies the statement of the proposition. We define Ψ as the composition

$$\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\Big(\widetilde{\mathrm{C}}(X)\Big) \xrightarrow{\cong} k \oplus \widetilde{\mathrm{C}}'(X) \xrightarrow{\Psi''' \amalg \left(\Psi' \circ i_2\right)} \Omega^{\bullet}_{k[X]/k}$$

in Mixed, where the first morphism is the inverse isomorphism of the second morphism in $(*)$. It remains to show that the morphism

$$\Psi''' \amalg \left(\Psi' \circ i_2\right) \colon k \oplus \widetilde{\mathrm{C}}'(X) \to \Omega^{\bullet}_{k[X]/k}$$

is a quasiisomorphism. But as Ψ'' and hence $\Psi'' \oplus \mathrm{id}_{\widetilde{\mathrm{C}}'(X)}$ is a quasiisomorphism, it suffices for this to show that

$$\left(\Psi''' \circ \Psi''\right) \amalg \left(\Psi' \circ i_2\right) \colon k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X) \to \Omega^{\bullet}_{k[X]/k}$$

is a quasiisomorphism. We know that $\Psi' = (\Psi' \circ i_1) \amalg (\Psi' \circ i_2)$ is a quasiisomorphism, so it would suffice to show that $(\Psi''' \circ \Psi'') \amalg (\Psi' \circ i_2)$ is chain homotopic to $(\Psi' \circ i_1) \amalg (\Psi' \circ i_2)$, for which it in turn suffices to show that $(\Psi''' \circ \Psi'')$ is chain homotopic to $(\Psi' \circ i_1)$. But this follows from existence of commutative diagram $(***)$, using that the underlying chain complex of k^{cof} is cofibrant by Proposition 4.2.2.12, while $\Omega^{\bullet}_{k[X]/k}$ is a fibrant chain complex, together with [Hov99, 1.2.10 (ii)] and Propositions A.1.0.1 and 4.1.4.2.

So to finish the proof it remains to show Claim 1, for which we need to unpack and rewrite the composition $\gamma_{\mathsf{Mixed}}(\Psi') \circ \gamma_{\mathsf{Mixed}}(i_1)$ that occurs in the square $(***)$ that we are to construct. Using the definition of Ψ' and $(*)$ and $(**)$ to unpack this composition we obtain that $\gamma_{\mathsf{Mixed}}(\Psi') \circ \gamma_{\mathsf{Mixed}}(i_1)$ is homotopic to the composition from the top left to the bottom right along the top row and right column of the following diagram in $\mathcal{M}\mathrm{ixed}$, which will be explained below.

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}}\right) \arrow[r, "\gamma_{\mathsf{Mixed}}(i_1)"] \arrow[d, "\gamma_{\mathsf{Mixed}}(j)"'] & \gamma_{\mathsf{Mixed}}\Big(k^{\mathrm{cof}} \oplus \widetilde{\mathrm{C}}'(X)\Big) \arrow[d, "\gamma_{\mathsf{Mixed}}(j \oplus \mathrm{id})"] \\
\gamma_{\mathsf{Mixed}}(k) \arrow[r, "\gamma_{\mathsf{Mixed}}(i_1)"] \arrow[d, "\mathrm{id}"'] & \gamma_{\mathsf{Mixed}}\Big(k \oplus \widetilde{\mathrm{C}}'(X)\Big) \arrow[d, "\simeq"] \\
\gamma_{\mathsf{Mixed}}(k) \arrow[r, "\gamma_{\mathsf{Mixed}}\left(\iota_{\widetilde{\mathrm{C}}(X)}\right)"] \arrow[d, "\simeq"'] & \gamma_{\mathsf{Mixed}}\Big(\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}\Big(\widetilde{\mathrm{C}}(X)\Big)\Big) \arrow[d, "\simeq"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[d, "\simeq"] \\
& \gamma_{\mathsf{Mixed}}\Big(\Omega^{\bullet}_{k[X]/k}\Big)
\end{tikzcd}$$

The top square is obtained by applying γ_{Mixed} to a commuting square in $\mathsf{Mixed}_{\mathrm{cof}}$. In the middle square we define the vertical morphism on the right as the equivalence induced by the isomorphism occurring in $(*)$. By definition this isomorphism is given on k by the unit morphism, which implies that this square also has a filler as it is given by γ_{Mixed} applied to a commuting square in $\mathsf{Mixed}_{\mathrm{cof}}$. The bottom square is given by applying the forgetful functor $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ to diagram (7.9) in Construction 7.4.2.5. Finally, the vertical equivalence at the bottom right is the one from Construction 7.3.11.3, which also occurs in $(**)$. Commutativity of the above diagram means that $\gamma_{\mathsf{Mixed}}(\Psi') \circ \gamma_{\mathsf{Mixed}}(i_1)$ is homotopic to the composition from the top left to the bottom right along the left column.

We now consider the following commutative diagram in $\mathcal{M}\mathrm{ixed}$, which we again explain below. The composition that we just showed is homotopic to $\gamma_{\mathsf{Mixed}}(\Psi') \circ \gamma_{\mathsf{Mixed}}(i_1)$ occurs as the composition from the top left to the bottom right while staying on the top and right side.

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}}\right) \arrow[d, "\gamma_{\mathsf{Mixed}}(j)"', "\simeq"] \arrow[dddd, bend right=60, "\psi''"', "\simeq"] & \\
\gamma_{\mathsf{Mixed}}(k) \arrow[d, "\simeq"'] & \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\iota_{k[X]}\right)"] \arrow[d, "\simeq"'] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(k[X]\right) \arrow[d, "\simeq"] \arrow[ddd, bend left=60, "\simeq"] \\
\gamma_{\mathsf{Mixed}}\left(\mathrm{C}(k)\right) \arrow[r, "\gamma_{\mathsf{Mixed}}\left(\mathrm{C}(\iota_{k[X]})\right)"] \arrow[d, "\simeq"'] & \gamma_{\mathsf{Mixed}}\left(\mathrm{C}\left(k[X]\right)\right) \arrow[d, "\simeq"] \\
\gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(k)\right) \arrow[r, "\gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(\iota_{k[X]})\right)"] & \gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}\left(k[X]\right)\right) \\
& \gamma_{\mathsf{Mixed}}\left(\Omega^{\bullet}_{k[X]/k}\right) \arrow[u, "\gamma_{\mathsf{Mixed}}\left(\epsilon_X^{(\bullet)}\right)", "\simeq"']
\end{tikzcd}$$

We start by just defining ψ'' as the composition of the equivalences in the left column (which will be explained in a moment); this shorthand will be useful to shorten notation later. The second morphism in the left column is obtained by applying the forgetful functor $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ to the top horizontal equivalence in diagram (7.9) in Construction 7.4.2.5. The top square arises from naturality of the equivalence between $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ and the standard Hochschild complex in Proposition 6.3.4.1. The bottom square arises from naturality of the quotient morphism from the standard Hochschild complex to the normalized standard Hochschild complex, see Proposition 6.3.1.10. The lower right vertical equivalence is the one induced by the strongly homotopy linear quasiisomorphism $\epsilon_X^{(\bullet)}$, see Proposition 7.3.11.2 and Construction 4.4.4.1. Finally, the long equivalence on the right is the one of Construction 7.3.11.3, which also occurs in $(**)$, and the right rectangle is obtained by unpacking its definition.

We have now shown that $\gamma_{\mathsf{Mixed}}(\Psi') \circ \gamma_{\mathsf{Mixed}}(i_1)$ is homotopic to the composition from the top left to the bottom right in the diagram above while staying to the left and bottom. Note that $\overline{\mathrm{C}}(k)$ is isomorphic to k as a strict mixed complex (as $\overline{k} = 0$), with an isomorphism given by the unit $\iota_{\overline{\mathrm{C}}(k)}$ of $\overline{\mathrm{C}}(k)$. As $\overline{\mathrm{C}}(\iota_{k[X]})$ is a morphism of differential graded algebras and equality of morphisms of strict mixed complex can be checked on the underlying morphisms of chain complexes we can conclude that

$$\overline{\mathrm{C}}\big(\iota_{k[X]}\big) \circ \iota_{\overline{\mathrm{C}}(k)} = \iota_{\overline{\mathrm{C}}(k[X])}$$

holds. We should comment here on why $\iota_{\overline{\mathrm{C}}(k)}$ and $\iota_{\overline{\mathrm{C}}(k[X])}$ are morphisms of strict mixed complexes. As $\overline{\mathrm{C}}(R)$ for a commutative k-algebra R is not in general an algebra in strict mixed complexes, it is not a purely formal fact that the unit morphism $k \to \overline{\mathrm{C}}(R)$ of the differential graded algebra structure is a morphism of strict mixed complexes rather than just a morphism of chain complexes. However, this is indeed the case, as one can check using the formula for d from Proposition 6.3.1.10.[39]

The upshot of the discussion so far is that there is a commutative diagram as follows in $\mathcal{M}\mathrm{ixed}$.

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\left(k^{\mathrm{cof}}\right) \arrow[d, "\psi''"', "\simeq"] \arrow[drr, "\gamma_{\mathsf{Mixed}}(\Psi')\circ\gamma_{\mathsf{Mixed}}(i_1)"] & & \\
\gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(k)\right) \arrow[r, "\gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(\iota_{k[X]})\right)"] & \gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(k[X])\right) & \gamma_{\mathsf{Mixed}}\left(\Omega^{\bullet}_{k[X]/k}\right) \arrow[l, "\gamma_{\mathsf{Mixed}}\left(\epsilon_X^{(\bullet)}\right)"', "\simeq"] \\
\gamma_{\mathsf{Mixed}}(k) \arrow[u, "\gamma_{\mathsf{Mixed}}\left(\iota_{\overline{\mathrm{C}}(k)}\right)", "\simeq"'] \arrow[ur, "\gamma_{\mathsf{Mixed}}\left(\iota_{\overline{\mathrm{C}}(k[X])}\right)"'] \arrow[urr, dashed] & &
\end{tikzcd}$$

As k^{cof} is a cofibrant object in Mixed we can lift the composition of the two equivalences on the left to a quasiisomorphism $\Psi''\colon k^{\mathrm{cof}} \to k$ in Mixed, and it remains to show that we can up to homotopy find a lift of the dashed composition in $\mathcal{M}\mathrm{ixed}$ to a strict morphism $\Psi'''\colon k \to \Omega^{\bullet}_{k[X]/k}$ (that such a lift exists is not automatic as k is not cofibrant in Mixed). We define Ψ''' as the unit morphism

$$\Psi''' := \iota_{\Omega^{\bullet}_{k[X]/k}} : k \to \Omega^{\bullet}_{k[X]/k}$$

[39] That this is not automatic is underlined by the fact that the analogous property does not hold if we had used $\mathrm{C}(R)$ instead of $\overline{\mathrm{C}}(R)$ – this is one of the reasons the *normalized* standard Hochschild complex is more convenient to work with.

which can be seen to be a morphism of strict mixed complexes from the definition of d on $\Omega^{\bullet}_{k[X]/k}$. It then suffices to show that the triangle

$$\begin{tikzcd}
\gamma_{\mathsf{Mixed}}\left(\Omega^{\bullet}_{k[X]/k}\right) \arrow[rr, "\gamma_{\mathsf{Mixed}}\left(\epsilon_X^{(\bullet)}\right)", "\simeq"'] & & \gamma_{\mathsf{Mixed}}\left(\overline{\mathrm{C}}(k[X])\right) \\
& \gamma_{\mathsf{Mixed}}(k) \arrow[ul, "\gamma_{\mathsf{Mixed}}\left(\Psi'''\right)"] \arrow[ur, "\gamma_{\mathsf{Mixed}}\left(\iota_{\overline{\mathrm{C}}(k[X])}\right)"'] &
\end{tikzcd}$$

commutes in $\mathcal{M}$ixed.

For this we unpack the definition of the lower horizontal equivalence $\gamma_{\mathsf{Mixed}}(\Omega^{\bullet}_{k[X]/k})$ from Construction 4.4.4.1. As ad hoc notation, let us denote the natural transformation coming with the functorial cofibrant replacement on Mixed by $q\colon -^{\mathrm{cof}} \to \mathrm{id}_{\mathsf{Mixed}}$. We will also use the notation that was in use in Construction 4.4.4.1. We need to show that there is a filler for the triangle at the bottom of the following diagram, where the top is the commutative rectangle from Construction 4.4.4.1. To make the diagram a bit cleaner we abbreviate γ_{Mixed} by γ_{M}, as well as $\Omega^{\bullet}_{k[X]/k}$ and $\overline{\mathrm{C}}(k[X])$ by Ω and $\overline{\mathrm{C}}$.

$$\begin{tikzcd}
\gamma_{\mathsf{M}}\left(\Omega^{\mathrm{cof}}\right) \arrow[r, "\gamma_{\mathsf{M}}\left(\left(\epsilon_X^{\mathrm{strict}}\right)^{\mathrm{cof}}\right)", "\simeq"'] \arrow[d, "\gamma_{\mathsf{M}}(q_\Omega)"', "\simeq"] & \gamma_{\mathsf{M}}\left(\left(\overline{\mathrm{C}}^{\mathrm{shl}}\right)^{\mathrm{cof}}\right) & \gamma_{\mathsf{M}}\left(\overline{\mathrm{C}}^{\mathrm{cof}}\right) \arrow[l, "\gamma_{\mathsf{M}}\left(\left(\iota^{\mathrm{shl}}_{\overline{\mathrm{C}}}\right)^{\mathrm{cof}}\right)"', "\simeq"] \arrow[d, "\simeq"', "\gamma_{\mathsf{M}}\left(q_{\overline{\mathrm{C}}}\right)"] \\
\gamma_{\mathsf{M}}(\Omega) \arrow[rr, "\gamma_{\mathsf{M}}\left(\epsilon_X^{(\bullet)}\right)", "\simeq"'] & & \gamma_{\mathsf{M}}\left(\overline{\mathrm{C}}\right) \\
& \gamma_{\mathsf{M}}(k) \arrow[ul, "\gamma_{\mathsf{M}}\left(\Psi'''\right)"] \arrow[ur, "\gamma_{\mathsf{M}}\left(\iota_{\overline{\mathrm{C}}}\right)"'] &
\end{tikzcd}$$

As all the morphism in the top rectangle are equivalences we can also partition the diagram differently and instead show that there is a morphism from $\gamma_{\mathsf{Mixed}}(k)$ to the object in the top middle such that the two shapes in the diagram below have a filler.

$$\begin{tikzcd}
\gamma_{\mathsf{M}}\left(\Omega^{\mathrm{cof}}\right) \arrow[r, "\gamma_{\mathsf{M}}\left(\left(\epsilon_X^{\mathrm{strict}}\right)^{\mathrm{cof}}\right)", "\simeq"'] \arrow[d, "\gamma_{\mathsf{M}}(q_\Omega)"', "\simeq"] & \gamma_{\mathsf{M}}\left(\left(\overline{\mathrm{C}}^{\mathrm{shl}}\right)^{\mathrm{cof}}\right) & \gamma_{\mathsf{M}}\left(\overline{\mathrm{C}}^{\mathrm{cof}}\right) \arrow[l, "\gamma_{\mathsf{M}}\left(\left(\iota^{\mathrm{shl}}_{\overline{\mathrm{C}}}\right)^{\mathrm{cof}}\right)"', "\simeq"] \arrow[d, "\simeq"', "\gamma_{\mathsf{M}}\left(q_{\overline{\mathrm{C}}}\right)"] \\
\gamma_{\mathsf{M}}(\Omega) & & \gamma_{\mathsf{M}}\left(\overline{\mathrm{C}}\right) \\
& \gamma_{\mathsf{M}}(k) \arrow[ul, "\gamma_{\mathsf{M}}\left(\Psi'''\right)"] \arrow[ur, "\gamma_{\mathsf{M}}\left(\iota_{\overline{\mathrm{C}}}\right)"'] \arrow[uu, dashed] &
\end{tikzcd}$$

Next we use that $q_k\colon k^{\mathrm{cof}} \to k$ is a quasiisomorphism to reduce to showing that there exists a dashed morphism as indicated in the diagram below such that the top two triangles have a filler, with the two squares having a filler by naturality of q.

$$\begin{tikzcd}
\gamma_{\mathsf{M}}\left(\Omega^{\mathrm{cof}}\right) \arrow[rr, "\gamma_{\mathsf{M}}\left(\left(\epsilon_X^{\mathrm{strict}}\right)^{\mathrm{cof}}\right)", "\simeq"'] \arrow[d, "\gamma_{\mathsf{M}}(q_\Omega)"', "\simeq"] & & \gamma_{\mathsf{M}}\left(\left(\overline{\mathrm{C}}^{\mathrm{shl}}\right)^{\mathrm{cof}}\right) & & \gamma_{\mathsf{M}}\left(\overline{\mathrm{C}}^{\mathrm{cof}}\right) \arrow[ll, "\gamma_{\mathsf{M}}\left(\left(\iota_{\overline{\mathrm{C}}}^{\mathrm{shl}}\right)^{\mathrm{cof}}\right)"', "\simeq"] \arrow[d, "\simeq"', "\gamma_{\mathsf{M}}\left(q_{\overline{\mathrm{C}}}\right)"] \\
\gamma_{\mathsf{M}}(\Omega) & & \gamma_{\mathsf{M}}\left(k^{\mathrm{cof}}\right) \arrow[ull, "\gamma_{\mathsf{M}}\left(\Psi'''^{\mathrm{cof}}\right)"] \arrow[u, dashed] \arrow[urr, "\gamma_{\mathsf{M}}\left(\iota_{\overline{\mathrm{C}}}^{\mathrm{cof}}\right)"'] \arrow[d, "\gamma_{\mathsf{M}}(q_k)"', "\simeq"] & & \gamma_{\mathsf{M}}\left(\overline{\mathrm{C}}\right) \\
& & \gamma_{\mathsf{M}}(k) \arrow[ull, "\gamma_{\mathsf{M}}\left(\Psi'''\right)"] \arrow[urr, "\gamma_{\mathsf{M}}\left(\iota_{\overline{\mathrm{C}}}\right)"'] & &
\end{tikzcd}$$

To show that the square formed by the two triangles has a filler in $\mathcal{M}$ixed it suffices to show that the square

$$\begin{tikzcd}
k^{\mathrm{cof}} \arrow[r, "\Psi'''^{\mathrm{cof}}"] \arrow[d, "\iota_{\overline{\mathrm{C}}(k[X])}^{\mathrm{cof}}"'] & \left(\Omega^{\bullet}_{k[X]/k}\right)^{\mathrm{cof}} \arrow[d, "\left(\epsilon_X^{\mathrm{strict}}\right)^{\mathrm{cof}}"] \\
\overline{\mathrm{C}}\left(k[X]\right)^{\mathrm{cof}} \arrow[r, "\left(\iota_{\overline{\mathrm{C}}(k[X])}^{\mathrm{shl}}\right)^{\mathrm{cof}}"'] & \left(\overline{\mathrm{C}}\left(k[X]\right)^{\mathrm{shl}}\right)^{\mathrm{cof}}
\end{tikzcd}$$

commutes in Mixed, for which it in turn suffices to show that the diagram

$$\begin{tikzcd}
k \arrow[r, "\Psi'''"] \arrow[d, "\iota_{\overline{\mathrm{C}}(k[X])}"'] & \Omega^{\bullet}_{k[X]/k} \arrow[d, "\epsilon_X^{\mathrm{strict}}"] \\
\overline{\mathrm{C}}\left(k[X]\right) \arrow[r, "\iota_{\overline{\mathrm{C}}(k[X])}^{\mathrm{shl}}"'] & \overline{\mathrm{C}}\left(k[X]\right)^{\mathrm{shl}}
\end{tikzcd}$$

commutes. This we can now check directly. As all morphisms are k-linear it suffices to check the image of the element 1 of k along the two compositions. We first consider the composition along the bottom left. $\iota_{\overline{\mathrm{C}}(k[X])}$ maps 1 to 1, which is then mapped by $\iota^{\mathrm{shl}}_{\overline{\mathrm{C}}(k[X])}$ to the tuple $(1, 0, 0, \ldots)$ of $\overline{\mathrm{C}}\big(k[X]\big)^{\mathrm{shl}}$, see Definition 4.2.3.3. In the composition along the top right Ψ''' maps 1 to 1, which is then mapped by $\epsilon^{\mathrm{strict}}_X$ to the tuple $\epsilon^{\mathrm{strict}}_X(1)$ that is defined as follows for $i \geq 0$, see Proposition 4.2.3.7 and Definition 4.2.3.8.

$$\epsilon^{\mathrm{strict}}_X(1)_{2i} = \epsilon^{(i)}_X(1)$$
$$\epsilon^{\mathrm{strict}}_X(1)_{2i+1} = \Big(\partial\epsilon^{(i+1)}_X - \epsilon^{(i+1)}_X\partial\Big)(1)$$

As $\partial(1) = 0$ we can simplify the odd case to $\epsilon^{\mathrm{strict}}_X(1)_{2i+1} = \partial(\epsilon^{(i+1)}_X(1))$. It thus suffices to show that $\epsilon^{(0)}_X(1) = 1$ and $\epsilon^{(i)}_X(1) = 0$ for $i > 0$. The former is clear as $\epsilon^{(0)}_X$ is a morphism of differential graded algebras by Proposition 7.2.2.2 (2). For the latter we check the definition of $\epsilon^{(i)}_X$ in Construction 7.3.1.1. Using the notation there, the element 1 implies that $m = 0$, and then $C(i, m)$ is empty[40], implying the claim. This finishes the proof. □

Definition 7.4.3.4. Let X be a totally ordered set. Then we choose once and for all a quasiisomorphism

$$\Psi_X\colon\ \mathrm{ev}^{\mathsf{Mixed}}_{\mathfrak{a}}\Big(\widetilde{\mathrm{C}}(X)\Big) \to \Omega^\bullet_{k[X]/k}$$

in Mixed, as exists by Proposition 7.4.3.3. ♢

7.4.4. Compatibility of Φ with d in the case of a single variable

In Section 7.4.3 we constructed two different comparison quasiisomorphisms between $\widetilde{\mathrm{C}}(X)$ and $\Omega^\bullet_{k[X]/k}$; one compatible with the strict mixed structure, and one compatible with the multiplicative structure. In this section we show that after possibly tweaking it slightly, the multiplicative morphism also preserves d in the special case of $X = \{t\}$.

Proposition 7.4.4.1. *There exists an element ν of $\{+1, -1\}$ such that the morphism*

$$\Phi'_{k,\{t\}}\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k\big(\{t\}\big)\Big) \to \Omega^\bullet_{k[t]/k}$$

from Construction 7.4.3.1 satisfies

$$\Phi'_{k,\{t\}}(\mathrm{d}\,y) = \nu \cdot \mathrm{d}\Big(\Phi'_{k,\{t\}}(y)\Big) \tag{7.11}$$

for every element y of $\widetilde{\mathrm{C}}_k(\{t\})$. ♡

Proof. By definition we can identify $\Phi'_{k,\{t\}}$ with $k \otimes_{\mathbb{Z}} \Phi'_{\mathbb{Z},\{t\}}$, and as the isomorphism (actually equality) $\widetilde{\mathrm{C}}_k(\{t\}) \cong k \otimes_{\mathbb{Z}} \widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$ is compatible with the strict mixed structure by definition and the isomorphism $\Omega^\bullet_{k[t]/k} \cong k \otimes_{\mathbb{Z}} \Omega^\bullet_{\mathbb{Z}[t]/\mathbb{Z}}$ that occurs in the definition of $\Phi'_{k,\{t\}}$ is compatible with the strict mixed structure by Remark 7.1.0.2, it suffices to

[40] As $i > 0$ we have that $1 \leq 1 \leq i$. Thus any element $\vec{c}$ of $C(i, m)$ must satisfy $c_1 + 1 \leq c_2 - 1$ while $1 \leq c_1, c_2 \leq 0 + 1 = 1$, which is not possible.

prove that there exists an element ν of $\{+1,-1\}$ such that (7.11) holds in the case of base ring $\mathbb{Z}$.

We next note that as $\Omega^{\bullet}_{\mathbb{Z}[t]/k}$ is concentrated in degrees 0 and 1, equation (7.11) is automatic no matter what we choose for ν if y is an element of a degree other than -1 or 0. As $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$ is concentrated in nonnegative degrees the equation also holds automatically for elements of degree -1, and every element of $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$ of degree 0 is a cycle. We are thus left showing that there exists an element ν of $\{+1,-1\}$ such that (7.11) holds for cycles y of degree 0 of $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$

As $\Phi'_{\mathbb{Z},\{t\}}$ is a quasiisomorphism and $\Omega^{\bullet}_{\mathbb{Z}[t]/\mathbb{Z}}$ has zero boundary operator, $\Phi'_{\mathbb{Z},\{t\}}$ must be surjective. We can thus lift the element t of $\Omega^{\bullet}_{\mathbb{Z}[t]/\mathbb{Z}}$ to an element $\overline{t}$ of $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$ of degree 0 such that $\Phi'_{\mathbb{Z},\{t\}}(\overline{t}) = t$. As $\Phi'_{\mathbb{Z},\{t\}}$ is multiplicative we then also have $\Phi'_{\mathbb{Z},\{t\}}(\overline{t}^n) = t^n$ for $n \geq 0$, so that we can conclude that the elements $[\overline{t}^n]$ for $n \geq 0$ form a $\mathbb{Z}$-basis for $\mathrm{H}_0(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\}))$. Let us assume for the moment that we found an element ν such that (7.11) holds for the elements $y = \overline{t}^n$ for $n \geq 0$. Then we claim (7.11) holds for all cycles y in degree 0. Indeed, any cycle y of degree 0 of $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$ must be of the form $y = \sum_{0\leq n} c_n \cdot \overline{t}^n + \partial z$ for some element z of degree 1 and elements c_n in $\mathbb{Z}$ for $n \geq 0$, only finitely many of which are nonzero. But then we have the following calculation, using that $\Omega^{\bullet}_{\mathbb{Z}[t]/\mathbb{Z}}$ has zero boundary operator and thus $\Phi'_{\mathbb{Z},\{t\}}$ maps boundaries to zero.

$$\begin{aligned}
\Phi'_{\mathbb{Z},\{t\}}(\mathrm{d}\, y) &= \Phi'_{\mathbb{Z},\{t\}}\left(\sum_{0\leq n} c_n \cdot \mathrm{d}\big(\overline{t}^n\big) - \partial(\mathrm{d}\, z)\right) = \sum_{0\leq n} c_n \cdot \Phi'_{\mathbb{Z},\{t\}}\Big(\mathrm{d}\big(\overline{t}^n\big)\Big) \\
&= \sum_{0\leq n} c_n \cdot \nu \cdot \mathrm{d}\Big(\Phi'_{\mathbb{Z},\{t\}}\big(\overline{t}^n\big)\Big) = \nu \cdot \mathrm{d}\left(\Phi'_{\mathbb{Z},\{t\}}\left(\sum_{0\leq n} c_n \cdot \overline{t}^n\right)\right) \\
&= \nu \cdot \mathrm{d}\left(\Phi'_{\mathbb{Z},\{t\}}\left(\sum_{0\leq n} c_n \cdot \overline{t}^n + \partial z\right)\right) = \nu \cdot \mathrm{d}\Big(\Phi'_{\mathbb{Z},\{t\}}(y)\Big)
\end{aligned}$$

It thus suffices to show that there exists an element ν of $\{+1,-1\}$ such that (7.11) holds for elements $y = \overline{t}^n$ for $n \geq 0$.

We now need some input on properties that d must satisfy on the homology of $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$. For this equip $\{t\}$ with the unique total order and let Ψ be as in Definition 7.4.3.4. Then Ψ being a quasiisomorphism as well as compatible with d, and $\Omega^{\bullet}_{\mathbb{Z}[t]/\mathbb{Z}}$ having zero boundary operator, implies that there is a commutative diagram

$$\begin{tikzcd}
\Omega^1_{\mathbb{Z}[t]/\mathbb{Z}} \arrow[r, dash, "\cong"] & \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \\
\Omega^0_{\mathbb{Z}[t]/\mathbb{Z}} \arrow[u, "\mathrm{d}"] \arrow[r, dash, "\cong"'] & \mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \arrow[u, "\mathrm{d}"']
\end{tikzcd}$$

of abelian groups where the two horizontal morphisms are isomorphisms[41]. A $\mathbb{Z}$-basis of $\Omega^0_{\mathbb{Z}[t]/\mathbb{Z}}$ is given by t^n for $n \geq 0$, and a $\mathbb{Z}$-basis of $\Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$ is given by $t^n \cdot \mathrm{d}\,t$ for $n \geq 0$. Combining this with $\mathrm{d}(t^n) = n{\cdot}t^{n-1}{\cdot}\mathrm{d}\,t$ for $n \geq 0$ one obtains the following two properties for d on $\Omega^\bullet_{\mathbb{Z}[t]/\mathbb{Z}}$.

(1) The morphism

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{d} \colon \mathbb{Q} \otimes_{\mathbb{Z}} \Omega^0_{\mathbb{Z}[t]/\mathbb{Z}} \to \mathbb{Q} \otimes_{\mathbb{Z}} \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$$

is surjective.

(2) The morphism

$$\mathrm{d} \colon \Omega^0_{\mathbb{Z}[t]/\mathbb{Z}} \to \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$$

is only divisible by units, i. e. if $\mathrm{d} = c{\cdot}\mathrm{d}'$ for another morphism $\mathrm{d}' \colon \Omega^0_{\mathbb{Z}[t]/\mathbb{Z}} \to \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$ and element c in $\mathbb{Z}$, then c must be a unit (so either $+1$ or -1).

Using the above commutative square we can conclude that the analogous properties hold for the homology $\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})$.

(1) The morphism

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{d} \colon \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big)$$

is surjective.

(2) The morphism

$$\mathrm{d} \colon \mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \to \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big)$$

is only divisible by units, i. e. if $\mathrm{d} = c \cdot \mathrm{d}'$ for another morphism

$$\mathrm{d}' \colon \mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \to \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big)$$

and element c in $\mathbb{Z}$, then c must be a unit (so either $+1$ or -1).

We now use property (1) to show that $\Phi'_{\mathbb{Z},\{t\}}(\mathrm{d}\,\bar{t}) = \nu \cdot \mathrm{d}\,t$ for a nonzero element ν in $\mathbb{Z}$. For this let a_m for $0 \leq m \leq s$ be elements of $\mathbb{Z}$ such that

$$\Phi'_{\mathbb{Z},\{t\}}(\mathrm{d}\,\bar{t}) = \sum_{0 \leq m \leq s} a_m \cdot t^m \cdot \mathrm{d}\,t$$

holds in $\Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$. We already noted that the elements $[\bar{t}^n]$ for $n \geq 0$ form a $\mathbb{Z}$-basis for $\mathrm{H}_0(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\}))$. Combining this with (1) we obtain that the elements $[\bar{t}^n \cdot \mathrm{d}\,\bar{t}]$ for $n \geq 0$ form a $\mathbb{Q}$-generating set for $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{H}_1(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\}))$. As $\Phi'_{\mathbb{Z},\{t\}}$ is a multiplicative quasiisomorphism it follows that the elements

$$\Phi'_{\mathbb{Z},\{t\}}\big(\bar{t}^n \cdot \mathrm{d}\,\bar{t}\big) = t^n \cdot \left(\sum_{0 \leq m \leq s} a_m \cdot t^m \cdot \mathrm{d}\,t \right) = \sum_{0 \leq m \leq s} a_m \cdot t^{n+m} \cdot \mathrm{d}\,t$$

[41] Induced by Ψ, but we do not actually care beyond them being isomorphisms.

for $n \geq 0$ form a $\mathbb{Q}$-linear generating set for $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$. In particular, there must exist elements b_n of $\mathbb{Q}$ for $0 \leq n \leq u$, such that

$$\mathrm{d}\, t = \sum_{0 \leq n \leq u} b_n \cdot \left(\sum_{0 \leq m \leq s} a_m \cdot t^{n+m} \cdot \mathrm{d}\, t \right)$$

holds in $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$. Note that if all a_m are zero or all b_n are zero, then the right hand side vanishes, which contradicts the equality, so we can without loss of generality assume that $0 \leq u$ and $0 \leq s$ are such that $b_u \neq 0$ and $a_s \neq 0$. But then rewriting the right hand side in terms of the $\mathbb{Q}$-basis $t^l \cdot \mathrm{d}\, t$ for $l \geq 0$ of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$ we will have a nonzero coefficient $b_u \cdot a_s$ for the summand associated to $t^{u+s} \cdot \mathrm{d}\, t$. This can only happen if $u + s = 0$, so in particular $s = 0$ so that we must have

$$\Phi'_{\mathbb{Z},\{t\}}(\mathrm{d}\, \bar{t}) = a_0 \cdot \mathrm{d}\, t$$

in $\Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}$ for a_0 a nonzero element of $\mathbb{Z}$.

Set $\nu = a_0$. Then we obtain the following calculation for $n \geq 0$.

$$\begin{aligned} \Phi'_{\mathbb{Z},\{t\}}\Big(\mathrm{d}\big(\bar{t}^n\big)\Big) &= \Phi'_{\mathbb{Z},\{t\}}\Big(n \cdot \bar{t}^{n-1} \cdot \mathrm{d}\, \bar{t}\Big) = n \cdot t^{n-1} \cdot (\nu \cdot \mathrm{d}\, t) \\ &= \nu \cdot \big(n \cdot t^{n-1} \cdot \mathrm{d}\, t\big) = \nu \cdot \mathrm{d}(t^n) = \nu \cdot \mathrm{d}\Big(\Phi'_{\mathbb{Z},\{t\}}\big(\bar{t}^n\big)\Big) \end{aligned}$$

We have thus shown that (7.11) holds for this choice of ν for the elements $y = \bar{t}^n$ for $n \geq 0$, but we still have to show that ν is an element of $\{+1, -1\}$. But note that as $[\bar{t}^n]$ for $n \geq 0$ is a $\mathbb{Z}$-basis for $\mathrm{H}_0(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\}))$, the calculation we just made implies that the composition

$$\mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \xrightarrow{\mathrm{d}} \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \xrightarrow{\mathrm{H}_1\left(\Phi'_{\mathbb{Z},\{t\}}\right)} \mathrm{H}_1\Big(\Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}\Big)$$

is ν times the composition $\mathrm{d} \circ \mathrm{H}_0(\Phi'_{\mathbb{Z},\{t\}})$, so the above composition is divisible by ν. As $\mathrm{H}_1(\Phi'_{\mathbb{Z},\{t\}})$ is an isomorphism this implies that also the morphism

$$\mathrm{d}\colon\ \mathrm{H}_0\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big) \to \mathrm{H}_1\Big(\widetilde{\mathrm{C}}_{\mathbb{Z}}(\{t\})\Big)$$

is divisible by ν. Finally, (2) implies that ν must then be either $+1$ or -1. $\square$

Definition 7.4.4.2. Let X be a set. We define a quasiisomorphism

$$\Phi_{k,X}\colon\ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \to \Omega^{\bullet}_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$ by

$$y \mapsto \nu^{\deg_{\mathsf{Ch}}(y)} \cdot \Phi'_{k,X}(y)$$

where $\Phi'_{k,X}$ is as in Construction 7.4.3.1 and ν as in Proposition 7.4.4.1. If k is clear from context we will also denote $\Phi_{k,X}$ by Φ_x. $\diamondsuit$

Proposition 7.4.4.3. *The morphism*

$$\Phi_{k,\{t\}}\colon \operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k\big(\{t\}\big)\Big) \to \Omega^\bullet_{k[t]/k}$$

from Definition 7.4.4.2 is compatible with d *and can thus be lifted to a morphism in* Alg(Mixed). ♡

Proof. Follows directly from the definition in combination with Proposition 7.4.4.1. □

7.4.5. A free resolution for de Rham forms

In this section we construct a cofibrant replacement of $\Omega^\bullet_{k[X]/k}$ in Alg(Mixed) for totally ordered sets X with $|X| \leq 2$, and prove some properties it satisfies. We know abstractly that a cofibrant replacement exists, but it will be crucial for applications that we have good control over the low degrees of the the cofibrant replacement that we use.

We will begin in Section 7.4.5.1 by giving a construction of a cofibrant replacement[42] that depends on the choice of certain sets $Y_0, Y_1, \ldots$. For our application we will need to make a specific choice for Y_0, Y_1, and Y_2, and we will describe those choices and show that they have the necessary properties in Section 7.4.5.2. Finally Section 7.4.5.3 will be concerned with proving that the object constructed in Section 7.4.5.1 actually is a cofibrant replacement of $\Omega^\bullet_{k[X]/k}$.

7.4.5.1. The general construction

In this section we give a general construction of a morphism $\Omega'^\bullet_{k[X]/k} \to \Omega^\bullet_{k[X]/k}$ in Alg(Mixed) that depends on the choice of certain sets $Y_0, Y_1, \ldots$.

Construction 7.4.5.1. Let X be a set. We will construct a commutative diagram

$$\begin{tikzcd}
A_0 \arrow[r, "\iota_0^1"] \arrow[drr, "\Theta_0"'] & A_1 \arrow[r, "\iota_1^2"] \arrow[dr, "\Theta_1"] & A_2 \arrow[r] \arrow[d, "\Theta_2"] & \ldots \arrow[r] & \Omega'^\bullet_{k[X]/k} \arrow[dll, "\Theta"] \\
& & \Omega^\bullet_{k[X]/k} & &
\end{tikzcd}$$

in Alg(Mixed), where the first line is a $\mathbb{Z}_{\geq 0}$-diagram and its colimit $\Omega'^\bullet_{k[X]/k}$. Beyond the notation indicated in the diagram, we will denote the morphism from A_n to $\Omega^\bullet_{k[X]/k}$ by ι_n, and the morphism from A_n to A_m for $m \geq n$ by ι_n^m. The objects A_n are going to be built up using free associative algebras in strict mixed complexes that are generated by strict mixed complexes that are themselves free, so to simplify notation we will use Notation 7.4.2.1. All morphisms ι_n^m are going to be levelwise injective, so if y is an element of A_n, we will also just write y for the element $\iota_n^m(y)$ of A_m.

[42] We will only construct the object and morphism to $\Omega^\bullet_{k[X]/k}$, but will not yet show that it indeed is a cofibrant replacement.

We begin by defining

$$A_0 \coloneqq \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(k \cdot X)$$

where by $k \cdot X$ we mean the chain complex that is free as a graded k-module on the set X, where we give every element of X chain degree 0.

Using the universal property of $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ and $k \cdot X$, we can now define Θ_0 as the unique morphism in $\mathrm{Alg}(\mathsf{Mixed})$ that maps an element x of X, considered as a basis element of $k \cdot X$, to the element x, considered as an element of $k[X]$ and thereby of $\Omega^0_{k[X]/k}$.

We next describe how to construct A_{n+1} from A_n, for $n \geq 0$. This will depend on the choice of a subset Y_n of $(A_n)_n$, i. e. elements of degree n in A_n. We note that we will later show that we can make some particular choices for some of these sets. The set Y_n has to satisfy the following conditions for every $n \geq 0$.

(a) Every element y of Y_n is a cycle in A_n.

(b) Every element y of Y_n is mapped to 0 by Θ_n.

(c) Let I be the graded ideal[43] in the graded k-algebra $\mathrm{H}_*(A_n)$ that is generated by the homology classes represented by elements of $Y_n \cup \{\, \mathrm{d}\, y \mid y \in Y_n \,\}$. Then we must have $I_n = \mathrm{Ker}(\mathrm{H}_n(\Theta_n))$[44].

Note that it is always possible to find a set Y_n satisfying all three requirements above, by starting with a generating set of $\mathrm{Ker}(\mathrm{H}_n(\Theta_n))$[45], and then for each of those homology classes choosing a cycle representing it. Note that as the boundary operator of $\Omega^\bullet_{k[X]/k}$ is zero, a cycle representing a homology class in the kernel of $\mathrm{H}_*(\Theta)$ must already be mapped to 0 by Θ, so (b) is then satisfied, and (a) and (c) hold by construction.

The idea behind the above requirements is that we want to divide out $\mathrm{Ker}(\mathrm{H}_n(\Theta_n))$ from A_n, but want to do so in an efficient fashion that does not create excessive new elements in homology. In particular, the assumption that the elements of Y_n all have degree n is needed to ensure that the connectivity of Θ_n increases with n.

Now let B'_n be the chain complex $B'_n \coloneqq k \cdot Y_n$, where we give elements of Y_n the the same chain degree as in A_n. If y is an element of Y_n, then we will denote the corresponding basis element of B'_n by y as well. Let $\underline{B'_n}$ the chain complex whose underlying graded k-module is given by $(k \cdot Y_n) \oplus (k \cdot Y_n)[1]$, where if y is an element of Y_n we will denote the corresponding basis element from the first summand by y again and the corresponding shifted[46] basis element of the second summand by $\underline{y}$, and where the boundary operator is determined by $\partial(\underline{y}) = y$. There is an evident morphism of chain complexes $j_n \colon B'_n \to \underline{B'_n}$ that maps y to y.

[43] That is, a subset that is closed under k-linear combinations as well as multiplication with any element of $\mathrm{H}_*(A_n)$ on either side.
[44] Note that (b) already implies that $I \subseteq \mathrm{Ker}(\mathrm{H}_*(\Theta_n))$.
[45] For example the very inefficient choice of *all* elements of $\mathrm{Ker}(\mathrm{H}_n(\Theta_n))$ works.
[46] One degree higher, see Definition 4.1.1.2.

We can now define A_{n+1} and ι_n^{n+1} as in the following pushout diagram in $\mathrm{Alg}(\mathsf{Mixed})$

$$\begin{tikzcd} B_n \coloneqq \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(B'_n) \arrow[rr, "\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(j_n)"] \arrow[d, "i_n"'] & & \underline{B}_n \coloneqq \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(\underline{B}'_n) \arrow[d, "\underline{i}_n"] \\ A_n \arrow[rr, "\iota_n^{n+1}"'] & & A_{n+1} \end{tikzcd} \tag{7.12}$$

where i_n is the morphism in $\mathrm{Alg}(\mathsf{Mixed})$ that extends the morphism of chain complexes $B'_n \to A_n$ given by mapping y considered as an element of B'_n to y considered as an element of A_n, for every element y of Y_n. The latter is a morphism of chain complexes due to (a).

We can define a morphism $\underline{\Theta}_n \colon \underline{B}_n \to \Omega^\bullet_{k[X]/k}$ in $\mathrm{Alg}(\mathsf{Mixed})$ as the one adjoint to the morphism of chain complexes $0 \colon \underline{B}'_n \to \Omega^\bullet_{k[X]/k}$ that maps y and $\underline{y}$ to 0 for every y in Y_n. If y is an element of Y_n, then by (b), $\Theta_n(i_n(y)) = 0$, so that $\Theta_n \circ i_n = \underline{\Theta}_n \circ \mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}(j_n)$, and hence, by the universal property of the pushout diagram in $\mathrm{Alg}(\mathsf{Mixed})$ above, we obtain a morphism $\Theta_{n+1} \colon A_{n+1} \to \Omega^\bullet_{k[X]/k}$ such that $\Theta_{n+1} \circ \iota_n^{n+1} = \Theta_n$ and $\Theta_{n+1} \circ \underline{i}_n = \underline{\Theta}_n$.

Finally, $\Omega'^\bullet_{k[X]/k}$ is defined as the colimit of the $\mathbb{Z}_{\geq 0}$-diagram

$$A_0 \xrightarrow{\iota_0^1} A_1 \xrightarrow{\iota_1^2} A_2 \xrightarrow{\iota_2^3} \dots$$

in $\mathrm{Alg}(\mathsf{Mixed})$, and $\Theta \colon \Omega'^\bullet_{k[X]/k} \to \Omega^\bullet_{k[X]/k}$ is defined using the universal property of the colimit such that $\Theta \circ \iota_n = \Theta_n$ for every $n \geq 0$. ♢

Remark 7.4.5.2. This remark concerns the situation of Construction 7.4.5.1. Let $n \geq 0$ be an integer. From Remark 7.4.2.2 it follows that the underlying graded k-algebra of A_{n+1} is given by the coproduct (in graded k-algebras) of A_n and the free graded k-algebra on elements $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_n$.

Inductively we can conclude that the underlying graded k-algebra of A_n is free on the elements x and $\mathrm{d}\,x$ for $x \in X$, and $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_m$ with $m < n$. As the forgetful functor from $\mathrm{Alg}(\mathsf{Mixed})$ to $\mathrm{Alg}\,\mathsf{Ch}(k)$ preserves filtered colimits by Proposition 4.2.2.12 we can also conclude that the colimit $\Omega'^\bullet_{k[X]/k}$ has an underlying graded k-algebra that is free on the elements x and $\mathrm{d}\,x$ for $x \in X$ and $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_m$ for $m \geq 0$.

Note that elements y of Y_m being of degree m implies that $\underline{y}$ is then of degree $m+1$, which is always positive. The only multiplicative generators of degree 0 are thus those of the form x for $x \in X$, and A_m is concentrated in nonnegative degrees for every $m \geq 0$. The above also implies that the morphisms $\iota_n^{n'}$ are isomorphisms in degrees smaller to or equal to n. ♢

7.4.5.2. Specific choices for Y_0, Y_1, and Y_2

In this section we discuss specific choices that we make for Y_0, Y_1, and Y_2 in Construction 7.4.5.1. We begin with a general remark explaining the maneuvers that we will make in all the proofs.

Remark 7.4.5.3. This remark concerns the situation of Construction 7.4.5.1, and we will use notation from there. In the proofs of Propositions 7.4.5.6, 7.4.5.7 and 7.4.5.8 we will for some $n \geq 0$ have defined sets $Y_0, \ldots, Y_{n-1}$ as in Construction 7.4.5.1 and shown that they satisfy (a), (b) and (c), and defined a set Y_n of elements of degree n in A_n for which we already showed that (a) and (b) holds, but we still have to show that (c) holds, i. e. that $I_n = \mathrm{Ker}(\mathrm{H}_n(\Theta_n))$, for I the graded ideal in $\mathrm{H}_*(A_n)$ that is generated by the homology classes represented by elements of $Y_n \cup \{ \mathrm{d}\, y \mid y \in Y_n \}$. In this remark we explain the general approach to proving this, in order to avoid repetition. Before we continue let us define J as the graded ideal in the graded k-algebra of cycles of A_n[47] that is generated by the elements y and $\mathrm{d}\, y$ for $y \in Y_n$[48].

Property (b) implies that $I_n \subseteq \mathrm{Ker}(\mathrm{H}_n(\Theta_n))$, so to show equality it only remains to show that every element in $\mathrm{Ker}(\mathrm{H}_n(\Theta_n))$ lies in I_n. Note that the set of homology classes represented by elements of J is exactly I. As $\Omega^{\bullet}_{k[X]/k}$ has zero boundary operator it also follows that a cycle in A_n represents a homology class in $\mathrm{Ker}(\mathrm{H}_n(\Theta_n))$ if and only if Θ_n maps it to 0. These two facts together imply that it suffices to show that every cycle in A_n of degree n that lies in the kernel of Θ_n is given as a sum of an element in J and a boundary.

The strategy we will employ to prove this will be by reducing step by step to the case of such cycles lying in increasingly restrictive submodules, by eliminating basis elements, as we now make more precise.

By Remark 7.4.5.2 the underlying $\mathbb{Z}$-graded k-algebra of A_n is free on the generators x and $\mathrm{d}\, x$ for $x \in X$, and $\underline{y}$ and $\mathrm{d}\, \underline{y}$ for $y \in Y_{n'}$ with $n' < n$. Let $\mathcal{G}$ be the set of generators just described, as a $\mathbb{Z}$-graded subset of A_n, and $\mathcal{B}$ the set of all words of degree n in $\mathcal{G}$. Then $\mathcal{B}$ is a k-basis of the underlying $\mathbb{Z}$-graded k-module of A_n. We will use a sequence of subsets

$$\mathcal{B} = \mathcal{B}_0 \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \cdots \supset \mathcal{B}_l$$

up to some subset $\mathcal{B}_l$ of $\mathcal{B}$ for $l > 0$ an integer. Suppose that we can show that one of the following two holds for every $0 \leq i < l$.

(I) For every element w of $\mathcal{B}_i \setminus \mathcal{B}_{i+1}$ there is a boundary in A_n or an element of J that, written in the basis $\mathcal{B}$, only has non-zero coefficients corresponding to the basis elements in $\mathcal{B}_{i+1}$, except for the basis element w, for which the coefficient is a unit in k. This implies that every element z of A_n of degree n that lies in the k-submodule generated by $\mathcal{B}_i$ is a sum of an element of J, a boundary, and an element z' in the k-submodule generated by $\mathcal{B}_{i+1}$. Note that every element of J and every boundary is a cycle, so z is a cycle if and only if z' is. Furthermore every element of J and every boundary is in the kernel of Θ_n, so z is in the kernel of Θ_n if and only if z' is.

(II) Every cycle z in A_n of degree n that satisfies $\Theta_n(z) = 0$ and that lies in the k-submodule generated by $\mathcal{B}_i$ already lies in the k-submodule generated by $\mathcal{B}_{i+1}$.

[47] Note that the Leibniz rule for ∂ implies that 1 is a cycle and that products of cycles are again cycles, so cycles form a sub-k-algebra of A_n.

[48] By (a) these elements are cycles.

In both cases this implies that if we can show that every cycle in A_n of degree n that lies in the kernel of Θ_n and also lies in the k-submodule generated by $\mathcal{B}_{i+1}$ is a sum of an element of J and a boundary, then the same statement follows for such cycles that lie in the k-submodule generated by $\mathcal{B}_i$. Inductively it then thus suffices to show that cycles in A_n that lie in the k-submodule generated by $\mathcal{B}_l$ and lie in the kernel of Θ_n are a sum of an element in J and a boundary. Usually $\mathcal{B}_l$ will be of such a form that we can already show that such a cycle must be zero, and we will explain how we usually show this further below.

In the propositions below we will not usually define $\mathcal{B}_i$ explicitly. Instead we will step by step describe the difference $\mathcal{B}_i \setminus \mathcal{B}_{i+1}$ and explain how to eliminate those basis elements using an element of J or boundary in A_n in the manner described above.

We make one remark about the elements of $\mathcal{B}$. It follows from Remark 7.4.5.2 $\mathcal{G}$ consists only of elements of nonnegative degree, with the only elements of degree 0 being the elements of X. A concrete implication of this that we will often use is that the number of factors that are not in X that can occur in a word in $\mathcal{G}$ of specified degree is bounded. For example words in $\mathcal{G}$ of degree 1 need to consists of precisely one factor of the form $\mathrm{d}\,x$ or $\underline{y}$ with y an element of Y_0, with the other factors all from X.

We will call products of elements of X, considered as elements of A_n, *words in* X. If we are given a total order on the set X then we say that a word in X is *ordered* if it is of the form $x_1^{i_1} \cdots x_a^{i_a}$ with $a \geq 0$ an integer, $i_1, \ldots, i_a \geq 1$ integers, and $x_1 < x_2 < \cdots < x_a$ elements of X. Similarly we will call products of elements of the form x and $\mathrm{d}\,x$ for $x \in X$ *words in* X *and* $\mathrm{d}\,X$, and call such a word *ordered* if it is of the form $x_1^{i_1} \cdots x_a^{i_a} \cdot \mathrm{d}\,x'_1 \cdots \mathrm{d}\,x'_b$ with $a, b \geq 0$ an integers, $i_1, \ldots, i_a \geq 1$ integers, and $x_1 < x_2 < \cdots < x_a$ and $x'_1 < \cdots < x'_b$ elements of X. We let $\mathcal{B}_X$ be the set of words in X and $\mathcal{B}_X^{\mathrm{ord}}$ the set of ordered words in X. Analogously, we let $\mathcal{B}_{X,\mathrm{d}\,X}$ be the ($\mathbb{Z}$-graded) set of words in X and $\mathrm{d}\,X$, and $\mathcal{B}_{X,\mathrm{d}\,X}^{\mathrm{ord}}$ the ($\mathbb{Z}$-graded) set of ordered words in X and $\mathrm{d}\,X$. We will often refer to the number of factors in a word w as its *length*, and denote it by $\mathrm{len}(w)$.

Now suppose that $\mathcal{B}_l$ is a subset of $\mathcal{B}_{X,\mathrm{d}\,X}^{\mathrm{ord}}$. Then the restriction of Θ_n to the sub-k-module with basis $\mathcal{B}_l$ is injective, so any element in the kernel of that restriction is already 0. The upshot is that if we can find

$$\mathcal{B} = \mathcal{B}_0 \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \cdots \supset \mathcal{B}_l$$

such that (I) or (II) holds for every $0 \leq i < l$ and such that $\mathcal{B}_l$ is a subset of $\mathcal{B}_{X,\mathrm{d}\,X}^{\mathrm{ord}}$, then this will complete the proof that $I_n = \mathrm{Ker}(\mathrm{H}_n(\Theta_n))$. $\diamondsuit$

Remark 7.4.5.4. Let X be a totally ordered set that is either $X = \emptyset$, $X = \{x_1\}$, or $X = \{x_1, x_2\}$ with $x_1 < x_2$. For reference we provide here a table with the multiplicative generators of A_0, A_1, A_2, A_3 with Y_0, Y_1, Y_2 as defined in Propositions 7.4.5.6, 7.4.5.7 and 7.4.5.8 below. The generators are given as for the case $X = \{x_1, x_2\}$, and to read off the case $X = \{x_1\}$ (the case $X = \emptyset$) one leaves out any element that involves x_2 (that involves x_1 or x_2) The first column contains the chain degree of the elements, the second lists their names, and the third column contains the first of A_0, A_1, A_2, A_3 that contains the element.

Deg.	Elements	In
0	x_1	A_0
0	x_2	A_0
1	$\mathrm{d}\,x_1$	A_0
1	$\mathrm{d}\,x_2$	A_0
1	$\underline{x_1x_2 - x_2x_1}$	A_1
2	$\mathrm{d}\,\underline{x_1x_2 - x_2x_1}$	A_1
2	$\underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}$	A_2
2	$\underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}$	A_2
2	$\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}$	A_2
2	$\underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}$	A_2
3	$\mathrm{d}\,\underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}$	A_2
3	$\mathrm{d}\,\underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}$	A_2
3	$\mathrm{d}\,\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}$	A_2
3	$\mathrm{d}\,\underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}$	A_2
3	$\underline{\mathrm{d}\,x_1 \cdot \mathrm{d}\,x_1}$	A_3
3	$\underline{\mathrm{d}\,x_2 \cdot \mathrm{d}\,x_2}$	A_3
3	$\underline{\mathrm{d}\,\underline{\underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}}}$	A_3
3	$\underline{\mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_2 \ldots}$	A_3
	$\underline{\ldots - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \cdot x_1 \ldots}$	
	$\underline{\ldots + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} \cdot x_2}$	
3	$\underline{\mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_1 \ldots}$	A_3
	$\underline{\ldots - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} \cdot x_1 \ldots}$	
	$\underline{\ldots + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} \cdot x_2}$	
4	$\mathrm{d}\,\underline{\mathrm{d}\,x_1 \cdot \mathrm{d}\,x_1}$	A_3
4	$\mathrm{d}\,\underline{\mathrm{d}\,x_2 \cdot \mathrm{d}\,x_2}$	A_3
4	$\mathrm{d}\,\underline{\mathrm{d}\,\underline{\underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}}}$	A_3
4	$\mathrm{d}\,\underline{\mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_2 \ldots}$	A_3
	$\underline{\ldots - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \cdot x_1 \ldots}$	
	$\underline{\ldots + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} \cdot x_2}$	
4	$\mathrm{d}\,\underline{\mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_1 \ldots}$	A_3
	$\underline{\ldots - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} \cdot x_1 \ldots}$	
	$\underline{\ldots + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} \cdot x_2}$	

This table is intended to be used to determine what the k-basis for A_n in a specific degree is. ♢

Before we actually define Y_0, Y_1, and Y_2, we first show a helper statement.

Proposition 7.4.5.5. *This proposition concerns Construction 7.4.5.1, and we use some notation from Remark 7.4.5.3.*

Let $X = \{x_1, x_2\}$. Then the elements

$$x_1^{a_1} x_2^{a_2} \cdot (x_1x_2 - x_2x_1) \cdot w$$

in A_0, with $a_1, a_2 \geq 0$ and $w \in \mathcal{B}_X$, are all pairwise distinct, and the set of all such elements is k-linearly independent. ♡

Proof. Suppose that $a_1, a_2, a_1', a_2' \geq 0$ and $w, w' \in \mathcal{B}_X$ such that

$$x_1^{a_1} x_2^{a_2} \cdot (x_1 x_2 - x_2 x_1) \cdot w = x_1^{a_1'} x_2^{a_2'} \cdot (x_1 x_2 - x_2 x_1) \cdot w'$$

Then as $\mathcal{B}_X$ is k-linearly independent and the left hand side has two summands in the basis $\mathcal{B}_X$ that both begin with $x_1^{a_1} x_2^{a_2}$, but where the next factor differs, the same must be true for the two summands of the right hands side, and vice versa. This implies $a_1' = a_1$ and $a_2' = a_2$, which in turn implies that $w' = w$.

Now suppose that

$$0 = \sum_{\substack{a_1, a_2 \geq 0, \\ w \in \mathcal{B}_X}} b_{a_1, a_2, w} \cdot x_1^{a_1} x_2^{a_2} \cdot (x_1 x_2 - x_2 x_1) \cdot w$$

with $b_{a_1,a_2,w}$ elements of k, all but finitely many zero. We have to show that all coefficients $b_{a_1,a_2,w}$ are already zero. If this is already the case, then we are done. So assume that there is a coefficient $b_{a_1,a_2,w}$ that is nonzero. Then let $\widetilde{a}_1 \geq 0$ and $\widetilde{a}_2 \geq 0$ and $\widetilde{w} \in \mathcal{B}_X$ be such that $b_{\widetilde{a}_1, \widetilde{a}_2, \widetilde{w}} \neq 0$ while first minimizing $\widetilde{a}_1$ and then (for that already fixed $\widetilde{a}_1$) maximizing $\widetilde{a}_2$.

Then it suffices to show that

$$x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot (x_1 x_2 - x_2 x_1) \cdot \widetilde{w}$$

is k-linearly independent of the k-submodule spanned by elements

$$x_1^{a_1} x_2^{a_2} \cdot (x_1 x_2 - x_2 x_1) \cdot w$$

for $a_1, a_2 \geq 0$ and $w \in \mathcal{B}_X$ such that $(a_1, a_2, w) \neq (\widetilde{a}_1, \widetilde{a}_2, \widetilde{w})$ and $a_1 \geq \widetilde{a}_1$, and $a_2 \leq \widetilde{a}_2$ if $a_1 = \widetilde{a}_1$.

So assume that

$$\begin{aligned} & c \cdot \left(x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot x_1 x_2 \cdot \widetilde{w} - x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot x_2 x_1 \cdot \widetilde{w} \right) \\ = & \sum_{\substack{a_1, a_2 \geq 0, \\ w \in \mathcal{B}_X, \\ a_1 \geq \widetilde{a}_1, \\ a_2 \leq \widetilde{a}_2 \text{ if } a_1 = \widetilde{a}_1, \\ (a_1, a_2, w) \neq (\widetilde{a}_1, \widetilde{a}_2, \widetilde{w})}} c_{a_1, a_2, w} \cdot \left(x_1^{a_1} x_2^{a_2} \cdot x_1 x_2 \cdot w - x_1^{a_1} x_2^{a_2} \cdot x_2 x_1 \cdot w \right) \end{aligned} \tag{$*$}$$

for c a nonzero element of k and $c_{a_1,a_2,w}$ elements of k, only finitely many of which are nonzero. We consider for which (a_1, a_2, w) as in the indexing set we can have that one of the following two equations holds.

$$x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot x_2 x_1 \cdot \widetilde{w} = x_1^{a_1} x_2^{a_2} \cdot x_1 x_2 \cdot w \qquad \text{or} \qquad x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot x_2 x_1 \cdot \widetilde{w} = x_1^{a_1} x_2^{a_2} \cdot x_2 x_1 \cdot w$$

We first note that the on left hand side of the equations the first $\widetilde{a}_1$ factors of x_1 are always followed by at least one factor of x_2. Thus it is not possible to have $a_1 > \widetilde{a}_1$. As by assumption $a_1 \geq \widetilde{a}_1$ we can thus conclude that $a_1 = \widetilde{a}_1$. Thus we must have $a_2 \leq \widetilde{a}_2$. The factor number $a_1 + a_2 + 1$ or $a_1 + a_2 + 2$ on the right hand side of the two equations is x_1. As factors $a_1 + 1$ up to $a_1 + \widetilde{a}_2 + 1$ on the left hand side are x_2 we must thus have $a_2 \geq \widetilde{a}_2$. As factor number $a_1 + \widetilde{a}_2 + 2$ on the left hand side is x_1 on the other hand we must have $a_2 \leq \widetilde{a}_2 + 1$. We are thus left with the two options $a_2 = \widetilde{a}_2$ and $a_2 = \widetilde{a}_2 + 1$. The former would imply that $w = \widetilde{w}$, which contradicts the assumption $(a_1, a_2, w) \neq (\widetilde{a}_1, \widetilde{a}_2, \widetilde{w})$. The latter contradicts the assumptions that $a_2 \leq \widetilde{a}_2$ if $a_1 = \widetilde{a}_1$. This shows that if we write both sides of equation $(*)$ in the basis $\mathcal{B}_X$, then the left hand side has a nonzero coefficient for the basis element $x_1^{\widetilde{a}_1} x_2^{\widetilde{a}_2} \cdot x_2 x_1 \cdot \widetilde{w}$ while the right hand side always has coefficient zero. This contradicts equation $(*)$, which implies all coefficients $b_{a_1,a_2,w}$ must have been zero, thereby showing the k-linear independence claim in the statement. $\square$

Proposition 7.4.5.6. *Let X be a totally ordered set. Then the subset Y_0 of $(A_0)_0$ in Construction 7.4.5.1 can be chosen as follows.*

$$Y_0 \coloneqq \left\{ x \cdot x' - x' \cdot x \;\middle|\; x, x' \in X \text{ such that } x < x' \right\} \qquad \heartsuit$$

Proof. *Condition (a)*: That the elements are cycles is clear as A_0 has zero boundary operator.

Condition (b): holds as $\Omega^\bullet_{k[X]/k}$ is commutative.

Condition (c): We are going to use the strategy explained in Remark 7.4.5.3 and also use notation from there. Elements of $\mathcal{B}$ are precisely words in X, and we can use elements of J to iteratively reorder the factors until we are left only with ordered words in X. $\square$

Proposition 7.4.5.7. *Let X be a subset of the totally ordered set $\{x_1 < x_2\}$. This proposition concerns Construction 7.4.5.1, and we let Y_0 be as in Proposition 7.4.5.6.*

Then the subset Y_1 of $(A_1)_1$ in Construction 7.4.5.1 can be chosen as follows.

$$Y_1 \coloneqq \left\{ x \cdot \mathrm{d}\, x' - \mathrm{d}\, x' \cdot x \;\middle|\; x, x' \in X \right\} \qquad \heartsuit$$

Proof. *Condition (a)*: All elements of Y_1 lie in A_0, which has zero boundary operator.

Condition (b): Holds as $\Omega^\bullet_{k[X]/k}$ is commutative.

Condition (c): We are going to use the strategy explained in Remark 7.4.5.3 and also use notation from there. The elements of $\mathcal{B}$ are words of one of the following two types, with the second only occurring if $|X| = 2$.

(1) A word in $\mathcal{G}$ with precisely one factor $\mathrm{d}\, x$ with $x \in X$ and the remaining factors in X.

(2) A word in $\mathcal{G}$ with precisely one factor $\underline{x_1 x_2 - x_2 x_1}$ and the remaining factors in X.

We first consider elements of type (1). We can first use elements of J to move the factor $\mathrm{d}\, x$ to the very end of the product, so that we are left with elements of the form

$w \cdot \mathrm{d}\, x$ with w a word in X. If $|X| < 2$ then w will already be ordered, and if $|X| = 2$ we can then use the boundary of elements of A_1 of the form

$$w' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w'' \cdot \mathrm{d}\, x$$

for w' and w'' words in X to reorder w, so that we can ultimately eliminate all basis elements of type (1) except those of the form $w \cdot \mathrm{d}\, x$ with w an ordered word in X and x an element of X.

We are thus left with basis elements of the following two types, with the second only occurring if $|X| = 2$.

(1') An element of $\mathcal{B}^{\mathrm{ord}}_{X, \mathrm{d}\, X}$.

(2') A word in $\mathcal{G}$ with precisely one factor $\underline{x_1 x_2 - x_2 x_1}$ and the remaining factors in X.

If $|X| < 2$ we are thus done per Remark 7.4.5.3. So now assume that $|X| = 2$.

Then let $w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w'$ be an element of $\mathcal{B}$ of type (2'), with w and w' words in X. Assume that w is not ordered. It is then possible to order w in a finite number of steps by swapping neighboring (nonequal) factors, and there also is a minimum number of such steps required, which in this case must be positive as we assumed that w is not already ordered. Then we can write w as $w = v \cdot x_2 \cdot x_1 \cdot v'$ such that v and v' are words in X, and such that the minimum number of swappings to order $v \cdot x_1 \cdot x_2 \cdot v'$ is smaller than the minimum number of swappings to order w. Consider the following boundary.

$$\begin{aligned}
&\partial\big(v \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w'\big) \\
&= v \cdot (x_1 x_2 - x_2 x_1) \cdot v' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' - v \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v' \cdot (x_1 x_2 - x_2 x_1) \cdot w' \\
&= v \cdot x_1 \cdot x_2 \cdot v' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' - v \cdot x_2 \cdot x_1 \cdot v' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \\
&\quad - v \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v' \cdot x_1 \cdot x_2 \cdot w' + v \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v' \cdot x_2 \cdot x_1 \cdot w'
\end{aligned}$$

Up to sign the second summand is the element we started with, the first has a word of the same length before $\underline{x_1 x_2 - x_2 x_1}$, but of smaller minimum number of swappings to order it, and the last two summands have a word of smaller length before the first factor $\underline{x_1 x_2 - x_2 x_1}$. By induction we can thus eliminate those elements from (2') where the word in X appearing before the factor $\underline{x_1 x_2 - x_2 x_1}$ is not ordered.

We are thus left with basis elements of the following two types.

(1") An element of $\mathcal{B}^{\mathrm{ord}}_{X, \mathrm{d}\, X}$.

(2") A product $x_1^{a_1} \cdot x_2^{a_2} \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w$ where w is a word in X.

To finish the proof it remains to eliminate the basis elements from (2"). We do this using method (II) from Remark 7.4.5.3. So let z' be a cycle in A_1 that is a k-linear combination of elements of type (1") and (2"). We have to show that z' is then already a k-linear combination of elements of type (1"). For this we write $z' = z'' + z$ with z'' a k-linear combination of elements of type (1") and z a k-linear combination of elements of type (2"). As every element of type (1") is a cycle this implies that z is a cycle. It now suffices to show that $z = 0$.

We can write z as

$$z = \sum_{\substack{a_1,a_2\geq 0 \\ w\in\mathcal{B}_X}} b_{a_1,a_2,w} \cdot x_1^{a_1}x_2^{a_2} \cdot \underline{x_1x_2 - x_2x_1} \cdot w$$

with $b_{a_1,a_2,w}$ elements of k, all but finitely many zero. The boundary of z is then given as follows.

$$\partial(z) = \sum_{\substack{a_1,a_2\geq 0, \\ w\in\mathcal{B}_X}} b_{a_1,a_2,w} \cdot x_1^{a_1}x_2^{a_2} \cdot x_1x_2 \cdot w - b_{a_1,a_2,w} \cdot x_1^{a_1}x_2^{a_2} \cdot x_2x_1 \cdot w$$

Now Proposition 7.4.5.5 directly implies that all coefficients $b_{a_1,a_2,w}$ must be zero, so $z = 0$.

We are thus now left with only basis elements of type (1”), which finishes the proof as explained in Remark 7.4.5.3. □

Proposition 7.4.5.8. *Let X be a subset of the totally ordered set $\{x_1 < x_2\}$. This proposition concerns Construction 7.4.5.1, and we let Y_0 be as in Proposition 7.4.5.6 and Y_1 as in Proposition 7.4.5.7.*

Then the subset Y_2 of $(A_2)_2$ in Construction 7.4.5.1 can be chosen as follows. If $|X| = 0$ we can let $Y_2 = \emptyset$, if $|X| = 1$ we can let $Y_2 = \{\mathrm{d}\,x_1 \cdot \mathrm{d}\,x_1\}$, and if $|x| = 2$ we can define Y_2 as follows.

$$\begin{aligned}
Y_2 \coloneqq\ & \big\{\mathrm{d}\,x_1 \cdot \mathrm{d}\,x_1, \mathrm{d}\,x_2 \cdot \mathrm{d}\,x_2\big\} \\
& \cup \big\{\mathrm{d}\,\underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big\} \\
& \cup \big\{\mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_2 \\
& \quad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \cdot x_1 \\
& \quad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} \cdot x_2\big\} \\
& \cup \big\{\mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_1 \\
& \quad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} \cdot x_1 \\
& \quad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} \cdot x_2\big\}
\end{aligned}$$

♡

Proof. To keep the proof shorter as it would otherwise be we mostly will implicitly work as if we had $|X| = 2$; the proof for $|X| < 2$ can be obtained by jumping over every element or argument that involves an element of $\{x_1, x_2\} \setminus X$. To shorten notation we also make the following definitions for this proof.

$$\begin{aligned}
D &\coloneqq \mathrm{d}\,\underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} \\
C_2 &\coloneqq \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_2 \\
&\quad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \cdot x_1 \\
&\quad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} \cdot x_2 \\
C_3 &\coloneqq \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\,x_1 \\
&\quad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2} \cdot x_1 \\
&\quad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} \cdot x_2
\end{aligned}$$

Condition (a): That the elements of Y_2 are cycles can be checked by direct calculation. For this, keep in mind the signs introduced by the Leibniz rule and $\partial \circ \mathrm{d} = -\mathrm{d} \circ \partial$.

Condition (b): Θ_2 maps $\mathrm{d}\,x_1 \cdot \mathrm{d}\,x_1$ and $\mathrm{d}\,x_2 \cdot \mathrm{d}\,x_2$ (if they are defined, depending on what X is) to zero as $\mathrm{d}\,x_1$ and $\mathrm{d}\,x_2$ square to zero in $\Omega^\bullet_{k[X]/k}$. The elements D, C_2, and C_3 are mapped to zero because every summand has a factor of the form y or $\mathrm{d}\,y$, with y an element of Y_0 or Y_1, and those elements are already mapped to zero.

Condition (c): We are going to use the strategy explained in Remark 7.4.5.3 and also use notation from there. The elements of $\mathcal{B}$ are words of one of the following types, with types (3), (4) and (5) only occurring for $|X| = 2$.

(1) A word in $\mathcal{G}$ with precisely two factors $\mathrm{d}\,x$ and $\mathrm{d}\,x'$ with $x, x' \in X$ (the case $x = x'$ is allowed) and the remaining factors in X.

(2) A word in $\mathcal{G}$ with precisely one factor $\underline{x \cdot x' - x' \cdot x}$ with $x, x' \in X$, and the remaining factors in X.

(3) A word in $\mathcal{G}$ with precisely one factor $\underline{x_1 x_2 - x_2 x_1}$, precisely one factor $\mathrm{d}\,x$ for $x \in X$, and the remaining factors in X.

(4) A word in $\mathcal{G}$ with precisely two factors $\underline{x_1 x_2 - x_2 x_1}$, and the remaining factors in X.

(5) A word in $\mathcal{G}$ with precisely one factor $\mathrm{d}\,\underline{x_1 x_2 - x_2 x_1}$, and the remaining factors in X.

As a first step the basis elements of type (5) can be eliminated using elements of J that involve a factor of D, so that we are only left with types (1), (2), (3) and (4).

For elements of type (4) we use a similar procedure as we did for elements of type (2') in Proposition 7.4.5.7. So let $w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w''$ be an element of type (4), with w, w', and w'' elements of $\mathcal{B}_X$. Assume that w' is not ordered. Then we can write w' as $w' = v \cdot x_2 \cdot x_1 \cdot v'$ such that v and v' are elements of $\mathcal{B}_X$ and such that the minimum number of swappings to order $v \cdot x_1 \cdot x_2 \cdot v'$ is smaller than the minimum number of swappings to order w'. Consider the following boundary.

$$\begin{aligned}
&\partial\big(w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w''\big) \\
= &+ w \cdot x_1 x_2 \cdot v \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w'' \\
&- w \cdot x_2 x_1 \cdot v \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w'' \\
&- w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v \cdot x_1 x_2 \cdot v' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w'' \\
&+ w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v \cdot x_2 x_1 \cdot v' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w'' \\
&+ w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v' \cdot x_1 x_2 \cdot w'' \\
&- w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v \cdot \underline{x_1 x_2 - x_2 x_1} \cdot v' \cdot x_2 x_1 \cdot w''
\end{aligned}$$

Up to sign the fourth summand is the element we started with, the third has a word in X between the two factors $\underline{x_1 x_2 - x_2 x_1}$ of same length as w' but with smaller minimum number of swappings to order it, and the remaining four summands have a word in X of smaller length between the two factors of $\underline{x_1x_2 - x_2x_1}$. By induction we can thus eliminate elements of type (4) where the word in X between the two factors of $\underline{x_1x_2 - x_2x_1}$ are not ordered.

We are thus left with the following types of basis elements.

(1') A word in $\mathcal{G}$ with precisely two factors $\mathrm{d}\,x$ and $\mathrm{d}\,x'$ with $x, x' \in X$ (the case $x = x'$ is allowed) and the remaining factors in X.

(2') A word in $\mathcal{G}$ with precisely one factor $\underline{x \cdot x' - x' \cdot x}$ with $x, x' \in X$, and the remaining factors in X.

(3') A word in $\mathcal{G}$ with precisely one factor $\underline{x_1x_2 - x_2x_1}$, precisely one factor $\mathrm{d}\,x$ for $x \in X$, and the remaining factors in X.

(4') $w \cdot \underline{x_1x_2 - x_2x_1} \cdot x_1^{a_1}x_2^{a_2} \cdot \underline{x_1x_2 - x_2x_1} \cdot w'$ with $w, w' \in \mathcal{B}_X$.

We next show that we can also eliminate the remaining elements of type (4') using method (II) from Remark 7.4.5.3. For this we first note that words in $\mathcal{G}$ that can occur[49] in the boundaries of elements of type (1'), (2'), (3') and (4') never have a factor $\underline{x_1x_2 - x_2x_1}$, but the boundary of elements of type (4') lies in the k-submodule spanned by words in $\mathcal{G}$ that have a factor $\underline{x_1x_2 - x_2x_1}$. To eliminate (4') it thus suffices to show that if z is a k-linear combination of elements of type (4'), with $\partial(z) = 0$, then already $z = 0$.

So let z be given by

$$z = \sum_{\substack{w,w' \in \mathcal{B}_X \\ a_1,a_2 \geq 0}} b_{w,a_1,a_2,w'} \cdot w \cdot \underline{x_1x_2 - x_2x_1} \cdot x_1^{a_1}x_2^{a_2} \cdot \underline{x_1x_2 - x_2x_1} \cdot w'$$

with $b_{w,a_1,a_2,w'}$ elements of k, only finitely many of which are nonzero, and assume that $\partial(z) = 0$. If all coefficients $b_{w,a_1,a_2,w'}$ are zero, then we already have $z = 0$ and are done, so assume that this is not the case. Then we can let $\widetilde{w} \in \mathcal{B}$ be such that there exist $a_1, a_2 \geq 0$ and $w' \in \mathcal{B}$ such that $b_{\widetilde{w},a_1,a_2,w'} \neq 0$ while minimizing $\operatorname{len}(\widetilde{w})$ with this property. The boundary $\partial(z)$ has the following form.

$$\begin{aligned} 0 &= \partial(z) \\ &= \sum_{\substack{w,w' \in \mathcal{B}_X \\ a_1,a_2 \geq 0 \\ \operatorname{len}(w) \geq \operatorname{len}(\widetilde{w})}} b_{w,a_1,a_2,w'} \cdot w \cdot (x_1x_2 - x_2x_1) \cdot x_1^{a_1}x_2^{a_2} \cdot \underline{x_1x_2 - x_2x_1} \cdot w' \\ &\quad - \sum_{\substack{w,w' \in \mathcal{B}_X \\ a_1,a_2 \geq 0 \\ \operatorname{len}(w) \geq \operatorname{len}(\widetilde{w})}} b_{w,a_1,a_2,w'} \cdot w \cdot \underline{x_1x_2 - x_2x_1} \cdot x_1^{a_1}x_2^{a_2} \cdot (x_1x_2 - x_2x_1) \cdot w' \end{aligned}$$

[49] By this we mean that writing the respective element in terms of the k-basis given by words in $\mathcal{G}$ the coefficient associated to that word is nonzero.

We now apply a k-linear morphism p to this equation. p is to be a morphism from $(A_2)_1$ to the k-submodule of $(A_2)_1$ that is spanned by words in $\mathcal{G}$ of degree 1 that begin with $\widetilde{w} \cdot \underline{x_1 x_2 - x_2 x_1}$. We define p on the basis given by words in $\mathcal{G}$ of degree 1 by mapping words that begin with $\widetilde{w} \cdot \underline{x_1 x_2 - x_2 x_1}$ to themselves, and all others to 0. Then the requirement $\mathrm{len}(w) \geq \mathrm{len}(\widetilde{w})$ implies that the summands

$$b_{w,a_1,a_2,w'} \cdot w \cdot (x_1 x_2 - x_2 x_1) \cdot x_1^{a_1} x_2^{a_2} \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w'$$

of the equation above are all mapped to 0 by p, and the summands

$$b_{w,a_1,a_2,w'} \cdot w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot x_1^{a_1} x_2^{a_2} \cdot (x_1 x_2 - x_2 x_1) \cdot w'$$

map to 0 unless $w = \widetilde{w}$. The upshot is that we obtain the following equality[50].

$$0 = \sum_{\substack{w' \in \mathcal{B}_X \\ a_1, a_2 \geq 0}} b_{\widetilde{w},a_1,a_2,w'} \cdot \widetilde{w} \cdot \underline{x_1 x_2 - x_2 x_1} \cdot x_1^{a_1} x_2^{a_2} \cdot (x_1 x_2 - x_2 x_1) \cdot w'$$

This implies that we must also have

$$0 = \sum_{\substack{w' \in \mathcal{B}_X \\ a_1, a_2 \geq 0}} b_{\widetilde{w},a_1,a_2,w'} \cdot x_1^{a_1} x_2^{a_2} \cdot (x_1 x_2 - x_2 x_1) \cdot w'$$

which by Proposition 7.4.5.5 implies that $b_{\widetilde{w},a_1,a_2,w'} = 0$ for all $a_1, a_2 \geq 0$ and $w' \in \mathcal{B}_X$. This however contradicts the assumption on $\widetilde{w}$, implying that z must have been zero after all.

Thus we can eliminate elements of type (4') and are left with basis elements of the following types.

(1') A word in $\mathcal{G}$ with precisely two factors $\mathrm{d}\, x$ and $\mathrm{d}\, x'$ with $x, x' \in X$ (the case $x = x'$ is allowed) and the remaining factors in X.

(2') A word in $\mathcal{G}$ with precisely one factor $\underline{x \cdot x' - x' \cdot x}$ with $x, x' \in X$, and the remaining factors in X.

(3') A word in $\mathcal{G}$ with precisely one factor $\underline{x_1 x_2 - x_2 x_1}$, precisely one factor $\mathrm{d}\, x$ for $x \in X$, and the remaining factors in X.

We now consider the basis elements of type (3'). We claim that we can eliminate those elements of type (3') that do not begin with the factor $\mathrm{d}\, x$. We can show this by induction on the number of factors before the factor $\mathrm{d}\, x$. There are two main cases, depending on what the preceding factor is. We first discuss the case in which the preceding factor is an element of X, say x'. Then we can write the element as either $w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \cdot x' \cdot \mathrm{d}\, x \cdot w''$

[50] We also multiplied with -1.

or $w \cdot x' \cdot \mathrm{d}\, x \cdot w' \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w''$ with $w, w', w'' \in \mathcal{B}_X$. We only discuss the first form, the second is completely analogous. Then consider the following boundary.

$$\begin{aligned}
&\partial\big(w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \cdot \underline{x' \cdot \mathrm{d}\, x - \mathrm{d}\, x \cdot x'} \cdot w''\big)\\
&= w \cdot x_1 x_2 \cdot w' \cdot \underline{x' \cdot \mathrm{d}\, x - \mathrm{d}\, x \cdot x'} \cdot w''\\
&\quad - w \cdot x_2 x_1 \cdot w' \cdot \underline{x' \cdot \mathrm{d}\, x - \mathrm{d}\, x \cdot x'} \cdot w''\\
&\quad - w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \cdot x' \cdot \mathrm{d}\, x \cdot w''\\
&\quad + w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w' \cdot \mathrm{d}\, x \cdot x' \cdot w''
\end{aligned}$$

Up to sign the third summand is the element we started with, the fourth is of type (3'), but with a smaller number of factors preceding $\mathrm{d}\, x$, and the other two are of type (2').

The other case to consider is when the factor preceding $\mathrm{d}\, x$ is $\underline{x_1 x_2 - x_2 x_1}$, so that the element is of the form $w \cdot \underline{x_1 x_2 - x_2 x_1} \cdot \mathrm{d}\, x \cdot w'$ for $w, w' \in \mathcal{B}_X$. We assume that $x = x_1$, the case $x = x_2$ is completely analogous by using C_2 instead of C_3. Then consider the following element in J.

$$\begin{aligned}
&w \cdot C_3 \cdot w'\\
&= w \cdot \mathrm{d}\, x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot w' + w \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\, x_1 \cdot w'\\
&\quad - w \cdot x_1 \cdot \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} \cdot w' + w \cdot \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} \cdot x_1 \cdot w'\\
&\quad + w \cdot x_2 \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} \cdot w' - w \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} \cdot x_2 \cdot w'
\end{aligned}$$

Up to sign the second summand is the element we started with, the first is of the (3'), but with a smaller number of factors preceding $\mathrm{d}\, x_1$, and the other four are of type (2').

We have now eliminated all elements of type (3') except those that start with $\mathrm{d}\, x$ as their first factor. Proceeding completely analogously to how we did with elements of type (2') in Proposition 7.4.5.7 we can now also eliminate those in which the word in X between $\mathrm{d}\, x$ and the factor $\underline{x_1 x_2 - x_2 x_1}$ is not ordered. We are thus left with the following basis elements.

(1") A word in $\mathcal{G}$ with precisely two factors $\mathrm{d}\, x$ and $\mathrm{d}\, x'$ with $x, x' \in X$ (the case $x = x'$ is allowed) and the remaining factors in X.

(2") A word in $\mathcal{G}$ with precisely one factor $\underline{x \cdot x' - x' \cdot x}$ with $x, x' \in X$, and the remaining factors in X.

(3") $\mathrm{d}\, x \cdot x_1^{a_1} x_2^{a_2} \cdot \underline{x_1 x_2 - x_2 x_1} \cdot w$ for $x \in X$, $a_1, a_2 \geq 0$, and $w \in \mathcal{B}_X$.

We now eliminate type (2") using method (II) from Remark 7.4.5.3. So assume that $z'' = z''' + z + z'$ is a cycle where z''' is a k-linear combination of basis elements of type (1"), z is a k-linear combination of basis elements of type (2") and z' is a k-linear combination of basis elements of type (3"). We have to show that then $z = 0$. We first note that as every element of type (1") is already a cycle we obtain that $z + z'$ is a cycle. We write

$$z = \sum_{\substack{w, w' \in \mathcal{B}_X,\\ x, x' \in X}} b_{w,x,x',w'} \cdot w \cdot \underline{x \cdot \mathrm{d}\, x' - \mathrm{d}\, x' \cdot x} \cdot w'$$

$$z' = \sum_{\substack{x\in X,\\ a_1,a_2\geq 0,\\ w\in\mathcal{B}_X}} c_{x,a_1,a_2,w} \cdot \mathrm{d}\,x \cdot x_1^{a_1}x_2^{a_2} \cdot \underline{x_1x_2 - x_2x_1} \cdot w$$

with $b_{w,x,x',w'}$ and $c_{x,a_1,a_2,w}$ elements of k, only finitely many of which are nonzero. If all coefficients $b_{w,x,x',w'}$ are zero, then we have $z = 0$ and are done. So assume that this is not the case. Then let $\widetilde{w}, \widetilde{w}' \in \mathcal{B}_X$ and $\widetilde{x}, \widetilde{x}' \in X$ be such that $b_{\widetilde{w},\widetilde{x},\widetilde{x}',\widetilde{w}'} \neq 0$ and choose $\widetilde{w}$ to be of maximum length with this property. From $\partial(z + z') = 0$ we then obtain the following equality.

$$\sum_{\substack{w,w'\in\mathcal{B}_X,\\ x,x'\in X,\\ \mathrm{len}(w)\leq\mathrm{len}(\widetilde{w})}} b_{w,x,x',w'} \cdot w \cdot x \cdot \mathrm{d}\,x' \cdot w' - \sum_{\substack{w,w'\in\mathcal{B}_X,\\ x,x'\in X,\\ \mathrm{len}(w)\leq\mathrm{len}(\widetilde{w})}} b_{w,x,x',w'} \cdot w \cdot \mathrm{d}\,x' \cdot x \cdot w'$$
$$= \sum_{\substack{x\in X,\\ a_1,a_2\geq 0,\\ w\in\mathcal{B}_X}} c_{x,a_1,a_2,w} \cdot \mathrm{d}\,x \cdot x_1^{a_1}x_2^{a_2} \cdot x_1x_2 \cdot w - \sum_{\substack{x\in X,\\ a_1,a_2\geq 0,\\ w\in\mathcal{B}_X}} c_{x,a_1,a_2,w} \cdot \mathrm{d}\,x \cdot x_1^{a_1}x_2^{a_2} \cdot x_2x_1 \cdot w$$

We now apply a k-linear morphism p' to this equation. p' is to be a morphism from $(A_2)_1$ to the k-submodule of $(A_2)_1$ that is spanned by the word $\widetilde{w} \cdot \widetilde{x} \cdot \mathrm{d}\,\widetilde{x}' \cdot \widetilde{w}'$ in $\mathcal{G}$ of degree 1. We define p on the basis given by words in $\mathcal{G}$ of degree 1 by mapping the just mentioned word to itself and all others to 0. Then note that all words are mapped to zero where the length of the word preceding a factor of the form $\mathrm{d}\,x$ is unequal to $\mathrm{len}(\widetilde{w}) + 1$. The condition $\mathrm{len}(w) \leq \mathrm{len}(\widetilde{w})$ on the left hand side of the above equation then implies that the second sum on the left hand side is mapped to zero. As all words in $\mathcal{G}$ occurring on the right hand side begin with an element of the form $\mathrm{d}\,x$ they are also all mapped to zero. We thus obtain that

$$b_{\widetilde{w},\widetilde{x},\widetilde{x}',\widetilde{w}'} \cdot \widetilde{w} \cdot \widetilde{x} \cdot \mathrm{d}\,\widetilde{x}' \cdot \widetilde{w}' = 0$$

which contradicts the assumption that $b_{\widetilde{w},\widetilde{x},\widetilde{x}',\widetilde{w}'} \neq 0$. Thus we must have $z = 0$ and can thus eliminate basis elements of type (2”).

We are thus left with the following basis elements.

(1”) A word in $\mathcal{G}$ with precisely two factors $\mathrm{d}\,x$ and $\mathrm{d}\,x'$ with $x, x' \in X$ (the case $x = x'$ is allowed) and the remaining factors in X.

(3”) $\mathrm{d}\,x \cdot x_1^{a_1}x_2^{a_2} \cdot \underline{x_1x_2 - x_2x_1} \cdot w$ for $x \in X$, $a_1, a_2 \geq 0$, and $w \in \mathcal{B}_X$.

We can now eliminate type (3”) in a manner that is completely analogous to the argument as we used to eliminate (2”) in Proposition 7.4.5.7. We are thus left with only type (1”). For this we can first use boundaries of words in $\mathcal{G}$ involving two factors $\mathrm{d}\,x$ and $\mathrm{d}\,x'$ as well as a factor $\underline{x_1x_2 - x_2x_1}$ with the other factors in X, as well as boundaries of words in $\mathcal{G}$ involving one factor $\mathrm{d}\,x''$ and one factor $\underline{x \cdot \mathrm{d}\,x' - \mathrm{d}\,x' \cdot x}$ with the remaining factors in X, to reorder the factors so that we are left with only elements of the form $x_1^{a_1}x_2^{a_2} \cdot \mathrm{d}\,x \cdot \mathrm{d}\,x'$ with $a_1, a_2 \geq 0$ and $x, x' \in X$. As a second step we can eliminate such elements with $x = x'$ by using elements of J that involve a factor of $\mathrm{d}\,x \cdot \mathrm{d}\,x$.

We are thus left with elements of the following two types.

(1*) $x_1^{a_1}x_2^{a_2} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2$ for $a_1, a_2 \geq 0$.

(2*) $x_1^{a_1}x_2^{a_2} \cdot \mathrm{d}\,x_2 \cdot \mathrm{d}\,x_1$ for $a_1, a_2 \geq 0$.

We can eliminate type (2*) using the following boundary.

$$\partial\left(x_1^{a_1}x_2^{a_2} \cdot \mathrm{d}\,\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}\right) = -x_1^{a_1}x_2^{a_2} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2 - x_1^{a_1}x_2^{a_2} \cdot \mathrm{d}\,x_2 \cdot \mathrm{d}\,x_1$$

We are thus left with only basis elements from (1*), which form a subset of $\mathcal{B}^{\mathrm{ord}}_{X,\mathrm{d}\,X}$, so we are done. □

Definition 7.4.5.9. Let X be a totally ordered set with $|X| \leq 2$. Then we define

$$\Theta_X \colon \Omega'^{\bullet}_{k[X]/k} \to \Omega^{\bullet}_{k[X]/k}$$

to be the morphism in $\mathrm{Alg}(\mathsf{Mixed})$ constructed in Construction 7.4.5.1 where we let Y_0 be as defined in Proposition 7.4.5.6, Y_1 as defined in Proposition 7.4.5.7, Y_2 as defined in Proposition 7.4.5.8, and where for $n > 2$ we just choose some subset Y_n of $(A_n)_n$ that satisfies (a), (b) and (c) of Construction 7.4.5.1 (we argued in Construction 7.4.5.1 that it is always possible to find Y_n satisfying this). ♢

7.4.5.3. Proof that the construction is a cofibrant resolution

In this section we show that Θ_X as defined in Definition 7.4.5.9 really is a cofibrant replacement of $\Omega^{\bullet}_{k[X]/k}$.

Proposition 7.4.5.10. *This proposition concerns Construction 7.4.5.1. Let X be a set and $n \geq 0$ an integer. Then*

$$\mathrm{H}_m(\Theta_n) \colon \mathrm{H}_m(A_n) \to \mathrm{H}_m\left(\Omega^{\bullet}_{k[X]/k}\right)$$

is an isomorphism for $m < n$ and surjective for every m. ♡

Proof. $\Omega^{\bullet}_{k[X]/k}$ is generated as a graded k-algebra by the elements x and $\mathrm{d}\,x$ for $x \in X$, so as every element of X is in the image of the morphism Θ_0 in $\mathrm{Alg}(\mathsf{Mixed})$, it follows that Θ_0 is surjective. As both A_0 and $\Omega^{\bullet}_{k[X]/k}$ have zero boundary operator, this implies that $\mathrm{H}_*(\Theta_0)$ and hence also $\mathrm{H}_*(\Theta_n)$ is surjective as well.

Now we show that $\mathrm{H}_m(\Theta_n)$ is even an isomorphism if $m < n$. We prove this by induction. The case $n = 0$ is clear, as both A_0 and $\Omega^{\bullet}_{k[X]/k}$ are concentrated in nonnegative degrees, so in particular have homology concentrated in nonnegative degrees.

So now assume that $n > 0$ and we already showed that $\mathrm{H}_m(\Theta_{n-1})$ is an isomorphism for $m < n-1$. By Remark 7.4.5.2 $\iota^n_{n-1} \colon A_{n-1} \to A_n$ is an isomorphism in degrees smaller than or equal to $n-1$. This implies that in the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_m(A_{n-1}) \arrow[rr, "\mathrm{H}_m(\iota^n_{n-1})"] \arrow[dr, "\mathrm{H}_m(\Theta_{n-1})"'] & & \mathrm{H}_m(A_n) \arrow[dl, "\mathrm{H}_m(\Theta_n)"] \\
& \mathrm{H}_m\left(\Omega^{\bullet}_{k[X]/k}\right) &
\end{tikzcd}$$

the top morphism is an isomorphism for $m \leq n-2$, and as the left morphism is an isomorphism in that range as well, it already follows that $\mathrm{H}_m(\Theta_n)$ is an isomorphism for $m \leq n-2$. For $m = n-1$ we still obtain that $\mathrm{H}_{n-1}(\iota_{n-1}^n)$ must be surjective[51]. In order to show that $\mathrm{Ker}(\mathrm{H}_{n-1}(\Theta_n)) \cong 0$ it thus suffices to show that $\mathrm{H}_{n-1}\big(\iota_{n-1}^n\big)$ maps $\mathrm{Ker}(\mathrm{H}_{n-1}(\Theta_{n-1}))$ to zero. But is precisely what condition (c) ensures. □

Proposition 7.4.5.11. *Let X be a totally ordered set with $|X| \leq 2$. This proposition concerns Θ_X as defined in Definition 7.4.5.9.*

The object

$$\Omega'^{\bullet}_{k[X]/k}$$

of Alg(Mixed) *is cofibrant, and the morphism*

$$\Theta_X \colon \Omega'^{\bullet}_{k[X]/k} \to \Omega^{\bullet}_{k[X]/k}$$

is a quasiisomorphism. ♡

Proof. $\mathrm{Free}^{\mathrm{Alg(Mixed)}}$ is a left Quillen functor by Definition 4.2.2.2, Proposition 4.2.2.9, and Theorem 4.2.2.1. As $k \cdot X$ is a cofibrant chain complex, this implies that A_0 is cofibrant in Alg(Mixed). Furthermore, for every $n \geq 0$, the morphism j_n is a cofibration in $\mathsf{Ch}(k)$ (it is a coproduct of generating cofibrations considered in [Hov99, 2.3.3 and 2.3.11]), so $\mathrm{Free}^{\mathrm{Alg(Mixed)}}(j_n)$ and thus also ι_n^{n+1} are cofibrations in Alg(Mixed). As cofibrations are closed under (transfinite) compositions, this implies that $\Omega'^{\bullet}_{k[X]/k}$ is cofibrant.

We now turn to showing that Θ_X is a quasiisomorphism. Remark 7.4.5.2 implies that $\iota_n^{n'} \colon A_n \to A_{n'}$ is an isomorphism in degrees smaller to or equal to n for all $0 \leq n < n'$. Combining this with the fact that the forgetful functor from Alg(Mixed) to $\mathsf{Ch}(k)$ preserves filtered colimits by Proposition 4.2.2.12 we obtain that $\iota_n \colon A_n \to \Omega'^{\bullet}_{k[X]/k}$ is an isomorphism in degrees smaller to or equal to n as well. In particular, in the diagram

$$\begin{tikzcd}
\mathrm{H}_m(A_n) \arrow[rr, "\mathrm{H}_m(\iota_n)"] \arrow[dr, "\mathrm{H}_m(\Theta_n)"'] & & \mathrm{H}_m\left(\Omega'^{\bullet}_{k[X]/k}\right) \arrow[dl, "\mathrm{H}_m(\Theta_X)"] \\
& \mathrm{H}_m\left(\Omega^{\bullet}_{k[t]/k}\right) &
\end{tikzcd}$$

the top morphism is an isomorphisms for $m < n$. As the left morphism is as isomorphism in that range as well by Proposition 7.4.5.10 we can conclude that $\mathrm{H}_m(\Theta_X)$ is an isomorphism for $m < n$ too. It follows that $\mathrm{H}_m(\Theta_X)$ is an isomorphism for all integers m, so Θ is a quasiisomorphism. □

[51] Given an element of $\mathrm{H}_{n-1}(A_n)$ we can represent it by a cycle of degree $n-1$ As ι_{n-1}^n is an isomorphism in degree $n-1$, there is an element z in A_{n-1} that is mapped to that cycle by ι_{n-1}^n. It thus remains to show that z is also a cycle and hence represents a homology class. But

$$\iota_{n-1}^n(\partial z) = \partial\big(\iota_{n-1}^n(z)\big) = 0$$

which implies $\partial z = 0$, as ι_{n-1}^n is also an isomorphism in degree l.

7.4.6. Naturality of ϵ

We explained in Warning 7.2.2.6 that the morphisms

$$\epsilon_X \colon \Omega^{\bullet}_{k[X]/k} \to \overline{\mathrm{C}}\big(k[X]\big)$$

of differential graded k-algebras that were defined in Construction 7.2.2.1 and Proposition 7.2.2.2 only assemble to a natural transformation of functors from Set to $\mathrm{Alg}(\mathsf{Ch}(k))$, but not to a natural transformation of functors from $\mathrm{CAlg}(\mathrm{LMod}_k(\mathsf{Ab}))$ to $\mathrm{Alg}(\mathsf{Ch}(k))$. In this section we show that a weaker statement is at least true in special cases: If X is a set with $|X| \leq 2$ and F a morphism of commutative algebras $F\colon k[t] \to k[X]$, then there is a filler for the square

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\epsilon_{\{t\}}\right)"] \arrow[d, "\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{F/k}\right)"'] & \mathrm{Alg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[t]\big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(F)\right)"] \\
\mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_X)"'] & \mathrm{Alg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[X]\big)\Big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$.

Proposition 7.4.6.1. *Let X be a totally ordered set satisfying $|X| \leq 2$, and f an element of $k[X]$. Denote the morphism of commutative k-algebras $k[t] \to k[X]$ that maps t to f by F.*

Then there is a filler for the square

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\epsilon_{\{t\}}\right)"] \arrow[d, "\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{F/k}\right)"'] & \mathrm{Alg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[t]\big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(F)\right)"] \\
\mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_X)"'] & \mathrm{Alg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[X]\big)\Big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where ϵ is as defined in Construction 7.2.2.1 and Proposition 7.2.2.2. ♡

Proof. Let the morphism

$$\Theta_{\{t\}} \colon \Omega'^{\bullet}_{k[t]/k} \to \Omega^{\bullet}_{k[t]/k}$$

in $\mathrm{Alg}(\mathsf{Mixed})$ be as in Definition 7.4.5.9. By Proposition 7.4.5.11 $\Omega'^{\bullet}_{k[t]/k}$ is a cofibrant object of $\mathrm{Alg}(\mathsf{Mixed})$, and thus has cofibrant underlying chain complex by Proposition 4.2.2.12. Furthermore, $\Theta_{\{t\}}$ is a quasiisomorphism, and thus induces an equivalence

$$\mathrm{Alg}(\gamma)\big(\Theta_{\{t\}}\big)\colon\ \mathrm{Alg}(\gamma)\Big(\Omega'^{\bullet}_{k[X]/k}\Big) \to \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big)$$

in $\mathrm{Alg}(\mathcal{D}(k))$. It thus suffices to show that there is a filler for the square

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\Omega'^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\epsilon_{\{t\}}\circ\Theta_{\{t\}}\right)"] \arrow[d, "\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{F/k}\circ\Theta_{\{t\}}\right)"'] & \mathrm{Alg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[t]\big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(F)\right)"] \\
\mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_X)"'] & \mathrm{Alg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[X]\big)\Big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$.

By Proposition 4.2.2.12 the underlying differential graded algebra of cofibrant objects in $\mathrm{Alg}(\mathsf{Mixed})$ is cofibrant, so $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant as an object in $\mathrm{Alg}(\mathsf{Ch}(k))$. Like every object of $\mathrm{Alg}(\mathsf{Ch}(k))$ also $\overline{\mathrm{C}}\big(k[X]\big)$ is fibrant. Combining this with Proposition A.1.0.1 and [Hov99, 1.2.10 (ii)] it suffices to show that there exists a homotopy in the model-category-theoretic sense between the two compositions in the following diagram in $\mathrm{Alg}(\mathsf{Ch}(k))$.

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[t]/k} \arrow[r, "\epsilon_{\{t\}}\circ\Theta_{\{t\}}"] \arrow[d, "\Omega^{\bullet}_{F/k}\circ\Theta_{\{t\}}"'] & \overline{\mathrm{C}}\big(k[t]\big) \arrow[d, "\overline{\mathrm{C}}(F)"] \\
\Omega^{\bullet}_{k[X]/k} \arrow[r, "\epsilon_X"'] & \overline{\mathrm{C}}\big(k[X]\big)
\end{tikzcd}$$

By Propositions 4.1.4.2 and 4.2.2.17 this means that we have to define a morphism of $\mathbb{Z}$-graded k-modules

$$h\colon \Omega'^{\bullet}_{k[t]/k} \to \overline{\mathrm{C}}\big(k[X]\big)$$

of degree 1 that satisfies

$$\partial\big(h(z)\big) + h\big(\partial(z)\big) = \Big(\overline{\mathrm{C}}(F)\circ\epsilon_{\{t\}}\circ\Theta_{\{t\}}\Big)(z) - \Big(\epsilon_X\circ\Omega^{\bullet}_{F/k}\circ\Theta_{\{t\}}\Big)(z)$$

for being a chain homotopy as well as the Leibniz rule for chain homotopies

$$h\big(z\cdot z'\big) = h(z)\cdot\Big(\epsilon_X\circ\Omega^{\bullet}_{F/k}\circ\Theta_{\{t\}}\Big)\big(z'\big) + (-1)^{\deg_{\mathsf{Ch}}(z)}\Big(\overline{\mathrm{C}}(F)\circ\epsilon_{\{t\}}\circ\Theta_{\{t\}}\Big)(z)\cdot h\big(z'\big)$$

for all elements z and z' of $\Omega'^{\bullet}_{k[t]/k}$.

In the following we will use notation from Construction 7.4.5.1. By definition, and using that the forgetful functor from $\mathrm{Alg}(\mathsf{Mixed})$ to $\mathrm{Alg}(\mathsf{Ch}(k))$ preserves filtered colimits by Proposition 4.2.2.12, we can identify $\Omega'^{\bullet}_{k[t]/k}$ as the colimit of the diagram

$$A_0 \xrightarrow{\iota_0^1} A_1 \xrightarrow{\iota_1^2} A_2 \xrightarrow{\iota_2^3} \ldots$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$. The forgetful functor to $\mathbb{Z}$-graded k-modules also preserves filtered colimits by Proposition 4.2.2.12, and together this implies that we can define h as above by defining a compatible system of homotopies h_n of morphisms of differential graded algebras from the restriction $\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_n$ to $\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_n$. We will do this by induction.

We begin with some general remarks on how the induction step will work. So assume that $n \geq 0$ and we already have constructed a homotopy h_n of morphisms of differential graded algebras $A_n \to \overline{\mathrm{C}}\big(k[X]\big)$ from $\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_n$ to $\epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_n$. We wish to extend h_n to h_{n+1}. For easier notation we will use the following shorthands.

$$\begin{aligned}
\varphi' &:= \overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_n \\
\varphi &:= \overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_{n+1} \\
\psi' &:= \epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_n \\
\psi &:= \epsilon_X \circ \Omega^\bullet_{F/k} \circ \Theta_{\{t\}} \circ \iota_{n+1}
\end{aligned}$$

By Remark 7.4.5.2 the underlying graded k-algebra of A_{n+1} is free on the elements t and $\mathrm{d}\,t$, and $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_m$ with $m \leq n$, while A_n is free on the same generators except the elements of $\underline{Y_n}$ and $\mathrm{d}\,\underline{Y_n}$. Let us denote by G_{n+1} the generators for A_{n+1} that were just mentioned, and by G_n those of A_n. For compatibility with h_n we are forced to define h_{n+1} as follows on G_n.

$$h_{n+1}(g) := h_n(g) \qquad \text{for } g \in G_n$$

On elements g in $G_{n+1} \backslash G_n$ we need to define h_{n+1} in such a way that h_{n+1} is a homotopy from φ to ψ, so we must have the following.

$$\partial\big(h_{n+1}(g)\big) = \varphi(g) - \psi(g) - h_{n+1}\big(\partial(g)\big) = -h_n\big(\partial(g)\big) \tag{$*$}$$

In the simplification we used that $\Theta_{\{t\}} \circ \iota_{n+1}$ is zero on g (and hence so are φ and ψ), and that $\partial(g)$ is an element of A_n. We claim that finding solutions to these lifting problems is the only obstacle to extending h_n to h_{n+1} as required. So assume that we can find values for $h_{n+1}(g)$ for every $g \in G_{n+1} \setminus G_n$ that satisfy $(*)$.

As we have already defined values of h_{n+1} on G_{n+1}, Proposition 4.2.2.18 implies that there is a unique way to extend this to a morphism h_{n+1} of $\mathbb{Z}$-graded k-modules from A_{n+1} to $\overline{\mathrm{C}}\big(k[X]\big)$ of that increases degree by 1 and that satisfies the Leibniz rule for homotopies of differential graded algebras from φ to ψ. As h_{n+1} agrees with h_n on G_n and h_n also satisfies the analogous Leibniz rule as a homotopy of differential graded algebras from φ' to ψ', and φ and ψ restrict to φ' and ψ', the uniqueness part of Proposition 4.2.2.18 then implies that h_{n+1} extends h_n. It remains to show that h_{n+1} satisfies $\partial \circ h_{n+1} + h_{n+1} \circ \partial = \varphi - \psi$. Again by Proposition 4.2.2.18 it suffices to check this on elements of G_{n+1}. On elements of $G_{n+1} \setminus G_n$ this holds by definition, and on elements of G_n this holds because it does for h_n.

We have now shown that the only obstruction to extending h_n to h_{n+1} with all the necessary properties is finding solutions for $h_{n+1}(g)$ for elements g of $G_{n+1} \setminus G_n$ to the equation $(*)$. We claim that such a solution can always be found if $n \geq 2$. So assume that $n \geq 2$ and we have already defined h_n. Let g be an element of $G_{n+1} \setminus G_n$. Then we first claim that the right hand side of equation $(*)$ is a cycle. For this we carry out the following calculation, using that h_n is a chain homotopy from φ' to ψ'.

$$\begin{aligned}
&\partial\Big(-h_n\big(\partial(g)\big)\Big)\\
&= h_n\Big(\partial\big(\partial(g)\big)\Big) - \varphi'\big(\partial(g)\big) + \psi'\big(\partial(g)\big)\\
&= -\varphi'\big(\partial(g)\big) + \psi'\big(\partial(g)\big)\\
&= 0
\end{aligned}$$

The last step needs a comment. The element g is either of the form $\underline{y}$ or $\mathrm{d}\,\underline{y}$ for $y \in Y_n$. Thus $\partial(g)$ is either y or $-\mathrm{d}\,y$ for a $y \in Y_n$, and $\Theta_{\{t\}}$, and thus also φ' and ψ', maps every element of Y_n (and hence also $\mathrm{d}\,Y_n$) to 0.

As the right hand sides of equation $(*)$ is a cycle, it represents a homology class, and finding a solution to the equation is equivalent to the homology class being zero. As the elements of Y_n are of degree n, the element g, and hence the right hand side of $(*)$, must be of degree $n+1$ or $n+2$.[52] Thus the obstructions to extending h_n to h_{n+1} are homology classes in degree $n+1$ and $n+2$. As ϵ_X is a quasiisomorphism by Proposition 7.2.2.2 (6) and $\Omega^\bullet_{k[X]/k}$ is concentrated in degrees less than or equal to 2 (this is where we use the assumption $|X| \leq 2$), the homology of $\overline{\mathrm{C}}\big(k[X]\big)$ is concentrated in degrees less than or equal to 2. Thus the homology classes obstructing extension of h_n to h_{n+1} are all trivially zero as we assumed $n \geq 2$, so that it is always possible to extend h_n to h_{n+1}.

By the above argument it thus suffices to construct h_2. Concretely, we first need to define $h_0(t)$ and $h_0(\mathrm{d}\,t)$ satisfying the following.[53]

$$\begin{aligned}
\partial\big(h_0(t)\big) &= \Big(\overline{\mathrm{C}}(F)\circ\epsilon_{\{t\}}\circ\Theta_{\{t\}}\circ\iota_0\Big)(t) - \Big(\epsilon_X\circ\Omega^\bullet_{F/k}\circ\Theta_{\{t\}}\circ\iota_0\Big)(t) \qquad (7.13)\\
\partial\big(h_0(\mathrm{d}\,t)\big) &= \Big(\overline{\mathrm{C}}(F)\circ\epsilon_{\{t\}}\circ\Theta_{\{t\}}\circ\iota_0\Big)(\mathrm{d}\,t) - \Big(\epsilon_X\circ\Omega^\bullet_{F/k}\circ\Theta_{\{t\}}\circ\iota_0\Big)(\mathrm{d}\,t)
\end{aligned}$$

For the set $\{t\}$ the set Y_0 is empty by Proposition 7.4.5.6, so $\iota_0^1\colon A_0 \to A_1$ is an isomorphism, and hence h_0 extends to h_1 for trivial reasons. Finally, to extend h_1 to h_2 we need to define $h_2(\underline{t\cdot\mathrm{d}\,t - \mathrm{d}\,t\cdot t})$ and $h_2(\mathrm{d}\,\underline{t\cdot\mathrm{d}\,t - \mathrm{d}\,t\cdot t})$ (see Proposition 7.4.5.7) satisfying the following.

$$\begin{aligned}
\partial\Big(h_2\big(\underline{t\cdot\mathrm{d}\,t - \mathrm{d}\,t\cdot t}\big)\Big) &= -h_0\Big(\partial\big(\underline{t\cdot\mathrm{d}\,t - \mathrm{d}\,t\cdot t}\big)\Big) \qquad (7.14)\\
\partial\Big(h_2\big(\mathrm{d}\,\underline{t\cdot\mathrm{d}\,t - \mathrm{d}\,t\cdot t}\big)\Big) &= -h_0\Big(\partial\big(\mathrm{d}\,\underline{t\cdot\mathrm{d}\,t - \mathrm{d}\,t\cdot t}\big)\Big)
\end{aligned}$$

However, the obstruction for the existence of a solution for $h_2(\mathrm{d}\,\underline{t\cdot\mathrm{d}\,t - \mathrm{d}\,t\cdot t})$ is a homology class in degree 3. By the same argument as the case of extensions from A_n to

[52] Recall that if y is an element of Y_n, then $\underline{y}$ is of degree $n+1$ and $\mathrm{d}\,\underline{y}$ is then of degree $n+2$.

[53] The argument that it suffices to define h_0 on t and $\mathrm{d}\,t$ satisfying the chain homotopy identity is completely analogous to the argument we gave for extending h_n to h_{n+1}, also using Proposition 4.2.2.18. This time the analogue of $(*)$ has slightly different form as $\Theta_{\{t\}}$ does not vanish on t and $\mathrm{d}\,t$, but t and $\mathrm{d}\,t$ are cycles in A_0.

A_{n+1} for $n \geq 2$ we thus already know abstractly that a solution can be found. To extend h_1 to h_2 it thus suffices to find a solution for $h_2\big(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}\big)$.

We begin by evaluating the right hand sides of (7.13), where we use the definitions in particular from Construction 7.4.5.1 and Construction 7.2.2.1. If $|X| = 2$ we denote the elements of X by $x_1 < x_2$, if $|X| = 1$ we denote the unique element by x_1.

$$\begin{aligned}
&\Big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\Big)(t) - \Big(\epsilon_X \circ \Omega^{\bullet}_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\Big)(t)\\
&= \Big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}}\Big)(t) - \Big(\epsilon_X \circ \Omega^{\bullet}_{F/k}\Big)(t)\\
&= \overline{\mathrm{C}}(F)(t) - \epsilon_X(f)\\
&= f - f\\
&= 0\\
&= \partial(0)
\end{aligned}$$

$$\begin{aligned}
&\Big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\Big)(\mathrm{d}\,t) - \Big(\epsilon_X \circ \Omega^{\bullet}_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\Big)(\mathrm{d}\,t)\\
&= \Big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}}\Big)(\mathrm{d}\,t) - \Big(\epsilon_X \circ \Omega^{\bullet}_{F/k}\Big)(\mathrm{d}\,t)\\
&= \overline{\mathrm{C}}(F)\big(1 \otimes \overline{t}\big) - \epsilon_X(\mathrm{d}\,f)\\
&= 1 \otimes \overline{f} - \epsilon_X(\mathrm{d}\,f)\\
&= \mathrm{d}(f) - \epsilon_X(\mathrm{d}\,f)\\
&= \mathrm{d}\big(\epsilon_X(f)\big) - \epsilon_X(\mathrm{d}\,f)
\end{aligned}$$

We can now use that $\epsilon_X^{(\bullet)}$ as defined in Construction 7.3.1.1 is a strongly homotopy linear morphism, see Proposition 7.3.11.2.

$$= -\partial\Big(\epsilon_X^{(1)}(f)\Big)$$

We can thus define $h_0(t) = 0$ and $h_0(\mathrm{d}\,t) = -\epsilon_X^{(1)}(f)$.

Now we evaluate the right hand side of (7.14).

$$\begin{aligned}
&- h_0\Big(\partial\big(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}\big)\Big)\\
&= -h_0(t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t)\\
&= -h_0(t) \cdot \Big(\epsilon_X \circ \Omega^{\bullet}_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\Big)(\mathrm{d}\,t) - \Big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\Big)(t) \cdot h_0(\mathrm{d}\,t)\\
&\quad + h_0(\mathrm{d}\,t) \cdot \Big(\epsilon_X \circ \Omega^{\bullet}_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\Big)(t) - \Big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\Big)(\mathrm{d}\,t) \cdot h_0(t)\\
&= -\Big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0\Big)(t) \cdot h_0(\mathrm{d}\,t) + h_0(\mathrm{d}\,t) \cdot \Big(\epsilon_X \circ \Omega^{\bullet}_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\Big)(t)\\
&= -\Big(\overline{\mathrm{C}}(F) \circ \epsilon_{\{t\}} \circ \Theta_{\{t\}} \circ \iota_0 - \epsilon_X \circ \Omega^{\bullet}_{F/k} \circ \Theta_{\{t\}} \circ \iota_0\Big)(t) \cdot h_0(\mathrm{d}\,t)\\
&= -\partial\big(h_0(t)\big) \cdot h_0(\mathrm{d}\,t)\\
&= 0
\end{aligned}$$

$$= \partial(0)$$

Thus we can define $h_2(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}) = 0$. □

As a significantly easier variant we can also show a analogous result to Proposition 7.4.6.1 where we consider morphisms into k.

Proposition 7.4.6.2. *Let X be a set and $F\colon k[X] \to k$ a morphism of commutative k-algebras.*

Then there is a filler for the square

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k[X]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma)(\epsilon_X)"] \arrow[d, "\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{F/k}\right)"'] & \mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}\left(k[X]\right)\right) \arrow[d, "\mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(F)\right)"] \\
\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{k/k}\right) \arrow[r, "\mathrm{Alg}(\gamma)\left(\epsilon_{\emptyset}\right)"'] & \mathrm{Alg}(\gamma)\left(\overline{\mathrm{C}}(k)\right)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$*, where ϵ is as defined in Construction 7.2.2.1 and Proposition 7.2.2.2.* ♡

Proof. It suffices to show that the diagram

$$\begin{tikzcd}
\Omega^{\bullet}_{k[X]/k} \arrow[r, "\epsilon_X"] \arrow[d, "\Omega^{\bullet}_{F/k}"'] & \overline{\mathrm{C}}\left(k[X]\right) \arrow[d, "\overline{\mathrm{C}}(F)"] \\
\Omega^{\bullet}_{k/k} \arrow[r, "\epsilon_{\emptyset}"'] & \overline{\mathrm{C}}(k)
\end{tikzcd}$$

commutes strictly. For this we note that as $\overline{k} \cong 0$, the lower right chain complex is concentrated in degree 0, so it suffices to check that the two compositions agree on elements of degree 0. But on degree 0 we can identify the diagram with

$$\begin{tikzcd}
k[X] \arrow[r, "\mathrm{id}_{k[X]}"] \arrow[d, "F"'] & k[X] \arrow[d, "F"] \\
k \arrow[r, "\mathrm{id}_k"'] & k
\end{tikzcd}$$

which commutes. □

7.4.7. Naturality of Φ

In Definition 7.4.4.2 we defined a quasiisomorphisms

$$\Phi_X \colon \operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \to \Omega^\bullet_{k[X]/k}$$

in $\operatorname{Alg}(\mathsf{Ch}(k))$, for every set X. While the morphisms Φ_X for different sets X do not assemble to a natural transformation from the category of commutative k-algebras to $\operatorname{Alg}(\mathsf{Ch}(k))$, we show in this section that a weaker naturality property holds with respect to some specific morphisms of commutative k-algebras.

Proposition 7.4.7.1. *Let X and Y be totally ordered sets satisfying one of the following.*

(1) $|X| = 1$ *and* $|Y| \leq 2$.

(2) $|Y| = 0$.

Let F be a morphism of commutative k-algebras $k[X] \to k[Y]$.
Then there is a filler for the square

$$\begin{tikzcd}
\operatorname{Alg}(\gamma)\Big(\operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big)\Big) \arrow[r, "\operatorname{Alg}(\gamma)(\Phi_X)"] \arrow[d, "\operatorname{Alg}(\gamma)\left(\operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})\left(\widetilde{\mathrm{C}}_k(F)\right)\right)"'] & \operatorname{Alg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[d, "\operatorname{Alg}(\gamma)\left(\Omega^\bullet_{F/k}\right)"] \\
\operatorname{Alg}(\gamma)\Big(\operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(Y)\Big)\Big) \arrow[r, "\operatorname{Alg}(\gamma)(\Phi_Y)"'] & \operatorname{Alg}(\gamma)\Big(\Omega^\bullet_{k[Y]/k}\Big)
\end{tikzcd}$$

in $\operatorname{Alg}(\mathcal{D}(k))$, where $\widetilde{\mathrm{C}}$ is as in Construction 7.4.2.5 and Φ_X and Φ_Y as in Definition 7.4.4.2. ♡

Proof. In the following we will omit the forgetful functor $\operatorname{Alg}(\mathrm{ev}_{\mathfrak{m}})$ from the notation to make diagrams more compact.

By Definition 7.4.4.2 Φ_X is the composition of Φ'_X with the quasiisomorphism mapping z to $\nu^{\deg_{\mathsf{Ch}}(z)} \cdot z$ (where ν is as in Proposition 7.4.4.1), and analogously for Φ_Y. As the diagram

$$\begin{tikzcd}
\Omega^\bullet_{k[X]/k} \arrow[r, "\nu^{\deg_{\mathsf{Ch}}(-)}\cdot_", "\simeq"'] \arrow[d, "\Omega^\bullet_{F/k}"'] & \Omega^\bullet_{k[X]/k} \arrow[d, "\Omega^\bullet_{F/k}"] \\
\Omega^\bullet_{k[Y]/k} \arrow[r, "\simeq", "\nu^{\deg_{\mathsf{Ch}}(-)}\cdot_"'] & \Omega^\bullet_{k[Y]/k}
\end{tikzcd}$$

commutes there is a filler for the right square in the following (non-commuting) diagram in $\mathrm{Alg}(\mathcal{D}(k))$.

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\widetilde{\mathrm{C}}_k(X)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Phi'_X)"] \arrow[d, "\mathrm{Alg}(\gamma)(\widetilde{\mathrm{C}}_k(F))"'] & \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\nu^{\deg_{\mathsf{Ch}}(-)}\cdot_)", "\simeq"'] \arrow[d, "\mathrm{Alg}(\gamma)(\Omega^\bullet_{F/k})"] & \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[d, "\mathrm{Alg}(\gamma)(\Omega^\bullet_{F/k})"] \\
\mathrm{Alg}(\gamma)\Big(\widetilde{\mathrm{C}}_k(Y)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Phi'_Y)"'] & \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[Y]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\nu^{\deg_{\mathsf{Ch}}(-)}\cdot_)", "\simeq"'] & \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[Y]/k}\Big)
\end{tikzcd}$$

It thus suffices to find a filler for the left square.

We now unpack the definition of Φ'_X, with Φ'_Y of course being completely analogous. By Proposition 7.4.3.2 $\mathrm{Alg}(\gamma)(\Phi'_X)$ is homotopic to the composition

$$\mathrm{Alg}(\gamma)\Big(\widetilde{\mathrm{C}}_k(X)\Big) \simeq \mathrm{HH}\big(k[X]\big) \simeq \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big)$$

where the first equivalence is obtained by applying the forgetful functor $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ to the equivalence at the bottom of diagram (7.9) in Construction 7.4.2.5 combined with compatibility of $\mathrm{ev}_{\mathfrak{a}}^{\mathsf{Mixed}}$ with $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})$ from Construction 4.4.1.1, and the second equivalence is the one from Corollary 7.2.2.3.

By definition that equivalence from Corollary 7.2.2.3 is given by the composition

$$\begin{tikzcd}
\mathrm{HH}\big(k[X]\big) \arrow[r, dash, "\simeq"] & \mathrm{Alg}(\gamma)\Big(\mathrm{C}\big(k[X]\big)\Big) \arrow[r, "\simeq"] & \mathrm{Alg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[X]\big)\Big) & \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[l, "\simeq", "\mathrm{Alg}(\gamma)(\epsilon_X)"']
\end{tikzcd}$$

where the left equivalence is the one from Proposition 6.3.4.3, the middle one is induced by the quotient morphism from Propositions 6.3.1.10 and 6.3.2.11, and the right equivalence is induced from ϵ_X as constructed in Construction 7.2.2.1.

In the following diagram in $\mathrm{Alg}(\mathcal{D}(k))$, we let the two columns be given by the composition the equivalences Φ'_X and Φ'_Y are defined as, as we just reviewed.

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\widetilde{\mathrm{C}}_k(X)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\widetilde{\mathrm{C}}_k(F))"] \arrow[d, no head, "\simeq"'] & \mathrm{Alg}(\gamma)\Big(\widetilde{\mathrm{C}}_k(Y)\Big) \arrow[d, no head, "\simeq"] \\
\mathrm{HH}\big(k[X]\big) \arrow[r, "\mathrm{HH}(F)"] \arrow[d, no head, "\simeq"'] & \mathrm{HH}\big(k[Y]\big) \arrow[d, no head, "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(\mathrm{C}\big(k[X]\big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\mathrm{C}(F))"] \arrow[d, "\simeq"'] & \mathrm{Alg}(\gamma)\Big(\mathrm{C}\big(k[Y]\big)\Big) \arrow[d, "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[X]\big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\overline{\mathrm{C}}(F))"] & \mathrm{Alg}(\gamma)\Big(\overline{\mathrm{C}}\big(k[Y]\big)\Big) \\
\mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[X]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Omega^\bullet_{F/k})"] \arrow[u, "\simeq"', "\mathrm{Alg}(\gamma)(\epsilon_X)"] & \mathrm{Alg}(\gamma)\Big(\Omega^\bullet_{k[Y]/k}\Big) \arrow[u, "\mathrm{Alg}(\gamma)(\epsilon_Y)"', "\simeq"]
\end{tikzcd}$$

There is a filler for the first square from the top by definition of $\widetilde{\mathrm{C}}_k(F)$, see Construction 7.4.2.5. The second square has a filler by naturality of the equivalence between HH and the standard Hochschild complex C in Proposition 6.3.4.3. The third square has a filler by naturality of the quotient map from the standard Hochschild complex to the normalized standard Hochschild complex, see Propositions 6.3.1.10 and 6.3.2.11. Finally, the bottom square has a filler by Proposition 7.4.6.1 (and Proposition 7.2.2.2 (3)) if we are in case (1) and by Proposition 7.4.6.2 if we are in case (2). □

7.4.8. Compatibility of Φ with d in degree 0

In Section 7.4.4 we showed that $\Phi_{\{t\}}$ is compatible with d (see Proposition 7.4.4.3). In this section we use the naturality statement from the previous Section 7.4.7 to deduce compatibility of Φ_X with d on elements of degree 0 as long as $|X| \leq 2$. Note that the following proposition still has content for $|X| = 1$. In this case it shows that the ν obtained in Proposition 7.4.4.1 is independent of the choices made along the way.

Proposition 7.4.8.1. *Let X be a totally ordered set satisfying $|X| \leq 2$. Then the quasiisomorphism*

$$\Phi_X\colon \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}_k(X)\Big) \to \Omega^\bullet_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Ch}(k))$ from Definition 7.4.4.2 satisfies

$$\Phi_X(\mathrm{d}\, z) = \mathrm{d}\big(\Phi_X(z)\big) \tag{7.15}$$

for every element z of degree 0 of $\widetilde{\mathrm{C}}_k(X)$. ♡

Proof. Let z and z' be elements of $\widetilde{\mathrm{C}}_k(X)$ of degree 0, and y an element of degree 1 such that $\partial(y) = z - z'$. Assume that

$$\Phi_X(\mathrm{d}\, z) = \mathrm{d}\big(\Phi_X(z)\big)$$

holds. Then we claim that

$$\Phi_X\big(\mathrm{d}\, z'\big) = \mathrm{d}\Big(\Phi_X\big(z'\big)\Big)$$

holds as well. Indeed, this follows from the following calculation.

$$\begin{aligned}
\Phi_X\big(\mathrm{d}\, z'\big) &= \Phi_X\Big(\mathrm{d}\big(z - \partial(y)\big)\Big) = \Phi_X\Big(\mathrm{d}\, z - \mathrm{d}\big(\partial(y)\big)\Big)\\
&= \Phi_X(\mathrm{d}\, z) + \Phi_X\Big(\partial\big(\mathrm{d}(y)\big)\Big) = \mathrm{d}\big(\Phi_X(z)\big) + \partial\Big(\Phi_X\big(\mathrm{d}(y)\big)\Big)\\
&= \mathrm{d}\big(\Phi_X(z)\big) = \mathrm{d}\Big(\Phi_X\big(z' + \partial(y)\big)\Big) = \mathrm{d}\Big(\Phi_X\big(z'\big) + \Phi_X\big(\partial(y)\big)\Big)\\
&= \mathrm{d}\Big(\Phi_X\big(z'\big) + \partial\big(\Phi_X(y)\big)\Big) = \mathrm{d}\Big(\Phi_X\big(z'\big)\Big)
\end{aligned}$$

As $\widetilde{\mathrm{C}}_k(X)$ is concentrated in nonnegative degrees by Construction 7.4.2.5 and Proposition 7.4.2.4, every element of degree 0 is a cycle. It thus suffices to show that for each homology class in $\mathrm{H}_0(\widetilde{\mathrm{C}}_k(X))$ there is a cycle representing it that satisfies (7.15).

As both sides of (7.15) are k-linear in z it even suffices to verify (7.15) on one cycle for each in a set of homology classes that generate $\mathrm{H}_0(\widetilde{\mathrm{C}}_k(X))$ as a k-module.

As Φ_X is a quasiisomorphism it is surjective, so that we can lift every element x of X, considered as an element of $\Omega^\bullet_{k[X]/k}$ of degree 0, to a cycle $\widetilde{x}$ in $\widetilde{\mathrm{C}}_k(X)$. Products[54] of elements of X form a k-basis for $\Omega^\bullet_{k[X]/k}$ and hence $\mathrm{H}_0(\Omega^\bullet_{k[X]/k})$. As Φ_X is a multiplicative quasiisomorphism this implies that products of elements of the form $\widetilde{x}$ for $x \in X$ are cycles representing homology classes that together form a generating set for $\mathrm{H}_0(\widetilde{\mathrm{C}}_k(X))$ as a k-module. It thus suffices to show that (7.15) is satisfied for products (with arbitrary many factors) of elements of the form $\widetilde{x}$ for $x \in X$.

Now suppose that z and z' are elements of degree 0 in $\widetilde{\mathrm{C}}_k(X)$ that both satisfy (7.15). Then we claim that the product $z \cdot z'$ satisfies (7.15) as well. This can be shown with the following simple calculation that uses that Φ_X is multiplicative and that d satisfies the Leibniz rule on both $\widetilde{\mathrm{C}}_k(X)$ and $\Omega^\bullet_{k[X]/k}$.

$$\begin{aligned}
\Phi_X\Big(\mathrm{d}\big(z \cdot z'\big)\Big) &= \Phi_X\Big(\mathrm{d}(z) \cdot z' + z \cdot \mathrm{d}\big(z'\big)\Big)\\
&= \Phi_X(\mathrm{d}\, z) \cdot \Phi_X\big(z'\big) + \Phi_X(z) \cdot \Phi_X\big(\mathrm{d}\, z'\big)\\
&= \mathrm{d}\big(\Phi_X(z)\big) \cdot \Phi_X\big(z'\big) + \Phi_X(z) \cdot \mathrm{d}\Big(\Phi_X\big(z'\big)\Big)\\
&= \mathrm{d}\Big(\Phi_X(z) \cdot \Phi_X\big(z'\big)\Big)\\
&= \mathrm{d}\Big(\Phi_X\big(z \cdot z'\big)\Big)
\end{aligned}$$

[54] With arbitrary (finite) number of factors, including zero factors.

Note that the element 1 satisfies (7.15) because $\mathrm{d}(1) = 0$ by the Leibniz rule in both $\widetilde{\mathrm{C}}_k(X)$ and $\Omega^\bullet_{k[X]/k}$, and $\Phi_X(1) = 1$. We are thus reduced to show that (7.15) holds for the specific elements $\widetilde{x}$ for $x \in X$.

So let x be an element of X and $F\colon k[t] \to k[X]$ the morphism of commutative k-algebras that maps t to x. By Proposition 7.4.7.1[55] there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\widetilde{\mathrm{C}}_k(\{t\})\big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Phi_{\{t\}})"] \arrow[d, "\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}(F))\right)"'] & \mathrm{Alg}(\gamma)\big(\Omega^\bullet_{k[t]/k}\big) \arrow[d, "\mathrm{Alg}(\gamma)\left(\Omega^\bullet_{F/k}\right)"] \\
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\widetilde{\mathrm{C}}_k(X)\big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(\Phi_X)"'] & \mathrm{Alg}(\gamma)\big(\Omega^\bullet_{k[X]/k}\big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$. As the underlying differential graded algebra of $\widetilde{\mathrm{C}}_k(\{t\})$ is cofibrant by Proposition 4.2.2.12 and every object is fibrant in $\mathrm{Alg}(\mathsf{Ch}(k))$, we obtain from [Hov99, 1.2.10 (ii)] and Proposition A.1.0.1 that the following diagram commutes up to chain homotopy of morphisms of differential graded algebras in the sense of Propositions 4.1.4.2 and 4.2.2.17.

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\widetilde{\mathrm{C}}_k(\{t\})\big) \arrow[r, "\Phi_{\{t\}}"] \arrow[d, "\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}(F))"'] & \Omega^\bullet_{k[t]/k} \arrow[d, "\Omega^\bullet_{F/k}"] \\
\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\widetilde{\mathrm{C}}_k(X)\big) \arrow[r, "\Phi_X"'] & \Omega^\bullet_{k[X]/k}
\end{tikzcd}$$

So let h be such a homotopy of morphisms of differential graded algebras from $\Omega^\bullet_{F/k} \circ \Phi_{\{t\}}$ to $\Phi_X \circ \mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}(F))$. Lift the element t in degree 0 of $\Omega^\bullet_{k[t]/k}$ to a cycle $\widetilde{t}$ in $\widetilde{\mathrm{C}}_k(\{t\})$. Then we have the following.

$$\begin{aligned}
\Phi_X\left(\widetilde{x} - \widetilde{\mathrm{C}}(F)\left(\widetilde{t}\right)\right) &= \Phi_X(\widetilde{x}) - \Phi_X\left(\widetilde{\mathrm{C}}(F)\left(\widetilde{t}\right)\right) \\
&= x + \partial\left(h\left(\widetilde{t}\right)\right) + h\left(\partial\left(\widetilde{t}\right)\right) - \Omega^\bullet_{F/k}\left(\Phi_{\{t\}}\left(\widetilde{t}\right)\right) \\
&= x + 0 + h(0) - \Omega^\bullet_{F/k}(t) \\
&= x - x \\
&= 0
\end{aligned}$$

Thus $\widetilde{x} - \widetilde{\mathrm{C}}(F)(\widetilde{t})$ is a cycle that represents a homology class that maps to 0 under $\mathrm{H}_0(\Phi_X)$. As Φ_X is a quasiisomorphism we must thus have that $\widetilde{x} - \widetilde{\mathrm{C}}(F)(\widetilde{t})$ is a boundary. By

[55] This is the part of this proof that uses the assumption that $|X| \leq 2$.

the argument we gave at the start of this proof it thus suffices to show that (7.15) holds for the element $\widetilde{\mathrm{C}}(F)(\widetilde{t})$. For this we use the following calculation, using that $\Phi_{\{t\}}$ is compatible with d by Proposition 7.4.4.3, and that $\Phi_X(\widetilde{\mathrm{C}}(F)(\widetilde{t})) = \Phi_X(\widetilde{x})$ by the above calculation.

$$\begin{aligned}
\Phi_X\left(\mathrm{d}\left(\widetilde{\mathrm{C}}(F)\left(\widetilde{t}\right)\right)\right) &= \Phi_X\left(\widetilde{\mathrm{C}}(F)\left(\mathrm{d}\,\widetilde{t}\right)\right)\\
&= \Omega^\bullet_{F/k}\left(\Phi_{\{t\}}\left(\mathrm{d}\,\widetilde{t}\right)\right) - \partial\left(h\left(\mathrm{d}\,\widetilde{t}\right)\right) - h\left(\partial\left(\mathrm{d}\,\widetilde{t}\right)\right)\\
&= \Omega^\bullet_{F/k}\left(\mathrm{d}\left(\Phi_{\{t\}}\left(\widetilde{t}\right)\right)\right) - 0 + h\left(\mathrm{d}\left(\partial\left(\widetilde{t}\right)\right)\right)\\
&= \Omega^\bullet_{F/k}(\mathrm{d}\,t) + h\big(\mathrm{d}(0)\big)\\
&= \mathrm{d}\,x\\
&= \mathrm{d}\left(\Phi_X\left(\widetilde{\mathrm{C}}(F)\left(\widetilde{t}\right)\right)\right)
\end{aligned}$$

□

7.4.9. Proof of Conjecture B for sets of cardinality at most 2

The goal of Section 7.4 is to show that Conjecture B holds for $|X| \leq 2$. This is what we do in this subsection, by combining all the ingredients from the previous subsections.

Construction 7.4.9.1. Let X be a totally ordered set with $|X| \leq 2$. We will construct a morphism

$$\Xi_X\colon \Omega'^\bullet_{k[X]/k} \to \widetilde{\mathrm{C}}(X)$$

in $\mathrm{Alg}(\mathsf{Mixed})$, where $\Omega'^\bullet_{k[X]/k}$ is as defined in Definition 7.4.5.9 and Construction 7.4.5.1, and $\widetilde{\mathrm{C}}(X)$ is as defined in Construction 7.4.2.5.

In this construction we will in particular use notation from Construction 7.4.5.1, and also make use of the multiplicative quasiisomorphism Φ_X from Definition 7.4.4.2 and the strict mixed quasiisomorphism Ψ_X from Definition 7.4.3.4.

By the universal property of the colimit it suffices to construct morphisms

$$\Xi_n\colon A_n \to \widetilde{\mathrm{C}}(X)$$

in $\mathrm{Alg}(\mathsf{Mixed})$ for every $n \geq 0$ such that $\Xi_{n+1} \circ \iota_n^{n+1} = \Xi_n$. By the universal property of pushouts and $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ this amounts to the following. To define Ξ_0 we need to prescribe a cycle as the value $\Xi_0(x)$ for every element x of X. If $n \geq 0$, then to lift Ξ_n to Ξ_{n+1} amounts to prescribing a value for $\Xi_{n+1}(\underline{y})$ for every element y of Y_n, under the constraint that

$$\partial\left(\Xi_{n+1}\left(\underline{y}\right)\right) = \Xi_n(y) \tag{$*$}$$

must hold. We will require one additional property that $\Xi_{n+1}(\underline{y})$ should satisfy, namely that

$$\Psi_X\left(\Xi_{n+1}\left(\underline{y}\right)\right) = 0 \tag{$**$}$$

where Ψ_X is as in Definition 7.4.3.4.

Let $n \geq 0$, let y be an element of Y_n, and assume that Ξ_n has already been defined. Note that $\Xi_n(y)$ is a cycle, as y is a cycle by (a) in Construction 7.4.5.1. We claim that if the homology class represented by $\Xi_n(y)$ is zero, then a value for $\Xi_{n+1}(\underline{y})$ can be found that satisfies both $(*)$ and $(**)$. So let z be an element of $\widetilde{\mathrm{C}}(X)$ so that $\partial(z) = \Xi_n(y)$. Then $\Psi_X(z)$ is a cycle (as every element of $\Omega^\bullet_{k[X]/k}$ is), so as Ψ_X is a quasiisomorphism and $\Omega^\bullet_{k[X]/k}$ has zero boundary operator we can lift $\Psi_X(z)$ to a cycle z' in $\widetilde{\mathrm{C}}(X)$ such that $\Psi_X(z') = \Psi_X(z)$. Now set $\Xi_{n+1}(\underline{y}) \coloneqq z - z'$. Then we immediately obtain

$$\Psi_X\left(\Xi_{n+1}\left(\underline{y}\right)\right) = \Psi_x(z) - \Psi_X\left(z'\right) = 0$$

and, using that z' is a cycle,

$$\partial\left(\Xi_{n+1}\left(\underline{y}\right)\right) = \partial\left(z - z'\right) = \partial(z) = \Xi_n(y)$$

so that this definition of $\Xi_{n+1}(\underline{y})$ satisfies both $(*)$ and $(**)$.

We now define Ξ_0 and then Ξ_n for $n > 0$ by induction, in such a way that $\Psi_X \circ \Xi_n$ maps $\underline{y}$ to 0 for all elements $y \in Y_{n'}$ for $n' < n$. By the argument above it suffices for the induction step in which we extend Ξ_n to Ξ_{n+1} for $n \geq 0$ to show that the homology class represented by $\Xi_n(y)$ is zero for every element y of Y_n. As Φ_X and Ψ_X are quasiisomorphisms it in turn suffices for this to show that each of those elements is mapped to zero by $\Phi_X \circ \Xi_0$ or $\Psi_X \circ \Xi_0$.

We thus start with Ξ_0. Let x be an element of X. We need to define a cycle $\Xi_0(x)$. For this we use that as Φ_X is a quasiisomorphism and $\Omega^\bullet_{k[X]/k}$ has zero boundary operator, we can lift the element x of $\Omega^0_{k[X]/k}$ to a cycle $\Xi_0(x)$ in $\widetilde{\mathrm{C}}(X)$. This defines Ξ_0 in such a way that

$$(\Phi_X \circ \Xi_0)(x) = x \tag{7.16}$$

holds for every element x of X.

To extend Ξ_0 to Ξ_1 and then Ξ_2 we use that $\Phi_X \circ \Xi_0$ maps the elements of Y_0 and Y_1 (note that Y_1 lies already in A_0) to zero. This is the case as all elements of Y_0 and Y_1 are given by commutators, so as $\Phi_X \circ \Xi_0$ is multiplicative those elements are mapped to zero as $\Omega^\bullet_{k[X]/k}$ is commutative.

We next extend Ξ_2 to Ξ_3. Let us denote the element(s) of X by $x_1 < \cdots < x_{|X|}$. Then by Proposition 7.4.5.8 the elements of Y_2 are given by the full list below for $|X| = 2$, consist of the first element of the list for $|X| = 1$, and $Y_1 = \emptyset$ for $|X| = 0$.

(1) $\mathrm{d}\,x_1 \cdot \mathrm{d}\,x_1$

(2) $\mathrm{d}\, x_2 \cdot \mathrm{d}\, x_2$

(3) $\mathrm{d}\, \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}$

(4) $\mathrm{d}\, x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\, x_2$
$- x_1 \cdot \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2} \cdot x_1$
$+ x_2 \cdot \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} \cdot x_2$

(5) $\mathrm{d}\, x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\, x_1$
$- x_1 \cdot \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} \cdot x_1$
$+ x_2 \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} \cdot x_2$

Elements (4) and (5) can be handled using Φ_X in the same way as we did with the elements of Y_0 and Y_1, as they are sums of commutators. Elements (1) and (2) can also be handled analogously with Φ_X, this time using that odd degree elements square to zero in $\Omega^\bullet_{k[X]/k}$. It remains to consider element (3). For this element we can use that $\Psi_X \circ \Xi_2$ maps it to 0, which is the case as by induction hypothesis $\Psi_X \circ \Xi_2$ maps every element of the form $\underline{y}$ for y an element of Y_0 or Y_1 to zero, and $\Psi_X \circ \Xi_2$ is also compatible with d.

Now let $n \geq 3$ and assume we have already constructed Ξ_n. To extend Ξ_n to Ξ_{n+1} it suffices to show that $\Phi_X \circ \Xi_n$ maps the elements of Y_n to zero. However the elements of Y_n are of degree n, and $\Omega^n_{k[X]/k} \cong 0$ as $|X| \leq 2 < 3 \leq n$, so this is automatically satisfied. $\diamondsuit$

Proposition 7.4.9.2. *Let X be a totally ordered set with $|X| \leq 2$. Then the morphism*

$$\Xi_X \colon \Omega'^\bullet_{k[X]/k} \to \widetilde{\mathrm{C}}(X)$$

in Alg(Mixed) *that was constructed in Construction 7.4.9.1 is a quasiisomorphism.* $\heartsuit$

Proof. Let us denote the element(s) of X by $x_1 < \cdots < x_{|X|}$. The morphism

$$\Theta_X \colon \Omega'^\bullet_{k[X]/k} \to \Omega^\bullet_{k[X]/k}$$

as defined in Definition 7.4.5.9 is a quasiisomorphism by Proposition 7.4.5.11. By construction Θ_X maps the cycle $x_1^{a_1} \cdots x_{|X|}^{a_{|X|}} \cdot \mathrm{d}\, x_1^{b_1} \cdots \mathrm{d}\, x_{|X|}^{b_{|X|}}$ of $\Omega'^\bullet_{k[X]/k}$ with $a_1, \ldots, a_{|X|} \geq 0$ and $b_1, \ldots, b_{|X|} \in \{0, 1\}$, to the element of $\Omega^\bullet_{k[X]/k}$ with the same name. As the homology classes of those cycles in $\Omega^\bullet_{k[X]/k}$ form a k-basis for the homology, the same must be true for $\Omega'^\bullet_{k[X]/k}$, i. e. the set

$$\left\{ \left[x_1^{a_1} \cdots x_{|X|}^{a_{|X|}} \cdot \mathrm{d}\, x_1^{b_1} \cdots \mathrm{d}\, x_{|X|}^{b_{|X|}} \right] \,\middle|\, a_1, \ldots, a_{|X|} \geq 0,\ b_1, \ldots, b_{|X|} \in \{0, 1\} \right\} \qquad (*)$$

forms a k-basis of the $\mathbb{Z}$-graded k-module $\mathrm{H}_*(\Omega'^\bullet_{k[X]/k})$.

To show that $\mathrm{H}_*(\Xi_X)$ is an isomorphism it suffices to show that $\mathrm{H}_*(\Phi_X \circ \Xi_X)$ is an isomorphism, where Φ_X is the quasiisomorphism defined in Definition 7.4.4.2. For this it now suffices to show that the basis $(*)$ of $\mathrm{H}_*(\Omega'^{\bullet}_{k[X]/k})$ is mapped to a basis of $\mathrm{H}_*(\Omega^{\bullet}_{k[X]/k})$ under $\mathrm{H}_*(\Phi_X \circ \Xi_X)$, and for this it is in turn enough to show that $\Phi_X \circ \Xi_X$ maps the element $x_1^{a_1} \cdots x_{|X|}^{a_{|X|}} \cdot \mathrm{d}\, x_1^{b_1} \cdots \mathrm{d}\, x_{|X|}^{b_{|X|}}$ of $\Omega'^{\bullet}_{k[X]/k}$ for $a_1, \ldots, a_{|X|} \geq 0$ and $b_1, \ldots, b_{|X|} \in \{0, 1\}$ to the element of $\Omega^{\bullet}_{k[X]/k}$ with the same name. As $\Phi_X \circ \Xi_X$ is multiplicative we only need to show that $\Phi_X \circ \Xi_X$ maps elements x to x and $\mathrm{d}\, x$ to $\mathrm{d}\, x$, for each element x in X. That $\Phi_X \circ \Xi_X$ maps elements x of X to x holds by construction of Ξ_X, see (7.16). We can also deduce from this that $\mathrm{d}\, x$ is mapped to $\mathrm{d}\, x$, as Ξ_X is compatible with d, and Φ_X is compatible with d on elements of degree 0 by Proposition 7.4.8.1. □

We can now sum up Section 7.4 as follows.

Corollary 7.4.9.3. *Let X be a totally ordered set with $|X| \leq 2$. Then there is a composite equivalence*

$$\begin{tikzcd}
\mathrm{HH}_{\mathrm{Mixed}}\big(k[X]\big) \arrow[r, "\simeq"] \arrow[drr, dashed, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(X)\Big) & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big) \arrow[l, "\simeq"', "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)"] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Theta_X)"', "\simeq"] \\
& & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[X]/k}\Big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, *where the first equivalence is the one at the bottom of diagram* (7.9) *in Construction 7.4.2.5, the second equivalence is induced by Ξ_X as constructed in Construction 7.4.9.1, and which is a quasiisomorphism by Proposition 7.4.9.2, and the third equivalence is induced by Θ_X as defined in Definition 7.4.5.9, which is a quasiisomorphism by Proposition 7.4.5.11.*

In particular, Conjecture B holds for X. ♡

Remark 7.4.9.4. Usage of Ψ_X is not really necessary in Construction 7.4.9.1, as we could also have arranged for

$$\Phi_X\Big(\Xi_2\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)\Big) = 0$$

and

$$\Phi_X\Big(\Xi_2\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\Big) = \Phi_X\Big(\Xi_1\Big(\mathrm{d}\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big)\Big)$$

instead of equation $(**)$ in Construction 7.4.9.1, and thereby also dealing with the problematic element

$$\mathrm{d}\, \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}$$

that we used Ψ_X to handle in Construction 7.4.9.1, by using Φ_X instead, having the contribution from the third summand exactly cancel out the uncontrollable (under Φ_X) first summand.

The reason Construction 7.4.9.1 was nevertheless written using Ψ_X is that it would not suffice to only use Φ_X anymore in the case $|X| = 3$, as in this case we would have to consider also obstructions to extend to generators of degree 4, and this would involve in particular an element like

$$\mathrm{d}\big(\underline{x \cdot \mathrm{d}\,x - \mathrm{d}\,x \cdot x}\big) + 2 \cdot \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x}$$

in degree 3 that can not be handled with the same idea using Φ_X only unless 2 is invertible in k. However, it is likely that the technique actually used in Construction 7.4.9.1 using Φ_X and Ψ_X extends to the three-variable case, so it would be an unnecessary assumption to assume that 2 is invertible in k.

The case $|X| = 5$ is expected to need different techniques for base rings such as $k = \mathbb{Z}$ in which 3 is not invertible, as the cofibrant resolution $\Omega'^{\bullet}_{k[X]/k}$ will have a generator in degree 6 with boundary of the form[56]

$$\begin{gathered}
x \cdot \mathrm{d}\,\underline{\mathrm{d}\,x \cdot \mathrm{d}\,x} - \mathrm{d}\,\underline{\mathrm{d}\,x \cdot \mathrm{d}\,x} \cdot x \\
+ \mathrm{d}\,x \cdot \mathrm{d}\,\underline{x \cdot \mathrm{d}\,x - \mathrm{d}\,x \cdot x + 2 \cdot \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x}} - \mathrm{d}\,\underline{x \cdot \mathrm{d}\,x - \mathrm{d}\,x \cdot x + 2 \cdot \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x}} \cdot \mathrm{d}\,x \\
+ \mathrm{d}\,\underline{x \cdot \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x} + \mathrm{d}\,x \cdot \underline{x \cdot \mathrm{d}\,x - \mathrm{d}\,x \cdot x} - \underline{x \cdot \mathrm{d}\,x - \mathrm{d}\,x \cdot x} \cdot \mathrm{d}\,x - \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x} \cdot x} \\
+3 \cdot \underline{\mathrm{d}\,x \cdot \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x} + \underline{\mathrm{d}\,x \cdot \mathrm{d}\,x} \cdot \mathrm{d}\,x}
\end{gathered}$$

which involves interactions of the multiplicative and strict mixed structure in a way that does not seem to be handleable using only Φ_X or Ψ_X (unless 3 is invertible). ◊

7.5. De Rham forms as a strict model in Alg(Mixed) and morphisms

In Section 7.4 we discussed Conjecture B, which asks for showing that for polynomial k-algebras de Rham forms are a strict model for Hochschild homology as an object in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. The next upgrade of such an objectwise equivalence would be showing that the morphism induced on de Rham forms by a morphism of polynomial k-algebras represents the induced morphism on Hochschild homology as well. We formulate this as the following conjecture.

Conjecture C. *Let X and Y be sets and $F\colon k[X] \to k[Y]$ a morphism of commutative k-algebras. Then there exists a commutative square*

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(\Omega^{\bullet}_{F/k}\big)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[Y]\big) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[Y]/k}\Big)
\end{tikzcd}$$

[56] The generators of $\Omega'^{\bullet}_{k[X]/k}$, in particular including this expected generator, were found using computer calculations.

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ *such that the horizontal morphisms are equivalences.*

We will often refer to the existence of such a commutative square for a specific F as "Conjecture C holds for F". ♧

Later in this section we will show that Conjecture C holds

- if $|X| = 0$ and $|Y| \leq 2$ by Proposition 7.5.1.1 in Section 7.5.1, and
- if $|X| = 1$ and $|Y| \leq 1$ by Proposition 7.5.2.6 in Section 7.5.2, and
- if $|X| = 1$ and $|Y| = 2$ and 2 is invertible in k by Proposition 7.5.2.6 in Section 7.5.2, and
- if $|X| = 2$ and $|Y| = 0$ by Proposition 7.5.4.1 in Section 7.5.4.

For applications we will need the following variant of Conjecture C, with two squares at once, with the same equivalence in the middle (so this is stronger than just two instances of Conjecture C).

Conjecture D. *Let X be a set and f an element of $k[X]$. Denote by $F\colon k[t] \to k[X]$ the morphism of commutative k-algebras that maps t to f and by $G\colon k[t] \to k$ the morphism of commutative k-algebras that maps t to 0. Then there exists a commutative diagram*

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k/k}\right) \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(k[t]\right) \arrow[r, "\simeq"] \arrow[u, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(G)"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[t]/k}\right) \arrow[u, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{G/k}\right)"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{F/k}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(k[X]\right) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{k[X]/k}\right)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ *such that the horizontal morphisms are equivalences.*

We will often refer to the existence of such a commutative diagram for a specific f as "Conjecture D holds for f". ♧

In Proposition 7.5.3.1 in Section 7.5.3 we will show that Conjecture D holds if $|X| \leq 1$ or $|X| = 2$ and 2 is invertible in k.

We will discuss Conjecture C for $|X| = 0$ in Section 7.5.1, for $|X| = 1$ in Section 7.5.2, and for $|X| = 2$ in Section 7.5.4. Conjecture D will be discussed in Section 7.5.3.

7.5.1. Conjecture C for zero variables in the domain

In this short section we prove Conjecture C in the case that the domain is a polynomial ring in zero variables, in which case Conjecture C is true for formal reasons.

Proposition 7.5.1.1. *Let X be totally ordered set satisfying $|X| \leq 2$. Then there exists a filler for the square*

$$\begin{tikzcd} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\iota_{k[X]})"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k/k}\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{\iota_{k[X]}/k}\right)"] \\ \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[X]/k}\right) \end{tikzcd} \tag{7.17}$$

in Alg($\mathcal{M}$ixed)*, where the horizontal equivalences are the ones from Corollary 7.4.9.3 (for the top horizontal equivalence applied to the empty set).*

In particular, Conjecture C holds for $F = \iota_{k[X]}$ if $|X| \leq 2$. ♡

Proof. $\Omega^{\bullet}_{k/k}$ is isomorphic to k, the monoidal unit of Mixed, considered as an object of Alg(Mixed). As γ_{Mixed} is symmetric monoidal (see Construction 4.4.1.1), k is mapped by Alg(γ_{Mixed}) to an initial object of Alg($\mathcal{M}$ixed) by [HA, 3.2.1.8]. That there is a filler for diagram (7.17) now follows purely from the universal property of initial objects. □

7.5.2. Conjecture C for one variable in the domain

In this section we turn to the much more involved proof that Conjecture C holds for morphisms $F\colon k[t] \to k[X]$ if $|X| \leq 1$ or $|X| = 2$ and 2 is invertible in k. Using that $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant in Alg(Mixed) it will be possible to obtain a morphism

$$\Omega'^{\bullet}_{F/k}\colon \Omega'^{\bullet}_{k[t]/k} \to \Omega'^{\bullet}_{k[X]/k}$$

in Alg(Mixed) so that there is a commutative diagram

$$\begin{tikzcd} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t]) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega'^{\bullet}_{k[t]/k}\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega'^{\bullet}_{F/k}\right)"] \\ \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega'^{\bullet}_{k[X]/k}\right) \end{tikzcd} \tag{7.18}$$

in Alg($\mathcal{M}$ixed). If we could then show that the square

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[t]/k} \arrow[r, "\Theta_{\{t\}}", "\simeq"'] \arrow[d, "\Omega'^{\bullet}_{F/k}"'] & \Omega^{\bullet}_{k[t]/k} \arrow[d, "\Omega^{\bullet}_{F/k}"] \\
\Omega'^{\bullet}_{k[X]/k} \arrow[r, "\simeq", "\Theta_X"'] & \Omega^{\bullet}_{k[X]/k}
\end{tikzcd} \tag{7.19}$$

in Alg(Mixed) commutes (perhaps up to homotopy of algebras in strict mixed complexes), then we would be finished. If $|X| \leq 1$, then it follows from Remark 7.4.5.2 that we only need to check that the two compositions map t to the same element (as the other generators must map to zero for degree reasons), and this is something that is actually true, both compositions mapping t to $F(t)$.

However, if $X = \{ x_1, x_2 \}$ (which we give the total order $x_1 < x_2$), then we also need to check that the two compositions agree on $\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}$. Unfortunately, this will not be the case in general. $\Omega^{\bullet}_{k[t]/k}$ is zero in degree 2, so the composition along the top right will map $\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}$ to zero, but this is not necessarily the case for the composition along the bottom left. The idea to deal with this is to replace Θ_X by a different quasiisomorphism of algebras in strict mixed complexes λ. For λ to be a quasiisomorphism and have the correct value on $\Omega'^{\bullet}_{F/k}(t)$ we will want to set $\lambda(x_i) := x_i$. We have a lot of choice in how we define λ on the higher generators $\underline{y}$ for $y \in Y_n$, which we can choose nearly arbitrarily, the only real restriction being that the following must hold.

$$\mathrm{d}\Big(\lambda\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big) + \lambda\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big) - \lambda\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big) \tag{7.20}$$

So how should we choose $\lambda(\underline{y})$ for $y \in Y_n$ for $n \geq 0$ in order to ensure that we have $\lambda(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})) = 0$ so that the analogue of diagram (7.19) commutes?

The main tool available to understand $\Omega'^{\bullet}_{F/k}$ is naturality of Φ as we showed it in Proposition 7.4.7.1, and we can use this to show that

$$\Phi_X\Big(\Xi_X\Big(\Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big)\Big)\Big) = 0 \tag{7.21}$$

holds. As $\Phi_X \circ \Xi_X$ is a quasiisomorphism and maps x_i to x_i we could thus set λ to $\Phi_X \circ \Xi_X$ if only it were a morphism in Alg(Mixed)! But unfortunately, Φ_X is only multiplicative but is not in general compatible with the strict mixed structure. What we could instead do is to try to define λ in such a way that $\Phi_X \circ \Xi_X$ and λ agree on $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$. As Φ_X is multiplicative and preserves d on elements of degree 0 by Proposition 7.4.8.1, $\Phi_X \circ \Xi_X$ and λ already agree on the $\mathbb{Z}$-graded k-subalgebra generated by elements x_i and $\mathrm{d}\, x_i$. If we for example choose

$$\lambda\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) := \Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big)$$

then the two morphisms would also agree on elements like $x_1 \cdot \mathrm{d}\, x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1}$.

However if terms involving $\mathrm{d}(\underline{x_1 \cdot x_2 - x_2 \cdot x_1})$ appeared in $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$, then we would not be able to deal with this, as we have no way of accessing where $\Phi_X \circ \Xi_X$ maps such an element. So as a first simplification step we need to make a particular choice for $\Omega'^{\bullet}_{F/k}$ for which $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$ is given by a k-linear combination of products of x_1, x_2, $\mathrm{d}\, x_1$, $\mathrm{d}\, x_2$, as well as elements of the form $\underline{y}$ for $y \in Y_n$, but without factors of the form $\mathrm{d}(\underline{y})$. This can be arranged as

$$\mathrm{d}\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) + \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}$$

is a boundary in $\Omega'^{\bullet}_{k[X]/k}$.

If we now just set $\lambda(\underline{y}) \coloneqq \Phi_X(\Xi_X(\underline{y}))$, then it would follow from (7.21) that

$$\lambda\Big(\Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big)\Big) = 0$$

holds as well, so that the analogue of diagram (7.19) commutes. However, the next hurdle is that (7.20) needs to be satisfied. So say if $\lambda(\underline{x_1 \cdot x_2 - x_2 \cdot x_1})$ had been defined in such a way as to be 0, then we must have

$$\lambda\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big) = \lambda\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)$$

and can not choose the two values independently. This is where the assumption that 2 is divisible in k comes in, because combining this assumption with choosing $\Omega'^{\bullet}_{F/k}$ such that $\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}$ and $\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}$ always contribute to $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$ in a pairwise manner we will be able to average out $\Phi_X(\Xi_X(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}))$ and $\Phi_X(\Xi_X(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}))$ between $\lambda\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)$ and $\lambda\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)$, and similarly deal with any possible contributions from $\mathrm{d}(\underline{x_1 \cdot x_2 - x_2 \cdot x_1})$.

We will begin putting this proof strategy into practice by first unpacking the data required to construct morphisms and homotopies with domain $\Omega'^{\bullet}_{k[t]/k}$ in Section 7.5.2.1. We will then be able to show existence of an appropriate morphism $\Omega'^{\bullet}_{F/k}$ in Section 7.5.2.2. Finally, we put everything together in Section 7.5.2.3 to prove that Conjecture C holds for morphisms $F\colon k[t] \to k[X]$ if $|X| \leq 1$ or $|X| = 2$ and 2 is invertible in k.

7.5.2.1. Morphisms and homotopies out of $\Omega'^{\bullet}_{k[t]/k}$

To put this proof strategy described in the introduction to Section 7.5.2 into practice we first need to construct a morphism $\Omega'^{\bullet}_{F/k}$ with the required properties. The next two propositions are helpful for that as they simplify the amount of data we need to provide and the amount of properties we need to check in order to construct morphisms out of $\Omega'^{\bullet}_{k[t]/k}$, and homotopies of such morphisms.

Proposition 7.5.2.1. *Let X be an object of* $\mathrm{Alg}(\mathsf{Mixed})$ *such that* $\mathrm{H}_*(X) \cong 0$ *for* $* > 2$ *and such that elements of* $\mathrm{H}_1(X)$ *square to zero. Let* $\Omega'^{\bullet}_{k[t]/k}$ *be as in Definition 7.4.5.9.*

Let F' be a map of $\mathbb{Z}$-graded sets from the subset $\{\, t, \underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t} \,\}$ *of* $\Omega'^{\bullet}_{k[t]/k}$ *to X, and assume that $F'(t)$ is a cycle and that F' satisfies the following equality.*

$$\partial\Big(F'\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big)\Big) = F'(t) \cdot \mathrm{d}\big(F'(t)\big) - \mathrm{d}\big(F'(t)\big) \cdot F'(t) \tag{7.22}$$

Then F can be extended to a morphism

$$F \colon \Omega'^{\bullet}_{k[t]/k} \to X$$

in $\mathrm{Alg}(\mathsf{Mixed})$. ♡

Proof. We are going to use notation from the construction of $\Omega'^{\bullet}_{k[t]/k}$ in Construction 7.4.5.1 in this proof.

By the universal property of $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ and $k\cdot -$ we obtain a morphism $F_0 \colon A_0 \to X$ in $\mathrm{Alg}(\mathsf{Mixed})$ that maps t to $F'(t)$, where we need to use that $F'(t)$ is a cycle. As Y_0 is empty the morphism ι_0^1 is an isomorphism, so we immediately obtain an extension of F_0 to $F_1 \colon A_1 \to X$. Again by the universal property of $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ as well as pushouts in $\mathrm{Alg}(\mathsf{Mixed})$, we can extend F_1 to a morphism $F_2 \colon A_2 \to X$ in $\mathrm{Alg}(\mathsf{Mixed})$ satisfying $F_2(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}) = F'(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$ if and only if

$$\partial\Big(F'\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big)\Big) = F_1(t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t)$$

holds. But this es precisely ensured by (7.22).

It now suffices to assume that $n \geq 2$ and $F_n \colon A_n \to X$ is a morphism in $\mathrm{Alg}(\mathsf{Mixed})$, and then to show that F_n can be extended to a morphism $F_{n+1} \colon A_{n+1} \to X$. Again by the universal property, this requires finding a value $F_{n+1}(\underline{y})$ for every $y \in Y_n$ such that

$$\partial\Big(F_{n+1}\big(\underline{y}\big)\Big) = F_n(y)$$

holds. But y is a cycle of degree n in A_n by Construction 7.4.5.1 (a), so $F_n(y)$ is a cycle in degree n of X, and such a solution exists if and only if the homology class represented by $F_n(y)$ is zero. If $n > 2$ then this must trivially be true as then $\mathrm{H}_n(X) \cong 0$ by assumption. If instead $n = 2$, then the only element of Y_2 is $\mathrm{d}\, t \cdot \mathrm{d}\, t$. As $\mathrm{d}\, t$ is already a cycle, the homology class $[F_n(\mathrm{d}\, t \cdot \mathrm{d}\, t)]$ is equal to the square of $[F_n(\mathrm{d}\, t)]$ and hence zero by assumption that elements of $H_1(X)$ square to zero. □

Proposition 7.5.2.2. *Let X be an object of* $\mathrm{Alg}(\mathsf{Mixed})$ *such that* $\mathrm{H}_*(X) \cong 0$ *for* $* > 2$. *and let $\Omega'^{\bullet}_{k[t]/k}$ be as in Definition 7.4.5.9.*

Let

$$F, G \colon \Omega'^{\bullet}_{k[t]/k} \to X$$

be two morphisms $\mathrm{Alg}(\mathsf{Mixed})$, *and assume that the elements*

$$F(t) - G(t) \qquad \textit{and} \qquad F(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}) - G(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$$

are boundaries in X.

Then there exists a homotopy of algebras of strict mixed complexes in the sense of Proposition 4.2.2.20 from F to G. ♡

Proof. We are going to use notation from the construction of $\Omega'^{\bullet}_{k[t]/k}$ in Construction 7.4.5.1 in this proof.

As the forgetful functor from Alg(Mixed) to $\mathbb{Z}$-graded k-modules preserves filtered colimits by Proposition 4.2.2.12 it suffices to construct compatible homotopies of algebras of strict mixed complexes h_n from $F \circ \iota_n$ to $G \circ \iota_n$ for every $n \geq 0$.

Let us begin by constructing the homotopy h_0. By Construction 7.4.5.1 the underlying $\mathbb{Z}$-graded k-algebra of A_0 is free on $\{\, t, \mathrm{d}\, t \,\}$. Define h_0 on $\{\, t \,\}$ by mapping t to an element whose boundary is $F(t) - G(t)$ (such an element exists by assumption). As t is a cycle Proposition 4.2.2.21 then immediately furnishes us with an extension to a homotopy of algebras of strict mixed complexes from $F \circ \iota_0$ to $G \circ \iota_0$.

We now assume that h_n has already been defined for $n \geq 0$, and show that h_n can be extended to h_{n+1}. By Proposition 4.2.2.21 and Remark 7.4.5.2 extending h_n to h_{n+1} amounts to finding a value for $h_{n+1}(\underline{y})$ for every element y in Y_n such that

$$\partial\Big(h_{n+1}\big(\underline{y}\big)\Big) = F\big(\underline{y}\big) - G\big(\underline{y}\big) - h_n(y) \tag{$*$}$$

holds. We now distinguish between the case $n = 0$, $n = 1$, and $n \geq 2$.

If $n = 0$, then Y_n is empty, so nothing needs to be done. If $n = 1$, then we have that $Y_n = \{\, t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t \,\}$, so we only need to consider the element $\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}$. By assumption $F(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}) - G(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t})$ is a boundary, so that it suffices to show that $h_0(t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t)$ is a boundary, which the following calculation does.

$$\begin{aligned}
&h_0(t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t)\\
&= h_0(t) \cdot G(\mathrm{d}\, t) + F(t) \cdot h_0(\mathrm{d}\, t) - h_0(\mathrm{d}\, t) \cdot G(t) + F(\mathrm{d}\, t) \cdot h_0(t)\\
&= h_0(t) \cdot \mathrm{d}\big(G(t)\big) - F(t) \cdot \mathrm{d}\big(h_0(t)\big) + \mathrm{d}\big(h_0(t)\big) \cdot G(t) + \mathrm{d}\big(F(t)\big) \cdot h_0(t)\\
&= h_0(t) \cdot \mathrm{d}\big(G(t)\big) + \mathrm{d}\big(F(t)\big) \cdot h_0(t) + \mathrm{d}\big(h_0(t)\big) \cdot G(t) - F(t) \cdot \mathrm{d}\big(h_0(t)\big)\\
&= h_0(t) \cdot \mathrm{d}\big(G(t)\big) - h_0(t) \cdot \mathrm{d}\big(F(t)\big) + \mathrm{d}\big(h_0(t)\big) \cdot G(t) - \mathrm{d}\big(h_0(t)\big) \cdot F(t)\\
&= -h_0(t) \cdot \mathrm{d}\big(F(t) - G(t)\big) - \mathrm{d}\big(h_0(t)\big) \cdot \big(F(t) - G(t)\big)\\
&= -h_0(t) \cdot \mathrm{d}\Big(\partial\big(h_0(t)\big)\Big) - \mathrm{d}\big(h_0(t)\big) \cdot \partial\big(h_0(t)\big)\\
&= h_0(t) \cdot \partial\Big(\mathrm{d}\big(h_0(t)\big)\Big) - \partial\big(h_0(t)\big) \cdot \mathrm{d}\big(h_0(t)\big)\\
&= -\partial\Big(h_0(t) \cdot \mathrm{d}\big(h_0(t)\big)\Big)
\end{aligned}$$

It remains to consider the case $n \geq 2$. Note that the right hand side of equation $(*)$ is in degree $n+1 > 2$, so as $\mathrm{H}_*(X)$ is concentrated in degrees $* \leq 2$ it suffices to show that the right hand side of equation $(*)$ is a cycle. This is shown via the following calculation, with $y \in Y_n$.

$$\begin{aligned}
&\partial\Big(F\big(\underline{y}\big) - G\big(\underline{y}\big) - h_n(y)\Big)\\
&= F\Big(\partial\big(\underline{y}\big)\Big) - G\Big(\partial\big(\underline{y}\big)\Big) - \partial\big(h_n(y)\big)
\end{aligned}$$

$$
\begin{aligned}
&= F(y) - G(y) - \partial\big(h_n(y)\big)\\
&= h_n\big(\partial(y)\big)\\
&= h_n(0)\\
&= 0
\end{aligned}
$$
□

7.5.2.2. Construction of $\Omega'^{\bullet}_{F/k}$

In this section we show the existence of a morphism $\Omega'^{\bullet}_{F/k}$ of appropriate form to put the proof strategy described in the introduction to Section 7.5.2 into practice. In order to be able to properly describe what kind of form $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$ is supposed to have we need to simplify $\Omega'^{\bullet}_{k[X]/k}$ by making it commutative. We thus introduce appropriate notation in the following definition.

Definition 7.5.2.3. Let X be a totally ordered set satisfying $|X| \leq 2$, and let $\Omega'^{\bullet}_{k[X]/k}$ be as in Definition 7.4.5.9.

Then we define

$$\xi_X \colon \Omega'^{\bullet}_{k[X]/k} \to \Omega''^{\bullet}_{k[X]/k}$$

to be the morphism of $\mathbb{Z}$-graded k-algebras that is given by quotienting out the commutator, i. e. ξ_X is initial among morphisms of $\mathbb{Z}$-graded k-algebras with commutative codomain. We will usually not use special notation to distinguish between elements of $\Omega'^{\bullet}_{k[X]/k}$ and their images under ξ_X, but make clear from context in which of the two they lie. It follows from Remark 7.4.5.2 that the $\mathbb{Z}$-graded commutative k-algebra $\Omega''^{\bullet}_{k[X]/k}$ is freely generated (as a *commutative* $\mathbb{Z}$-graded k-algebra) by the elements x and $\mathrm{d}\,x$ for $x \in X$ and $\underline{y}$ and $\mathrm{d}\,\underline{y}$ for $y \in Y_n$ for $n \geq 0$. ♢

We wish to show that there exists a morphism $\Omega'^{\bullet}_{F/k}$ fitting into a square (7.18) and such that $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$ has a specific form[57]. As $\Omega'^{\bullet}_{F/k}$ has to be a morphism of algebras of strict mixed complexes we already know that the boundary will have to be of the following form.

$$\partial\Big(\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})\Big) = \Omega'^{\bullet}_{F/k}(t) \cdot \mathrm{d}\Big(\Omega'^{\bullet}_{F/k}(t)\Big) - \mathrm{d}\Big(\Omega'^{\bullet}_{F/k}(t)\Big) \cdot \Omega'^{\bullet}_{F/k}(t)$$

The strategy to obtain $\Omega'^{\bullet}_{F/k}$ where $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$ is of a specified form will be to first show that every commutator as on the right hand side of the equation is the boundary of an element E of degree 2 of $\Omega'^{\bullet}_{k[X]/k}$ that is of a certain form, and then show that, up to some small adjustments, we can construct $\Omega'^{\bullet}_{k[X]/k}$ in such a way that $\Omega'^{\bullet}_{F/k}(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t})$ is precisely given by E. While the following proposition does not yet refer to $\Omega'^{\bullet}_{F/k}$ it is however the crucial preparatory result in its construction, ensuring that such an E of appropriate form exists.

Proposition 7.5.2.4. *Let X be the set $X = \{ x_1, x_2 \}$ equipped with the total order $x_1 < x_2$. In this proposition we are going to use Definitions 7.4.5.9 and 7.5.2.3.*

[57] For example not involving $\mathrm{d}\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)$.

Let J be the $\mathbb{Z}$-graded subset of $\Omega''^{\bullet}_{k[X]/k}$ that consists of elements of degree 1 of the form $g \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1}$ with g an element of $k[X]$ and of elements of degree 2 of the form

$$\begin{aligned}
&g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \\
&+ g_{\mathrm{both}} \cdot \left(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\right) \\
&+ g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} + g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}
\end{aligned}$$

with $g_{\mathrm{d}\,x_1}$, $g_{\mathrm{d}\,x_2}$, g_{both}, g_{same,x_1}, and g_{same,x_2} elements in $k[X]$. Denote by I the $\mathbb{Z}$-graded subset of $\Omega'^{\bullet}_{k[X]/k}$ that is the preimage of J under ξ_X.

Then the following holds.

(1) Every element of the form $w \cdot w' - w' \cdot w$ for w and w' elements of degree 0 in $\Omega'^{\bullet}_{k[X]/k}$ is the boundary of an element in I.

(2) Every element of the form $w \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot w + w' \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w'$ for w and w' elements of degree 0 in $\Omega'^{\bullet}_{k[X]/k}$ is the boundary of an element of I.

(3) Every element of the form $w \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w$ for w an element of degree 0 in $\Omega'^{\bullet}_{k[X]/k}$ is the boundary of an element of I. ♡

Proof. In this proof we will make use of notation from Construction 7.4.5.1 as well as repeatedly use Remark 7.4.5.2 without further comment.

Before proving the claims let us note that I is closed under k-linear combinations as well as multiplying from either side with an element of X. Furthermore, the product of an element of I of degree 1 with $\mathrm{d}\,x_1$ or $\mathrm{d}\,x_2$ is an element of I again.

If w and w' are as in (1), then we will say that E is a *lift associated to w and w' as in (1)* to mean that E is an element of I such that $\partial(E) = w \cdot w' - w' \cdot w$. We use the analogous convention for (2) and (3).

We now begin by proving (1). For this we note that as $\Omega'^{\bullet}_{k[X]/k}$ is concentrated in nonnegative degrees, the element $w \cdot w' - w' \cdot w$ of degree 0 is a cycle. As $\Omega^{\bullet}_{k[X]/k}$ is commutative, the commutator $w \cdot w' - w' \cdot w$ must be mapped to 0 by Θ_0. $\mathrm{H}_0(\Theta_1)$ is an isomorphism by Proposition 7.4.5.10, so this implies that there is an element E' in degree 1 of A_1 whose boundary is $w \cdot w' - w' \cdot w$. By Remark 7.4.5.2 E' can be written as $E' = E'' + E$ where E'' is an element of A_0 and E is in the k-submodule generated by words in X with one extra factor $\underline{x_1 \cdot x_2 - x_2 \cdot x_1}$, so that E is an element of I. As $\partial(E'') = 0$ we already have $\partial(E) = w \cdot w' - w' \cdot w$, which finishes the proof of (1).

Now we show claim (2), which we will do by reducing to more and more specific w and w', and using claim (1). First assume that w_1, w_2, w_1' and w_2' are elements of degree 0 of $\Omega'^{\bullet}_{k[X]/k}$ such that (2) holds for the pair (w_1, w_1') with associated lift E_{11}, for (w_1, w_2') with associated lift E_{12}, for (w_2, w_1') with associated lift E_{21}, and for (w_2, w_2') with associated lift E_{22}. Let a_1 and a_2 be elements of k. Then we claim that (2) also holds for the pair $(a_1 \cdot w_1 + a_2 \cdot w_2, a_1 \cdot w_1' + a_2 \cdot w_2')$, with associated lift $E = a_1 \cdot a_1 \cdot E_{11} + a_1 \cdot a_2 \cdot E_{12} + a_1 \cdot a_2 \cdot E_{21} + a_2 \cdot a_2 \cdot E_{22}$. That E is again an element of

I follows from the argument at the start of this proof, and that the boundary is what it should be is verified by the following calculation.

$$
\begin{aligned}
&(a_1 \cdot w_1 + a_2 \cdot w_2) \cdot \mathrm{d}\big(a_1 \cdot w_1' + a_2 \cdot w_2'\big) - \mathrm{d}\big(a_1 \cdot w_1' + a_2 \cdot w_2'\big) \cdot (a_1 \cdot w_1 + a_2 \cdot w_2) \\
&\quad + \big(a_1 \cdot w_1' + a_2 \cdot w_2'\big) \cdot \mathrm{d}(a_1 \cdot w_1 + a_2 \cdot w_2) - \mathrm{d}(a_1 \cdot w_1 + a_2 \cdot w_2) \cdot \big(a_1 \cdot w_1' + a_2 \cdot w_2'\big) \\
&= a_1 \cdot a_1 \cdot w_1 \cdot \mathrm{d}\big(w_1'\big) + a_1 \cdot a_2 \cdot w_1 \cdot \mathrm{d}\big(w_2'\big) + a_2 \cdot a_1 \cdot w_2 \cdot \mathrm{d}\big(w_1'\big) + a_2 \cdot a_2 \cdot w_2 \cdot \mathrm{d}\big(w_2'\big) \\
&\quad - a_1 \cdot a_1 \cdot \mathrm{d}\big(w_1'\big) \cdot w_1 - a_1 \cdot a_2 \cdot \mathrm{d}\big(w_1'\big) \cdot w_2 - a_2 \cdot a_1 \cdot \mathrm{d}\big(w_2'\big) \cdot w_1 - a_2 \cdot a_2 \cdot \mathrm{d}\big(w_2'\big) \cdot w_2 \\
&\quad + a_1 \cdot a_1 \cdot w_1' \cdot \mathrm{d}(w_1) + a_1 \cdot a_2 \cdot w_1' \cdot \mathrm{d}(w_2) + a_2 \cdot a_1 \cdot w_2' \cdot \mathrm{d}(w_1) + a_2 \cdot a_2 \cdot w_2' \cdot \mathrm{d}(w_2) \\
&\quad - a_1 \cdot a_1 \cdot \mathrm{d}(w_1) \cdot w_1' - a_1 \cdot a_2 \cdot \mathrm{d}(w_1) \cdot w_2' - a_2 \cdot a_1 \cdot \mathrm{d}(w_2) \cdot w_1' - a_2 \cdot a_2 \cdot \mathrm{d}(w_2) \cdot w_2' \\
&= a_1 \cdot a_1 \cdot w_1 \cdot \mathrm{d}\big(w_1'\big) - a_1 \cdot a_1 \cdot \mathrm{d}\big(w_1'\big) \cdot w_1 + a_1 \cdot a_1 \cdot w_1' \cdot \mathrm{d}(w_1) - a_1 \cdot a_1 \cdot \mathrm{d}(w_1) \cdot w_1' \\
&\quad + a_1 \cdot a_2 \cdot w_1 \cdot \mathrm{d}\big(w_2'\big) - a_1 \cdot a_2 \cdot \mathrm{d}\big(w_2'\big) \cdot w_1 + a_1 \cdot a_2 \cdot w_2' \cdot \mathrm{d}(w_1) - a_1 \cdot a_2 \cdot \mathrm{d}(w_1) \cdot w_2' \\
&\quad + a_1 \cdot a_2 \cdot w_2 \cdot \mathrm{d}\big(w_1'\big) - a_1 \cdot a_2 \cdot \mathrm{d}\big(w_1'\big) \cdot w_2 + a_1 \cdot a_2 \cdot w_1' \cdot \mathrm{d}(w_2) - a_1 \cdot a_2 \cdot \mathrm{d}(w_2) \cdot w_1' \\
&\quad + a_2 \cdot a_2 \cdot w_2 \cdot \mathrm{d}\big(w_2'\big) - a_2 \cdot a_2 \cdot \mathrm{d}\big(w_2'\big) \cdot w_2 + a_2 \cdot a_2 \cdot w_2' \cdot \mathrm{d}(w_2) - a_2 \cdot a_2 \cdot \mathrm{d}(w_2) \cdot w_2' \\
&= \partial(a_1 \cdot a_1 \cdot E_{11} + a_1 \cdot a_2 \cdot E_{12} + a_1 \cdot a_2 \cdot E_{21} + a_2 \cdot a_2 \cdot E_{22})
\end{aligned}
$$

By the above argument it not suffices to show claim (2) for pairs (w, w') of elements of degree 0 of $\Omega'^{\bullet}_{k[X]/k}$ that are in a k-basis. By Remark 7.4.5.2 it thus suffices to consider the case in which w and w' are words in X. If w is a word of length 0 (i. e. $w = 1$), then $w \cdot \mathrm{d}(w') - \mathrm{d}(w') \cdot w + w' \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w' = 0$, so that we can use 0 as an associated lift. Now assume that we have shown (2) for pairs (w, w') where the length of w is smaller or equal to n, for $n \geq 1$, and that w is a word of length n and x an element of X. Then we claim that (2) also holds for $(x \cdot w, w')$. Indeed, let $E_{w,w'}$ be a lift associated to the pair (w, w') and $E_{x,w'}$ a lift associated to the pair (x, w') as in (2), and let $E_{w',w}$ be a lift of $w' \cdot w - w \cdot w'$ and $E_{w',x}$ be a lift of $w' \cdot x - x \cdot w'$ as in (1). Then $E = x \cdot E_{w,w'} + E_{x,w'} \cdot w + \mathrm{d}(x) \cdot E_{w',w} + E_{w',x} \cdot \mathrm{d}(w)$ is again in I and the following calculation then shows that this E is a lift associated to to the pair $(x \cdot w, w')$ as in (2).

$$
\begin{aligned}
&x \cdot w \cdot \mathrm{d}\big(w'\big) - \mathrm{d}\big(w'\big) \cdot x \cdot w + w' \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot w' \\
&= x \cdot w \cdot \mathrm{d}\big(w'\big) - \mathrm{d}\big(w'\big) \cdot x \cdot w \\
&\quad + w' \cdot \mathrm{d}(x) \cdot w + w' \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot w' - x \cdot \mathrm{d}(w) \cdot w' \\
&= x \cdot \Big(w \cdot \mathrm{d}\big(w'\big) - \mathrm{d}\big(w'\big) \cdot w + w' \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w'\Big) \\
&\quad + x \cdot \mathrm{d}\big(w'\big) \cdot w - x \cdot w' \cdot \mathrm{d}(w) - \mathrm{d}\big(w'\big) \cdot x \cdot w \\
&\quad + w' \cdot \mathrm{d}(x) \cdot w + w' \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot w' \\
&= x \cdot \Big(w \cdot \mathrm{d}\big(w'\big) - \mathrm{d}\big(w'\big) \cdot w + w' \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w'\Big) \\
&\quad + \Big(x \cdot \mathrm{d}\big(w'\big) - \mathrm{d}\big(w'\big) \cdot x + w' \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot w'\Big) \cdot w \\
&\quad + \mathrm{d}(x) \cdot w' \cdot w - x \cdot w' \cdot \mathrm{d}(w) + w' \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot w'
\end{aligned}
$$

$$\begin{aligned}
&= x \cdot \Big(w \cdot \mathrm{d}\big(w'\big) - \mathrm{d}\big(w'\big) \cdot w + w' \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w' \Big) \\
&\quad + \Big(x \cdot \mathrm{d}\big(w'\big) - \mathrm{d}\big(w'\big) \cdot x + w' \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot w' \Big) \cdot w \\
&\quad + \mathrm{d}(x) \cdot \big(w' \cdot w - w \cdot w' \big) + \big(w' \cdot x - x \cdot w' \big) \cdot \mathrm{d}(w) \\
&= \partial\big(x \cdot E_{w,w'} + E_{x,w'} \cdot w + \mathrm{d}(x) \cdot E_{w',w} + E_{w',x} \cdot \mathrm{d}(w) \big) \\
&= \partial(E)
\end{aligned}$$

It now remains to show (2) for pairs (x, w') where x is an element of X and w' is a word in X. With a completely analogous argument as the one we just carried out, this time for w' instead of w, we can even reduce to the case of pairs (x, x') with x and x' elements of X. But for such pairs

$$E = \underline{x \cdot \mathrm{d}(x') - \mathrm{d}(x') \cdot x} + \underline{x' \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x'}$$

works as an associated lift.

We now turn to showing claim (3), which we do using a similar strategy as (2). First assume that w and w' are elements of degree 0 of $\Omega'^{\bullet}_{k[X]/k}$ such that (3) holds for w with associated lift E_w, and for w' with associated lift $E_{w'}$. Let a and a' be elements of k and let $E_{w,w'}$ be a lift associated to the pair (w, w') as in (2). Then we claim that (3) also holds for $a \cdot w + a' \cdot w'$ with associated lift $E = a \cdot a \cdot E_w + a' \cdot a' \cdot E_{w'} + a \cdot a' \cdot E_{w,w'}$. That E is again an element of I is covered by the argument at the start of the proof, and the following calculation checks that the boundary is correct as well.

$$\begin{aligned}
&\big(a \cdot w + a' \cdot w'\big) \cdot \mathrm{d}\big(a \cdot w + a' \cdot w'\big) - \mathrm{d}\big(a \cdot w + a' \cdot w'\big) \cdot \big(a \cdot w + a' \cdot w'\big) \\
&= a \cdot a \cdot w \cdot \mathrm{d}(w) + a \cdot a' \cdot w \cdot \mathrm{d}\big(w'\big) + a' \cdot a \cdot w' \cdot \mathrm{d}(w) + a' \cdot a' \cdot w' \cdot \mathrm{d}\big(w'\big) \\
&\quad - a \cdot a \cdot \mathrm{d}(w) \cdot w - a \cdot a' \cdot \mathrm{d}(w) \cdot w' - a' \cdot a \cdot \mathrm{d}\big(w'\big) \cdot w - a' \cdot a' \cdot \mathrm{d}\big(w'\big) \cdot w' \\
&= a \cdot a \cdot w \cdot \mathrm{d}(w) - a \cdot a \cdot \mathrm{d}(w) \cdot w + a' \cdot a' \cdot w' \cdot \mathrm{d}\big(w'\big) - a' \cdot a' \cdot \mathrm{d}\big(w'\big) \cdot w' \\
&\quad + a \cdot a' \cdot w \cdot \mathrm{d}\big(w'\big) - a \cdot a' \cdot \mathrm{d}\big(w'\big) \cdot w + a \cdot a' \cdot w' \cdot \mathrm{d}(w) - a \cdot a' \cdot \mathrm{d}(w) \cdot w' \\
&= \partial\big(a \cdot a \cdot E_w + a' \cdot a' \cdot E_{w'} + a \cdot a' \cdot E_{w,w'}\big)
\end{aligned}$$

It now suffices to show (3) for words in X. Assume that we have already shown (3) for words in X of length smaller or equal to n, and that $n \geq 1$. Let x be an element of X and w a word in X of length n. Let $E_{xw,x}$ be a lift for the pair $(x \cdot w, x)$ as in (2), E_x a lift for x as in (3), E_w a lift for w as in (3), and $E_{w,x}$ a lift for the pair (w, x) as in (1). Then $E = E_{xw,x} \cdot w - E_x \cdot w \cdot w + x \cdot x \cdot E_w + x \cdot E_{w,x} \cdot \mathrm{d}(w)$ is again in I and the following calculation shows that E is a lift for $x \cdot w$ as in (3).

$$\begin{aligned}
&x \cdot w \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x \cdot w \\
&= x \cdot w \cdot \mathrm{d}(x) \cdot w + x \cdot w \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot x \cdot w - x \cdot \mathrm{d}(w) \cdot x \cdot w \\
&= \big(x \cdot w \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x \cdot w + x \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x\big) \cdot w \\
&\quad + \mathrm{d}(x) \cdot x \cdot w \cdot w - x \cdot \mathrm{d}(x \cdot w) \cdot w + \mathrm{d}(x \cdot w) \cdot x \cdot w \\
&\quad + x \cdot w \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot x \cdot w - x \cdot \mathrm{d}(w) \cdot x \cdot w
\end{aligned}$$

$$
\begin{aligned}
&= \big(x \cdot w \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x \cdot w + x \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x\big) \cdot w \\
&\quad + \mathrm{d}(x) \cdot x \cdot w \cdot w - x \cdot \mathrm{d}(x) \cdot w \cdot w - x \cdot x \cdot \mathrm{d}(w) \cdot w \\
&\quad + \mathrm{d}(x) \cdot w \cdot x \cdot w + x \cdot \mathrm{d}(w) \cdot x \cdot w \\
&\quad + x \cdot w \cdot x \cdot \mathrm{d}(w) - \mathrm{d}(x) \cdot w \cdot x \cdot w - x \cdot \mathrm{d}(w) \cdot x \cdot w \\
&= \big(x \cdot w \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x \cdot w + x \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x\big) \cdot w \\
&\quad + \mathrm{d}(x) \cdot x \cdot w \cdot w - x \cdot \mathrm{d}(x) \cdot w \cdot w - x \cdot x \cdot \mathrm{d}(w) \cdot w + x \cdot w \cdot x \cdot \mathrm{d}(w) \\
&= \big(x \cdot w \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x \cdot w + x \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x\big) \cdot w \\
&\quad - \big(x \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x\big) \cdot w \cdot w + x \cdot x \cdot \big(w \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w\big) \\
&\quad - x \cdot x \cdot w \cdot \mathrm{d}(w) + x \cdot w \cdot x \cdot \mathrm{d}(w) \\
&= \big(x \cdot w \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x \cdot w + x \cdot \mathrm{d}(x \cdot w) - \mathrm{d}(x \cdot w) \cdot x\big) \cdot w \\
&\quad - \big(x \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x\big) \cdot w \cdot w + x \cdot x \cdot \big(w \cdot \mathrm{d}(w) - \mathrm{d}(w) \cdot w\big) \\
&\quad + x \cdot (w \cdot x - x \cdot w) \cdot \mathrm{d}(w) \\
&= \partial\big(E_{xw,x} \cdot w - E_x \cdot w \cdot w + x \cdot x \cdot E_w + x \cdot E_{w,x} \cdot \mathrm{d}(w)\big)
\end{aligned}
$$

It thus only remains to show (3) for the elements 1, x_1, and x_2. For 1 we obtain $1 \cdot \mathrm{d}(1) - \mathrm{d}(1) \cdot 1 = 0$, so that we can use 0 as a lift. For x either x_1 or x_2 we can use $x \cdot \mathrm{d}(x) - \mathrm{d}(x) \cdot x$ as a lift. □

With the preparation of Proposition 7.5.2.4 we can now construct a morphism $\Omega'^{\bullet}_{F/k}$ with the required properties in the following proposition.

Proposition 7.5.2.5. *Let X be a totally ordered set satisfying $|X| \leq 2$, and denote the elements of X by $x_1 < \cdots < x_{|X|}$. Let f be an element of $k[X]$, and denote by $F\colon k[t] \to k[X]$ the morphism of commutative k-algebras that maps t to f.*

Then there exists a morphism

$$\Omega'^{\bullet}_{F/k} \colon \Omega'^{\bullet}_{k[t]/k} \to \Omega'^{\bullet}_{k[X]/k}$$

in $\mathrm{Alg}(\mathsf{Mixed})$ such that there exists a commutative diagram

$$
\begin{tikzcd}
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_{\{t\}})", "\simeq"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega'^{\bullet}_{F/k}\right)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(\{t\})\Big) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(F)\right)"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big) \arrow[r, "\simeq", "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(X)\Big)
\end{tikzcd}
$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ where $\Xi_{\{t\}}$ and Ξ_X are as in Construction 7.4.9.1, and such that $\xi_X \circ \Omega'^{\bullet}_{F/k}$ maps t to f (see Definition 7.5.2.3 for a definition of ξ_X).

If $|X| = 2$*, then* $\Omega'^{\bullet}_{F/k}$ *can furthermore be chosen such that there additionally exist elements* $g_{\mathrm{d}\,x_1}$*,* $g_{\mathrm{d}\,x_2}$*,* g_{both}*,* g_{same,x_1}*,* g_{same,x_2}*, and* g_{obs} *in* $k[X]$ *such that*

$$\begin{aligned}
&\xi_X\left(\Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}\big)\right) \\
&= g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \\
&\quad + g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big) \\
&\quad + g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} + g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \\
&\quad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2
\end{aligned} \tag{7.23}$$

holds in $\Omega''^{\bullet}_{k[X]/k}$. ♡

Proof. As $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant as an object of Alg(Mixed) by Proposition 7.4.5.11, we can lift the composition

$$\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)^{-1} \circ \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(F)\right) \circ \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Xi_{\{t\}}\right)$$

to a morphism

$$G\colon \Omega'^{\bullet}_{k[t]/k} \to \Omega'^{\bullet}_{k[X]/k}$$

that thus comes with a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega'^{\bullet}_{k[t]/k}\right) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_{\{t\}})", "\simeq"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(G)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(\{t\})\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\widetilde{\mathrm{C}}(F))"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega'^{\bullet}_{k[X]/k}\right) \arrow[r, "\simeq", "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\widetilde{\mathrm{C}}(X)\right)
\end{tikzcd} \tag{7.24}$$

in Alg($\mathcal{M}$ixed). It now suffices by [Hov99, 1.2.10 (ii)] and Propositions A.1.0.1 and 4.2.2.20 to show that there is a homotopy of algebras of strict mixed complexes from G to a morphism $\Omega'^{\bullet}_{F/k}$ that takes the required form on the elements t and $\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}$.

We begin by showing that G already maps t to an acceptable value. For this we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\Omega'^{\bullet}_{k[t]/k}\big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)(G)"] \arrow[d, "\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_{\{t\}})\right)", "\simeq"'] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\Omega'^{\bullet}_{k[X]/k}\big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_{X})\right)"', "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\widetilde{\mathrm{C}}(\{t\})\big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}(F))\right)"] \arrow[d, "\mathrm{Alg}(\gamma)(\Phi_{\{t\}})", "\simeq"'] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\widetilde{\mathrm{C}}(X)\big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)(\Phi_X)"', "\simeq"] \\
\mathrm{Alg}(\gamma)\big(\Omega^{\bullet}_{k[t]/k}\big) \arrow[r, "\mathrm{Alg}(\gamma)(\Omega^{\bullet}_{F/k})"] & \mathrm{Alg}(\gamma)\big(\Omega^{\bullet}_{k[X]/k}\big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where the top square is obtained from the transpose of diagram (7.24) by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ and using compatibility with γ_{Mixed} (see Construction 4.4.1.1), and the bottom square is the one from Proposition 7.4.7.1. The underlying differential graded k-algebra of $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant by Propositions 7.4.5.11 and 4.2.2.12, so we can conclude by [Hov99, 1.2.10 (ii)], Propositions A.1.0.1 and 4.2.2.17 that there exists a homotopy of differential graded k-algebras h from $\Phi_X \circ \Xi_X \circ G$ to $\Omega^{\bullet}_{F/k} \circ \Phi_{\{t\}} \circ \Xi_{\{t\}}$ (we omit forgetful functors in the notation here) in the sense of Proposition 4.2.2.17. We can then carry out the following calculation, where we use that $(\Phi_{\{t\}} \circ \Xi_{\{t\}})(t) = t$ by definition of $\Xi_{\{t\}}$, see around equation (7.16) of Construction 7.4.9.1.

$$\begin{aligned}
(\Phi_X \circ \Xi_X \circ G)(t) &= \Big(\Omega^{\bullet}_{F/k} \circ \Phi_{\{t\}} \circ \Xi_{\{t\}}\Big)(t) + \partial\big(h(t)\big) + h\big(\partial(t)\big) \\
&= \Omega^{\bullet}_{F/k}\Big(\big(\Phi_{\{t\}} \circ \Xi_{\{t\}}\big)(t)\Big) + 0 + h(0) \\
&= \Omega^{\bullet}_{F/k}(t) \\
&= f
\end{aligned}$$

By the universal property of ξ_X there exists a commutative diagram

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[X]/k} \arrow[r, "\Xi_X"] \arrow[d, "\xi_X"'] & \widetilde{\mathrm{C}}(X) \arrow[r, "\Phi_X"] & \Omega^{\bullet}_{k[X]/k} \\
\Omega''^{\bullet}_{k[X]/k} \arrow[urr, dashed] & &
\end{tikzcd}$$

of $\mathbb{Z}$-graded k-algebras, and as $\Phi_X \circ \Xi_X$ maps elements x_i of X to x_i by Construction 7.4.9.1, it follows from Remark 7.4.5.2 that the dashed morphism is an isomorphism in degree 0, mapping x_i to x_i. That $(\Phi_X \circ \Xi_X)(G(t)) = f$ thus implies that $\xi_X(G(t)) = f$.

If $|X| < 2$ we can now define $\Omega'^{\bullet}_{F/k} \coloneqq G$ and are finished. So from now on we will assume that $X = \{ x_1, x_2 \}$. Unfortunately the value of G at $\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}$ is not automatically of the right form, so we will need to replace G by a homotopic morphism that takes a different value at $\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}$, but the same one at t.

By Proposition 7.5.2.4 (3) we can let E be an element of degree 2 in $\Omega'^{\bullet}_{k[X]/k}$ satisfying the following two properties.

(1) $\partial(E) = G(t) \cdot \mathrm{d}\big(G(t)\big) - \mathrm{d}\big(G(t)\big) \cdot G(t)$

(2) There exist elements $g_{\mathrm{d}\, x_1}$, $g_{\mathrm{d}\, x_2}$, g_{both}, g_{same,x_1}, and g_{same,x_2} in $k[X]$ such that

$$\begin{aligned}\xi_X(E) = {} & g_{\mathrm{d}\, x_1} \cdot \mathrm{d}\, x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\, x_2} \cdot \mathrm{d}\, x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \\ & + g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big) \\ & + g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} + g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2}\end{aligned}$$

holds.

We first note that

$$\partial\Big(G\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big) - E\Big) = G(t) \cdot \mathrm{d}\big(G(t)\big) - \mathrm{d}\big(G(t)\big) \cdot G(t) - \partial(E) = 0$$

so that $G\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big) - E$ is a cycle. As Θ_0 (see the construction of $\Omega'^{\bullet}_{k[X]/k}$ in Construction 7.4.5.1) is surjective on homology by Proposition 7.4.5.10, we can find a cycle z in A_0 such that the homology classes represented by z and $G\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big) - E$ map to the same homology class in $\Omega'^{\bullet}_{k[X]/k}$ under Θ_X. As Θ_X is a quasiisomorphism by Proposition 7.4.5.11 this implies that

$$G\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big) - E - z \tag{$**$}$$

must be a boundary.

We now want to apply Proposition 7.5.2.1 to obtain a morphism

$$\Omega'^{\bullet}_{F/k} \colon \Omega'^{\bullet}_{k[t]/k} \to \Omega'^{\bullet}_{k[X]/k}$$

in Alg(Mixed) with $\Omega'^{\bullet}_{F/k}(t) = G(t)$ and $\Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big) = E + z$. We first note that as $\mathrm{H}_*(\Phi_X \circ \Xi_X)$ is a multiplicative isomorphism by Definition 7.4.4.2 and Proposition 7.4.9.2 it holds that $\mathrm{H}_*(\Omega'^{\bullet}_{k[X]/k})$ is zero above degree 2 and that odd degree elements square to zero. That $G(t)$ is a cycle is clear as G is a morphism of chain complexes and t is a cycle in $\Omega'^{\bullet}_{k[t]/k}$. Finally, (7.22) holds in this context, as this follows from (1) above combined with z being a cycle. Thus we can apply Proposition 7.5.2.1 to obtain a morphism $\Omega'^{\bullet}_{F/k}$ with the prescribed values.

We next show that $\Omega'^{\bullet}_{F/k}$ is indeed homotopic to G. For this we use Proposition 7.5.2.2, so that we have to show that

$$G(t) - \Omega'^{\bullet}_{F/k}(t) \qquad \text{and} \qquad G\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big) - \Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big)$$

are boundaries. The first term is 0 by definition, and that the second is a boundary was ensured around ($**$) (we chose z specifically so that this would hold). Thus Proposition 7.5.2.1 applies to show that there indeed exists a homotopy of algebras in strict mixed complexes from G to $\Omega'^{\bullet}_{F/k}$.

It remains to show that the two values of $\xi_X \circ \Omega'^{\bullet}_{F/k}$ are as required. For t this is clear as

$$\xi_X\left(\Omega'^{\bullet}_{F/k}(t)\right) = \xi_X\big(G(t)\big) = f$$

holds, as we discussed at the start of this proof. For $\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}$ we note that the image of $\xi_X \circ \Theta_0$ is generated by the multiplicative generators x_1, x_2, $\mathrm{d}\, x_1$, and $\mathrm{d}\, x_2$. Thus the element z of degree 2 in A_0 must map to an element of the form $g_{\mathrm{obs}} \cdot \mathrm{d}\, x_1 \cdot \mathrm{d}\, x_2$ with g_{obs} an element of $k[x_1, x_2]$. Then we obtain the following by combining the definition just made with (2).

$$\begin{aligned}
&\xi_X\left(\Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big)\right)\\
&= \xi_X(E) + \xi_X(z)\\
&= g_{\mathrm{d}\, x_1} \cdot \mathrm{d}\, x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\, x_2} \cdot \mathrm{d}\, x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1}\\
&\quad + g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\\
&\quad + g_{\mathrm{same}, x_1} \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} + g_{\mathrm{same}, x_2} \cdot \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2}\\
&\quad + g_{\mathrm{obs}} \cdot \mathrm{d}\, x_1 \cdot \mathrm{d}\, x_2
\end{aligned}$$

□

7.5.2.3. Conclusion

Having constructed $\Omega'^{\bullet}_{F/k}$ in the preceding Section 7.5.2.2 we can now use it to show Conjecture C in certain cases using the strategy sketched in the introduction to Section 7.5.2. Note that what we show is actually slightly stronger than Conjecture C, as we show that there is a specific top horizontal equivalence in diagram (7.25) that is independent of X and f. This is what allows us to even deduce Conjecture D from this, as we do in Proposition 7.5.3.1 in Section 7.5.3.

Proposition 7.5.2.6. *Let X be a set, let f be an element of $k[X]$, and denote by $F\colon k[t] \to k[X]$ the morphism of commutative k-algebras that maps t to f. Assume that one of the following holds.*

(1) $|X| \leq 1$.

(2) $|X| = 2$ *and 2 is invertible in k.*

Then there exists a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathrm{Mixed}}\big(k[t]\big) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathrm{Mixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[t]/k}\right) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{F/k}\right)"] \\
\mathrm{HH}_{\mathrm{Mixed}}\big(k[X]\big) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[X]/k}\right)
\end{tikzcd} \qquad (7.25)$$

in Alg($\mathcal{M}$ixed) *such that the top horizontal equivalence is the one from Corollary 7.4.9.3 and the bottom horizontal morphism is an equivalence*[58]. *In particular, Conjecture C holds for F.* ♡

Proof. We begin by equipping X with a total order, and will denote the elements of X by $x_1 < \cdots < x_{|X|}$. Consider the following (non-commuting) diagram in Alg($\mathcal{M}$ixed), that will be explained below.

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[t]\big) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"] \arrow[d, dash, "\simeq"] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}\big(\{t\}\big)\Big) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(\widetilde{\mathrm{C}}(F)\big)"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(X)\Big) \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(\Omega'^{\bullet}_{F/k}\big)"] \arrow[u, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_{\{t\}})", "\simeq"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Theta_{\{t\}})"', "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big) \arrow[u, "\simeq", "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)"'] \arrow[d, dashed, "\simeq"'] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(\Omega^{\bullet}_{F/k}\big)"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[X]/k}\Big)
\end{tikzcd} \tag{$*$}$$

The top square has a filler given by the (transpose of) commutative diagram (7.10) from the definition of $\widetilde{\mathrm{C}}(F)$ in Construction 7.4.2.5, $\Xi_{\{t\}}$, Ξ_X, and $\Theta_{\{t\}}$ are as in Construction 7.4.9.1 and Definition 7.4.5.9, and $\Omega'^{\bullet}_{F/k}$ is as in Proposition 7.5.2.5 so that the middle square has a filler as well.

By Corollary 7.4.9.3 the vertical composition on the left is the top horizontal equivalence in diagram (7.25) from the statement. As the top and middle square have fillers it thus suffices to construct a quasiisomorphism of algebras in strict mixed complexes

$$\lambda\colon \Omega'^{\bullet}_{k[X]/k} \to \Omega^{\bullet}_{k[X]/k}$$

such that the diagram

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[t]/k} \arrow[r, "\Theta_{\{t\}}", "\simeq"'] \arrow[d, "\Omega'^{\bullet}_{F/k}"'] & \Omega^{\bullet}_{k[t]/k} \arrow[d, "\Omega^{\bullet}_{F/k}"] \\
\Omega'^{\bullet}_{k[X]/k} \arrow[r, dashed, "\simeq", "\lambda"'] & \Omega^{\bullet}_{k[X]/k}
\end{tikzcd} \tag{$**$}$$

[58] We do *not* claim that there exists a filler for such a square where also the bottom horizontal equivalence is given by the one from Corollary 7.4.9.3.

in Alg(Mixed) commutes strictly.

Suppose for the moment that we have defined a λ. Using notation from Construction 7.4.5.1, it follows from Remark 7.4.5.2 that for checking strict commutativity of $(**)$ it suffices to check that the diagram commutes on the element t as well as elements of the form $\underline{y}$ for y an element of one of the sets Y_m for $m \geq 0$ used in the definition of $\Omega'^{\bullet}_{k[t]/k}$. But elements of Y_m have degree m so that $\underline{y}$ is of degree $m+1$. As we assume $|X| \leq 2$, we have that $\Omega^{\bullet}_{k[X]/k}$ is concentrated in degrees at most 2, so diagram $(**)$ will commute on elements $\underline{y}$ for y an element of Y_m for $m \geq 2$ automatically, and if even $|X| \leq 1$ then it will commute automatically on such elements for $m \geq 1$. As Y_0 is empty by Definition 7.4.5.9 and Proposition 7.4.5.6 and Y_1 has only one element $t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t$ by Definition 7.4.5.9 and Proposition 7.4.5.7, this means that it suffices to check that the following two equations hold if $|X| = 2$, and only that the first one holds if $|X| \leq 1$.

$$\begin{aligned}
\lambda\Big(\Omega'^{\bullet}_{F/k}(t)\Big) &= \Omega^{\bullet}_{F/k}\big(\Theta_{\{t\}}(t)\big) \\
\lambda\Big(\Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big)\Big) &= \Omega^{\bullet}_{F/k}\Big(\Theta_{\{t\}}\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big)\Big)
\end{aligned}$$

We can evaluate the right hand sides. By definition $\Theta_{\{t\}}$ maps t to t and $\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}$ to 0. Thus we need to define λ such that it is a quasiisomorphism and show that both of the following equations hold if $|X| = 2$, and that the first one holds if $|X| \leq 1$.

$$\begin{aligned}
\lambda\Big(\Omega'^{\bullet}_{F/k}(t)\Big) &= f \\
\lambda\Big(\Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\, t - \mathrm{d}\, t \cdot t}\big)\Big) &= 0
\end{aligned} \tag{$***$}$$

We can now already show the statement under the assumption that $|X| \leq 1$. In that case, we let λ be the quasiisomorphism of algebras in strict mixed complexes Θ_X from Definition 7.4.5.9. We only need to verify that the first equation of $(***)$ holds for this choice of λ. As the underlying $\mathbb{Z}$-graded k-algebra of $\Omega^{\bullet}_{k[X]/k}$ is commutative, the underlying morphism of λ factors as in the following diagram of $\mathbb{Z}$-graded k-algebras.

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[X]/k} \arrow[rr, "\lambda"] \arrow[d, "\xi_X"'] & & \Omega^{\bullet}_{k[X]/k} \\
\Omega''^{\bullet}_{k[X]/k} \arrow[urr, dashed, "\lambda''"'] & &
\end{tikzcd}$$

As λ and ξ_X map the elements x_i of X to x_i (considered as elements of the respective $\mathbb{Z}$-graded k-algebras), the same holds for λ'', so that in particular $\lambda''(f) = f$. By Proposition 7.5.2.5 we know that $\xi_X\Big(\Omega'^{\bullet}_{F/k}(t)\Big) = f$, so it follows that

$$\lambda\Big(\Omega'^{\bullet}_{F/k}(t)\Big) = \lambda''\Big(\xi_X\Big(\Omega'^{\bullet}_{F/k}(t)\Big)\Big) = \lambda''(f) = f$$

holds.

We now consider the case $|X| = 2$, and thus assume that 2 is invertible in k. In this case setting λ to Θ_X will unfortunately not work in general. We will in the following use notation from the construction of $\Omega'^{\bullet}_{k[X]/k}$ in Construction 7.4.5.1, as well as the concrete choices for Y_0, Y_1 and Y_2 in Definition 7.4.5.9. We will define λ using the universal property of the definition of $\Omega'^{\bullet}_{k[X]/k}$ as a colimit by constructing a compatible system of morphisms $\lambda_n \colon A_n \to \Omega^{\bullet}_{k[X]/k}$ in $\mathrm{Alg}(\mathsf{Mixed})$ for every $n \geq 0$.

We will begin by defining λ_0 using the universal property of $\mathrm{Free}^{\mathrm{Alg}(\mathsf{Mixed})}$ by prescribing $\lambda_0(x_i) = x_i$. We first note that the argument that we used above in the case $|X| \leq 1$ to show that the first equation of $(***)$ holds did not use that $\lambda = \Theta_X$, but only that λ maps x_i to x_i, and hence this argument is still applicable. Thus it only remains to show that λ_0 can be extended to a morphism $\lambda \colon \Omega'^{\bullet}_{k[X]/k} \to \Omega^{\bullet}_{k[X]/k}$ in $\mathrm{Alg}(\mathsf{Mixed})$ that is a quasiisomorphism and that is such that the second equation of $(***)$ holds.

We claim that *any* extension of λ_0 to λ is automatically a quasiisomorphism. For this we note that the $\mathbb{Z}$-graded subset

$$\left\{ \left[x_1^{a_1} \cdot x_2^{a_2} \cdot \mathrm{d}(x_1)^{b_1} \cdot \mathrm{d}(x_2)^{b_2} \right] \;\middle|\; a_1, a_2 \geq 0,\ b_1, b_2 \in \{0, 1\} \right\}$$

of $\mathrm{H}_*(\Omega'^{\bullet}_{k[X]/k})$ forms a k-basis of $\mathrm{H}_*(\Omega'^{\bullet}_{k[X]/k})$, as $\mathrm{H}_*(\Theta_X)$ is an isomorphism and maps this set to the set with the same description (see Construction 7.4.5.1 and Proposition 7.4.5.11). As this subset is also mapped by $\mathrm{H}_*(\lambda)$ to the same subset of $\mathrm{H}_*(\Omega^{\bullet}_{k[X]/k})$ it follows that λ is a quasiisomorphism as well.

It thus suffices to show that there is some extension of λ_0 to λ such that the second equation of $(***)$ holds. We will now inductively assume that λ_n has already been defined for $n \geq 0$ and then extend λ_n to λ_{n+1}. By construction such an extension amounts to defining a value for $\lambda_{n+1}(\underline{y})$ for every element y of Y_n, and showing that

$$\partial\Big(\lambda_{n+1}(\underline{y})\Big) = \lambda_n(y)$$

holds in $\Omega^{\bullet}_{k[X]/k}$. As $\Omega^{\bullet}_{k[X]/k}$ has zero boundary operator the left hand side is always zero and in particular does not depend on what we chose for $\lambda_{n+1}(\underline{y})$. So for an extension to λ_{n+1} to exist λ_n must map all elements of Y_n to zero, and then we are free to prescribe any value for $\lambda_{n+1}(\underline{y})$ for elements y of Y_n. Note that $\lambda_n(y)$ lies in $\Omega^{n}_{k[X]/k}$, so as we assumed $|X| = 2$ this is automatically zero if $n \geq 3$, and hence we can already conclude that an extension of λ_3 to λ exists.

To extend λ_0 to λ_1 we need to check that

$$\lambda_0(x_1 \cdot x_2 - x_2 \cdot x_1) = 0$$

which is clear as $\Omega^{\bullet}_{k[X]/k}$ is commutative, and can then set the following value.

$$\lambda_1\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) \coloneqq \Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big)$$

Next, to extend λ_1 to λ_2 we need to check

$$\begin{aligned}
\lambda_1(x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1) &= 0\\
\lambda_1(x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1) &= 0\\
\lambda_1(x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2) &= 0\\
\lambda_1(x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2) &= 0
\end{aligned}$$

all of which are clear as $\Omega^\bullet_{k[X]/k}$ is commutative, and can then prescribe the following values.

$$\begin{aligned}
&\lambda_2\big(\underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1}\big) \coloneqq \Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1}\big)\Big)\\
&\lambda_2\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big) \coloneqq\\
&\qquad \frac{1}{2}\bigg(\Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)\Big) + \Phi_X\Big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\Big)\bigg)\\
&\qquad - \frac{1}{2}\,\mathrm{d}\bigg(\Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big)\bigg)\\
&\lambda_2\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big) \coloneqq\\
&\qquad \frac{1}{2}\bigg(\Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)\Big) + \Phi_X\Big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\Big)\bigg)\\
&\qquad + \frac{1}{2}\,\mathrm{d}\bigg(\Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big)\bigg)\\
&\lambda_2\big(\underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2}\big) \coloneqq \Phi_X\Big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2}\big)\Big)
\end{aligned}$$

Finally, we need to extend λ_2 to λ_3. For this we need to check the following.

$$\begin{aligned}
&\lambda_2\Big(\mathrm{d}\, x_1 \cdot \mathrm{d}\, x_1\Big) = 0\\
&\lambda_2\Big(\mathrm{d}\, x_2 \cdot \mathrm{d}\, x_2\Big) = 0\\
&\lambda_2\Big(\mathrm{d}\,\underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\Big) = 0\\
&\lambda_2\Big(\mathrm{d}\, x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\, x_2\\
&\quad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_2} \cdot x_1\\
&\quad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} \cdot x_2\Big) = 0\\
&\lambda_2\Big(\mathrm{d}\, x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \cdot \mathrm{d}\, x_1\\
&\quad - x_1 \cdot \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} + \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2} \cdot x_1\\
&\quad + x_2 \cdot \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} - \underline{x_1 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_1} \cdot x_2\Big) = 0
\end{aligned}$$

The first two equations are satisfied as odd degree elements in $\Omega^\bullet_{k[X]/k}$ square to zero and the last two as $\Omega^\bullet_{k[X]/k}$ is commutative. It remains to show that middle equation, which is shown by the following calculation. The values for $\lambda_2(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1})$ and $\lambda_2(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2})$ were chosen precisely so as to make this work out, and this is why we needed that 2 is invertible in k.

$$\begin{aligned}
&\lambda_2\Big(\mathrm{d}\,\underline{x_1 \cdot x_2 - x_2 \cdot x_1} + \underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1} - \underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\Big)\\
&= \mathrm{d}\Big(\Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big)\Big)\\
&\quad + \frac{1}{2}\Big(\Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)\Big) + \Phi_X\Big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\Big)\Big)\\
&\quad - \frac{1}{2}\mathrm{d}\Big(\Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big)\Big)\\
&\quad - \frac{1}{2}\Big(\Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot \mathrm{d}\, x_2 - \mathrm{d}\, x_2 \cdot x_1}\big)\Big) + \Phi_X\Big(\Xi_X\big(\underline{x_2 \cdot \mathrm{d}\, x_1 - \mathrm{d}\, x_1 \cdot x_2}\big)\Big)\Big)\\
&\quad - \frac{1}{2}\mathrm{d}\Big(\Phi_X\Big(\Xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big)\Big)\\
&= 0
\end{aligned}$$

Thus we can extend λ_2 to λ_3 by mapping $\underline{y}$ to 0 for y an element of Y_2.

As already mentioned λ_3 can be further be extended to λ. It now only remains to show that the second equation of $(***)$ holds. Again as the underlying $\mathbb{Z}$-graded k-algebra of $\Omega^{\bullet}_{k[X]/k}$ is commutative, the underlying morphism of λ factors as in the following diagram of $\mathbb{Z}$-graded k-algebras.

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[X]/k} \arrow[r, "\lambda"] \arrow[d, "\xi_X"'] & \Omega^{\bullet}_{k[X]/k} \\
\Omega''^{\bullet}_{k[X]/k} \arrow[ur, dashed, bend right, "\lambda''"'] &
\end{tikzcd}$$

as we already had above. Similarly we can factor $\Phi_X \circ \Xi_X$ as follows.

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[X]/k} \arrow[r, "\Phi_X \circ \Xi_X"] \arrow[d, "\xi_X"'] & \Omega^{\bullet}_{k[X]/k} \\
\Omega''^{\bullet}_{k[X]/k} \arrow[ur, dashed, bend right, "\Phi''_X"'] &
\end{tikzcd}$$

We now begin with the following calculation, where we let $g_{\mathrm{d}\, x_1}$, $g_{\mathrm{d}\, x_2}$, g_{both}, g_{same, x_1}, g_{same, x_2}, and g_{obs} be elements in $k[X]$ as in Proposition 7.5.2.5 so that (7.23) holds. Note that as λ maps x_i to x_i and hence also $\mathrm{d}\,x_i$ to $\mathrm{d}\,x_i$, the same is true for λ''.

$$
\begin{aligned}
&\lambda\Big(\Omega'^{\bullet}_{F/k}\big(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}\big)\Big)\\
&= \lambda''\Big(\xi_X\Big(\Omega'^{\bullet}_{F/k}\big(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}\big)\Big)\Big)\\
&= \lambda''\Big(g_{\mathrm{d}\,x_1}\cdot \mathrm{d}\,x_1\cdot \underline{x_1\cdot x_2 - x_2\cdot x_1} + g_{\mathrm{d}\,x_2}\cdot \mathrm{d}\,x_2\cdot \underline{x_1\cdot x_2 - x_2\cdot x_1}\\
&\qquad + g_{\mathrm{both}}\cdot \big(\underline{x_1\cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2\cdot x_1} + \underline{x_2\cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1\cdot x_2}\big)\\
&\qquad + g_{\mathrm{same},x_1}\cdot \underline{x_1\cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1\cdot x_1} + g_{\mathrm{same},x_2}\cdot \underline{x_2\cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2\cdot x_2}\\
&\qquad + g_{\mathrm{obs}}\cdot \mathrm{d}\,x_1\cdot \mathrm{d}\,x_2\Big)\\
&= g_{\mathrm{d}\,x_1}\cdot \mathrm{d}\,x_1\cdot \lambda\big(\underline{x_1\cdot x_2 - x_2\cdot x_1}\big) + g_{\mathrm{d}\,x_2}\cdot \mathrm{d}\,x_2\cdot \lambda\big(\underline{x_1\cdot x_2 - x_2\cdot x_1}\big)\\
&\quad + g_{\mathrm{both}}\cdot \lambda\big(\underline{x_1\cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2\cdot x_1}\big)\\
&\quad + g_{\mathrm{both}}\cdot \lambda\big(\underline{x_2\cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1\cdot x_2}\big)\\
&\quad + g_{\mathrm{same},x_1}\cdot \lambda\big(\underline{x_1\cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1\cdot x_1}\big) + g_{\mathrm{same},x_2}\cdot \lambda\big(\underline{x_2\cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2\cdot x_2}\big)\\
&\quad + g_{\mathrm{obs}}\cdot \mathrm{d}\,x_1\cdot \mathrm{d}\,x_2\\
&= g_{\mathrm{d}\,x_1}\cdot \mathrm{d}\,x_1\cdot \Phi_X\Big(\Xi_X\big(\underline{x_1\cdot x_2 - x_2\cdot x_1}\big)\Big) + g_{\mathrm{d}\,x_2}\cdot \mathrm{d}\,x_2\cdot \Phi_X\Big(\Xi_X\big(\underline{x_1\cdot x_2 - x_2\cdot x_1}\big)\Big)\\
&\quad + g_{\mathrm{both}}\cdot \frac{1}{2}\Big(\Phi_X\Big(\Xi_X\big(\underline{x_1\cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2\cdot x_1}\big)\Big) + \Phi_X\Big(\Xi_X\big(\underline{x_2\cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1\cdot x_2}\big)\Big)\Big)\\
&\quad - g_{\mathrm{both}}\cdot \frac{1}{2}\,\mathrm{d}\Big(\Phi_X\Big(\Xi_X\big(\underline{x_1\cdot x_2 - x_2\cdot x_1}\big)\Big)\Big)\\
&\quad + g_{\mathrm{both}}\cdot \frac{1}{2}\Big(\Phi_X\Big(\Xi_X\big(\underline{x_1\cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2\cdot x_1}\big)\Big) + \Phi_X\Big(\Xi_X\big(\underline{x_2\cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1\cdot x_2}\big)\Big)\Big)\\
&\quad + g_{\mathrm{both}}\cdot \frac{1}{2}\,\mathrm{d}\Big(\Phi_X\Big(\Xi_X\big(\underline{x_1\cdot x_2 - x_2\cdot x_1}\big)\Big)\Big)\\
&\quad + g_{\mathrm{same},x_1}\cdot \Phi_X\Big(\Xi_X\big(\underline{x_1\cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1\cdot x_1}\big)\Big) + g_{\mathrm{same},x_2}\cdot \Phi_X\Big(\Xi_X\big(\underline{x_2\cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2\cdot x_2}\big)\Big)\\
&\quad + g_{\mathrm{obs}}\cdot \mathrm{d}\,x_1\cdot \mathrm{d}\,x_2\\
&= g_{\mathrm{d}\,x_1}\cdot \mathrm{d}\,x_1\cdot \Phi_X\Big(\Xi_X\big(\underline{x_1\cdot x_2 - x_2\cdot x_1}\big)\Big) + g_{\mathrm{d}\,x_2}\cdot \mathrm{d}\,x_2\cdot \Phi_X\Big(\Xi_X\big(\underline{x_1\cdot x_2 - x_2\cdot x_1}\big)\Big)\\
&\quad + g_{\mathrm{both}}\cdot \Big(\Phi_X\Big(\Xi_X\big(\underline{x_1\cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2\cdot x_1}\big)\Big) + \Phi_X\Big(\Xi_X\big(\underline{x_2\cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1\cdot x_2}\big)\Big)\Big)\\
&\quad + g_{\mathrm{same},x_1}\cdot \Phi_X\Big(\Xi_X\big(\underline{x_1\cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1\cdot x_1}\big)\Big) + g_{\mathrm{same},x_2}\cdot \Phi_X\Big(\Xi_X\big(\underline{x_2\cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2\cdot x_2}\big)\Big)\\
&\quad + g_{\mathrm{obs}}\cdot \mathrm{d}\,x_1\cdot \mathrm{d}\,x_2
\end{aligned}
$$

Now we use that $\Phi \circ \Xi_X = \Phi''_X \circ \xi_X$.

$$
\begin{aligned}
&= g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \Phi''_X\Big(\xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big) + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \Phi''_X\Big(\xi_X\big(\underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big)\Big) \\
&\quad + g_{\mathrm{both}} \cdot \bigg(\Phi''_X\Big(\xi_X\big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1}\big)\Big) + \Phi''_X\Big(\xi_X\big(\underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big)\Big)\bigg) \\
&\quad + g_{\mathrm{same},x_1} \cdot \Phi''_X\Big(\xi_X\big(\underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}\big)\Big) + g_{\mathrm{same},x_2} \cdot \Phi''_X\Big(\xi_X\big(\underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}\big)\Big) \\
&\quad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2
\end{aligned}
$$

We now use that Φ''_X is multiplicative and maps x_i to x_i and $\mathrm{d}\,x_i$ to $\mathrm{d}\,x_i$. The latter two properties follow from $\Phi_X \circ \Xi_X$ mapping x_i to x_i by construction of Ξ_X (see Construction 7.4.9.1), and then also mapping $\mathrm{d}\,x_i$ to $\mathrm{d}\,x_i$ by Proposition 7.4.8.1. Furthermore we can evaluate ξ_X.

$$
\begin{aligned}
&= \Phi''_X\big(g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) + \Phi''_X\big(g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1}\big) \\
&\quad + \Phi''_X\Big(g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big)\Big) \\
&\quad + \Phi''_X\big(g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1}\big) + \Phi''_X\big(g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2}\big) \\
&\quad + \Phi''_X\big(g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2\big) \\
&= \Phi''_X\Big(g_{\mathrm{d}\,x_1} \cdot \mathrm{d}\,x_1 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} + g_{\mathrm{d}\,x_2} \cdot \mathrm{d}\,x_2 \cdot \underline{x_1 \cdot x_2 - x_2 \cdot x_1} \\
&\qquad + g_{\mathrm{both}} \cdot \big(\underline{x_1 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_1} + \underline{x_2 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_2}\big) \\
&\qquad + g_{\mathrm{same},x_1} \cdot \underline{x_1 \cdot \mathrm{d}\,x_1 - \mathrm{d}\,x_1 \cdot x_1} + g_{\mathrm{same},x_2} \cdot \underline{x_2 \cdot \mathrm{d}\,x_2 - \mathrm{d}\,x_2 \cdot x_2} \\
&\qquad + g_{\mathrm{obs}} \cdot \mathrm{d}\,x_1 \cdot \mathrm{d}\,x_2\Big) \\
&= \Phi''_X\bigg(\xi_X\Big(\Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}\big)\Big)\bigg) \\
&= \Phi_X\bigg(\Xi_X\Big(\Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}\big)\Big)\bigg)
\end{aligned}
$$

It thus only remains to show that

$$
\Phi_X\bigg(\Xi_X\Big(\Omega'^{\bullet}_{F/k}\big(\underline{t \cdot \mathrm{d}\,t - \mathrm{d}\,t \cdot t}\big)\Big)\bigg) = 0
$$

holds. Note that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\Omega'^{\bullet}_{k[t]/k}\Big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\Omega'^{\bullet}_{F/k}\right)"] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_{\{t\}})\right)"] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big)\Big) \arrow[d, "\simeq", "\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_X)\right)"'] \\
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}(\{t\})\Big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\widetilde{\mathrm{C}}(F))\right)"] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma)\left(\Phi_{\{t\}}\right)"] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\Big(\widetilde{\mathrm{C}}(X)\Big)\Big) \arrow[d, "\simeq", "\mathrm{Alg}(\gamma)(\Phi_X)"'] \\
\mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[t]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{F/k}\right)"] & \mathrm{Alg}(\gamma)\Big(\Omega^{\bullet}_{k[X]/k}\Big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where the top square is obtained from the transpose of the diagram from Proposition 7.5.2.5 by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ and using compatibility with γ_{Mixed} (see Construction 4.4.1.1), and the bottom square is the one from Proposition 7.4.7.1. The underlying differential graded k-algebra of $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant by Propositions 7.4.5.11 and 4.2.2.12, so we can conclude by [Hov99, 1.2.10 (ii)], Propositions A.1.0.1 and 4.2.2.17 that there exists a homotopy of differential graded k-algebras h from $\Phi_X \circ \Xi_X \circ \Omega'^{\bullet}_{F/k}$ to $\Omega^{\bullet}_{F/k} \circ \Phi_{\{t\}} \circ \Xi_{\{t\}}$ (we omit forgetful functors in the notation here) in the sense of Proposition 4.2.2.17. We can thus conclude the proof with the following calculation.

$$\Phi_X\Big(\Xi_X\Big(\Omega'^{\bullet}_{F/k}\big(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}\big)\Big)\Big)$$
$$= \Omega^{\bullet}_{F/k}\Big(\Phi_{\{t\}}\Big(\Xi_{\{t\}}\big(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}\big)\Big)\Big) + h\Big(\partial\big(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}\big)\Big) + \partial\Big(h\big(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}\big)\Big)$$

$\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t}$ is an element of degree 2, while $\Omega^2_{k[t]/k} = 0$. Thus purely for degree reasons we have $\Phi_{\{t\}}(\Xi_{\{t\}}(\underline{t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t})) = 0$ so that the first summand is zero.

$$= 0 + h(t\cdot \mathrm{d}\,t - \mathrm{d}\,t\cdot t) + 0$$
$$= h(t)\cdot\Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(\mathrm{d}\,t) + \Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(t)\cdot h(\mathrm{d}\,t)$$
$$\quad - h(\mathrm{d}\,t)\cdot\Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(t) + \Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(\mathrm{d}\,t)\cdot h(t)$$
$$= h(t)\cdot\Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(\mathrm{d}\,t) - h(t)\cdot\Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(\mathrm{d}\,t)$$
$$\quad + h(\mathrm{d}\,t)\cdot\Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(t)\cdot - h(\mathrm{d}\,t)\cdot\Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(t)$$
$$= h(t)\cdot\Big(\Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(\mathrm{d}\,t) - \Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(\mathrm{d}\,t)\Big)$$
$$\quad + h(\mathrm{d}\,t)\cdot\Big(\Big(\Phi_X\circ\Xi_X\circ\Omega'^{\bullet}_{F/k}\Big)(t) - \Big(\Omega^{\bullet}_{F/k}\circ\Phi_{\{t\}}\circ\Xi_{\{t\}}\Big)(t)\Big)$$

$$\begin{aligned}
&= h(t)\cdot\Big(-\partial\big(h(\mathrm{d}\,t)\big) - h\big(\partial(\mathrm{d}\,t)\big)\Big)\\
&\quad + h(\mathrm{d}\,t)\cdot\Big(\partial\big(h(t)\big) + h\big(\partial(t)\big)\Big)\\
&= h(t)\cdot\big(-0 - h(0)\big) + h(\mathrm{d}\,t)\cdot\big(0 + h(0)\big)\\
&= 0
\end{aligned}$$

□

7.5.3. Conjecture D

In this short section we deduce Conjecture D in certain cases from Proposition 7.5.2.6.

Proposition 7.5.3.1. *Let X be a set and f an element of $k[X]$. Assume that one of the following holds.*

(1) $|X| \leq 1$.

(2) $|X| = 2$ *and* 2 *is invertible in* k.

Then Conjecture D holds for f. ♡

Proof. Apply Proposition 7.5.2.6 for f as well as for the element 0 of k (as the polynomial ring generated by an empty set of variables) and combine the commutative squares. Note that it is crucial here that Proposition 7.5.2.6 constructs the commutative square (7.25) with the top horizontal equivalence not depending on X or f, which is what allows us to glue the two squares together. □

7.5.4. Conjecture C for two variables in the domain

In this section we show Conjecture C for morphisms out of polynomial k-algebras in two variables into k using some of the same arguments that also went into Proposition 7.5.2.6.

Proposition 7.5.4.1. *Let X be a totally ordered set satisfying $|X| \leq 2$ and $F\colon k[X] \to k$ a morphism of commutative k-algebras.*

Then there exists a commutative square

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(\Omega^{\bullet}_{F/k}\big)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k/k}\Big)
\end{tikzcd} \tag{7.26}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ *such that the horizontal equivalences are the ones from Corollary 7.4.9.3. In particular, Conjecture C holds for F.* ♡

Proof. The cases $|X| = 0$ and $|X| = 1$ are already contained in Proposition 7.5.1.1 and Proposition 7.5.2.6, respectively. For the case $|X| = 1$ this requires a small argument for why the lower horizontal equivalence we obtain from Proposition 7.5.2.6 is homotopic to the one from Corollary 7.4.9.3, but as $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$ is an initial object of $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ (see the proof of Proposition 7.5.1.1) this is automatic.

So now assume that $|X| = 2$ and denote the elements of X by $x_1 < x_2$. As in Proposition 7.5.2.6, we begin by considering the following (non-commuting) diagram in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, that will be explained below.

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[X]\big) \arrow[r, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"] \arrow[d, dash, "\simeq"] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[d, dash, "\simeq"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(X)\Big) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(\widetilde{\mathrm{C}}(F)\big)"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(\emptyset)\Big) \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k[X]/k}\Big) \arrow[u, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)", "\simeq"'] \arrow[r, dashed] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Theta_X)"', "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega'^{\bullet}_{k/k}\Big) \arrow[u, "\simeq", "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_\emptyset)"'] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Theta_\emptyset)"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k[X]/k}\Big) \arrow[r, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(\Omega^{\bullet}_{F/k}\big)"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^{\bullet}_{k/k}\Big)
\end{tikzcd} \qquad (*)$$

The top square has a filler given by the (transpose of) commutative diagram (7.10) from the definition of $\widetilde{\mathrm{C}}(F)$ in Construction 7.4.2.5, and Ξ and Θ are as in Construction 7.4.9.1 and Definition 7.4.5.9. By Corollary 7.4.9.3 the vertical compositions are the horizontal equivalences in diagram (7.26) from the statement, so that it suffices to find a filler for the lower rectangle in the above diagram.

As $\Omega'^{\bullet}_{k[X]/k}$ is cofibrant as an object of $\mathrm{Alg}(\mathsf{Mixed})$ by Proposition 7.4.5.11, we can lift the composition

$$\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_\emptyset)^{-1} \circ \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\widetilde{\mathrm{C}}(F)\Big) \circ \mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Xi_X)$$

to a morphism

$$\Omega'^{\bullet}_{F/k} \colon \Omega'^{\bullet}_{k[X]/k} \to \Omega'^{\bullet}_{k/k}$$

so that if we let the dashed morphism in the above diagram be $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(\Omega'^{\bullet}_{F/k})$ there will be a filler for the middle square of diagram $(*)$. It thus suffices to show that

$$\begin{tikzcd}
\Omega'^{\bullet}_{k[X]/k} \arrow[r, "\Omega'^{\bullet}_{F/k}"] \arrow[d, "\Theta_X"'] & \Omega'^{\bullet}_{k/k} \arrow[d, "\Theta_{\emptyset}"] \\
\Omega^{\bullet}_{k[X]/k} \arrow[r, "\Omega^{\bullet}_{F/k}"'] & \Omega^{\bullet}_{k/k}
\end{tikzcd}$$

commutes strictly. Note that as $\Omega^{\bullet}_{k/k}$ is concentrated in degree 0, it suffices to check that the two compositions agree on elements of degree 0, and as both compositions are multiplicative it even suffices to check the values on elements x in X. The composition over the bottom left maps x to $F(x)$, so this boils down to showing that $\Omega'^{\bullet}_{k[X]/k}(x) = F(x)$ for every element x in X.

For this we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\Omega'^{\bullet}_{k[X]/k}\big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\Omega'^{\bullet}_{k[X]/k}\right)"] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_X)\right)"] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\Omega'^{\bullet}_{k/k}\big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})(\Xi_{\emptyset})\right)"', "\simeq"] \\
\mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\widetilde{\mathrm{C}}(X)\big)\Big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\left(\widetilde{\mathrm{C}}(F)\right)\right)"] \arrow[d, "\simeq"', "\mathrm{Alg}(\gamma)(\Phi_X)"] & \mathrm{Alg}(\gamma)\Big(\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})\big(\widetilde{\mathrm{C}}(\emptyset)\big)\Big) \arrow[d, "\mathrm{Alg}(\gamma)(\Phi_{\emptyset})"', "\simeq"] \\
\mathrm{Alg}(\gamma)\big(\Omega^{\bullet}_{k[X]/k}\big) \arrow[r, "\mathrm{Alg}(\gamma)\left(\Omega^{\bullet}_{F/k}\right)"] & \mathrm{Alg}(\gamma)\big(\Omega^{\bullet}_{k/k}\big)
\end{tikzcd}$$

in $\mathrm{Alg}(\mathcal{D}(k))$, where the top square is obtained from the middle square of diagram $(*)$ by applying the forgetful functor $\mathrm{Alg}(\mathrm{ev}_{\mathfrak{m}})$ and using compatibility with γ_{Mixed} (see Construction 4.4.1.1), and the bottom square is (the transpose of) the one from Proposition 7.4.7.1. The underlying differential graded k-algebra of $\Omega'^{\bullet}_{k[t]/k}$ is cofibrant by Propositions 7.4.5.11 and 4.2.2.12, so we can conclude by [Hov99, 1.2.10 (ii)], Propositions A.1.0.1 and 4.2.2.17 that there exists a homotopy of differential graded k-algebras h from $\Phi_{\emptyset} \circ \Xi_{\emptyset} \circ \Omega'^{\bullet}_{k[X]/k}$ to $\Omega^{\bullet}_{F/k} \circ \Phi_X \circ \Xi_X$ (we omit forgetful functors in the notation here) in the sense of Proposition 4.2.2.17.

We can then carry out the following calculation for x an element of X, where we use that $\Phi_X \circ \Xi_X$ by definition in Construction 7.4.9.1 maps x to x.

$$\begin{aligned}
(\Phi_{\emptyset} \circ \Xi_{\emptyset})\Big(\Omega'^{\bullet}_{k[X]/k}(x)\Big) &= \Omega^{\bullet}_{F/k}\big((\Phi_X \circ \Xi_X)(x)\big) + h\big(\partial(x)\big) + \partial\big(h(x)\big) \\
&= \Omega^{\bullet}_{F/k}(x) + h(0) + 0
\end{aligned}$$

$$= F(x)$$

Note that $\Omega'^{\bullet}_{k/k}$ is by Remark 7.4.5.2 given by $k \cdot \{1\}$ in degree 0, so that also using the analogous identification for degree 0 of $\Omega^{\bullet}_{k/k}$ we obtain that $\Phi_{\emptyset} \circ \Xi_{\emptyset}$ is given by the identity in degree 0. Hence we can conclude that $\Omega'^{\bullet}_{k[X]/k}(x) = F(x)$ holds for every element x in X. □

Chapter 8.

Hochschild homology of certain quotients of commutative algebras

The goal of this chapter can be roughly summarized as giving a concrete formula for a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1, \ldots, x_n))$ as an object of $\mathcal{M}\mathrm{ixed}$, where R is a commutative k-algebra and $x_1, \ldots, x_n$ elements of R satisfying some conditions, given a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$.

More specifically, we require Conjecture C to hold for the morphism of k-algebras $k[t_1, \ldots, t_n] \to k$ mapping t_i to 0[1]. Furthermore we need as input an object M in $\mathrm{RMod}_{\Omega^\bullet_{k[t_1,\ldots,t_n]/k}}(\mathsf{Mixed}_{\mathrm{cof}})$ that represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ as an object in the ∞-category $\mathrm{RMod}_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])}(\mathcal{M}\mathrm{ixed})$, where the action is induced by the action of $k[t_1, \ldots, t_n]$ on R, where t_i acts by multiplication by x_i. Assuming Conjecture C as above and given such an object M, Proposition 8.3.0.1 can be roughly summarized as saying that (under some further conditions on R and $x_1, \ldots, x_n$), $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1, \ldots, x_n))$ is represented by a strict mixed complex that can be described as

$$M \otimes \Lambda(s_1, \ldots, s_n) \otimes \Gamma(\mathrm{d}\, s_1, \ldots, \mathrm{d}\, s_n)$$

with s_i of degree 1, $\mathrm{d}\, s_i$ of degree 2, and ∂ and d described by formulas given in Proposition 8.3.0.1. In particular, if m is a cycle in M_0 representing the unit 1 of R, then $\partial(m \otimes s_i) = (m \cdot t_i) \otimes 1$ and $\partial(m \otimes \mathrm{d}\, s_i^{[1]}) = -(m \cdot \mathrm{d}\, t_i) \otimes 1$, so we can think of s_i and $\mathrm{d}\, s_i^{[1]}$ as adding the relations that make x_i and $\mathrm{d}\, x_i$ zero.

To obtain such a formula, we proceed as follows. In Section 8.1 we start by showing that – under some conditions – we can write the quotient $R/(x_1, \ldots, x_n)$ as a derived tensor product $R \otimes_{k[t_1,\ldots,t_n]} k$, with t_i acting by multiplication with x_i on the left and by multiplication with 0 on the right. Using that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is compatible with relative tensor products we then obtain an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1, \ldots, x_n)) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R) \otimes_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

so that the task becomes to find strict models for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n])$ (as an algebra in $\mathcal{M}\mathrm{ixed}$) as well as for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ and $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$ (the latter two as modules over the strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n])$), and then calculating the derived relative tensor product. Assuming Conjecture B for $\{t_1, \ldots, t_n\}$ we can use $\Omega^\bullet_{k[t_1,\ldots,t_n]/k}$ as a strict model

[1]This is the case for $n \leq 2$ by Proposition 7.5.4.1.

for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n])$, and assuming that even Conjecture C holds for the morphism of commutative k-algebras $k[t_1, \ldots, t_n] \to k$ that maps t_i to 0 we can also use $\Omega^\bullet_{k/k}$ as a module over $\Omega^\bullet_{k[t_1,\ldots,t_n]/k}$ as a model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$. In order to be able to calculate the derived tensor product as an ordinary, underived tensor product, it will then be useful to replace $\Omega^\bullet_{k/k}$ with a weakly equivalent module over $\Omega^\bullet_{k[t_1,\ldots,t_n]/k}$ that is sufficiently cofibrant. Constructing such a module will be the goal of Section 8.2, and we will put everything together in Section 8.3.

8.1. Hochschild homology of certain quotients as relative tensor products

In Section 8.1.1 we will show that if R is a commutative k-algebra and $x_1, \ldots, x_n$ are elements of R satisfying some conditions[2], then $\gamma(R/(x_1, \ldots, x_n))$ is equivalent to a relative tensor product $\gamma(R) \otimes_{\gamma(k[t_1,\ldots,t_n])} k$ in $\mathrm{CAlg}(\mathcal{D}(k))$. Using compatibility of $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ with relative tensor products, we can thus write $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\gamma(R/(x_1, \ldots, x_n)))$ as a relative tensor product as well, as we will make explicit in Section 8.1.2.

8.1.1. Certain quotients as relative tensor products

Proposition 8.1.1.1. *Let R be a commutative algebra in $\mathsf{Ch}(k)$ and let $x_1, \ldots, x_n$ be elements of R_0. We obtain a morphism of commutative algebras in $\mathsf{Ch}(k)$*

$$k[t_1, \ldots, t_n] \to R, \qquad t_i \mapsto x_i$$

that determines a $k[t_1, \ldots, t_n]$-module structure on R (see Construction E.8.0.4). Assume that R is cofibrant as an object of $\mathrm{RMod}_{k[t_1,\ldots,t_n]}(\mathsf{Ch}(k))$ with respect to the model structure of Theorem 4.2.2.1.

Consider the commutative diagram

$$\begin{tikzcd} k[t_1, \ldots, t_n] \arrow[r, "t_i \mapsto x_i"] \arrow[d, "t_i \mapsto 0"'] & R \arrow[d] \\ k \arrow[r] & R/(x_1, \ldots, x_n) \end{tikzcd} \tag{8.1}$$

in $\mathrm{CAlg}(\mathsf{Ch}(k))$, where the right vertical morphism is the canonical quotient morphism. Then the following hold.

(1) Diagram (8.1) is a pushout diagram in $\mathrm{CAlg}(\mathsf{Ch}(k))$.

(2) All four objects in diagram (8.1) have cofibrant underlying chain complex.

[2]Roughly, $x_1, \ldots, x_n$ need to act sufficiently nicely on R by multiplication.

(3) *The functor*

$$\mathrm{CAlg}(\gamma)\colon \mathrm{CAlg}\Big(\mathsf{Ch}(k)^{\mathrm{cof}}\Big) \to \mathrm{CAlg}\big(\mathcal{D}(k)\big)$$

maps diagram (8.1) *to a pushout diagram in* $\mathrm{CAlg}(\mathcal{D}(k))$.

(4) *There is an equivalence*

$$\mathrm{CAlg}(\gamma)(R/(x_1,\ldots,x_n)) \simeq \mathrm{CAlg}(\gamma)(R) \otimes_{\mathrm{CAlg}(\gamma)(k[t_1,\ldots,t_n])} k$$

in $\mathrm{CAlg}(\mathcal{D}(k))$, *where the module structures used for the relative tensor product arise from the morphisms* $k[t_1,\ldots,t_n] \to R$ *and* $k[t_1,\ldots,t_n] \to k$ *in* (8.1) *by applying* $\mathrm{CAlg}(\gamma)$, *Construction E.8.0.4, and identifying* $\mathrm{CAlg}(\gamma)(k)$ *with* k. ♡

Proof. Proof of claim (1): This is well-known and can be shown by repeatedly applying the $n = 1$ case[3], which can be shown using Proposition E.8.0.5[4].

[3]For this one decomposes the (transposed) square (8.1) as

$$\begin{tikzcd}
k[t_1,\ldots,t_n] \arrow[r, "t_i \mapsto \begin{cases} 0 & i=1 \\ t_i & i>1 \end{cases}"] \arrow[d, "t_i\mapsto x_i"'] & k[t_2,\ldots,t_n] \arrow[r] \arrow[d, "t_i\mapsto x_i"'] & \cdots \arrow[r] \arrow[d] & k \arrow[d] \\
R \arrow[r] & R/(x_1) \arrow[r] & \cdots \arrow[r] & R/(x_1,\ldots,x_n)
\end{tikzcd}$$

so that it suffices to show that for each $1 \le j \le n$ the square

$$\begin{tikzcd}
k[t_j,\ldots,t_n] \arrow[r] \arrow[d] & k[t_{j+1},\ldots,t_n] \arrow[d] \\
R/(x_1,\ldots,x_{j-1}) \arrow[r] & R/(x_1,\ldots,x_j)
\end{tikzcd}$$

is a pushout square. The transpose of this square is the right square in the following commutative diagram.

$$\begin{tikzcd}
k[t_j] \arrow[r] \arrow[d] & k[t_j,\ldots,t_n] \arrow[r] \arrow[d] & R/(x_1,\ldots,x_{j-1}) \arrow[d] \\
k \arrow[r] & k[t_{j+1},\ldots,t_n] \arrow[r] & R/(x_1,\ldots,x_j)
\end{tikzcd}$$

It thus suffices to show that the outer rectangle and the left square are pushouts, but as $k[t_j,\ldots,t_n]/(t_j) \cong k[t_{j+1},\ldots,t_n]$ and $(R/(x_1,\ldots,x_{j-1}))/(x_j) \cong R/(x_1,\ldots,x_j)$, this follows from the $n = 1$ case.

[4]Using Proposition E.8.0.5, it suffices to show that the morphism

$$R \to R \otimes_{k[t_1]} k$$

exhibits $R \otimes_{k[t_1]} k$ as the quotient $R/(x_1)$. As the forgetful functor $\mathrm{CAlg}(\mathsf{Ch}(k)) \to \mathsf{Ch}(k)$ is conservative and preserves relative tensor products (see Proposition E.8.0.1), we can take the relative tensor product in $\mathsf{Ch}(k)$.

There is a short exact sequence

$$0 \longrightarrow k[t_1] \xrightarrow{1\mapsto t_1} k[t_1] \xrightarrow{1\mapsto 1} k \longrightarrow 0$$

Proof of claim (2): $k[t_1, \ldots, t_n]$ and k are free as k-modules and hence cofibrant as chain complexes [Hov99, 2.3.6]. We assumed that R is cofibrant as an object of $\mathrm{RMod}_{k[t_1,\ldots,t_n]}(\mathsf{Ch}(k))$, and as the underlying chain complex of $k[t_1, \ldots, t_n]$ is cofibrant as just mentioned, Theorem 4.2.2.1 (8) implies that the underlying chain complex of R is cofibrant as well. By (1) and Proposition E.8.0.5 the underlying chain complex of $R/(x_1, \ldots, x_n)$ is isomorphic to the relative tensor product $R \otimes_{k[t_1,\ldots,t_n]} k$, which is cofibrant as a chain complex by Proposition 6.3.3.3.

Proof of claim (3) and (4): Combining (1) and (2) with Proposition E.8.0.5 (applied to both $\mathsf{Ch}(k)^{\mathrm{cof}}$ as well as $\mathcal{D}(k)$) we only need to show that $\mathrm{CAlg}(\gamma)$ preserves the relative tensor product $R \otimes_{k[t_1,\ldots,t_n]} k$. As the forgetful functors $\mathrm{CAlg}(\mathsf{Ch}(k)^{\mathrm{cof}}) \to \mathsf{Ch}(k)^{\mathrm{cof}}$ and $\mathrm{CAlg}(\mathcal{D}(k)) \to \mathcal{D}(k)$ are conservative and preserve relative tensor products by Proposition E.8.0.1 and [HA, 3.2.3.1 (4)], it suffices to show that $\gamma\colon \mathsf{Ch}(k)^{\mathrm{cof}} \to \mathcal{D}(k)$ preserves this relative tensor product, which follows from Proposition 6.3.3.3[5]. $\square$

8.1.2. Application to Hochschild homology

Combining Proposition 8.1.1.1 with $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserving relative tensor products by Proposition 6.2.3.1 we obtain the following result.

Proposition 8.1.2.1. *Let R and $x_1, \ldots, x_n$ be as in Proposition 8.1.1.1. Then we can consider R as an object in $\mathrm{RMod}_{k[t_1,\ldots,t_n]}(\mathsf{Ch}(k)^{\mathrm{cof}})$, with t_i acting by multiplication with x_i, and k as an object in $\mathrm{LMod}_{k[t_1,\ldots,t_n]}(\mathsf{Ch}(k)^{\mathrm{cof}})$, with t_i acting by multiplication with 0.*

As $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is a monoidal functor, $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ obtains the structure of an object in $\mathrm{RMod}_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])}(\mathcal{M}\mathrm{ixed})$ and similarly $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$ obtains the structure of an object in $\mathrm{LMod}_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])}(\mathcal{M}\mathrm{ixed})$.

Let P_n be an object of $\mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})$ coming with an equivalence

$$\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(P_n) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n]) \tag{8.2}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. Let furthermore M be a right-P_n-module and A_n a left-P_n-module in $\mathsf{Mixed}_{\mathrm{cof}}$ such that there are equivalences

$$\mathrm{RMod}(\gamma_{\mathsf{Mixed}})(M) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R) \qquad \text{and} \qquad \mathrm{LMod}(\gamma_{\mathsf{Mixed}})(A_n) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \tag{8.3}$$

in $\mathrm{RMod}(\mathcal{M}\mathrm{ixed})$ and $\mathrm{LMod}(\mathcal{M}\mathrm{ixed})$ such that the underlying equivalences of algebras are given by equivalence (8.2). Assume that A_n is cofibrant as an object in[6]. $\mathrm{LMod}_{P_n}(\mathsf{Ch}(k))$.

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of left-$k[t_1]$-modules in $\mathsf{Ch}(k)$, so as $R \otimes_{k[t_1]} -$ is right exact [Wei94, 2.6.2], we obtain an exact sequence

$$R \otimes_{k[t_1]} k[t_1] \longrightarrow R \otimes_{k[t_1]} k[t_1] \longrightarrow R \otimes_{k[t_1]} k \longrightarrow 0$$

that can be identified with

$$R \xrightarrow{x_1 \cdot -} R \longrightarrow R \otimes_{k[t_1]} k \longrightarrow 0$$

which shows the claim.

[5]It is here were we really use the assumption that R is cofibrant as a $k[t_1, \ldots, t_n]$-module.

[6]We are using here that the forgetful functor $\mathrm{ev}_{\mathsf{m}}\colon \mathsf{Mixed} \to \mathsf{Ch}(k)$ is monoidal.

Then the underlying chain complex of the relative tensor product $M \otimes_{P_n} A_n$ (taken in Mixed*) is cofibrant. Furthermore, there is an equivalence*

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(R/(x_1, \ldots, x_n)\big) \simeq \gamma_{\mathsf{Mixed}}(M \otimes_{P_n} A_n)$$

in $\mathcal{M}\mathrm{ixed}$*.* ♡

Proof. By Proposition E.8.0.1 the forgetful functor $\mathrm{ev}_{\mathfrak{m}} \colon \mathsf{Mixed} \to \mathsf{Ch}(k)$ preserves relative tensor products, so cofibrancy of the underlying chain complex of $M \otimes_{P_n} A_n$ follows from Proposition 6.3.3.3.

By Proposition 8.1.1.1 (4) there is an equivalence

$$\mathrm{CAlg}(\gamma)(R/(x_1, \ldots, x_n)) \simeq \mathrm{CAlg}(\gamma)(R) \otimes_{\mathrm{CAlg}(\gamma)(k[t_1,\ldots,t_n])} k$$

in $\mathrm{CAlg}(\mathcal{D}(k))$. As $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ preserves relative tensor products by Proposition 6.2.3.1 we obtain an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(R/(x_1, \ldots, x_n)\big) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R) \otimes_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

in $\mathcal{M}\mathrm{ixed}$, and the equivalences (8.2) and (8.3) induce an equivalence in $\mathcal{M}\mathrm{ixed}$ as follows.

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R) \otimes_{\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1,\ldots,t_n])} \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \simeq \gamma_{\mathsf{Mixed}}(M) \otimes_{\gamma_{\mathsf{Mixed}}(P_n)} \gamma_{\mathsf{Mixed}}(A_n)$$

There is a comparison morphism

$$\gamma_{\mathsf{Mixed}}(M) \otimes_{\gamma_{\mathsf{Mixed}}(P_n)} \gamma_{\mathsf{Mixed}}(A_n) \to \gamma_{\mathsf{Mixed}}(M \otimes_{P_n} A_n)$$

in $\mathcal{M}\mathrm{ixed}$ just like in Remark 6.3.3.2, and it suffices to show that this is an equivalence. As the forgetful functors $\mathrm{ev}_{\mathfrak{m}} \colon \mathcal{M}\mathrm{ixed} \to \mathcal{D}(k)$ and $\mathrm{ev}_{\mathfrak{m}} \colon \mathsf{Mixed} \to \mathsf{Ch}(k)$ are conservative and preserve relative tensor products by Proposition E.8.0.1, it suffices to show that the comparison morphism

$$\gamma(M) \otimes_{\gamma(P_n)} \gamma(A_n) \to \gamma(M \otimes_{P_n} A_n)$$

in $\mathcal{D}(k)$ from Remark 6.3.3.2 is an equivalence. But this is precisely what we obtain from Proposition 6.3.3.3, as A_n was assumed to be cofibrant as a left-P_n-module. □

8.2. A sufficiently cofibrant strict model of k

Proposition 7.5.4.1 implies that the morphism

$$\Omega^{\bullet}_{k[t_1,\ldots,t_n]/k} \to \Omega^{\bullet}_{k/k}$$

in $\mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})$, induced by the morphism of commutative algebras $k[t_1, \ldots, t_n] \to k$ that sends t_i to 0, represents the morphism

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n]) \to \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

in Alg($\mathcal{M}$ixed) induced by the same morphism, as long as $n \leq 2$. For $n > 2$ we have encapsulated this statement as Conjecture C for this morphism, and we will assume that Conjecture C holds for the results of this chapter.

Unfortunately, we can not directly use $\Omega^\bullet_{k/k}$, as the left-module P_n over $A_n = \Omega^\bullet_{k[t_1,\ldots,t_n]/k}$ as in Proposition 8.1.2.1, as this would require $\Omega^\bullet_{k/k}$ to be cofibrant as a module over $\Omega^\bullet_{k[t_1,\ldots,t_n]/k}$ in chain complexes, which is not necessarily the case.

The goal of this section is thus to construct a commutative diagram

$$\begin{tikzcd} & A_n \arrow[dd] \\ \Omega^\bullet_{k[t_1,\ldots,t_n]/k} \arrow[ur] \arrow[dr] & \\ & \Omega^\bullet_{k/k} \end{tikzcd}$$

in Alg($\mathsf{Mixed}_{\mathrm{cof}}$) such that the lower morphism is the one discussed above, the vertical morphism on the right is a quasiisomorphism, and such that A_n is cofibrant when considered as an object in $\mathrm{LMod}_{\Omega^\bullet_{k[t_1,\ldots,t_n]/k}}(\mathsf{Ch}(k))$. We will construct A_n and morphisms as in the diagram above in Section 8.2.1, show that A_k has the required cofibrancy property in Section 8.2.2, and show that the right vertical morphism is a quasiisomorphism in Section 8.2.3.

8.2.1. Construction of the strict model

Before we construct A_n we need a small result on the Leibniz rule and compositions.

Proposition 8.2.1.1. *Let R be a commutative differential graded algebra, and let f and g be two operators of odd degree on R that both satisfy the Leibniz rule. Then $f \circ f$ as well as $fg + gf$ satisfy the Leibniz rule as well[7].* ♡

Proof. Let x and y be two elements in R. Then we can calculate as follows.

$$\begin{aligned} f\big(g(x \cdot y)\big) &= f\Big(g(x)y + (-1)^{\deg_{\mathsf{Ch}}(x)} x g(y)\Big) \\ &= f\big(g(x)\big)y + (-1)^{\deg_{\mathsf{Ch}}(x)+1} g(x)f(y) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)g(y) \\ &\quad + (-1)^{\deg_{\mathsf{Ch}}(x)+\deg_{\mathsf{Ch}}(x)} x f\big(g(y)\big) \\ &= f\big(g(x)\big)y + x f\big(g(y)\big) \\ &\quad - (-1)^{\deg_{\mathsf{Ch}}(x)} g(x)f(y) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x)g(y) \end{aligned}$$

Applying this to $g = f$ we immediately obtain the claim for $f \circ f$. For $fg + gf$ there is the following calculation.

$$(fg + gf)(x \cdot y) = f\big(g(x)\big)y + x f\big(g(y)\big)$$

[7]Note that $f \circ f$ and $fg + gf$ will be of even degree, so there will be no sign.

$$\begin{aligned}
&- (-1)^{\deg_{\mathsf{Ch}}(x)} g(x) f(y) + (-1)^{\deg_{\mathsf{Ch}}(x)} f(x) g(y) \\
&+ g\big(f(x)\big) y + x g\big(f(y)\big) \\
&- (-1)^{\deg_{\mathsf{Ch}}(x)} f(x) g(y) + (-1)^{\deg_{\mathsf{Ch}}(x)} g(x) f(y) \\
= {}& (fg + gf)(x) y + x (fg + gf)(y)
\end{aligned}$$
$\square$

Construction 8.2.1.2. We define P_1 and A_1 to be the strict commutative graded k-modules given by[8]

$$P_1 \coloneqq k[t] \otimes \Lambda(\mathrm{d}\, t) \qquad \text{and} \qquad A_1 \coloneqq k[t] \otimes \Lambda(\mathrm{d}\, t) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$$
$$\deg_{\mathsf{Ch}}(t) = 0, \quad \deg_{\mathsf{Ch}}(\mathrm{d}\, t) = 1, \quad \deg_{\mathsf{Ch}}(s) = 1, \quad \deg_{\mathsf{Ch}}(\mathrm{d}\, s^{[m]}) = 2m$$

and let $g_1 \colon P_1 \to A_1$ be the inclusion. Note that there is a commutative triangle of commutative graded k-modules

$$\begin{tikzcd}
 & A_1 \arrow[d, "p_1"] \\
P_1 \arrow[ur, "g_1"] \arrow[r, "g_1'"'] & k
\end{tikzcd} \tag{8.4}$$

where g_1' and p_1 map t, $\mathrm{d}\, t$, s, and $\mathrm{d}\, s^{[m]}$ to 0.

We will now upgrade diagram (8.4) to a commutative diagram in CAlg(Mixed). For this we define ∂ and d on P_1 and A_1 by

$$\partial(t) = 0, \qquad \partial(\mathrm{d}\, t) = 0, \qquad \partial(s) = t, \qquad \partial\Big(\mathrm{d}\, s^{[m]}\Big) = -\,\mathrm{d}\, t\, \mathrm{d}\, s^{[m-1]}$$
$$\mathrm{d}(t) = \mathrm{d}\, t, \qquad \mathrm{d}(\mathrm{d}\, t) = 0, \qquad \mathrm{d}(s) = \mathrm{d}\, s^{[1]}, \qquad \mathrm{d}\Big(\mathrm{d}\, s^{[m]}\Big) = 0$$

and extending by k-linearity and the Leibniz rule. It is clear that if this equips A_1 with the structure of a commutative algebra in strict mixed complexes, then this structure restricts to P_1 and makes g_1 into a morphism in CAlg(Mixed). What we need to show is that this definition of $\partial(\mathrm{d}\, s^{[m]})$ and $\mathrm{d}(\mathrm{d}\, s^{[m]})$ is well-defined[9] and that d and ∂ satisfy $\partial \circ \partial = 0$, $\mathrm{d} \circ \mathrm{d} = 0$, and $\mathrm{d}\partial + \partial\mathrm{d} = 0$ on A_1, see Remark 4.2.1.4 and Remark 4.2.1.12.

But first, let us state the formulas for d and ∂ for a k-linear basis of A_1 (obtained by applying k-linearity and the Leibniz rule), so we may refer to them later[10].

$$t^{n_1}\, \mathrm{d}\, t^{\epsilon_1} s^{\eta_1}\, \mathrm{d}\, s^{[m_1]} \cdot t^{n_2}\, \mathrm{d}\, t^{\epsilon_2} s^{\eta_2}\, \mathrm{d}\, s^{[m_2]} = (-1)^{\eta_1 \cdot \epsilon_2} \binom{m_1 + m_2}{m_1} t^{n_1+n_2}\, \mathrm{d}\, t^{\epsilon_1+\epsilon_2} s^{\eta_1+\eta_2}\, \mathrm{d}\, s^{[m_1+m_2]}$$
$$\mathrm{d}\Big(t^n\, \mathrm{d}\, t^{\epsilon} s^{\eta}\, \mathrm{d}\, s^{[m]}\Big) = n \cdot t^{n-1}\, \mathrm{d}\, t^{\epsilon+1} s^{\eta}\, \mathrm{d}\, s^{[m]} + (-1)^{\epsilon} \cdot \eta \cdot (1+m) \cdot t^n\, \mathrm{d}\, t^{\epsilon}\, \mathrm{d}\, s^{[1+m]} \tag{8.5}$$

[8] For now $\mathrm{d}\, t$ and $\mathrm{d}\, s$ are just names, but we will in a moment define a strict mixed complex structure that will justify this notation.

[9] I. e. compatible with the relation $\mathrm{d}\, s^{[m_1]} \cdot \mathrm{d}\, s^{[m_2]} = \binom{m_1+m_2}{m_1} \mathrm{d}\, s^{[m_1+m_2]}$.

[10] In the formulas, some summands may contain factors that are undefined, such as $\mathrm{d}\, s^{[-1]}$. Those summands are to be interpreted as 0.

$$\partial\Big(t^n \,\mathrm{d}\, t^\epsilon s^\eta \,\mathrm{d}\, s^{[m]}\Big) = (-1)^\epsilon \cdot \eta \cdot t^{n+\eta} \,\mathrm{d}\, t^\epsilon \,\mathrm{d}\, s^{[m]} - t^n \,\mathrm{d}\, t^{\epsilon+1} s^\eta \,\mathrm{d}\, s^{[m-1]}$$

For well-definedness, nothing needs to be done for d. For ∂, evaluating on

$$\mathrm{d}\, s^{[m_1]} \cdot \mathrm{d}\, s^{[m_2]} = \binom{m_1 + m_2}{m_1} \mathrm{d}\, s^{[m_1+m_2]}$$

using the left hand side and the Leibniz rule we obtain

$$\begin{aligned} &\Big(-\mathrm{d}\, t \,\mathrm{d}\, s^{[m_1-1]}\Big) \,\mathrm{d}\, s^{[m_2]} + \mathrm{d}\, s^{[m_1]} \Big(-\mathrm{d}\, t \,\mathrm{d}\, s^{[m_2-1]}\Big) \\ = -\mathrm{d}\, t &\left(\binom{m_1+m_2-1}{m_1-1} \mathrm{d}\, s^{[m_1+m_2-1]} + \binom{m_1+m_2-1}{m_1} \mathrm{d}\, s^{[m_1+m_2-1]}\right) \end{aligned}$$

and using the right hand side we obtain

$$-\binom{m_1+m_2}{m_1} \mathrm{d}\, t \,\mathrm{d}\, s^{[m_1+m_2-1]}$$

which are equal by the well-known binomial identity $\binom{m_1+m_2-1}{m_1-1} + \binom{m_1+m_2-1}{m_1} = \binom{m_1+m_2}{m_1}$.

We now check $\partial \circ \partial = 0$, $\mathrm{d} \circ \mathrm{d} = 0$, and $\mathrm{d}\partial + \partial\mathrm{d} = 0$. Note that Proposition 8.2.1.1 implies that we only need to check this on multiplicative generators. That $\mathrm{d} \circ \mathrm{d} = 0$ on multiplicative generators is clear from the definition, and for $\partial \circ \partial = 0$ the only case to consider is

$$\partial\Big(\partial\Big(\mathrm{d}\, s^{[m]}\Big)\Big) = \partial\Big(-\mathrm{d}\, t \,\mathrm{d}\, s^{[m-1]}\Big) = \mathrm{d}\, t \,\mathrm{d}\, t \,\mathrm{d}\, s^{[m-2]}$$

which is 0 as $(\mathrm{d}\, t)^2 = 0$. Finally, we verify that $\mathrm{d}\partial + \partial\mathrm{d} = 0$.

$$\begin{aligned} (\mathrm{d}\partial + \partial\mathrm{d})(t) &= 0 + \partial(\mathrm{d}\, t) = 0 \\ (\mathrm{d}\partial + \partial\mathrm{d})(\mathrm{d}\, t) &= 0 + 0 \\ (\mathrm{d}\partial + \partial\mathrm{d})(s) &= \mathrm{d}(t) + \partial\Big(\mathrm{d}\, s^{[1]}\Big) = \mathrm{d}\, t - \mathrm{d}\, t = 0 \\ (\mathrm{d}\partial + \partial\mathrm{d})\Big(\mathrm{d}\, s^{[m]}\Big) &= \mathrm{d}\Big(-\mathrm{d}\, t \,\mathrm{d}\, s^{[m-1]}\Big) = -\mathrm{d}(\mathrm{d}\, t) \,\mathrm{d}\, s^{[m-1]} + \mathrm{d}\, t \,\mathrm{d}\Big(\mathrm{d}\, s^{[m-1]}\Big) = 0 + 0 = 0 \end{aligned}$$

It is clear that the two morphisms to k in diagram (8.4) are compatible with d and ∂, so (8.4) is a commutative diagram in $\mathrm{CAlg}(\mathsf{Mixed})$.

For n a positive integer we denote by

$$A_n \coloneqq A_1^{\otimes n} = k[t_1, \ldots, t_n] \otimes \Lambda(\mathrm{d}\, t_1, \ldots, \mathrm{d}\, t_n) \otimes \Lambda(s_1, \ldots, s_n) \otimes \Gamma(\mathrm{d}\, s_1, \ldots, \mathrm{d}\, s_n)$$

the n-fold tensor product of A_1 in $\mathrm{CAlg}(\mathsf{Mixed})$. We will also let

$$P_n \coloneqq P_1^{\otimes n} = k[t_1, \ldots, t_n] \otimes \Lambda(\mathrm{d}\, t_1, \ldots, \mathrm{d}\, t_n)$$

be the n-fold tensor product of P_1. The n-fold tensor product of diagram (8.4) then yields a commutative diagram

$$\begin{tikzcd} & A_n \arrow[dd, "p_n"] \\ P_n \arrow[ur, "g_n"] \arrow[dr, "g'_n"'] & \\ & k \end{tikzcd} \tag{8.6}$$

in CAlg(Mixed). ♢

8.2.2. Cofibrancy

Proposition 8.2.2.1. *Let n be a positive integer. Then A_n from Construction 8.2.1.2 is cofibrant (with respect to the model structure from Theorem 4.2.2.1) as an object in $\mathrm{LMod}_{P_n}(\mathsf{Ch}(k))$, where the module structure is the one arising from the morphism of differential graded algebras g_n from Construction 8.2.1.2.* ♡

Proof. Considered first as just a graded module over the graded algebra P_n, it is clear that A_n is a free P_n-module and that

$$\mathcal{B} := \left\{ s^{\vec{\epsilon}} \, \mathrm{d}\, s^{[\vec{i}]} \;\middle|\; \vec{i} \in \mathbb{Z}^n_{\geq 0},\ \vec{\epsilon} \in \{0,1\}^n \right\}$$

forms a basis.

Let $\preceq$ be the lexicographic[11] well-order on $(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n \times \{0,1\}^n$. For an element $(\vec{j}, \vec{\eta})$ in $(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n \times \{0,1\}^n$ we define

$$\mathcal{B}_{\vec{j},\vec{\eta}} := \left\{ s^{\vec{\epsilon}} \, \mathrm{d}\, s^{[\vec{i}]} \;\middle|\; \vec{i} \in \mathbb{Z}^n_{\geq 0},\ \vec{\epsilon} \in \{0,1\}^n,\ (\vec{i}, \vec{\epsilon}) \preceq (\vec{j}, \vec{\eta}) \right\}$$

and let $X_{\vec{j},\vec{\eta}}$ be the sub-P_n-module (still as just a graded module over a graded algebra) generated by $\mathcal{B}_{\vec{j},\vec{\eta}}$. It is clear from the definition of the differential on A_n that $X_{\vec{j},\vec{\eta}}$ is actually a subcomplex of A_n, and that $A_n = X_{(\infty,\ldots,\infty),(1,\ldots,1)}$.

Considering $(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n \times \{0,1\}^n$ as a category with a unique morphism $(\vec{i}, \vec{\epsilon}) \to (\vec{j}, \vec{\eta})$ if and only if $(\vec{i}, \vec{\epsilon}) \preceq (\vec{j}, \vec{\eta})$, we obtain a functor

$$(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n \times \{0,1\}^n \to \mathrm{LMod}_{P_n}\big(\mathsf{Ch}(k)\big) \tag{$*$}$$

that sends $(\vec{j}, \vec{\eta})$ to $X_{\vec{j},\vec{\eta}}$ and the morphisms to the respective inclusions. One can see that this functor is colimit-preserving, which boils down to the fact that

$$\mathcal{B}_{\vec{j},\vec{0}} = \bigcup_{(\vec{i},\vec{\epsilon}) \prec (\vec{j},\vec{0})} \mathcal{B}_{\vec{i},\vec{\epsilon}}$$

[11] In $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ we let ∞ be greater than any integer. The lexicographic order is then defined such that $(\vec{i}, \vec{\epsilon}) \preceq (\vec{j}, \vec{\eta})$ if and only if there is an index $1 \leq l \leq n$ with $i_1 = j_1, \ldots, i_{l-1} = j_{l-1}$ and $i_l < j_l$, or $\vec{i} = \vec{j}$ and there is an index $1 \leq l \leq n$ with $\epsilon_1 = \eta_1, \ldots, \epsilon_{l-1} = \eta_{l-1}$ and $\epsilon_l < \eta_l$.

for every $\vec{j}$ in $(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n$ such that there is an $1 \leq l \leq n$ with $j_l = \infty$ and $j_{l'} = 0$ for $l' > l$[12]. Thus the functor $(*)$ exhibits A_n as a transfinite composition, and so to show that A_n is cofibrant in $\mathrm{LMod}_{P_n}(\mathsf{Ch}(k))$ it suffices to show that $X_{\vec{0},\vec{0}}$ is cofibrant, and that for each $(\vec{i}, \vec{\epsilon})$ and $(\vec{j}, \vec{\eta})$ in $(\mathbb{Z}_{\geq 0} \cup \{\infty\})^n \times \{0,1\}^n$, such that $(\vec{j}, \vec{\eta})$ is the successor of $(\vec{i}, \vec{\epsilon})$, the inclusion $X_{\vec{i},\vec{\epsilon}} \to X_{\vec{j},\vec{\eta}}$ is a cofibration.

As $X_{\vec{0},\vec{0}}$ is isomorphic to P_n, and hence free on the cofibrant chain complex $k[0]$ as a P_n-module in $\mathsf{Ch}(k)$, it is cofibrant. Furthermore, with $(\vec{i}, \vec{\epsilon})$ and $(\vec{j}, \vec{\eta})$ as above, the difference $\mathcal{B}_{\vec{j},\vec{\eta}} \setminus \mathcal{B}_{\vec{i},\vec{\epsilon}}$ consists of precisely the element $s^{\vec{\eta}} \,\mathrm{d}\, s^{[\vec{j}]}$. The diagram

$$\begin{tikzcd}
\mathrm{Free}^{\mathrm{LMod}_{P_n}}\left(S_{2\deg_{\mathsf{Ch}}(\vec{j})+\deg_{\mathsf{Ch}}(\vec{\eta})-1}\right) \arrow[r] \arrow[d] & X_{\vec{i},\vec{\epsilon}} \arrow[d] \\
\mathrm{Free}^{\mathrm{LMod}_{P_n}}\left(D_{2\deg_{\mathsf{Ch}}(\vec{j})+\deg_{\mathsf{Ch}}(\vec{\eta})}\right) \arrow[r] & X_{\vec{j},\vec{\eta}}
\end{tikzcd}$$

is a pushout, where we use use the notation from [Hov99, 2.3.3][13], the morphism on the left is induced by the usual inclusion[14]. The morphism on the top sends the generator 1 in degree $2\deg_{\mathsf{Ch}}(\vec{j}) + \deg_{\mathsf{Ch}}(\vec{\eta}) - 1$ to $\partial\left(s^{\vec{\eta}} \,\mathrm{d}\, s^{[\vec{j}]}\right)$, and the morphism at the bottom sends the new generator in degree $2\deg_{\mathsf{Ch}}(\vec{j}) + \deg_{\mathsf{Ch}}(\vec{\eta})$ to $s^{\vec{\eta}} \,\mathrm{d}\, s^{[\vec{j}-\vec{\eta}]}$. It is crucial here that even though $s^{\vec{\eta}} \,\mathrm{d}\, s^{[\vec{j}]}$ is not an element of $X_{\vec{i},\vec{\epsilon}}$, its boundary is. □

8.2.3. Quasiisomorphism

Proposition 8.2.3.1. *Let n be a positive integer. Then the morphism*

$$p_n \colon A_n \to k$$

from Construction 8.2.1.2 is a quasiisomorphism. ♡

Proof. By Proposition 8.2.2.1 and Theorem 4.2.2.1 (8)[15], A_m is cofibrant as a chain complex for every positive integer m. By the pushout-product property for $\mathsf{Ch}(k)$ (see Fact 4.1.3.1) and Ken Brown's lemma [Hov99, 1.1.12], the tensor product of a cofibrant chain complex with a quasiisomorphism between cofibrant chain complexes is again a quasiisomorphism. Writing $p_n \colon A_n \to k$ as the composition

$$A_1 \otimes A_1^{n-1} \xrightarrow{p_1 \otimes \mathrm{id}_{A_1^{n-1}}} k \otimes A_1 \otimes A_1^{n-2} \xrightarrow{\mathrm{id}_k \otimes p_1 \otimes \mathrm{id}_{A_1^{n-2}}} k \otimes k \otimes A_1^{n-2} \to \cdots \to k^n \cong k$$

it suffices to show that $p_1 \colon A_1 \to k$ is a quasiisomorphism.

---

[12] I. e. we consider those $(\vec{j}, \vec{\eta})$ that are not successors or $(\vec{0}, \vec{0})$.

[13] So S_l is the complex with k concentrated in degree l and D_l is the acyclic complex with k in degree l and $l-1$, with boundary operator the identity.

[14] Which is the identity in degree $2\deg_{\mathsf{Ch}}(\vec{j}) + \deg_{\mathsf{Ch}}(\vec{\eta})$.

[15] This is applicable because P_m has cofibrant underlying chain complex by [Hov99, 2.3.6], as P_m is concentrated in nonnegative degrees and free as a graded k-module.

As a morphism of chain complexes p_1 has a section ι that maps 1 to 1, so it suffices to give an homotopy ϑ between the id_{A_1} and $\iota \circ p_1$. As a graded abelian group, A_1 is free with basis $\left\{ t^n \,\mathrm{d}\, t^\epsilon s^\eta \,\mathrm{d}\, s^{[m]} \;\middle|\; n, m \in \mathbb{Z}_{\geq 0},\; \epsilon, \eta \in \{0,1\} \right\}$, and we will define ϑ on this basis. Define

$$\vartheta(t^n \,\mathrm{d}\, t^\epsilon s^\eta \,\mathrm{d}\, s^{[m]}) = \begin{cases} (-1)^\epsilon t^{n-1} \,\mathrm{d}\, t^\epsilon s^{\eta+1} \,\mathrm{d}\, s^{[m]} & \text{if } n > 0 \\ -\,\mathrm{d}\, s^{[m+1]} & \text{if } n = 0,\ \eta = 0, \text{ and } \epsilon = 1 \\ 0 & \text{otherwise} \end{cases}$$

We now check that $\vartheta\partial + \partial\vartheta = \iota p_1$ on basis elements $t^n \,\mathrm{d}\, t^\epsilon s^\eta \,\mathrm{d}\, s^{[m]}$ by distinguishing a couple of cases.

Case $n > 0, \eta = 0$:

$$\begin{aligned} (\vartheta\partial + \partial\vartheta)(t^n \,\mathrm{d}\, t^\epsilon \,\mathrm{d}\, s^{[m]}) &= \vartheta\Big((-1) \cdot (-1)^\epsilon t^n \,\mathrm{d}\, t^{\epsilon+1} \,\mathrm{d}\, s^{[m-1]}\Big) + \partial\Big((-1)^\epsilon t^{n-1} \,\mathrm{d}\, t^\epsilon s \,\mathrm{d}\, s^{[m]}\Big) \\ &= (-1) \cdot (-1)^\epsilon \cdot (-1)^{\epsilon+1} t^{n-1} \,\mathrm{d}\, t^{\epsilon+1} s \,\mathrm{d}\, s^{[m-1]} \\ &\quad + (-1)^\epsilon \cdot (-1)^\epsilon t^n \,\mathrm{d}\, t^\epsilon \,\mathrm{d}\, s^{[m]} \\ &\quad + (-1)^\epsilon \cdot (-1) \cdot (-1)^\epsilon t^{n-1} \,\mathrm{d}\, t^{\epsilon+1} s \,\mathrm{d}\, s^{[m-1]} \\ &= t^n \,\mathrm{d}\, t^\epsilon \,\mathrm{d}\, s^{[m]} \end{aligned}$$

Case $n > 0, \eta = 1$:

$$\begin{aligned} (\vartheta\partial + \partial\vartheta)(t^n \,\mathrm{d}\, t^\epsilon s \,\mathrm{d}\, s^{[m]}) &= \vartheta\Big((-1)^\epsilon t^{n+1} \,\mathrm{d}\, t^\epsilon \,\mathrm{d}\, s^{[m]} - t^n \,\mathrm{d}\, t^{\epsilon+1} s \,\mathrm{d}\, s^{[m-1]}\Big) + \partial(0) \\ &= (-1)^\epsilon \cdot (-1)^\epsilon t^n \,\mathrm{d}\, t^\epsilon s \,\mathrm{d}\, s^{[m]} - 0 + 0 \\ &= t^n \,\mathrm{d}\, t^\epsilon s \,\mathrm{d}\, s^{[m]} \end{aligned}$$

Case $n = 0, \eta = 0, \epsilon = 1$:

$$\begin{aligned} (\vartheta\partial + \partial\vartheta)(\mathrm{d}\, t \,\mathrm{d}\, s^{[m]}) &= \vartheta(0) + \partial\Big(-\,\mathrm{d}\, s^{[m+1]}\Big) \\ &= \mathrm{d}\, t \,\mathrm{d}\, s^{[m]} \end{aligned}$$

Case $n = 0, \eta = 0, \epsilon = 0$:

$$\begin{aligned} (\vartheta\partial + \partial\vartheta)(\mathrm{d}\, s^{[m]}) &= \vartheta\Big(-\,\mathrm{d}\, t \,\mathrm{d}\, s^{[m-1]}\Big) + \partial(0) \\ &= \begin{cases} \mathrm{d}\, s^{[m]} & \text{if } m > 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Note that the case $n = m = \eta = \epsilon = 0$ is special, as 1 is the only basis element on which ιp is acts as the identity, rather than zero, so this is the expected result.

Case $n = 0, \eta = 1$:

$$\begin{aligned} (\vartheta\partial + \partial\vartheta)(\mathrm{d}\, t^\epsilon s \,\mathrm{d}\, s^{[m]}) &= \vartheta\Big((-1)^\epsilon t \,\mathrm{d}\, t^\epsilon \,\mathrm{d}\, s^{[m]} - \mathrm{d}\, t^{\epsilon+1} s \,\mathrm{d}\, s^{[m-1]}\Big) + \partial(0) \\ &= (-1)^\epsilon \cdot (-1)^\epsilon \,\mathrm{d}\, t^\epsilon s \,\mathrm{d}\, s^{[m]} + 0 \\ &= \mathrm{d}\, t^\epsilon s \,\mathrm{d}\, s^{[m]} \end{aligned}$$

□

8.3. A formula for Hochschild homology of certain quotients

In this section we combine Sections 8.1 and 8.2 to obtain a somewhat more concrete formula for a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of certain quotients than in Proposition 8.1.2.1.

Proposition 8.3.0.1. *Let $n \geq 1$ be an integer and assume*[16] *that Conjecture C holds for the morphism of commutative k-algebras $T\colon k[t_1, \ldots, t_n] \to k$ that maps t_i to 0, and fix a commutative square*

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(k[t_1,\ldots,t_n]\big) \arrow[r, "\simeq"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(T)"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^\bullet_{k[t_1,\ldots,t_n]/k}\Big) \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^\bullet_{T/k}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^\bullet_{k/k}\Big)
\end{tikzcd} \tag{8.7}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ such that the horizontal morphisms are equivalences.

Let R be a commutative algebra in $\mathsf{Ch}(k)$ and let $x_1, \ldots, x_n$ be elements of R_0. Assume that R is cofibrant as an object of $\mathrm{RMod}_{k[t_1,\ldots,t_n]}(\mathsf{Ch}(k))$ with respect to the model structure of Theorem 4.2.2.1, where t_i acts by multiplication with x_i. Note that as $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ is monoidal, $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$ obtains an induced structure of a right module over $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n])$ in $\mathcal{M}\mathrm{ixed}$.

Let $P_n = k[t_1, \ldots, t_n] \otimes \Lambda(\mathrm{d}\, t_1, \ldots, \mathrm{d}\, t_n)$ be as in Construction 8.2.1.2 and M a right-P_n-module in $\mathsf{Mixed}_{\mathrm{cof}}$ such that there is an equivalence

$$\mathrm{RMod}(\gamma_{\mathsf{Mixed}})(M) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R)$$

in $\mathrm{RMod}(\mathcal{M}\mathrm{ixed})$ such that the underlying equivalence of algebras is the composition

$$\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(P_n) \simeq \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^\bullet_{k[t_1,\ldots,t_n]/k}\Big) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n]) \tag{8.8}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$, where the first equivalence is induced by the identification

$$\Omega^\bullet_{k[t_1,\ldots,t_n]/k} \cong k[t_1, \ldots, t_n] \otimes \Lambda(\mathrm{d}\, t_1, \ldots, \mathrm{d}\, t_n)$$

from the start of Section 7.1 and the second equivalence is the one from (8.7).

Then there is an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(R/(x_1, \ldots, x_n)\big) \simeq \gamma_{\mathsf{Mixed}}\big(M'\big)$$

in $\mathcal{M}\mathrm{ixed}$, where M' is the strict mixed complex defined as follows. As a graded k-module, M' is given by

$$M' := M \otimes \Lambda(s_1, \ldots, s_n) \otimes \Gamma(\mathrm{d}\, s_1, \ldots, \mathrm{d}\, s_n)$$

[16] If $n \leq 2$ this holds by Proposition 7.5.4.1, making this result unconditional.

with $s_1, \ldots, s_n$ *of degree* 1 *and* $\mathrm{d}\, s_1, \ldots, \mathrm{d}\, s_n$ *of degree* 2*. The boundary operator* ∂ *and differential* d *are given by* k*-linearly extending the following formulas for* $m \in M$*,* $\vec{\epsilon} \in \{0,1\}^n$*, and* $\vec{i} \in \mathbb{Z}^n_{\geq 0}$*.*

$$\begin{aligned}
\partial\left(m \otimes s^{\vec{\epsilon}}\,\mathrm{d}\, s^{[\vec{i}]}\right) &= \left(\partial^M(m) \otimes s^{\vec{\epsilon}}\,\mathrm{d}\, s^{[\vec{i}]}\right)\\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(m)} \sum_{a=1}^{n} (-1)^{\sum_{b=1}^{a-1}\epsilon_b} \left(m \cdot t_a \otimes s^{\vec{\epsilon}-\vec{e_a}}\,\mathrm{d}\, s^{[\vec{i}]}\right)\\
&\quad - (-1)^{\deg_{\mathsf{Ch}}(m)} \sum_{a=1}^{n} \left(m \cdot \mathrm{d}\, t_a \otimes s^{\vec{\epsilon}}\,\mathrm{d}\, s^{[\vec{i}-\vec{e_a}]}\right)\\
\mathrm{d}\left(m \otimes s^{\vec{\epsilon}}\,\mathrm{d}\, s^{[\vec{i}]}\right) &= \left(\mathrm{d}^M(m) \otimes s^{\vec{\epsilon}}\,\mathrm{d}\, s^{[\vec{i}]}\right)\\
&\quad + (-1)^{\deg_{\mathsf{Ch}}(m)} \sum_{a=1}^{n} (-1)^{\sum_{b=1}^{a-1}\epsilon_b} (i_a + 1) \left(m \otimes s^{\vec{\epsilon}-\vec{e_a}}\,\mathrm{d}\, s^{[\vec{i}+\vec{e_a}]}\right)
\end{aligned}$$

In the above formulas, summands in which a vector occurs with a component that is negative are to be interpreted as zero. ♡

Proof. We first apply Proposition 8.1.2.1, where we are using the specific model A_n constructed in Section 8.2 for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$ as a module over $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t_1, \ldots, t_n])$. To do so, we only need to check that A_n has the properties required of it in Proposition 8.1.2.1. Concretely, we need an equivalence

$$\mathrm{LMod}(\gamma_{\mathsf{Mixed}})(A_n) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

in $\mathrm{LMod}(\mathcal{M}\mathrm{ixed})$ such that the underlying equivalence of algebras is (8.8), and we need that A_n is cofibrant as an object of $\mathrm{LMod}_{P_n}(\mathsf{Ch}(k))$. The latter is precisely Proposition 8.2.2.1, and for the former we use the following composite equivalence.

$$\mathrm{LMod}(\gamma_{\mathsf{Mixed}})(A_n) \xrightarrow{\simeq} \mathrm{LMod}(\gamma_{\mathsf{Mixed}})(k) \simeq \mathrm{LMod}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k/k}\right) \simeq \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k)$$

The first morphism is induced by the morphism of P_n-algebras $p_n\colon A_n \to k$ as defined in Construction 8.2.1.2, and lies over the identity morphism of $\mathrm{Alg}(\gamma_{\mathsf{Mixed}})(P_n)$ in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$. The second equivalences uses naturality of the isomorphism from Section 7.1, which ensures that the underlying equivalence of algebras is the first equivalence in (8.8). Finally, the third equivalence arises from the commutative square (8.7), and the underlying equivalence of algebras is the second one in (8.8).

By Proposition 8.1.2.1 we now obtain an equivalence

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(R/(x_1, \ldots, x_n)) \simeq \gamma_{\mathsf{Mixed}}(M \otimes_{P_n} A_n)$$

in $\mathcal{M}\mathrm{ixed}$. It thus remains to evaluate the relative tensor product $M \otimes_{P_n} A_n$ in Mixed.

As the forgetful functor from strict mixed complexes to graded k-modules is conservative, symmetric monoidal, and preserves colimits, we obtain an isomorphism of underlying graded k-modules[17]

$$\begin{aligned} &M \otimes_{P_n} A_n \\ &= M \otimes_{k[t_1,\dots,t_n]\otimes\Lambda(\mathrm{d}\,t_1,\dots,\mathrm{d}\,t_n)} k[t_1,\dots,t_n] \otimes \Lambda(\mathrm{d}\,t_1,\dots,\mathrm{d}\,t_n) \otimes \Lambda(s_1,\dots,s_n) \otimes \Gamma(\mathrm{d}\,s_1,\dots,\mathrm{d}\,s_n) \\ &\cong M \otimes \Lambda(s_1,\dots,s_n) \otimes \Gamma(\mathrm{d}\,s_1,\dots,\mathrm{d}\,s_n) \end{aligned}$$

where the isomorphism maps an element of the form $m \otimes t^{\vec{i}}\,\mathrm{d}\,t^{\vec{\epsilon}} s^{\vec{\eta}}\,\mathrm{d}^{[\vec{j}]}$ to $m \cdot (t^{\vec{i}}\,\mathrm{d}\,t^{\vec{\epsilon}}) \otimes s^{\vec{\eta}}\,\mathrm{d}^{[\vec{j}]}$. We can lift this isomorphism to an isomorphism of strict mixed complex, and it then remains to determine d and ∂, for which we use the morphism of strict mixed complexes

$$M \otimes A_n \to M \otimes_{P_n} A_n \cong M \otimes \Lambda(s_1,\dots,s_n) \otimes \Gamma(\mathrm{d}\,s_1,\dots,\mathrm{d}\,s_n)$$

where the first morphism is the canonical one and the isomorphism the one just described. One can then read off the formulas claimed in the statement using Definition 4.1.2.1, Remark 4.2.1.10, and Construction 8.2.1.2 □

[17] The point is that in graded k-modules, $k[t_1,\dots,t_n] \otimes \Lambda(\mathrm{d}\,t_1,\dots,\mathrm{d}\,t_n) \otimes \Lambda(s_1,\dots,s_n) \otimes \Gamma(\mathrm{d}\,s_1,\dots,\mathrm{d}\,s_n)$ really is the tensor product of $k[t_1,\dots,t_n] \otimes \Lambda(\mathrm{d}\,t_1,\dots,\mathrm{d}\,t_n)$ and $\Lambda(s_1,\dots,s_n) \otimes \Gamma(\mathrm{d}\,s_1,\dots,\mathrm{d}\,s_n)$, whereas this is not the case as chain complexes.

Chapter 9.

Hochschild homology of certain quotients of polynomial algebras

In Chapter 8 we obtained a general result that helps to produce strict mixed complexes that represent $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of some quotients of commutative algebras. In this chapter we specialize to quotients of polynomial algebras by a single monic polynomial f of positive degree. The crucial input that we will need for this is that Conjecture D holds for f. After verifying the necessary requirements to apply the result, we will in Section 9.2 be able to specialize Proposition 8.3.0.1 to the case $k[x_1, \ldots, x_n]/f$ for n a positive integer and f a monic polynomial of positive degree satisfying Conjecture D, obtaining a strict mixed complex X_f that is a model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{M}\mathrm{ixed}$. The underlying graded k-module of X_f is of the form

$$X_f \coloneqq k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$$

where x_i, $\mathrm{d}\, x_i$, s, and $\mathrm{d}\, s$ are of degree 0, 1, 1, and 2, respectively.

In our goal to obtain a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ that is as small as possible, this is already a significant improvement on the standard Hochschild complex $\mathrm{C}(k[x_1, \ldots, x_n]/f)$ that we discussed in Section 6.3.1. To underline this, note that X_f can be given the structure of a graded $k[x_1, \ldots, x_n]$-module, with $k[x_1, \ldots, x_n]$ acting through the leftmost tensor factor. X_f is then degreewise free as a $k[x_1, \ldots, x_n]$-module, so we can consider the rank[1]. We find that $\mathrm{rank}_{k[x_1,\ldots,x_n]}((X_f)_i)$ (where i is an integer) is finite, and furthermore bounded, i. e. there is an integer r such that

$$\mathrm{rank}_{k[x_1,\ldots,x_n]}\Big(\big(X_f\big)_i\Big) \leq r$$

for all integers i. This is very far from the situation for the standard Hochschild complex $\mathrm{C}(k[x_1, \ldots, x_n]/f)$. While $k[x_1, \ldots, x_n]$ doesn't act freely on the leftmost tensor factor, $k[x_1, \ldots, x_n]/f$ does, and

$$\begin{aligned}
&\mathrm{rank}_{k[x_1,\ldots,x_n]/f}\Big(\mathrm{C}\big(k[x_1, \ldots, x_n]/f\big)_i\Big)\\
&= \mathrm{rank}_{k[x_1,\ldots,x_n]/f}\Big(\big(k[x_1, \ldots, x_n]/f\big)^{\otimes(i+1)}\Big)
\end{aligned}$$

[1] If we wanted to make the following discussion regarding ranks precise, we would define bases for the various modules and discuss their cardinalities (the modules we consider all have a relatively obvious basis to use for this). We omit such a detour, as this discussion is only for purpose of motivation.

$$= \operatorname{rank}_k\Big(\big(k[x_1,\dots,x_n]/f\big)^{\otimes(i)}\Big)$$
$$= \operatorname{rank}_k\Big(\big(k[x_1,\dots,x_n]/f\big)\Big)^i$$

for $i \geq 0$. For $n > 1$, $\operatorname{rank}_k\Big(\big(k[x_1,\dots,x_n]/f\big)\Big)$ will already be infinite, and additionally it would also be reasonable to consider the rank to grow exponentially in the degree i.

So X_f is an improvement over the standard Hochschild complex. It is though certainly not optimal for specific polynomials. For example, for $f = x_1$ the quotient $k[x_1]/f$ is isomorphic to k, so we can by Corollary 7.4.9.3 use $\Omega^\bullet_{k/k} \cong k$ as a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1]/f)$, and k is significantly smaller than $X_f = k[x_1] \otimes \Lambda(\mathrm{d}\,x_1) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\,s)$.

The main goal of this chapter will thus be to improve on the size of X_f while relaxing what the result covers. This can be done in two directions: Firstly, we can reduce the amount of structure we consider, which we do by asking only for a sub-chain-complex of X_f that represents $\mathrm{HH}(k[x_1,\dots,x_n]/f)$ as an object of $\mathcal{D}(k)$, rather than as a mixed complex, which we will do in Section 9.3. Secondly, we can insist on a sub-strict-mixed-complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\dots,x_n]/f)$ as an object of $\mathcal{M}\mathrm{ixed}$, while reducing the set of polynomials f that we consider. This will be done in Section 9.5.

The results of this chapter should themselves also only be considered as stepping stones, just like Proposition 8.3.0.1 and X_f was a stepping stone for the results of this chapter. So for actual calculations that need a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\dots,x_n]/f)$ for specific polynomials f, one would begin with the strict mixed complex obtained in Section 9.5 (if the relevant result is applicable) and then simplify it further, making use of the specific form of f. In Chapter 10 we will discuss the concrete example of $f = x_1^2 - x_2x_3$ in details along those lines[2].

Let us now say some more on the individual sections of this chapter.

As we stated at the beginning of this introduction, we will consider *monic* multivariable polynomials f to divide out of a polynomial algebra. For polynomials in a single variable there is precisely one standard definition of what it means to be monic, but this is not the case for multivariable polynomials, where there are multiple sensible definitions. What we will mean by *monic* is *monic with respect to a chosen monomial order*, and this notion will be introduced in Section 9.1. It will also be very important in this chapter to have a good handle of moving back and forth between $k[x_1,\dots,x_n]$ and $k[x_1,\dots,x_n]/f$, for example by producing canonical representatives in $k[x_1,\dots,x_n]$ of elements in the quotient $k[x_1,\dots,x_n]/f$. For this we will also discuss division with remainder for multivariable polynomials in Section 9.1.

In Section 9.2 we will then combine previous results to obtain X_f as a strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\dots,x_n]/f)$ as an object of $\mathcal{M}\mathrm{ixed}$, assuming that Conjecture D holds for f. Heavily using constructions discussed in Section 9.1 that are built on top of the division with remainder for multivariable polynomials, we will also describe a new basis for X_f as well as calculate some formulas for the boundary operator and differential in terms of that new basis.

[2] This polynomial has however so far not been proven to satisfy Conjecture D. The strict mixed complex X_f can nevertheless be constructed.

In Section 9.3 we will discuss $\mathrm{HH}(k[x_1,\ldots,x_n]/f)$ as only an object of $\mathcal{D}(k)$. A chain complex representing it has already been obtained in the previous work of the Buenos Aires Cyclic Homology Group in [BACH]. For k a commutative ring and f an element of $k[x_1,\ldots,x_n]$ satisfying relatively mild conditions, they give a quite small differential graded algebra together with a multiplicative inclusion into the normalized standard Hochschild complex for $\overline{\mathrm{C}}(k[x_1,\ldots,x_n]/f)$, as well as a homotopy inverse to this inclusion, as a morphism of chain complexes. Using the basis for X_f and the formulas for the boundary operator in this basis obtained in Section 9.2, it will be relatively straightforward in Section 9.3 to define a subcomplex $X^e_{f,0}$ of X_f such that the inclusion into X_f is a quasiisomorphism, thereby obtaining a smaller chain complex than X_f representing $\mathrm{HH}(k[x_1,\ldots,x_n]/f)$ as an object of $\mathcal{D}(k)$. We will also show that $X^e_{f,0}$ is isomorphic to the chain complex described in [BACH]. Assuming that Conjecture D holds for f, the rest of the assumptions we need to make for f are the same as in [BACH], so this amounts to giving a new proof for one of the main results of [BACH], using a quite different approach, for the range in which Conjecture D has been proven, so $n \leq 2$ as long as 2 is invertible in k by Proposition 7.5.3.1.

Unfortunately the definition of the comparison morphisms used in [BACH] between the smaller chain complex and the normalized standard Hochschild complex are quite complicated, making them difficult to unwrap for transferring additional structure. Trying to transfer the strict mixed complex structure to the smaller chain complex from the normalized standard Hochschild complex additionally runs into the problem that one does not obtain a strict mixed complex structure; the necessary identities will only be satisfied up to homotopy for general f, and it is not possible to upgrade either of the two quasiisomorphisms between the small chain complex and the normalized standard Hochschild complex to a morphism of strict mixed complexes, as we show in Section 9.6.

However, for some polynomials f, the strict mixed structure on X_f restricts to $X^e_{f,0}$, so that $X^e_{f,0}$ even represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$ as an object of $\mathcal{M}$ixed. To properly formulate a condition for when the strict mixed structure restricts we introduce the notions of logarithm and the *log dimension* for multivariable polynomials in Section 9.4. In particular, we will prove a criterion that can be easily checked for multivariable polynomials f and that implies that $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$.

In Section 9.5 we will then show that if f satisfies $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$, then the strict mixed structure of X_f restricts to $X^e_{f,0}$, making the inclusion of $X^e_{f,0}$ into X_f into a morphism of strict mixed complexes that is a weak equivalence.

Under some stronger assumptions on f a strict mixed complex isomorphic to $X^e_{f,0}$ was already constructed by Larsen in [Lar95]. In the two-variable case Larsen furthermore constructs a strongly homotopy linear quasiisomorphism[3] from this strict mixed complex into the normalized standard Hochschild complex. The result in Section 9.5 can thus be seen as a generalization of one of the main results of [Lar95].[4] A number of constructions relating to polynomials that we use in order to simplify X_f are inspired by their use in

[3]See Definition 4.2.3.1 for a definition. By Remark 4.4.4.2 a strongly homotopy linear quasiisomorphism induces an equivalence in $\mathcal{M}$ixed.

[4]However introducing the new assumption that 2 is invertible in k.

[Lar95].

In Section 9.7 we discuss the relationship between our results and the main result of [Lar95], as well as how, assuming Conjecture D, our results provide an affirmative answer to a question posed by Larsen in [Lar95].

9.1. Prerequisites on polynomials and dividing with remainder

Given the non-zero polynomial f in n variables by which we want to divide the polynomial algebra $k[x_1, \dots, x_n]$, it will be important for us to define uniquely determined remainders of dividing an arbitrary polynomial P by f, i. e. we would like to have a procedure obtain a unique decomposition of P as $P = Q \cdot f + R$ for other polynomials Q and R. In the one-variable case with f an element of $k[x]$ it is relatively straightforward to come up with an idea of how this decomposition should look like: We would like P to uniquely decompose as $P = Q \cdot f + R$ where R has smaller degree than f. It is not difficult to see that if the leading coefficient of f is not a zero-divisor, then this determines Q and R uniquely as long as such a decomposition exists. However, such a decomposition may not exist for all f and P – as a counterexample consider $f = 2$ and $P = 3$ for $k = \mathbb{Z}$. However it turns out that such a decomposition does exist if the polynomial f is *monic*, that is the leading coefficient is 1. In that case, one can perform the Euclidean algorithm, iteratively eliminating the highest power of x remaining with the leading term of f, i. e. if we have given $f = x^n + f'$ with f' of degree less than n, and $P = \sum_{i=0}^{m} a_i x^i$ with $m \geq n$, then the first step will be to write

$$P = (a_m x^{m-n}) \cdot f + \left(\left(\sum_{i=0}^{m} a_i x^i \right) - (a_m x^{m-n}) \cdot f' \right)$$

and in this decomposition the term in brackets is of degree less than m, so iterating this process will eventually come to a stop.

If we wish to generalize this procedure to the multi-variable case, we are confronted with an obvious question: Which term of P should we start eliminating? What is the leading term of f that we should use to do so? There is no obviously correct choice for a definition of leading terms of multivariable polynomials but multiple equally good competing ones. Thus we will have to codify what we require of such a definition to be nice enough to allow us to define the kind of decompositions described, and then require that f be monic with respect to that choice. The results will then also depend on that choice.

We will start in Section 9.1.1 by discussing *monomial orders*, which provide a consistent way of determining which of two monomials is to be considered the larger one. This will allow us to define a notion of degree of a multivariable polynomial in Section 9.1.2. Finally, we will discuss division with remainder for multivariable polynomials in Section 9.1.3.

9.1.1. Monomial orders

In this section we introduce the concept of monomial orders and discuss some easy consequences of the definition. We start in Section 9.1.1.1 by recalling the notions of partial, total, and well-orders. The important example of the pointwise partial order on $\mathbb{Z}^n_{\geq 0}$ will be discussed in Section 9.1.1.2, before we define monomial orders in Section 9.1.1.3. We end this section by proving some easy properties of monomial orders in Section 9.1.1.4.

9.1.1.1. Partial, total, and well-orders

We recall the following notions.

Definition 9.1.1.1. Let X be a set and $\preceq$ a binary relation on X. Recall the following properties that $\preceq$ may have.

Antisymmetry For any $a, b \in X$, if $a \preceq b$ and $b \preceq a$, then $a = b$.

Transitivity For any $a, b, c \in X$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

Reflexivity For any $a \in X$ it holds that $a \preceq a$.

Connectivity For any $a, b \in X$, it holds that $a \preceq b$ or $b \preceq a$.

Well-foundedness If X' is nonempty subset of X, then X' has a least element, that is an element $x \in X'$ such that for all $y \in X'$ it holds that $x \preceq y$.

Note that connectivity implies reflexivity.

The relation $\preceq$ is called a

partial order if it is antisymmetric, transitive, and reflexive.

total order if it is antisymmetric, transitive, and connected.

well-order if it is antisymmetric, transitive, connected, and well-founded.

A set equipped with a partial order (total order, well-order) on it will be called a *partially ordered set* (*totally ordered set*, *well-ordered set*). $\diamondsuit$

Notation 9.1.1.2. Let X be a set and $\preceq$ a binary relation on X. If x and y are elements of X such that $x \preceq y$ and $x \neq y$, then we will say that x is smaller than y and y is bigger than x.

We will use the notation $x \succeq y$ to mean $y \preceq x$. Furthermore, we will use $x \succ y$ and $y \prec x$ to mean $y \preceq x$ and $x \neq y$. $\diamondsuit$

Remark 9.1.1.3. The important consequence of well-foundedness is that we can prove statements about every element of X by transfinite induction: If we prove that any element of X has some property if every smaller element has that property, then it follows that *every* element of X has that property[5]. $\diamondsuit$

[5]Proof: Let $X' \subseteq X$ be the subset of X of elements that do *not* have the property in question. By

9.1.1.2. The standard partial order on $\mathbb{Z}^n_{\geq 0}$

We now define an important example of a partial order on $\mathbb{Z}^n_{\geq 0}$.

Definition 9.1.1.4. Let n be a positive integer. We define a relation $\leq$ on $\mathbb{Z}^n_{\geq 0}$ by letting $\vec{a} \leq \vec{b}$ if and only if $a_i \leq b_i$ for all $1 \leq i \leq n$. $\diamondsuit$

Remark 9.1.1.5. The relation $\leq$ as defined in Definition 9.1.1.4 is a partial order.

Note that a monomial $x^{\vec{i}}$ divides $x^{\vec{j}}$ for $\vec{i}, \vec{j} \in \mathbb{Z}^n_{\geq 0}$ if and only if $\vec{i} \leq \vec{j}$. This is the reason why the partial order $\leq$ is of relevance for us. $\diamondsuit$

Proposition 9.1.1.6. *Let n be a positive integer. For the partial order $\leq$ defined on $\mathbb{Z}^n_{\geq 0}$ as in Definition 9.1.1.4 and $\vec{a}, \vec{b}, \vec{c} \in \mathbb{Z}^n_{\geq 0}$, if $\vec{a} \leq \vec{b}$, then $\vec{a} + \vec{c} \leq \vec{b} + \vec{c}$.* $\heartsuit$

Proof. Follows directly from the definition. $\square$

9.1.1.3. Definition of monomial orders

The partial order $\leq$ encodes intuition on how some monomials definitely should compare: Certainly the monomial $x^{\vec{j}}$ should be "bigger" than $x^{\vec{i}}$ if $x^{\vec{i}}$ divides $x^{\vec{j}}$, or equivalently if $\vec{i} \leq \vec{j}$. But what if neither $\vec{i} \leq \vec{j}$ nor $\vec{j} \leq \vec{i}$? In order to be able to define notions such as degrees and leading terms for all elements of $k[x_1, \ldots, x_n]$, we are thus led to ask for a total order $\preceq$ on $\mathbb{Z}^n_{\geq 0}$ that extends $\leq$.

A finite subset of a totally ordered set has a maximum element. If we have a total order $\preceq$ on $\mathbb{Z}^n_{\geq 0}$ given, then we can now provisionally define what the leading term of a polynomial $f \in k[x_1, \ldots, x_n]$ should be: If f is given by

$$f = \sum_{\vec{i} \in \mathbb{Z}^n_{\geq 0}} f_{\vec{i}} x^{\vec{i}}$$

then we can declare $f_{\vec{j}} x^{\vec{j}}$ to be the leading term of f if $\vec{j}$ is the maximal element of $\left\{ \vec{i} \in \mathbb{Z}^n_{\geq 0} \,\middle|\, f_{\vec{i}} \neq 0 \right\}$.

However this is not quite enough to obtain the kind of decomposition we described in the introduction to Section 9.1. Firstly, in the one-variable case the procedure to iteratively eliminate the highest degree has to eventually terminate because there is no infinite strictly decreasing sequence of nonnegative integers. For the multivariable case we should thus require that $\preceq$ is a well-order. Secondly, in the one-variable case we need to argue that if f' has degree smaller than m, then $x^{l-m} \cdot f'$ has degree smaller than l, and we need an analogue of this in the multivariable case as well. This leads us to the following definition, which is also used in [BACH, 2.2].

well-foundedness, if X' were non-empty, it would need to have a least element x. But this would mean that every element smaller than x has the property, so x must have had it as well, so X' must have been empty.

Definition 9.1.1.7. Let n be a positive integer. A *monomial order (for n variables)* is a well-order $\preceq$ on $\mathbb{Z}^n_{\geq 0}$ satisfying the following property: For every $\vec{a}, \vec{b}, \vec{c} \in \mathbb{Z}^n_{\geq 0}$ such that $\vec{a} \preceq \vec{b}$ it also holds that $\vec{a} + \vec{c} \preceq \vec{b} + \vec{c}$. ♢

That a monomial order indeed extends $\leq$ will follow from this, and is shown below in Proposition 9.1.1.8.

9.1.1.4. Properties of monomial orders

Proposition 9.1.1.8. *Let n be a positive integer and $\preceq$ a monomial order for n variables. Then the following hold.*

(1) Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{Z}^n_{\geq 0}$ such that $\vec{a} + \vec{c} \preceq \vec{b} + \vec{c}$. Then it also holds that $\vec{a} \preceq \vec{b}$.

(2) $\vec{0}$ is minimal in $\mathbb{Z}^n_{\geq 0}$ with respect to $\preceq$, i. e. for every $\vec{a} \in \mathbb{Z}^n_{\geq 0}$ it holds that $\vec{0} \preceq \vec{a}$.

(3) $\preceq$ extends $\leq$, i. e. if $\vec{a}, \vec{b} \in \mathbb{Z}^n_{\geq 0}$ such that $\vec{a} \leq \vec{b}$, then $\vec{a} \preceq \vec{b}$. ♡

Proof. Proof of claim (1): If it is not true that $\vec{a} \preceq \vec{b}$, then we must have $\vec{a} \succeq \vec{b}$ by connectivity, and so $\vec{a} + \vec{c} \succeq \vec{b} + \vec{c}$ as $\preceq$ is a monomial order. But by antisymmetry this implies that $\vec{a} + \vec{c} = \vec{b} + \vec{c}$ and so $\vec{a} = \vec{b}$, from which $\vec{a} \preceq \vec{b}$ follows by reflexivity.

Proof of claim (2): Let $\vec{m}$ be an element of $\mathbb{Z}^n_{\geq 0}$. We need to show that $\vec{0} \preceq \vec{m}$, but by connectivity and reflexivity it suffices to show that if $\vec{0} \succeq \vec{m}$, then $\vec{m} = \vec{0}$. So assume that $\vec{0} \succeq \vec{m}$. By adding $l \cdot \vec{m}$ to this inequality we obtain $l \cdot \vec{m} \succeq (l+1) \cdot \vec{m}$, so that we obtain an infinite descending chain

$$\vec{0} \succeq \vec{m} \succeq 2 \cdot \vec{m} \succeq \cdots$$

in $\mathbb{Z}^n_{\geq 0}$. Well-foundedness of $\preceq$ implies that this chain must eventually stabilize, so there must be an $l \geq 0$ with $(l+1) \cdot \vec{m} = l \cdot \vec{m}$, which implies $\vec{m} = 0$.

Proof of (3): $\vec{a} \leq \vec{b}$ implies that $\vec{b} - \vec{a}$ still lies in $\mathbb{Z}^n_{\geq 0}$. Applying (2) we obtain $\vec{0} \preceq \vec{b} - \vec{a}$, and adding $\vec{a}$ to this inequality we conclude that $\vec{a} \preceq \vec{b}$. □

Remark 9.1.1.9. If $\preceq$ is a monomial order for 1 variable, then Proposition 9.1.1.8 (3) implies that $\preceq$ is equal to $\leq$. ♢

Remark 9.1.1.10. Let n be a positive integer. The assumptions made on the binary relation $\leq_T$ on $\mathbb{Z}^n_{\geq 0}$ considered in [BACH, 2.2] are that $\leq_T$ is a monomial order in the sense of Definition 9.1.1.7, and that $\leq_T$ extends $\leq$. Proposition 9.1.1.8 (3) shows that the latter assumption is unnecessary. ♢

Construction 9.1.1.11. Let n be a positive integer and $\preceq$ a monomial order for n variables. Let $m \leq n$ be another positive integer and $\varphi\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ an injection. Then we can define an additive injection $\mathbb{Z}^m_{\geq 0} \to \mathbb{Z}^n_{\geq 0}$ as follows.

$$\psi\colon \mathbb{Z}^m_{\geq 0} \to \mathbb{Z}^n_{\geq 0}\,, \qquad \psi(\vec{a})_i := \begin{cases} a_j & \text{if } \psi(j) = i \\ 0 & \text{if } i \text{ is not in the image of } \psi \end{cases}$$

For example if φ is the inclusion of $\{1\}$ into $\{1,2\}$, then ψ maps (a) to $(a,0)$.

We can then define a binary relation $\preceq$ on $\mathbb{Z}^m_{\geq 0}$ as follows. For $\vec{a}, \vec{b} \in \mathbb{Z}^m_{\geq 0}$ we let $\vec{a} \preceq \vec{b}$ if and only if $\psi(\vec{a}) \preceq \psi(\vec{b})$. It follows immediately from ψ being additive and injective that this defines a monomial order for m variables, which we will call the *restricted* monomial order.

Let $\{i_1, \ldots, i_{n-m}\}$ be the elements of $\{1, \ldots, n\}$ that are not in the image of φ. Define k' to be the commutative k-algebra $k' = k[x_{i_1}, \ldots, x_{i_{n-m}}]$. Then there is an isomorphism of k-algebras

$$k'[y_1, \ldots, y_m] \xrightarrow{\cong} k[x_1, \ldots, x_n]$$

that maps x_{i_j} to x_{i_j} and y_j to $x_{\varphi(j)}$. Note that this morphism then maps $y^{\vec{j}}$ to $x^{\psi(\vec{j})}$. We will make use of this isomorphism on some occasions when inducting on the number of variables. ♢

9.1.2. Degrees for multivariable polynomials

In this section we define a notation of degree of multivariable polynomials, dependent on a monomial order.

Definition 9.1.2.1. Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f \in k[x_1, \ldots, x_n]$ a polynomial. We define

$$\deg_{\preceq}(f) = \begin{cases} \max\left\{\vec{i} \in \mathbb{Z}^n_{\geq 0} \,\middle|\, f_{\vec{i}} \neq 0\right\} & \text{if } f \neq 0 \\ -\infty & \text{if } f = 0 \end{cases}$$

where the maximum is taken with respect to the order $\preceq$. We call $\deg_{\preceq}(f)$ the *degree of f (with respect to the monomial order $\preceq$)*. We call $f_{\deg_{\preceq}(f)} x^{\deg_{\preceq}(f)}$ the *leading term* and $f_{\deg_{\preceq}(f)}$ the *leading coefficient* of f (with respect to the monomial order $\preceq$).

If $f, g \in k[x_1, \ldots, x_n]$, then we write $f \preceq g$ if $\deg_{\preceq}(f) \preceq \deg_{\preceq}(g)$. ♢

Remark 9.1.2.2. It follows from Remark 9.1.1.9 and the definition that the degree as defined in Definition 9.1.2.1 recovers the usual notion in the case $n = 1$. ♢

The degree of multivariable polynomials as defined above satisfies the usual properties with respect to addition and multiplication of polynomials, as we record below.

Proposition 9.1.2.3. *Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f, g \in k[x_1, \ldots, x_n]$. Then the following hold.*

(1) $\deg_{\preceq}(f+g) \preceq \max\left\{\deg_{\preceq}(f), \deg_{\preceq}(g)\right\}$.

(2) If $\deg_{\preceq}(f) \succ \deg_{\preceq}(g)$, *then* $\deg_{\preceq}(f+g) = \deg_{\preceq}(f)$.

(3) $\deg_{\preceq}(f \cdot g) \preceq \deg_{\preceq}(f) + \deg_{\preceq}(g)$.

(4) *If at least one of f or g is zero, or both are nonzero and $f_{\deg_{\preceq}(f)} \cdot g_{\deg_{\preceq}(g)} \neq 0$, then* $\deg_{\preceq}(f \cdot g) = \deg_{\preceq}(f) + \deg_{\preceq}(g)$.

With respect to max *we interpret $-\infty$ as smaller than all elements of $\mathbb{Z}^n_{\geq 0}$, and we interpret the sum of $-\infty$ with $-\infty$ or an integer to be $-\infty$ again.* ♡

Proof. Proof of claim (1): By definition

$$f_{\deg_{\preceq}(f+g)} + g_{\deg_{\preceq}(f+g)} = (f+g)_{\deg_{\preceq}(f+g)} \neq 0$$

holds, so one of $f_{\deg_{\preceq}(f+g)}$ and $g_{\deg_{\preceq}(f+g)}$ must be non-zero, which directly implies that $\deg_{\preceq}(f) \succeq \deg_{\preceq}(f+g)$ or $\deg_{\preceq}(g) \succeq \deg_{\preceq}(f+g)$.

Proof of claim (2): In this case $\max\{\deg_{\preceq}(f), \deg_{\preceq}(g)\} = \deg_{\preceq}(f)$, so using (1) it suffices to show that $\deg_{\preceq}(f+g) \succeq \deg_{\preceq}(f)$. The assumption $\deg_{\preceq}(f) \succ \deg_{\preceq}(g)$ also implies $g_{\deg_{\preceq}(f)} = 0$ and thus $(f+g)_{\deg_{\preceq}(f)} = f_{\deg_{\preceq}(f)} + g_{\deg_{\preceq}(f)} = f_{\deg_{\preceq}(f)} \neq 0$, from which $\deg_{\preceq}(f+g) \succeq \deg_{\preceq}(f)$ follows.

Proof of claim (3) and (4): We can write

$$f = \sum_{\vec{i} \preceq \deg_{\preceq}(f)} f_{\vec{i}} x^{\vec{i}} \qquad \text{and} \qquad g = \sum_{\vec{j} \preceq \deg_{\preceq}(g)} g_{\vec{j}} x^{\vec{j}}$$

and thus obtain the following description of the product fg.

$$f \cdot g = \sum_{\substack{\vec{i} \preceq \deg_{\preceq}(f) \\ \vec{j} \preceq \deg_{\preceq}(g)}} f_{\vec{i}} f_{\vec{j}} x^{\vec{i}+\vec{j}}$$

As $\preceq$ is not just a well-order, but a monomial order, it follows from $\vec{i} \preceq \deg_{\preceq}(f)$ and $\vec{j} \preceq \deg_{\preceq}(g)$ that $\vec{i} + \vec{j} \preceq \deg_{\preceq}(f) + \deg_{\preceq}(g)$, and if one (or both) of the former two inequalities is strict, then so is the latter inequality. This implies both claims. □

Proposition 9.1.2.4. *Assume we are in the situation of Construction 9.1.1.11. Let f be an element of $k[x_1, \dots, x_n]$, and assume that $\deg_{\preceq}(f)$ is in the image of ψ. Let f' be the element of $k'[y_1, \dots, y_m]$ corresponding to f under the isomorphism from Construction 9.1.1.11. Then*

$$\deg_{\preceq}(f) = \psi\Big(\deg_{\preceq}(f')\Big)$$

where on the right hand side $\preceq$ refers to the restricted monomial order as defined in Construction 9.1.1.11. Furthermore, $f'_{\deg_{\preceq}(f')}$ is an element of k and the leading coefficients of f and f' agree, i. e. $f'_{\deg_{\preceq}(f')} = f_{\deg_{\preceq}(f)}$. ♡

Proof. Let $\vec{j} \in \mathbb{Z}^m_{\geq 0}$ be such that $\psi(\vec{j}) = \deg_{\preceq}(f)$. Then $f_{\psi(\vec{j})} \neq 0$ implies that $f'_{\vec{j}} \neq 0$ and hence $\deg_{\preceq}(f') \succeq \vec{j}$, from which we can conclude that $\psi(\deg_{\preceq}(f')) \succeq \deg_{\preceq}(f)$. On the other hand, $f'_{\deg_{\preceq}(f')} \neq 0$, so there must be some $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ with $i_l = 0$ for l in the image of φ such that

$$f_{\psi\left(\deg_{\preceq}(f')\right)+\vec{i}} = (f'_{\deg_{\preceq}(f')})_{\vec{i}} \neq 0$$

from which

$$\deg_{\preceq}(f) \succeq \psi\Big(\deg_{\preceq}(f')\Big) + \vec{i} \succeq \psi\Big(\deg_{\preceq}(f')\Big) \tag{$*$}$$

follows. Antisymmetry now implies that $\deg_{\preceq}(f) = \psi\Big(\deg_{\preceq}(f')\Big)$.

Furthermore, this implies that if $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ with $i_l = 0$ for l in the image of φ such that $(f'_{\deg_{\preceq}(f')})_{\vec{i}} \neq 0$, then $\vec{i}$ must actually be $\vec{0}$, as otherwise the inequality ($*$) would by *strict* by Proposition 9.1.1.8 ((2)). It follows that $f'_{\deg_{\preceq}(f')}$ is in k and that $f'_{\deg_{\preceq}(f')} = f_{\deg_{\preceq}(f)}$ as elements of k. □

9.1.3. Dividing multivariable polynomials with remainder

In this section we discuss a generalization of division with remainder of polynomials from the one-variable case as discussed in the introduction to Section 9.1 to the multivariable case. If we want to have a chance of dividing polynomials P with remainder by some polynomial f, then we should require that f is *monic*, and we discuss the multivariable notion of monic polynomials that we will use in Section 9.1.3.1. If f is a monic polynomial, then division with remainder will yield a decomposition of P as $P = Qf + R$, where R is in some sense “small” with respect to f. In the one-variable case, R will have smaller degree than f. In the multivariable case, R will be f*-reduced*, and we discuss what this means in Section 9.1.3.2. We will then be able to tackle division with remainder for multivariable polynomials in Section 9.1.3.3, and discuss decomposing P as $P = \sum_{i\geq 0} r^i_f(P) f^i$ with $r^i_f(P)$ being f-reduced polynomials in Section 9.1.3.4.

9.1.3.1. Monic polynomials

After the discussions in Sections 9.1.1 and 9.1.2, we can now give a definition of monic polynomials that generalizes the usual definition for the univariable case.

Definition 9.1.3.1. Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f \in k[x_1, \ldots, x_n]$ a polynomial. Then f is *monic with respect to* $\preceq$ if $f_{\deg_{\preceq}(f)} = 1$. In particular a monic polynomial is nonzero. ♢

Convention 9.1.3.2. From here on we will introduce a monomial order $\preceq$ in statements which depend on one, but will drop reference to $\preceq$ when this will not cause confusion. For example we will write “Let f be a monic polynomial.” rather than “Let f be a monic polynomial with respect to $\preceq$.” when there is only one polynomial degree order in context. ♢

Remark 9.1.3.3. If $n = 1$, then f is monic as defined in Definition 9.1.3.1 if and only if it is monic in the usual sense. See Remarks 9.1.1.9 and 9.1.2.2. ♢

Proposition 9.1.3.4. *Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f, g \in k[x_1, \ldots, x_n]$ monic polynomials. Then $f \cdot g$ is also monic.* ♡

Proof. Follows immediately from Proposition 9.1.2.3 (4). □

Proposition 9.1.3.5. *Assume we are in the situation of Construction 9.1.1.11, and that f and f' are as in Proposition 9.1.2.4. Then f is monic with respect to the monomial order on $\mathbb{Z}^n_{\geq 0}$ if and only if f' is monic with respect to the restricted monomial order on $\mathbb{Z}^m_{\geq 0}$.* ♡

Proof. Follows immediately from Proposition 9.1.2.4. □

We end this section with a useful statement we will use later.

Proposition 9.1.3.6. *Let n be a positive integer, $\preceq$ a monomial order for n variables, $f \in k[x_1, \ldots, x_n]$ a monic polynomial, and $g \in k[x_1, \ldots, x_n]$ any polynomial. Then $g = 0$ if and only if $fg = 0$.* ♡

Proof. It is clear that $g = 0$ implies $fg = 0$, so it remains to show that $g \neq 0$ implies $fg \neq 0$. But if $g \neq 0$, then we can apply Proposition 9.1.2.3 (4)[6], to conclude that

$$\deg_{\preceq}(fg) = \deg_{\preceq}(f) + \deg_{\preceq}(g)$$

where the right hand side, and thus also the left hand side, is a nonnegative integer. Thus fg must be nonzero. □

9.1.3.2. Reduced polynomials

Let f be a monic polynomial in a single variable, i. e. an element of $k[x]$. Then we can write any polynomial $P \in k[x]$ as $P = Q \cdot f + R$ for $Q, R \in k[x]$ such that the degree of R is smaller than the degree of f. If we want to generalize this to the multivariable case we should find an analogous condition for R. A first guess might be to use the condition that $\deg_{\preceq}(R) \prec \deg_{\preceq}(f)$, but this turns out not to work. Consider for example the case of two variables and the lexicographic order, so where $(a_1, a_2) \preceq (b_1, b_2)$ if $a_1 < b_1$ or if $a_1 = b_1$ and $a_2 \leq b_2$. If we then consider $f = x_1 x_2$ and $P = x_1^2$, then it is impossible to find a decomposition $P = Q \cdot f + R$ such that $\deg_{\preceq}(R) \prec \deg_{\preceq}(f)$. So this condition is too strong. The reason is that we can only eliminate the lead term of P if $\deg_{\preceq}(f) \leq \deg_{\preceq}(P)$. We should thus ask R to be f-reduced in the following sense.

Definition 9.1.3.7. Let n be a positive integer, $\preceq$ a monomial order for n variables, $\vec{j} \in \mathbb{Z}^n_{\geq 0}$, and $P \in k[x_1, \ldots, x_n]$ a polynomial. P is called *$\vec{j}$-reduced* if $P_{\vec{i}} = 0$ for all $\vec{i} \geq \vec{j}$.

If $f \in k[x_1, \ldots, x_n]$ is a nonzero polynomial, then P is called f-reduced if and only if P is $\deg_{\preceq}(f)$-reduced. ♢

Remark 9.1.3.8. If $f \neq 0$ and P are elements of $k[x]$, then P is f-reduced in the sense of Definition 9.1.3.7 if and only if the degree of P is smaller than the degree of f. ♢

Remark 9.1.3.9. Assume that we are in the situation of Construction 9.1.1.11. Let f and P be elements of $k[x_1, \ldots, x_n]$ and assume that $\deg_{\preceq}(f)$ is in the image of ψ. Let f' and P' be the elements of $k'[y_1, \ldots, y_m]$ corresponding to f and P under the isomorphism from Construction 9.1.1.11.

Then P is f-reduced if and only if P' is f'-reduced. This can be seen by combining Proposition 9.1.2.4 with arguments very similar to those used in the proof of Proposition 9.1.2.4. ♢

[6] Both f and g are nonzero, and as f is monic we also have $f_{\deg_{\preceq}(f)} \cdot g_{\deg_{\preceq}(g)} = g_{\deg_{\preceq}(g)} \neq 0$.

9.1.3.3. Division with remainder

We are now ready to discuss division with remainders for multivariable polynomials.

Proposition 9.1.3.10. *Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f \in k[x_1, \ldots, x_n]$ a monic polynomial. Let $P \in k[x_1, \ldots, x_n]$ be another polynomial. Then there exist unique polynomials $Q, R \in k[x_1, \ldots, x_n]$ such that $P = Q \cdot f + R$ and R is f-reduced.* ♡

Proof. We first prove uniqueness. Assume that

$$P = Q_1 \cdot f + R_1 \qquad \text{and} \qquad P = Q_2 \cdot f + R_2$$

are two such decompositions. Then the equation

$$(Q_1 - Q_2) \cdot f = R_2 - R_1 \tag{$*$}$$

holds. We have to show that $Q_1 = Q_2$ and $R_1 = R_2$, but applying Proposition 9.1.3.6 to $(*)$ it suffices to show that $R_1 = R_2$.

We show $R_1 = R_2$ by contradiction and assume that $R_1 \neq R_2$. Without loss of generality we can additionally assume that $R_1 \prec R_2$. By Proposition 9.1.3.6 $Q_1 - Q_2 \neq 0$, so we can apply Proposition 9.1.2.3 (4) to $(*)$ and obtain the following formula relating the degrees.

$$\deg_{\preceq}(R_2 - R_1) = \deg_{\preceq}(Q_1 - Q_2) + \deg_{\preceq}(f)$$

As we assumed $R_1 \prec R_2$, we can also apply Proposition 9.1.2.3 (2) to obtain

$$\deg_{\preceq}(R_2 - R_1) = \deg_{\preceq}(R_2)$$

which implies that

$$\deg_{\preceq}(R_2) = \deg_{\preceq}(Q_1 - Q_2) + \deg_{\preceq}(f)$$

and thus in particular $\deg_{\preceq}(R_2) \geq \deg_{\preceq}(f)$, contradicting the assumption that R_2 is f-reduced.

It remains to show existence of the claimed decomposition. So for every polynomial $P \in k[x_1, \ldots, x_n]$ we have to prove the following claim.

Claim There exist $Q, R \in k[x_1, \ldots, x_n]$ such that R is f-reduced and $P = Qf + R$.

To do so, we first define the map

$$\begin{aligned} \Theta\colon k[x_1, \ldots, x_n] &\to \mathbb{Z}^n_{\geq 0} \cup \{-\infty\} \\ P &\mapsto \max \left\{ \vec{i} \in \mathbb{Z}^n_{\geq 0} \;\middle|\; P_{\vec{i}} \neq 0 \text{ and } \vec{i} \geq \deg_{\preceq}(f) \right\} \end{aligned}$$

where the maximum is to be interpreted as $-\infty$ if the set is empty, and the set the maximum is taken over is always finite[7], so the maximum exists if the set is nonempty.

[7] As polynomials only have finitely many nonzero components.

Note that $R \in k[x_1, \ldots, x_n]$ is f-reduced if and only if $\Theta(R) = -\infty$. We can extend the well-order $\preceq$ on $\mathbb{Z}^n_{\geq 0}$ to $\mathbb{Z}^n_{\geq 0} \cup \{-\infty\}$ by letting $-\infty$ be the minimal element, and will prove the claim stated above for every element P of $k[x_1, \ldots, x_n]$ by transfinite induction on $\Theta(P)$.

So we let P be an element of $k[x_1, \ldots, x_n]$ and assume that the claim holds for any $P' \in k[x_1, \ldots, x_n]$ with $\Theta(P') \preceq \Theta(P)$. We have to show that then P also satisfies the claim.

If $\Theta(P) = -\infty$, then P itself is reduced and so we can take $Q = 0$, $R = P$ and are done.

So assume now that $\Theta(P) \neq -\infty$. Note that the definition of $\Theta(P)$ and the assumption that $\Theta(P) \neq -\infty$ together imply that $\Theta(P) \geq \deg_{\preceq}(f)$, so that in particular $\Theta(P) - \deg_{\preceq}(f)$ is an element of $\mathbb{Z}^n_{\geq 0}$. We can thus define a new polynomial P' as follows.

$$P' = P - P_{\Theta(P)} \cdot x^{\Theta(P) - \deg_{\preceq}(f)} \cdot f \tag{$**$}$$

We claim that $\Theta(P') \prec \Theta(P)$. Let us for the moment assume this and explain how the claim for P follows. As $\Theta(P') \prec \Theta(P)$ we can use the induction hypothesis and obtain $Q', R' \in k[x_1, \ldots, x_n]$ such that R' is f-reduced and $P' = Q'f + R'$. Combining this with ($**$) we obtain

$$P = \left(Q' + P_{\Theta(P)} x^{\Theta(P) - \deg_{\preceq}(f)}\right) \cdot f + R'$$

so that setting $Q = Q' + P_{\Theta(P)} x^{\Theta(P) - \deg_{\preceq}(f)}$ and $R = R'$ shows the claim for P.

We are left to show that $\Theta(P') \prec \Theta(P)$. Note that ($**$) implies that for $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ we have

$$P'_{\vec{i}} = P_{\vec{i}} - P_{\Theta(P)} \cdot f_{\vec{i} + \det_{\preceq}(f) - \Theta(P)} \tag{$***$}$$

where $f_{\vec{i} + \det_{\preceq}(f) - \Theta(P)}$ is to be interpreted as 0 if $\vec{i} + \det_{\preceq}(f) - \Theta(P)$ is not in $\mathbb{Z}^n_{\geq 0}$. Plugging in $\vec{i} = \Theta(P)$ we obtain

$$P'_{\Theta(P)} = P_{\Theta(P)} - P_{\Theta(P)} \cdot f_{\det_{\preceq}(f)} = P_{\Theta(P)} - P_{\Theta(P)} = 0$$

so if $\Theta(P') \succeq \Theta(P)$ then we actually must have $\Theta(P') \succ \Theta(P)$. So now assume that $\vec{i}$ is an element of $\mathbb{Z}^n_{\geq 0}$ such that the following holds.

(1) $\vec{i} \geq \deg_{\preceq}(f)$

(2) $\vec{i} \succ \Theta(P)$

What we have to show is that then $P'_{\vec{i}} = 0$. The two assumptions imply that $P_{\vec{i}} = 0$, so if $\vec{i} + \det_{\preceq}(f) - \Theta(P)$ is not in $\mathbb{Z}^n_{\geq 0}$, then equation ($***$) implies $P'_{\vec{i}} = 0$. So assume that $\vec{i} + \det_{\preceq}(f) - \Theta(P)$ is in $\mathbb{Z}^n_{\geq 0}$. (2) implies that

$$\vec{i} + \deg_{\preceq}(f) \succ \Theta(P) + \deg_{\preceq}(f)$$

which – using that $\vec{i} + \det_{\preceq}(f) - \Theta(P)$ is in $\mathbb{Z}^n_{\geq 0}$ – implies that

$$\vec{i} + \deg_{\preceq}(f) - \Theta(P) \succ \deg_{\preceq}(f)$$

from which we can deduce that $f_{\vec{i}+\deg_{\preceq}(f)-\Theta(P)} = 0$. It again follows from equation $(***)$ that $P'_{\vec{i}} = 0$. $\square$

Remark 9.1.3.11. Assume that we are in the situation of Construction 9.1.1.11. Let f and P be elements of $k[x_1, \ldots, x_n]$ and assume that $\deg_{\preceq}(f)$ is in the image of ψ. Let f' and P' be the elements of $k'[y_1, \ldots, y_m]$ corresponding to f and P under the isomorphism from Construction 9.1.1.11. Then the decompositions of P and P' with respect to f and f' correspond to each other under the isomorphism from Construction 9.1.1.11. Concretely, if Q, R are elements of $k[x_1, \ldots, x_n]$ such that $P = Qf + R$ and R is f-reduced, and Q' and R' are the elements of $k'[y_1, \ldots, y_m]$ corresponding to Q and R under the isomorphism from Construction 9.1.1.11, then $P' = Q'f + R$ as the isomorphism is an isomorphism of R-algebras, and R' is f'-reduced by Remark 9.1.3.9. $\diamondsuit$

9.1.3.4. Full sum decomposition

If f is a monic polynomial and P any polynomial, we saw in Proposition 9.1.3.10 that we can divide P by f with remainder to obtain a decomposition $P = Qf + R_0$ for polynomials Q and R_0 such that R_0 is f-reduced. We can then also divide Q by f with remainder and obtain a decomposition of Q as $Q = Q'f + R_1$, so that we can write P as $P = Q'f^2 + R_1 f + R_0$. We would like this process to eventually stop (i. e. eventually arrive at an R_i that is already f-reduced), to obtain a decomposition of P as $P = \sum_{i\geq 0} R_i \cdot f^i$, such that each R_i is f-reduced and all but finitely many are zero. For this we however need one extra assumption: If $f = 1$, then the decomposition from Proposition 9.1.3.10 will be $P = P \cdot 1 + 0$, so iterating this process will never yield an f-reduced R_i unless $P = 0$. We thus arrive at the following proposition.

Proposition 9.1.3.12. *Let n be a positive integer, $\preceq$ a monomial order for n variables, and $f \in k[x_1, \ldots, x_n]$ a monic polynomial with $\deg_{\preceq}(f) > \vec{0}$ (equivalently, $f \neq 1$). Let $P \in k[x_1, \ldots, x_n]$ be another polynomial. Then there exist unique $R_i \in k[x_1, \ldots, x_n]$ for $i \in \mathbb{Z}_{\geq 0}$ of which all but finitely many are zero such that*

$$P = \sum_{i\geq 0} R_i \cdot f^i$$

and all R_i are f-reduced. $\heartsuit$

Proof. We first show uniqueness. So assume we are given two such decompositions as follows.

$$P = \sum_{i\geq 0} R_i \cdot f^i \qquad \text{and} \qquad P = \sum_{i\geq 0} R'_i \cdot f^i$$

We can rewrite this as

$$\left(\sum_{i\geq 1} R_i \cdot f^{i-1}\right) \cdot f + R_0 = \left(\sum_{i\geq 1} R'_i \cdot f^{i-1}\right) \cdot f + R'_0$$

and hence by Proposition 9.1.3.10 we can conclude that $R_0 = R'_0$ and

$$\sum_{i\geq 1} R_i \cdot f^{i-1} = \sum_{i\geq 1} R'_i \cdot f^{i-1}$$

as well. Iterating this argument now yields $R_1 = R'_1$, $R_2 = R'_2$, and so on.

We prove existence by transfinite induction on $\deg_{\preceq}(P)$ and assume that the statement has already been proven for all polynomials P' with $\deg_{\preceq}(P') \prec \deg_{\preceq}(P)$. By Proposition 9.1.3.10 there are polynomials Q and R_0 such that $P = Qf + R_0$ and R_0 is f-reduced. If $Q = 0$ we are already done, so assume that $Q \neq 0$. As R_0 is f-reduced we must have $(R_0)_{\deg_{\preceq}(Q)+\deg_{\preceq}(f)} = 0$ and hence, using Proposition 9.1.2.3 (4),

$$P_{\deg_{\preceq}(Q)+\deg_{\preceq}(f)} = (Qf)_{\deg_{\preceq}(Q)+\deg_{\preceq}(f)} \neq 0$$

so that we can conclude that $\deg_{\preceq}(P) \succeq \deg_{\preceq}(Q)+\deg_{\preceq}(f)$. As we assumed $\vec{0} \prec \deg_{\preceq}(f)$ this implies the following inequality.

$$\deg_{\preceq}(Q) \prec \deg_{\preceq}(Q) + \deg_{\preceq}(f) \preceq \deg_{\preceq}(P)$$

By the induction hypothesis we can thus find f-reduced polynomials R_i for $i \geq 1$, all but finitely many zero, such that

$$Q = \sum_{i\geq 1} R_i f^{i-1}$$

which implies that

$$P = Q \cdot f + R_0 = \left(\sum_{i\geq 1} R_i f^{i-1}\right) \cdot f + R_0 = \sum_{i\geq 0} R_i f^i$$

and thus proves the claim. □

The assumptions made in Proposition 9.1.3.12 will be used a lot in the rest of this chapter. To improve readability and reduce unnecessary repetitions, we thus package them together.

Assumption MonOrdMonicPoly. *Wherever we invoke this assumption, we let n be a positive integer, $\preceq$ a monomial order for n variables, and $f \in k[x_1, \ldots, x_n]$ a monic polynomial with $\deg_{\preceq}(f) > 0$.* ♢

We next introduce some notation to help us refer to the polynomials R_i occurring in the decomposition from Proposition 9.1.3.12.

Definition 9.1.3.14. Assume MonOrdMonicPoly. We define maps

$$r_f^j, r_f^{\leq j}, r_f^{<j}, q_f^j\colon k[x_1,\ldots,x_n] \to k[x_1,\ldots,x_n]$$

for each integer j in the following way.

For $P \in k[x_1,\ldots,x_n]$, let

$$P = \sum_{i\geq 0} R_i f^i$$

be the decomposition from Proposition 9.1.3.12, i. e. R_i is an f-reduced element of $k[x_1,\ldots,x_n]$ for each $i \geq 0$. We then define r_f^j, $r_f^{\leq j}$, $r_f^{<j}$, and q_f^j for $j \geq 0$ as follows.

$$r_f^j(P) \coloneqq R_j \qquad r_f^{\leq j}(P) \coloneqq \sum_{i=0}^{j} r_f^i(P) f^i \qquad r_f^{<j}(P) \coloneqq \sum_{i=0}^{j-1} r_f^i(P) f^i$$

$$q_f^j(P) \coloneqq \sum_{i\geq j} r_f^i(P)\cdot f^{i-j} \qquad \diamondsuit$$

If $j < 0$, then we define r_f^j, $r_f^{\leq j}$, and $r_f^{<j}$ to map P to 0, and define $q_f^j(P) \coloneqq P \cdot f^{-j}$.

9.1.3.5. Properties of remainders

We collect a number of useful properties of the maps from Definition 9.1.3.14.

Proposition 9.1.3.15. *Assume MonOrdMonicPoly. Then the following hold for each $i, j \geq 0$ and $P, Q \in k[x_1,\ldots,x_n]$.*

(1) $r_f^j(P)$ *is f-reduced.*

(2) $P = q_f^j(P)\cdot f^j + r_f^{<j}(P)$.

(3) r_f^j, $r_f^{\leq j}$, $r_f^{<j}$, *and* q_f^j *are k-linear.*

(4) $r_f^j(P\cdot f^i) = r_f^{j-i}(P)$ *and* $q_f^j(P\cdot f^i) = q_f^{j-i}(P)$.

(5) $r_f^j(P\cdot Q) = \sum_{a+b+c=j} r_f^a\Big(r_f^b(P)\cdot r_f^c(Q)\Big)$.

(6) $q_f^i\Big(q_f^j(P)\Big) = q_f^{i+j}(P)$. $\heartsuit$

Proof. Proof of claims (1), (2), and (4): Clear by definition.

Proof of claim (3): Follows immediately from uniqueness of the decomposition in Proposition 9.1.3.12, as k-linear combinations of f-reduced polynomials are again f-reduced.

Proof of claim (5): First note that both sides are k-linear in both P and Q. It hence suffices to consider the case $P = R\cdot f^e$, $Q = R'\cdot f^{e'}$ with f-reduced polynomials R and R' and nonnegative integers e and e'. In this case we can read off

$$r_f^b(P) = \begin{cases} R & \text{if } b = e \\ 0 & \text{otherwise}\end{cases} \qquad \text{and} \qquad r_f^c(Q) = \begin{cases} R' & \text{if } c = e' \\ 0 & \text{otherwise}\end{cases}$$

so that we obtain

$$\sum_{a+b+c=j} r_f^a\Big(r_f^b(P)\cdot r_f^c(Q)\Big) = r_f^{j-e-e'}\big(RR'\big)$$

which is equal to $r_f^j(P\cdot Q) = r_f^j(RR'f^{e+e'})$ by (4).

Proof of claim (6): This follows from the previous claims, as in the following calculation.

$$\begin{aligned} q_f^{i+j}(P) &= q_f^{i+j}\Big(q_f^j(P)f^j + r_f^{<j}(P)\Big) \\ &= q_f^{i+j}\Big(q_f^j(P)f^j\Big) + q_f^{i+j}\Big(r_f^{<j}(P)\Big) \\ &= q_f^i\Big(q_f^j(P)\Big) + 0 \end{aligned}$$

□

As r_f^j, $r_f^{\leq j}$, $r_f^{<j}$, and q_f^j are k-linear by Proposition 9.1.3.15 (3), we can extend their definitions as follows.

Convention 9.1.3.16. Assume MonOrdMonicPoly. Let M be a (graded) k-module. Then for any integer j we obtain a morphism of (graded) k-modules

$$r_f^j \otimes_k \mathrm{id}_M : k[x_1,\ldots,x_n]\otimes_k M \to k[x_1,\ldots,x_n]\otimes_k M \tag{9.1}$$

which we will also call r_f^j. Similarly for $r_f^{\leq j}, r_f^{<j}$, and q_f^j. ♢

9.2. A strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of medium size

In this section we will give a description of a strict mixed complex that represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$ as an object of $\mathcal{M}\mathrm{ixed}$ under assumptions MonOrdMonicPoly and Conjecture D for f.

We will start in Section 9.2.1 by showing that $k[x_1,\ldots,x_n]$ satisfies the necessary conditions as a module over $k[t]$ in order to apply the more general result Proposition 8.3.0.1 on a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$ of quotients. We will then spell out Proposition 8.3.0.1 specialized to $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$ in Section 9.2.2. While there is an obvious basis of the resulting strict mixed complex, that basis is not well adapted to further simplification steps that we will want to do in later sections. We thus describe a new, more useful, basis in Section 9.2.3.

9.2.1. $k[x_1,\ldots,x_n]$ as a module over $k[t]$

In this short section we show that multiplication with f acts on $k[x_1,\ldots,x_n]$ in a way that satisfies the requirements to apply Proposition 8.3.0.1.

Proposition 9.2.1.1. *Assume MonOrdMonicPoly. Then the subset*

$$\left\{ x^{\vec{i}} \;\middle|\; \vec{i}\in\mathbb{Z}^n_{\geq 0},\ \vec{i}\not\geq \deg_{\preceq}(f) \right\} \tag{9.2}$$

of $k[x_1,\ldots,x_n]$ is a basis of $k[x_1,\ldots,x_n]$ as a right-$k[t]$-module, where t acts by multiplication with f. In particular, $k[x_1,\ldots,x_n]$ is free as a right-$k[t]$-module. ♡

Proof. The sub-k-module of $k[x_1, \ldots, x_n]$ spanned by $x^{\vec{i}}$ for $\vec{i} \ngeq \deg_{\preceq}(f)$ is a basis of the sub-k-module of f-reduced polynomials, so it follows from Proposition 9.1.3.12 that (9.2) generates $k[x_1, \ldots, x_n]$ as a right-$k[t]$-module.

For linear independence, assume that $p_{\vec{i}}$ are elements of $k[t]$ for each $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ such that $\vec{i} \ngeq \deg_{\preceq}(f)$, with all but finitely many $p_{\vec{i}}$ zero. We can write $p_{\vec{i}}$ as $p_{\vec{i}} = \sum_{j\geq 0} a_{\vec{i},j} t^j$, with $a_{\vec{i},j}$ elements of k, all but finitely many (for fixed $\vec{i}$) zero. Furthermore, assume that the following holds.

$$\sum_{\substack{\vec{i}\in\mathbb{Z}^n_{\geq 0},\\ \vec{i}\ngeq\deg_{\preceq}(f)}} x^{\vec{i}} \cdot \left(\sum_{j\geq 0} a_{\vec{i},j} f^j \right) = 0$$

Then the uniqueness part of Proposition 9.1.3.12 implies

$$\sum_{\substack{\vec{i}\in\mathbb{Z}^n_{\geq 0},\\ \vec{i}\ngeq\deg_{\preceq}(f)}} a_{\vec{i},j} x^{\vec{i}} = 0$$

for every $j \geq 0$, but as the $x^{\vec{i}}$ are k-linearly independent, this implies that all $a_{\vec{i},j}$ are zero. □

Proposition 9.2.1.2. *Assume MonOrdMonicPoly. Then $k[x_1, \ldots, x_n]$ is cofibrant as an object in $\mathrm{RMod}_{k[t]}(\mathsf{Ch}(k))$, where t acts by multiplication with f.* ♡

Proof. As $k[x_1, \ldots, x_n]$ is free as a right-$k[t]$-module by Proposition 9.2.1.1, this follows from Theorem 4.2.2.1 (5) and [Hov99, 2.3.6]. □

9.2.2. A strict model for $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}$

We can now specialize Proposition 8.3.0.1 to obtain a first strict mixed complex X_f that represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$. While the result is conditional on Conjecture D holding for f, we can construct X_f in greater generality.

Construction 9.2.2.1. Assume MonOrdMonicPoly. We will construct a strict mixed complex X_f. As a $\mathbb{Z}$-graded k-module[8], X_f is given by

$$X_f \coloneqq k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$$

with $x_1, \ldots, x_n$ of degree 0, $\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n$ and s of degree 1 and $\mathrm{d}\, s$ of degree 2. The boundary operator ∂ and differential d are given by k-linearly extending the following formulas for $\vec{i} \in \mathbb{Z}^n_{\geq 0}$, $\vec{\epsilon} \in \{0,1\}^n$, and $m \geq 0$.

$$\partial\Big(x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}} s\, \mathrm{d}\, s^{[m]}\Big) = (-1)^{|\vec{\epsilon}|} x^{\vec{i}} f\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m]} - (-1)^{|\vec{\epsilon}|} x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, f \cdot s\, \mathrm{d}\, s^{[m-1]}$$

[8] We will use the structure of a commutative $\mathbb{Z}$-graded k-algebra on X_f to write down elements, but X_f itself is only considered as a strict mixed complex.

$$\begin{aligned}
\partial\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big) &= -(-1)^{|\vec{\epsilon}|}x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]}\\
\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}s\,\mathrm{d}\,s^{[m]}\Big) &= \mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}s\,\mathrm{d}\,s^{[m]} + (-1)^{|\vec{\epsilon}|}(m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m+1]}\\
\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big) &= \mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}
\end{aligned}$$

In the formulas above, d applied to elements of $k[x_1,\dots,x_n]\otimes\Lambda(\mathrm{d}\,x_1,\dots,\mathrm{d}\,x_n)$ is defined as in $\Omega^{\bullet}_{k[x_1,\dots,x_n]/k}$[9], and $\mathrm{d}\,s^{[-1]}$ is to be interpreted as zero.

To see that ∂ and d as defined really upgrade X_f to a strict mixed complex we need to check that ∂ and d square to 0, and that $\partial\circ\mathrm{d}+\mathrm{d}\circ\partial=0$ holds. We check all of these on basis elements. Using that $\mathrm{d}\,f\cdot\mathrm{d}\,f=0$ in the $\mathbb{Z}$-graded k-algebra underlying X_f we obtain the following calculations for $\vec{i}\in\mathbb{Z}^n_{\geq 0}$, $\vec{\epsilon}\in\{0,1\}^n$, and $m\geq 0$.

$$\begin{aligned}
&\partial\Big(\partial\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}s\,\mathrm{d}\,s^{[m]}\Big)\Big)\\
&=\partial\Big((-1)^{|\vec{\epsilon}|}x^{\vec{i}}f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]} - (-1)^{|\vec{\epsilon}|}x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\cdot s\,\mathrm{d}\,s^{[m-1]}\Big)\\
&= -(-1)^{|\vec{\epsilon}|}(-1)^{|\vec{\epsilon}|}x^{\vec{i}}f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]}\\
&\quad - (-1)^{|\vec{\epsilon}|}\Big((-1)^{|\vec{\epsilon}|+1}x^{\vec{i}}f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]} - (-1)^{|\vec{\epsilon}|+1}x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\cdot\mathrm{d}\,f\cdot s\,\mathrm{d}\,s^{[m-1]}\Big)\\
&= -x^{\vec{i}}f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]} + x^{\vec{i}}f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]} - 0\\
&= 0
\end{aligned}$$

$$\begin{aligned}
&\partial\Big(\partial\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big)\Big)\\
&=\partial\Big(-(-1)^{|\vec{\epsilon}|}x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]}\Big)\\
&=\Big(-(-1)^{|\vec{\epsilon}|}\Big)\cdot\Big(-(-1)^{|\vec{\epsilon}|+1}\Big)\cdot x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-2]}\\
&=0
\end{aligned}$$

Using that d squares to 0 in $\Omega^{\bullet}_{k[x_1,\dots,x_n]/k}$ we obtain the following calculations.

$$\begin{aligned}
&\mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}s\,\mathrm{d}\,s^{[m]}\Big)\Big)\\
&=\mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}s\,\mathrm{d}\,s^{[m]} + (-1)^{|\vec{\epsilon}|}(m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m+1]}\Big)\\
&=\mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}s\,\mathrm{d}\,s^{[m]} + (-1)^{|\vec{\epsilon}|+1}(m+1)\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m+1]}\\
&\quad + (-1)^{|\vec{\epsilon}|}(m+1)\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m+1]}\\
&=0
\end{aligned}$$

[9] So extending from $\mathrm{d}(x_i)\coloneqq\mathrm{d}\,x_i$ using k-linearity and the Leibniz rule.

$$
\begin{aligned}
&\mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big)\Big)\\
&= \mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big)\\
&= \mathrm{d}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\\
&= 0
\end{aligned}
$$

Finally, using that d satisfies the Leibniz rule on $\Omega^{\bullet}_{k[x_1,\ldots,x_n]/k}$ we can carry out the following calculations showing that $\partial \circ \mathrm{d} + \mathrm{d} \circ \partial = 0$.

$$
\begin{aligned}
&\partial\Big(\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}s\,\mathrm{d}\,s^{[m]}\Big)\Big) + \mathrm{d}\Big(\partial\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}s\,\mathrm{d}\,s^{[m]}\Big)\Big)\\
&= \partial\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}s\,\mathrm{d}\,s^{[m]} + (-1)^{|\vec{\epsilon}|}(m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m+1]}\Big)\\
&\quad + \mathrm{d}\Big((-1)^{|\vec{\epsilon}|}x^{\vec{i}}f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]} - (-1)^{|\vec{\epsilon}|}x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\cdot s\,\mathrm{d}\,s^{[m-1]}\Big)\\
&= (-1)^{|\vec{\epsilon}|+1}\,\mathrm{d}\Big(x^{\vec{i}}\Big)f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]} - (-1)^{|\vec{\epsilon}|+1}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\cdot s\,\mathrm{d}\,s^{[m-1]}\\
&\quad - (-1)^{|\vec{\epsilon}|}(-1)^{|\vec{\epsilon}|}(m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\\
&\quad + (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\cdot f\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\\
&\quad - (-1)^{|\vec{\epsilon}|}\Big(\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\Big)\cdot s\,\mathrm{d}\,s^{[m-1]} + (-1)^{|\vec{\epsilon}|+1}\cdot m\cdot x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\Big)\\
&= -(-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\Big)f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]} + (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\cdot s\,\mathrm{d}\,s^{[m-1]}\\
&\quad - (m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\\
&\quad + (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\cdot f\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]} + (-1)^{|\vec{\epsilon}|}x^{\vec{i}}\cdot\mathrm{d}(f)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\\
&\quad - (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\cdot s\,\mathrm{d}\,s^{[m-1]} + mx^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\\
&= -(m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]} + (-1)^{|\vec{\epsilon}|}x^{\vec{i}}\cdot\mathrm{d}(f)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]} + mx^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\\
&= -(m+1)x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]} + x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]} + mx^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m]}\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&\partial\Big(\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big)\Big) + \mathrm{d}\Big(\partial\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big)\Big)\\
&= \partial\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,s^{[m]}\Big) - (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]}\Big)\\
&= -(-1)^{|\vec{\epsilon}|+1}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]} - (-1)^{|\vec{\epsilon}|}\,\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\,x^{\vec{\epsilon}}\,\mathrm{d}\,f\,\mathrm{d}\,s^{[m-1]}\\
&= 0
\end{aligned}
$$

Note that as X_f is free as a $\mathbb{Z}$-graded k-module, it follows from [Hov99, 2.3.6] that the underlying chain complex of X_f is cofibrant. ♢

Proposition 9.2.2.2. *Assume MonOrdMonicPoly and that Conjecture D*[10] *holds for f. Then there is an equivalence*

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/(f)) \simeq \gamma_{\mathsf{Mixed}}(X_f)$$

in $\mathcal{M}\mathrm{ixed}$, *where X_f is as in Construction 9.2.2.1.* ♡

Proof. This is a specialization of Proposition 8.3.0.1 for $R = k[x_1, \ldots, x_n]$, the x_1 from Proposition 8.3.0.1 being f and the n from Proposition 8.3.0.1 being 1. The requirement on R was verified with Proposition 9.2.1.2. That Conjecture D holds for f yields a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k) \arrow[r, dash, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k/k}\right) \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t]) \arrow[u, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(G)"] \arrow[d, "\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(F)"'] \arrow[r, dash, "\simeq"] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[t]/k}\right) \arrow[u, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{G/k}\right)"'] \arrow[d, "\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{F/k}\right)"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[X]) \arrow[r, dash, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\left(\Omega^{\bullet}_{k[X]/k}\right)
\end{tikzcd} \tag{$*$}$$

in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$ such that the horizontal morphisms are equivalences. We can use the top square as the one witnessing Conjecture C for Proposition 8.3.0.1.

Naturality of the identification at the start of Section 7.1 yields a commutative diagram

$$\begin{tikzcd}
k[t] \otimes \Lambda(\mathrm{d}\, t) \arrow[r] \arrow[d, dash, "\cong"'] & k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots \mathrm{d}\, x_n) \arrow[d, dash, "\cong"] \\
\Omega^{\bullet}_{k[t]/k} \arrow[r] & \Omega^{\bullet}_{k[x_1,\ldots,x_n]/k}
\end{tikzcd}$$

in $\mathrm{Alg}(\mathsf{Mixed}_{\mathrm{cof}})$ with the vertical morphisms the isomorphisms from Section 7.1 and the horizontal morphisms induced by $t \mapsto f$. Combining this with the bottom square in

[10] Note that Conjecture D holds if $n = 1$ or $n = 2$ with 2 invertible in k by Proposition 7.5.3.1.

diagram ($*$), we obtain a commutative diagram as follows in $\mathrm{Alg}(\mathcal{M}\mathrm{ixed})$

$$\begin{tikzcd}
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(k[t] \otimes \Lambda(\mathrm{d}\, t)\big) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\big(k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)\big) \arrow[d, "\simeq"] \\
\mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^\bullet_{k[t]/k}\Big) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Alg}(\gamma_{\mathsf{Mixed}})\Big(\Omega^\bullet_{k[x_1,\ldots,x_n]/k}\Big) \arrow[d, "\simeq"] \\
\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[t]) \arrow[r] & \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n])
\end{tikzcd}$$

where the left column is precisely (8.8), and the horizontal morphisms are all induced by $t \mapsto f$. Letting M be $k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots \mathrm{d}\, x_n)$, as a right-$k[t] \otimes \Lambda(\mathrm{d}\, t)$-module in $\mathsf{Mixed}_{\mathrm{cof}}$, with the module action arising from the above morphism of algebras, M thus satisfies the requirements for applying Proposition 8.3.0.1. $\square$

9.2.3. A basis for the strict model

In this section we describe a new basis for $k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$ in which the formulas for ∂ and d will take a form that will make it easier to construct smaller strict models in later sections.

9.2.3.1. Interaction of q_f^1 with d and multiplication

We will need two small results on the interaction of q_f^1 and q_f^2 with products and the differentiation.

Proposition 9.2.3.1. *Assume MonOrdMonicPoly. Then the following hold for P and Q elements of the strict mixed complex $k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)$ (see Section 7.1).*

(1) If P is f-reduced, then $\mathrm{d}\, P$ is f-reduced as well.

(2) $-q_f^1(\mathrm{d}\, f \cdot \mathrm{d}\, P) = q_f^1\Big(\mathrm{d}\, f \cdot q_f^1(\mathrm{d}\, f \cdot P)\Big) + \mathrm{d}\Big(q_f^1(\mathrm{d}\, f \cdot P)\Big)$

(3) $q_f^2(PQ) = q_f^1\Big(P \cdot q_f^1(Q)\Big) + q_f^2\Big(P \cdot r_f^0(Q)\Big)$ ♡

Proof. Proof of claim (1): It suffices to consider the case $P = x^{\vec{i}}$ for $\vec{i} \in \mathbb{Z}^n_{\geq 0}$. In this case, $\mathrm{d}\, P = \sum_{j=1}^n i_j x^{\vec{i} - \vec{e_j}}$, and the claim follows from $\vec{i} - \vec{e_j} \leq \vec{i}$.

Proof of claim (2): By definition we have

$$\mathrm{d}\, f \cdot P = f \cdot q_f^1(\mathrm{d}\, f \cdot P) + r_f^0(\mathrm{d}\, f \cdot P)$$

so that applying d yields the following.

$$-\mathrm{d}\, f \cdot \mathrm{d}\, P = \mathrm{d}\, f \cdot q_f^1(\mathrm{d}\, f \cdot P) + f\, \mathrm{d}\Big(q_f^1(\mathrm{d}\, f \cdot P)\Big) + \mathrm{d}\Big(r_f^0(\mathrm{d}\, f \cdot P)\Big)$$

We can now apply q_f^1, to obtain the following.

$$-q_f^1(\mathrm{d}\, f \cdot \mathrm{d}\, P) = q_f^1\Big(\mathrm{d}\, f \cdot q_f^1(\mathrm{d}\, f \cdot P)\Big) + q_f^1\bigg(f\,\mathrm{d}\Big(q_f^1(\mathrm{d}\, f \cdot P)\Big)\bigg) + q_f^1\bigg(\mathrm{d}\Big(r_f^0(\mathrm{d}\, f \cdot P)\Big)\bigg)$$

$r_f^0(\mathrm{d}\, f \cdot P)$ is f-reduced, so the third summand is zero by (1). We use Proposition 9.1.3.15 (4) for the second summand.

$$= q_f^1\Big(\mathrm{d}\, f \cdot q_f^1(\mathrm{d}\, f \cdot P)\Big) + \mathrm{d}\Big(q_f^1(\mathrm{d}\, f \cdot P)\Big)$$

Proof of claim (3): By definition of q_f^1 and r_f^0, the following holds.

$$Q = q_f^1(Q) \cdot f + r_f^0(Q)$$

We can now multiply with P on the left.

$$PQ = P \cdot q_f^1(Q) \cdot f + P \cdot r_f^0(Q)$$

Applying q_f^2 and using Proposition 9.1.3.15 (4) on the first summand on the right hand side we obtain the following.

$$q_f^2(PQ) = q_f^1\Big(P \cdot q_f^1(Q)\Big) + q_f^2\Big(P \cdot r_f^0(Q)\Big) \qquad \square$$

9.2.3.2. The basis

Definition 9.2.3.2. Assume MonOrdMonicPoly and let m be an integer. We define two k-linear maps

$C^{[m]}\colon k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \to k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$
and
$E^{[m]}\colon k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \to k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$

as follows. If $m < 0$, then we let $C^{[m]}$ and $E^{[m]}$ be constant with value 0. If $m \geq 0$, then we define them as follows.

$$C^{[m]}(g) \coloneqq sg\,\mathrm{d}\, s^{[m]}$$
$$E^{[m]}(g) \coloneqq g\,\mathrm{d}\, s^{[m]} + sq_f^1(\mathrm{d}\, f \cdot g)\,\mathrm{d}\, s^{[m-1]} = g\,\mathrm{d}\, s^{[m]} + C^{[m-1]}\Big(q_f^1(\mathrm{d}\, f \cdot g)\Big)$$

In the formulas above, we interpret $\mathrm{d}\, s^{[-1]}$ as zero.

Let $\mathfrak{J}$ be the defined as

$$\mathfrak{J} \coloneqq \left\{ \Big(\vec{i}, l, \vec{\epsilon}, m\Big) \in \mathbb{Z}_{\geq 0}^n \times \mathbb{Z}_{\geq 0} \times \{0,1\}^n \times \mathbb{Z}_{\geq 0} \;\middle|\; \vec{i} \ngeq \deg_{\preceq}(f) \right\}$$

and for $(\vec{i}, l, \vec{\epsilon}, m) \in \mathfrak{J}$, define $c_{\vec{i},l,\vec{\epsilon},m}$ and $e_{\vec{i},l,\vec{\epsilon},m}$ as follows.

$$c_{\vec{i},l,\vec{\epsilon},m} \coloneqq C^{[m]}\Big(x^{\vec{i}} f^l\,\mathrm{d}\, x^{\vec{\epsilon}}\Big) = sx^{\vec{i}} f^l\,\mathrm{d}\, x^{\vec{\epsilon}}\,\mathrm{d}\, s^{[m]}$$
$$e_{\vec{i},l,\vec{\epsilon},m} \coloneqq E^{[m]}\Big(x^{\vec{i}} f^l\,\mathrm{d}\, x^{\vec{\epsilon}}\Big) = x^{\vec{i}} f^l\,\mathrm{d}\, x^{\vec{\epsilon}}\,\mathrm{d}\, s^{[m]} + C^{[m-1]}\bigg(q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\bigg) \qquad \diamondsuit$$

Proposition 9.2.3.3. *Assume MonOrdMonicPoly. Then*

$$\left\{ c_{\vec{i},l,\vec{\epsilon},m} \;\middle|\; \left(\vec{i}, l, \vec{\epsilon}, m\right) \in \mathfrak{J} \right\} \cup \left\{ e_{\vec{i},l,\vec{\epsilon},m} \;\middle|\; \left(\vec{i}, l, \vec{\epsilon}, m\right) \in \mathfrak{J} \right\}$$

is a k-basis for the graded k-module $k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$. ♡

Proof. The set

$$\left\{ \mathrm{d}\, s^{[m]} \;\middle|\; m \in \mathbb{Z}_{\geq 0} \right\} \cup \left\{ s\, \mathrm{d}\, s^{[m]} \;\middle|\; m \in \mathbb{Z}_{\geq 0} \right\}$$

is a k-basis for $\Lambda(s) \otimes \Gamma(\mathrm{d}\, s)$, so there is a sum decomposition as follows.

$$k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\, s) \cong \bigoplus_{m \geq 0} \mathrm{Im}\left(C^{[m]}\right) \oplus \bigoplus_{m \geq 0} \mathrm{Im}\left(E^{[m]}\right)$$

As $C^{[m]}$ and $E^{[m]}$ are clearly injective for $m \geq 0$, it thus suffices to show that

$$\left\{ x^{\vec{i}} f^l\, \mathrm{d}\, x^{\vec{\epsilon}} \;\middle|\; \vec{i} \in \mathbb{Z}^n_{\geq 0}, \vec{i} \ngeq \deg_{\preceq}(f), l \in \mathbb{Z}_{\geq 0}, \vec{\epsilon} \in \{0, 1\}^n \right\}$$

is a k-basis of $k[x_1, \ldots, x_n] \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n)$, which follows from Proposition 9.2.1.1. □

9.2.3.3. Description of boundary and differential

Proposition 9.2.3.4. *Assume MonOrdMonicPoly, recall the notation from Definition 9.2.3.2, and let $(\vec{i}, l, \vec{\epsilon}, m) \in \mathfrak{J}$. Then the following formulas hold in the strict mixed complex X_f from Construction 9.2.2.1.*

$$\begin{aligned}
\partial\left(c_{\vec{i},l,\vec{\epsilon},m}\right) &= e_{\vec{i},l+1,\vec{\epsilon},m} \\
\partial\left(e_{\vec{i},l,\vec{\epsilon},m}\right) &= \begin{cases} -E^{[m-1]}\left(r^0_f\left(\mathrm{d}\, f \cdot x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\right)\right) & \text{if } l = 0 \\ 0 & \text{if } l > 0 \end{cases} \\
\mathrm{d}\left(e_{\vec{i},l,\vec{\epsilon},m}\right) &= E^{[m]}\left(\mathrm{d}\left(x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}} + m q^1_f\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}}\right) \\
&\quad + (m-1) C^{[m-1]}\left(q^2_f\left(\mathrm{d}\, f \cdot r^0_f\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\, \mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\right)
\end{aligned}$$

♡

Proof. We start with $\partial\left(c_{\vec{i},l,\vec{\epsilon},m}\right)$ and obtain the following by reordering the factors and applying the formula from Construction 9.2.2.1.

$$\begin{aligned}
\partial\left(c_{\vec{i},l,\vec{\epsilon},m}\right) &= \partial\left(s x^{\vec{i}} f^l\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m]}\right) \\
&= \partial\left((-1)^{|\vec{\epsilon}|} x^{\vec{i}} f^l\, \mathrm{d}\, x^{\vec{\epsilon}} s\, \mathrm{d}\, s^{[m]}\right) \\
&= (-1)^{|\vec{\epsilon}|}\left((-1)^{|\vec{\epsilon}|} x^{\vec{i}} f^{l+1}\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, s^{[m]} - (-1)^{|\vec{\epsilon}|} x^{\vec{i}} f^l\, \mathrm{d}\, x^{\vec{\epsilon}}\, \mathrm{d}\, f \cdot s\, \mathrm{d}\, s^{[m-1]}\right)
\end{aligned}$$

$$= x^{\vec{i}} f^{l+1} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]} - x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \cdot s \,\mathrm{d}\, s^{[m-1]}$$
$$= x^{\vec{i}} f^{l+1} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]} + s \,\mathrm{d}\, f x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m-1]}$$

If follows from Proposition 9.1.3.15 (4) that

$$q_f^1(\mathrm{d}\, f x^{\vec{i}} f^{l+1} \,\mathrm{d}\, x^{\vec{\epsilon}}) = q_f^0(\mathrm{d}\, f x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}) = \mathrm{d}\, f x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}$$

so that we obtain the following (continuing for $\partial(c_{\vec{i},l,\vec{\epsilon},m})$).

$$= x^{\vec{i}} f^{l+1} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]} + s q_f^1\left(\mathrm{d}\, f x^{\vec{i}} f^{l+1} \,\mathrm{d}\, x^{\vec{\epsilon}}\right) \mathrm{d}\, s^{[m-1]}$$
$$= e_{\vec{i},l+1,\vec{\epsilon},m}$$

We next consider $\partial(e_{\vec{i},l,\vec{\epsilon},m})$.

$$\begin{aligned}
&\partial\left(e_{\vec{i},l,\vec{\epsilon},m}\right)\\
&= \partial\left(x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]} + s q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right) \mathrm{d}\, s^{[m-1]}\right)\\
&= -(-1)^{|\vec{\epsilon}|} x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \,\mathrm{d}\, s^{[m-1]}\\
&\quad + (-1)^{1+|\vec{\epsilon}|} \partial\left(q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right) s \,\mathrm{d}\, s^{[m-1]}\right)\\
&= -(-1)^{|\vec{\epsilon}|} x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, f \,\mathrm{d}\, s^{[m-1]}\\
&\quad + (-1)^{2(1+|\vec{\epsilon}|)}\left(q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right) f \,\mathrm{d}\, s^{[m-1]} - q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right) \mathrm{d}\, f \cdot s \cdot \mathrm{d}\, s^{[m-2]}\right)\\
&= \left(-\,\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}} + q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right) f\right) \cdot \mathrm{d}\, s^{[m-1]}\\
&\quad + s \,\mathrm{d}\, f q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right) \mathrm{d}\, s^{[m-2]}
\end{aligned}$$

Before we continue with $\partial(e_{\vec{i},l,\vec{\epsilon},m})$, we carry out the following small calculation.

$$q_f^1\left(\mathrm{d}\, f \cdot \left(-\,\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}} + q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right) f\right)\right)$$

Using that $\mathrm{d}\, f$ squares to 0.

$$= q_f^1\left(\mathrm{d}\, f \cdot q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right) f\right)$$

Applying Proposition 9.1.3.15 (4) to the outer q_f^1.

$$= \mathrm{d}\, f \cdot q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right)$$

Note that by definition we also have the following equality.

$$-\,\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}} + q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right) f = -r_f^0\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^{l} \,\mathrm{d}\, x^{\vec{\epsilon}}\right)$$

Continuing with $\partial(e_{\vec{i},l,\vec{\epsilon},m})$, we can plug in the above two calculations to obtain the following.

$$
\begin{aligned}
&\partial(e_{\vec{i},l,\vec{\epsilon},m})\\
&= -r_f^0\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\,\mathrm{d}\, s^{[m-1]} - s q_f^1\Big(\mathrm{d}\, f \cdot r_f^0\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\,\mathrm{d}\, s^{[m-2]}\\
&= -E^{[m-1]}\Big(r_f^0\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)
\end{aligned}
$$

It follows from Proposition 9.1.3.15 (4) that this is zero for $l > 0$.

We now turn towards the mixed structure.

$$
\begin{aligned}
&\mathrm{d}\Big(e_{\vec{i},l,\vec{\epsilon},m}\Big)\\
&= \mathrm{d}\Big(x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]} + s q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\,\mathrm{d}\, s^{[m-1]}\Big)\\
&= \mathrm{d}\Big(x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]}\Big)\\
&\quad + (-1)^{1+|\vec{\epsilon}|}\,\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big) s \,\mathrm{d}\, s^{[m-1]}\Big)
\end{aligned}
$$

Applying the definition in Construction 9.2.2.1.

$$
\begin{aligned}
&= \mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}} \,\mathrm{d}\, s^{[m]}\\
&\quad + (-1)^{1+|\vec{\epsilon}|}\,\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big) s \,\mathrm{d}\, s^{[m-1]} + m q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\,\mathrm{d}\, s^{[m]}\\
&= \Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}} + m q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\,\mathrm{d}\, s^{[m]}\\
&\quad - s\,\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\,\mathrm{d}\, s^{[m-1]}
\end{aligned}
$$

Replacing the first summand by $E^{[m]} - C^{[m-1]}(q_f^1(\mathrm{d}\, f \cdot -))$ and the second summand by $C^{[m-1]}$.

$$
\begin{aligned}
&= E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}} + m q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\\
&\quad - C^{[m-1]}\Big(q_f^1\Big(\mathrm{d}\, f \cdot \mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}} + m\,\mathrm{d}\, f \cdot q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\\
&\quad - C^{[m-1]}\Big(\mathrm{d}\Big(q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big)\\
&= E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}} + m q_f^1\Big(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\Big)\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\\
&\quad - C^{[m-1]}\Big(q_f^1\Big(\mathrm{d}\, f \cdot \mathrm{d}\Big(x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big)
\end{aligned}
$$

$$- C^{[m-1]}\left(q_f^1\left(m \,\mathrm{d}\, f \cdot q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}}\right)\right) - C^{[m-1]}\left(\mathrm{d}\left(q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\right)$$

We now apply Proposition 9.2.3.1 (2) to the second summand, for $P = x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}$.

$$\begin{aligned}
&= E^{[m]}\left(\mathrm{d}\left(x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}} + m q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}}\right)\\
&\quad + C^{[m-1]}\left(q_f^1\left(\mathrm{d}\, f \cdot q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\right) + C^{[m-1]}\left(\mathrm{d}\left(q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\right)\\
&\quad - C^{[m-1]}\left(q_f^1\left(m \,\mathrm{d}\, f \cdot q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}}\right)\right) - C^{[m-1]}\left(\mathrm{d}\left(q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\right)\\
&= E^{[m]}\left(\mathrm{d}\left(x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}} + m q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}}\right)\\
&\quad - (m-1) C^{[m-1]}\left(q_f^1\left(\mathrm{d}\, f \cdot q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\right)
\end{aligned}$$

We apply Proposition 9.2.3.1 (3) to the second summand for $P = \mathrm{d}\, f$ and $Q = x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}$

$$\begin{aligned}
&= E^{[m]}\left(\mathrm{d}\left(x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}} + m q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}}\right)\\
&\quad - (m-1) C^{[m-1]}\left(q_f^2\left(\mathrm{d}\, f \cdot \mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\\
&\quad + (m-1) C^{[m-1]}\left(q_f^2\left(\mathrm{d}\, f \cdot r_f^0\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\right)
\end{aligned}$$

Finally, we use that $\mathrm{d}\, f$ squares to 0.

$$\begin{aligned}
&= E^{[m]}\left(\mathrm{d}\left(x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}} + m q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l\right) \mathrm{d}\, x^{\vec{\epsilon}}\right)\\
&\quad + (m-1) C^{[m-1]}\left(q_f^2\left(\mathrm{d}\, f \cdot r_f^0\left(\mathrm{d}\, f \cdot x^{\vec{i}} f^l \,\mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\right)
\end{aligned}$$

□

9.3. A smaller strict model for the underlying complex

Assume MonOrdMonicPoly and that Conjecture D holds for the polynomial f. Then Proposition 9.2.2.2 shows that the strict mixed complex X_f constructed in Construction 9.2.2.1 represents the Hochschild homology $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$. This strict mixed model is significantly "smaller" than the standard Hochschild complex that we discussed in Section 6.3.1, but we would nevertheless like to obtain an even smaller model.

There are two ways in which we can relax the problem in the hope of being able to make progress on this. We could impose stronger conditions on f (so make the result less general), or we could consider less structure. It is the latter that we do in this section. Instead of asking for a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$ as an object in $\mathcal{M}\mathrm{ixed}$, we merely ask for a chain complex representing $\mathrm{HH}(k[x_1,\ldots,x_n]/f)$ as an object in $\mathcal{D}(k)$.

Such a chain complex was already given in [BACH], obtained by identifying a decomposition of the normalized standard Hochschild complex[11] as a sum of a small chain complex with a very large acyclic chain complex.

We will instead start from the chain complex X_f from Construction 9.2.2.1 and Propositions 9.2.2.2 and 9.2.3.4, and similarly show that a chain complex isomorphic to the one obtained in [BACH] is a subcomplex and that the inclusion is a quasiisomorphism. This gives a new, different proof of the result in [BACH] (albeit requiring the additional assumption of Conjecture D, which we only showed for $n = 1$ and $n = 2$, additionally assuming that 2 is invertible in k, in Proposition 7.5.3.1).

We will describe the smaller model as a subcomplex of the complex X_f from Construction 9.2.2.1 in Section 9.3.1, and then show that this subcomplex is isomorphic to the one described in [BACH] in Section 9.3.2.

9.3.1. The smaller strict model as a subcomplex

In this section we define a subcomplex of X_f from Construction 9.2.2.1 and show that the inclusion of this subcomplex is a quasiisomorphism.

Definition 9.3.1.1. Assume MonOrdMonicPoly. Let

$$X_f := k[x_1,\ldots,x_n] \otimes \Lambda(\mathrm{d}\,x_1,\ldots,\mathrm{d}\,x_n) \otimes \Lambda(s) \otimes \Gamma(\mathrm{d}\,s)$$

be the strict mixed complex from Construction 9.2.2.1.

We then define the following sub-graded-k-modules of X_f for every integer $l \geq 0$, where $c_{\vec{i},l',\vec{\epsilon},m}$ and $e_{\vec{i},l',\vec{\epsilon},m}$ are the basis elements defined in Definition 9.2.3.2.

$$X^c_{f,l} := \bigoplus_{\substack{(\vec{i},l',\vec{\epsilon},m)\in\mathcal{I}\\ l'=l}} k\cdot c_{\vec{i},l',\vec{\epsilon},m} \qquad X^c_{f,\geq l} := \bigoplus_{l'\geq l} X^c_{f,l'} \qquad X^c_{f,\leq l} := \bigoplus_{l'\leq l} X^c_{f,l'}$$

$$X^e_{f,l} := \bigoplus_{\substack{(\vec{i},l',\vec{\epsilon},m)\in\mathcal{I}\\ l'=l}} k\cdot e_{\vec{i},l',\vec{\epsilon},m} \qquad X^e_{f,\geq l} := \bigoplus_{l'\geq l} X^e_{f,l'} \qquad X^e_{f,\leq l} := \bigoplus_{l'\leq l} X^e_{f,l'} \qquad \diamondsuit$$

Proposition 9.3.1.2. *Assume MonOrdMonicPoly and let $l \geq 0$. Then the following hold for the sub-graded-k-modules of the strict mixed complex X_f from Construction 9.2.2.1 that were defined in Definition 9.3.1.1.*

$$\partial\left(X^c_{f,l}\right) \subseteq X^e_{f,l+1}$$

[11] See Section 6.3.1.5.

$$\partial\left(X^e_{f,0}\right) \subseteq X^e_{f,0}$$
$$\partial\left(X^e_{f,l}\right) \subseteq 0 \qquad \text{if } l > 0$$

In particular, $X^e_{f,0}$ as well as the sum

$$X^c_{f,\geq l} \oplus X^e_{f,\geq l+1}$$

are closed under the boundary operator and hence subcomplexes of X_f. Both of these chain complexes are cofibrant, and $X^c_{f,\geq l} \oplus X^e_{f,\geq l+1}$ is acyclic. ♡

Proof. The statement about the images of the differential follow immediately from the description of ∂ in Proposition 9.2.3.4.

That $X^e_{f,0}$ and $X^c_{f,\geq l} \oplus X^e_{f,\geq l+1}$ are cofibrant as chain complexes follows from [Hov99, 2.3.6], as they are concentrated in nonnegative degree and by definition free as graded k-modules.

Finally, that $X^c_{f,\geq l} \oplus X^e_{f,\geq l+1}$ is acyclic also immediately follows from the description of ∂ in Proposition 9.2.3.4;

$$\begin{aligned} e_{\vec{i},l',\vec{\epsilon},m} &\mapsto c_{\vec{i},l'-1,\vec{\epsilon},m} && \text{for } \left(\vec{i},l',\vec{\epsilon},m\right) \in \mathfrak{I},\ l' \geq l+1 \\ c_{\vec{i},l',\vec{\epsilon},m} &\mapsto 0 && \text{for } \left(\vec{i},l',\vec{\epsilon},m\right) \in \mathfrak{I},\ l' \geq l \end{aligned}$$

defines a contracting homotopy, see Definition 9.2.3.2 and Propositions 9.2.3.3 and 9.2.3.4. □

Proposition 9.3.1.3. *Assume MonOrdMonicPoly and that Conjecture D*[12] *holds for f. Then there is an equivalence*

$$\mathrm{HH}\left(k[x_1,\ldots,x_n]/f\right) \simeq \gamma\left(X^e_{f,0}\right)$$

in $\mathcal{D}(k)$, where $X^e_{f,0}$ is the cofibrant chain complex defined in Definition 9.3.1.1 and Proposition 9.3.1.2. ♡

Proof. It follows from Proposition 9.2.3.3 that, as a graded k-module, X_f decomposes as the direct sum of $X^e_{f,0}$ and $X^c_{f,\geq 0} \oplus X^e_{f,\geq 1}$. As both summands are subcomplexes of X_f by Proposition 9.3.1.2, with the latter chain complex acyclic, it follows that the inclusion

$$X^e_{f,0} \to X_f$$

is a quasiisomorphism. We hence obtain equivalences

$$\gamma\left(X^e_{f,0}\right) \simeq \gamma\left(X_f\right) \simeq \mathrm{HH}\left(k[x_1,\ldots,x_n]/f\right)$$

in $\mathcal{D}(k)$, where the first equivalence is induced by the just mentioned quasiisomorphism, and the second equivalence is the one from Proposition 9.2.2.2. □

[12]Note that Conjecture D holds if $n = 1$ or $n = 2$ with 2 invertible in k by Proposition 7.5.3.1.

9.3.2. A different description of the smaller model

In Proposition 9.3.1.3 we showed that the chain complex $X^e_{f,0}$ defined in Definition 9.3.1.1 is a model for $\mathrm{HH}(k[x_1, \ldots, x_n]/f)$ as an object in $\mathcal{D}(k)$, assuming some conditions on f. As $X^e_{f,0}$ was defined as a subcomplex of X_f generated by some basis elements, it is slightly unexplicit, and in this section we give a somewhat more direct description of this complex. In particular, our description will be nearly the same as the one in [BACH, 2.3 and 3.2][13].

Construction 9.3.2.1. Assume MonOrdMonicPoly.

We let

$$p\colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/f$$

be the canonical quotient map. Note that p is a morphism of k-algebras. If M is a graded k-module, then we will also denote the morphism of graded k-modules

$$p \otimes \mathrm{id}_M\colon k[x_1, \ldots, x_n] \otimes M \to k[x_1, \ldots, x_n]/f \otimes M$$

by p again.

Consider the commutative graded k-algebra

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

with x_i of degree 0, $\mathrm{d}\, x_i$ of degree 1 and t of degree 2. We define an operator ∂ decreasing degree by 1 by extending the following formulas by k-linearity and the Leibniz rule, where $P \in k[x_1, \ldots, x_n]/f$, $1 \leq i \leq n$, and $m \geq 0$.

$$\partial(P) = 0, \qquad \partial(\mathrm{d}\, x_i) = 0, \qquad \partial\left(t^{[m]}\right) = -p(\mathrm{d}\, f)t^{[m-1]}$$

To show that ∂ is well-defined we need to verify that the formula for $\partial(t^{[m]})$ is compatible with the Leibniz rule, so as for $m, m' \geq 0$ we have

$$t^{[m]} \cdot t^{[m']} = \binom{m+m'}{m} t^{[m+m']}$$

we have to show that the following equality holds.

$$-p(\mathrm{d}\, f)t^{[m-1]} \cdot t^{[m']} - t^{[m]} \cdot p(\mathrm{d}\, f)t^{[m'-1]} = -\binom{m+m'}{m} p(\mathrm{d}\, f)t^{[m+m'-1]} \tag{$*$}$$

The left hand side is given by

$$-p(\mathrm{d}\, f)t^{[m-1]} \cdot t^{[m']} - t^{[m]} \cdot p(\mathrm{d}\, f)t^{[m'-1]}$$

[13] The complex constructed here differs from the one in [BACH] in the very minor detail that our external generators are the the additive inverses of the external generators they consider. We do this because we will in Section 9.5 also define a mixed structure on this complex, and prefer the exterior generators to be given by $\mathrm{d}\, x_i$ rather than $-\mathrm{d}\, x_i$.

$$= -p(\mathrm{d}\, f)\Big(t^{[m-1]} \cdot t^{[m']} + t^{[m]} \cdot t^{[m'-1]}\Big)$$
$$= -p(\mathrm{d}\, f)\left(\binom{m+m'-1}{m-1} t^{[m+m'-1]} + \binom{m+m'-1}{m} t^{[m+m'-1]}\right)$$

so $(*)$ follows from $\binom{m+m'-1}{m-1} + \binom{m+m'-1}{m} = \binom{m+m'}{m}$.

As $\mathrm{d}\, f \cdot \mathrm{d}\, f = 0$, the operator ∂ squares to zero, and thus makes

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

into a commutative differential graded k-algebra. It is isomorphic to the one considered in [BACH, 2.3 and 3.2][14], where it is shown that this complex is quasiisomorphic to the normalized standard Hochschild complex for $k[x_1, \ldots, x_n]/f$.

Now let

$$\varphi\colon X^{e}_{f,0} \to k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

be the morphism of graded k-modules defined on basis elements as follows.

$$\varphi\Big(e_{\vec{i},0,\vec{\epsilon},m}\Big) := p\Big(x^{\vec{i}}\Big)\, \mathrm{d}\, x^{\vec{\epsilon}} t^{[m]} \qquad \text{for } \Big(\vec{i}, 0, \vec{\epsilon}, m\Big) \in \mathfrak{J} \qquad \diamondsuit$$

Proposition 9.3.2.2. *Assume MonOrdMonicPoly. Then the morphism of graded k-modules φ from Construction 9.3.2.1 is an isomorphism of chain complexes.* ♡

Proof. We first check that φ is compatible with the differential, letting $(\vec{i}, 0, \vec{\epsilon}, m) \in \mathfrak{J}$.

$$\varphi\Big(\partial\Big(e_{\vec{i},0,\vec{\epsilon},m}\Big)\Big)$$

We first use Proposition 9.2.3.4.

$$= \varphi\Big(-E^{[m-1]}\Big(r^0_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\, \mathrm{d}\, x^{\vec{\epsilon}}\Big)\Big)\Big) = -p\Big(r^0_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\Big)\Big)\, \mathrm{d}\, x^{\vec{\epsilon}} t^{[m-1]}$$

We can now use that p sends the ideal generated by f to 0 and hence satisfies $p \circ r^0_f = p$, and furthermore that p is multiplicative.

$$= -p\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\Big)\, \mathrm{d}\, x^{\vec{\epsilon}} t^{[m-1]} = -p(\mathrm{d}\, f) p\Big(x^{\vec{i}}\Big)\, \mathrm{d}\, x^{\vec{\epsilon}} t^{[m-1]}$$
$$= \partial\Big(p\Big(x^{\vec{i}}\Big)\, \mathrm{d}\, x^{\vec{\epsilon}} t^{[m]}\Big) = \partial\Big(\varphi\Big(e_{\vec{i},0,\vec{\epsilon},m}\Big)\Big)$$

It now remains to show that φ is an isomorphism of graded k-modules. For this it is enough to show that the restriction of the quotient map

$$p\colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/f$$

[14] As noted before, our description deviates in the signs of the external generators, but this does not change the fact that the differential graded k-algebras themselves are isomorphic, via an isomorphism from our complex to the one in [BACH, 2.3 and 3.2] mapping x_i to X_i, $\mathrm{d}\, x_i$ to $-e_i$, and $t^{[m]}$ to $t^{(m)}$.
.

to the sub-graded-k-module of f-reduced polynomials is an isomorphism. But this follows immediately from Proposition 9.1.3.10, which shows that every element of $k[x_1, \ldots, x_n]/f$ has a unique f-reduced representative in $k[x_1, \ldots, x_n]$. □

The following corollary alternatively follows easily from the main result of [BACH], without requiring the assumption that Conjecture D holds for f. Our approach gives a different, independent, proof for those cases in which Conjecture D holds for f.

Corollary 9.3.2.3. *Assume MonOrdMonicPoly and that Conjecture D[15] holds for f. Then there is an equivalence*

$$\mathrm{HH}\big(k[x_1, \ldots, x_n]/f\big) \simeq \gamma\big(k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\,x_1, \ldots, \mathrm{d}\,x_n) \otimes \Gamma(t)\big)$$

in $\mathcal{D}(k)$, where

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\,x_1, \ldots, \mathrm{d}\,x_n) \otimes \Gamma(t)$$

is the cofibrant chain complex defined in Construction 9.3.2.1. ♡

Proof. Combine Proposition 9.3.1.3 with Proposition 9.3.2.2. □

9.4. Logarithmic dimension of polynomials

Assume MonOrdMonicPoly and that Conjecture D holds for f. In Section 9.3.1 we constructed a subcomplex $X^e_{f,0}$ of the strict mixed complex X_f from Construction 9.2.2.1 such that the inclusion is a quasiisomorphism, which implied that $X^e_{f,0}$ represents the Hochschild homology $\mathrm{HH}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{D}(k)$.

We would like to show that the strict mixed structure on X_f restricts to $X^e_{f,0}$, which would allow us to conclude that $X^e_{f,0}$ even represents $\mathrm{HH}_{\mathrm{Mixed}}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{M}\mathrm{ixed}$.

Unfortunately the formula for d we obtained in Proposition 9.2.3.4 is somewhat more complicated than those we obtained for ∂ and it is not obvious that $X^e_{f,0}$ is closed under d. In particular, there is a term of the form

$$C^{[m-1]}\left(q_f^2\left(\mathrm{d}\,f \cdot r_f^0\left(\mathrm{d}\,f \cdot x^{\vec{i}}\,\mathrm{d}\,x^{\vec{\epsilon}}\right)\right)\right) \tag{9.3}$$

that we would need to vanish, and there is no reason to assume this is always the case. Indeed, the following example shows that this term can be nonzero.

Example 9.4.0.1. Let $k = \mathbb{Z}$, $n = 2$, and consider the polynomial $f = x_1x_2 - x_2^2$. If we let $\preceq$ be the lexicographic monomial order[16], then f is monic and of degree $(1, 1)$.

We claim that

$$q_f^2\left(\mathrm{d}\,f \cdot r_f^0\left(\mathrm{d}\,f \cdot x_1^2\right)\right)$$

[15] Note that Conjecture D holds if $n = 1$ or $n = 2$ and 2 is invertible in k by Proposition 7.5.3.1.
[16] So $(i_1, i_2) \preceq (j_1, j_2)$ if $i_1 < j_1$ or $i_1 = j_1$ and $i_2 < j_2$.

is nonzero, even though x_1^2 is f-reduced. Let us calculate this step by step.

$$r_f^0\big(\mathrm{d}\, f \cdot x_1^2\big) = r_f^0\big(x_1^2 x_2 \,\mathrm{d}\, x_1 + x_1^3 \,\mathrm{d}\, x_2 - 2x_1^2 x_2 \,\mathrm{d}\, x_2\big)$$

To calculate for example $r_f^0(x_1^2 x_2)$ we start by writing $x_1^2 x_2 = x_1 f + x_1 x_2^2$ and then continue with $x_1 x_2^2 = x_2 f + x_2^3$.

$$= x_2^3 \,\mathrm{d}\, x_1 + x_1^3 \,\mathrm{d}\, x_2 - 2x_2^3 \,\mathrm{d}\, x_2$$

We next need to multiply by $\mathrm{d}\, f$, and obtain the following.

$$\mathrm{d}\, f \cdot r_f^0\big(\mathrm{d}\, f \cdot x_1^2\big) = \big(x_1^3 x_2 - 2x_2^4 - x_1 x_2^3 + 2x_2^4\big) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2$$

Applying q_f^2 amounts to applying q_f^1 twice by Proposition 9.1.3.15 (6), so we obtain the following.

$$\begin{aligned}
&q_f^2\Big(\mathrm{d}\, f \cdot r_f^0\big(\mathrm{d}\, f \cdot x_1^2\big)\Big) \\
&= q_f^1\Big(q_f^1\big(x_1^3 x_2 - 2x_2^4 - x_1 x_2^3 + 2x_2^4\big)\Big) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \\
&= q_f^1\Big(\big(x_1^2 + x_1 x_2 + x_2^2\big) - 2 \cdot (0) - \big(x_2^2\big) + 2 \cdot (0)\Big) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \\
&= q_f^1\big(x_1^2 + x_1 x_2\big) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \\
&= (0 + 1) \,\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 = \mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \neq 0
\end{aligned} \qquad \diamondsuit$$

The goal of this section is to describe a criterion for f that is easy to check and that implies that terms of the form (9.3) that need to be zero for $X_{f,0}^e$ to be closed under d are indeed zero. For this we will generalize $r_f^0(\mathrm{d}\, f \cdot x^{\vec{i}} \,\mathrm{d}\, x^{\vec{e}})$ to an arbitrary f-reduced polynomial R and ask what the largest integer i is such that $q_f^i(\mathrm{d}\, f \cdot R)$ can be nonzero for an f-reduced polynomial R (with f fixed). We will call this number the *log dimension of* $\mathrm{d}\, f$ *to basis* f and will give an easy to check criterion that implies that this number is at most 1 in Proposition 9.4.2.5 and Corollary 9.4.2.6.

We will start this section with Section 9.4.1, where we discuss the logarithm for polynomials, before we turn towards the log dimension in Section 9.4.2.

9.4.1. Logarithm for polynomials

In this section we introduce a notion of logarithm for multivariable polynomials and point out some basic properties and consistency results.

Definition 9.4.1.1. Assume MonOrdMonicPoly. We define a map

$$\log_f \colon k[x_1, \ldots, x_n] \to \mathbb{Z}_{\geq 0}$$

as follows. For P an element of $k[x_1, \ldots, x_n]$, we let

$$\log_f(P) := \max\left(\left\{\, i \in \mathbb{Z}_{\geq 0} \;\middle|\; r_f^i(P) \neq 0 \,\right\}\right)$$

and call $\log_f(P)$ the *logarithm to base f of P (with respect to the monomial order $\preceq$)*. Note that the set over which we take the maximum is finite, as all but finitely many summands in the decomposition from Proposition 9.1.3.12 are zero, so attains a maximum in $\mathbb{Z}_{\geq 0}$. $\diamondsuit$

Remark 9.4.1.2. Assume MonOrdMonicPoly and let P be an element of $k[x_1, \ldots, x_n]$. Then P is f-reduced if and only if $\log_f(P) = 0$. $\diamondsuit$

Remark 9.4.1.3. Assume MonOrdMonicPoly and that we are in the situation of Construction 9.1.1.11 and that $\deg_\preceq(f)$ is in the image of ψ. Let P be an element of $k[x_1, \ldots, x_n]$ and let f' and P' be the elements of $k'[y_1, \ldots, y_m]$ corresponding to f and P under the isomorphism of Construction 9.1.1.11. It then follows from Remark 9.1.3.11 that $\log_{f'}(P') = \log_f(P)$. $\diamondsuit$

Proposition 9.4.1.4. *Assume MonOrdMonicPoly and let P and Q be elements of $k[x_1, \ldots, x_n]$. Then the following holds.*

$$\log_f(P+Q) \leq \max\Big(\Big\{\log_f(P), \log_f(Q)\Big\}\Big) \qquad \heartsuit$$

Proof. By Proposition 9.1.3.15 (3), r^i_f is additive for every $i \geq 0$, so if $r^i_f(P+Q) \neq 0$ for some $i \geq 0$, then at least one of $r^i_f(P)$ and $r^i_f(Q)$ must be nonzero as well. $\square$

9.4.2. Logarithmic dimension for polynomials

Let f be an element of $\mathbb{R}_{>1}$, i. e. a real number bigger than 1, and let us for a moment consider the logarithm function

$$\log_f\colon \mathbb{R}_{>0} \to \mathbb{R}$$

for the real numbers. This function satisfies a compatibility relation with multiplication; if P and Q are positive real numbers, then $\log_f(P \cdot Q) = \log_f(P) + \log_f(Q)$. In Section 9.4.1 we defined a logarithm for (multivariable) polynomials, and we would like to better understand how the logarithm of products relates to the individual logarithms as well. The logarithm for polynomials does not take real values, so to improve the analogy we should first replace $\log_f$ with the function

$$\log'_f\colon \mathbb{R}_{>0} \to \mathbb{Z}_{\geq 0}, \qquad x \mapsto \begin{cases} \lfloor \log_f(x) \rfloor & \text{if } \log_f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

so we round down the logarithm, but set it to 0 should it be negative. The rounding destroys the precise property of the logarithm of a product being the sum of the logarithms, but as for any real number x it holds that $x - 1 < \lfloor x \rfloor \leq x$, we still obtain an inequality

$$\log'_f(P) + \log'_f(Q) \leq \log'_f(P \cdot Q) \leq \log'_f(P) + \log'_f(Q) + 1 \tag{9.4}$$

for every P and Q in $\mathbb{R}_{>0}$.

If we now let f be an element of $k[x_1]$ that is a monic polynomial of positive degree, and P and Q any elements of $k[x_1]$, then the analogue of (9.4) holds, at least as long k is an integral domain. Indeed, for one-variable polynomials, it is actually not difficult to see that

$$\log_f(P) = \left\lfloor \frac{\deg(P)}{\deg(f)} \right\rfloor$$

from which the inequality

$$\log_f(P) + \log_f(Q) \leq \log_f(P \cdot Q) \leq \log_f(P) + \log_f(Q) + 1$$

follows as long as k is an integral domain. The inequality

$$\log_f(P \cdot Q) \leq \log_f(P) + \log_f(Q) + 1$$

holds for any commutative ring k. We can restate this as saying that the expression

$$\log_f(P \cdot Q) - \log_f(P) - \log_f(Q) \tag{9.5}$$

is bounded above by 1 as we let f, P, and Q vary.

Let us now consider multivariable polynomials and assume MonOrdMonicPoly. The first question we can then ask is whether (9.5) is still bounded above while letting f, P, and Q range over $k[x_1, \ldots, x_n]$ with f satisfying the assumptions in MonOrdMonicPoly.

Unfortunately, this is not the case as soon as $n \geq 2$. Consider the example $f = x_1 x_2$, $P = x_1^m$, $Q = x_2^m$, where $m \geq 1$. In this case, $\log_f(P) = \log_f(Q) = 0$, but $\log_f(P \cdot Q) = m$, so the value of (9.5) is unbounded if we let f, P, and Q vary.

However, if we fix f, then it is not difficult to find examples where the value of (9.5) *is* bounded while letting P and Q range over $k[x_1, \ldots, x_n]$. For example consider $f = x_1$. In this case the value of $\log_f(P)$ is given by the highest exponent of x_1 appearing in the monomials of P, and the value of (9.5) is bounded above by 0.

So we can instead ask, given fixed f, whether the value of (9.5), as P and Q range over the elements of $k[x_1, \ldots, x_n]$, is bounded above, and if so, what the maximum value is. In this section we go one step further, and fix both f as well as P, and consider the supremum of (9.5) when varying Q, calling it the *log dimension to base f of P*. In particular, we will establish a condition that ensures that the log dimension of a polynomial is at most 1.

Definition 9.4.2.1. Assume MonOrdMonicPoly. For P an element of $k[x_1, \ldots, x_n]$ we let $\operatorname{logdim}_f(P)$ be the element of $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ defined as

$$\operatorname{logdim}_f(P) \coloneqq \sup\left(\left\{ \log_f(P \cdot Q) - \log_f(P) - \log_f(Q) \;\middle|\; Q \in k[x_1, \ldots, x_n] \right\} \right)$$

and call $\operatorname{logdim}_f(P)$ the *log dimension to base f of P*. ♢

Remark 9.4.2.2. Assume MonOrdMonicPoly, that we are in the situation of Construction 9.1.1.11, and that $\deg_{\preceq}(f)$ is in the image of ψ. Let P be an element of $k[x_1, \ldots, x_n]$ and let f' and P' be the elements of $k'[y_1, \ldots, y_m]$ corresponding to f and P under the isomorphism of Construction 9.1.1.11. It then follows from Remark 9.4.1.3 and Remark 9.1.3.9 that $\operatorname{logdim}_{f'}(P') = \operatorname{logdim}_f(P)$. ♢

Proposition 9.4.2.3. *Assume MonOrdMonicPoly and let $P \in k[x_1, \dots, x_n]$ be a polynomial. Then it suffices to consider f-reduced polynomials Q in the definition of* $\operatorname{logdim}_f(P)$*, i. e. there is an equality as follows.*

$$\operatorname{logdim}_f(P) = \sup\left(\left\{ \log_f(P \cdot R) - \log_f(P) \;\middle|\; R \in k[x_1, \dots, x_n],\ R \textit{ is } f\textit{-reduced} \right\}\right) \quad \heartsuit$$

Proof. For the moment let us denote the right hand side of the equality in the statement by $\operatorname{logdim}_f^{\mathrm{red}}(P)$. The inequality $\operatorname{logdim}_f^{\mathrm{red}}(P) \leq \operatorname{logdim}_f(P)$ is clear, so it suffices to show that $\operatorname{logdim}_f(P) \leq \operatorname{logdim}_f^{\mathrm{red}}(P)$ also holds.

So let Q be any element of $k[x_1, \dots, x_n]$. It suffices to find an f-reduced polynomial R such that

$$\log_f(P \cdot Q) - \log_f(P) - \log_f(Q) \leq \log_f(P \cdot R) - \log_f(P)$$

holds, which is equivalent to the following inequality.

$$\log_f(P \cdot Q) - \log_f(Q) \leq \log_f(P \cdot R)$$

For this, let us write Q as

$$Q = \sum_{i=0}^{\log_f(Q)} r_f^i(Q) f^i$$

so that we obtain the following chain of inequalities.

$$\begin{aligned}
&\log_f(P \cdot Q) - \log_f(Q) \\
&= \log_f\left(\sum_{i=0}^{\log_f(Q)} P \cdot r_f^i(Q) \cdot f^i\right) - \log_f(Q)
\end{aligned}$$

Using Proposition 9.4.1.4.

$$\leq \max\left(\left\{ \log_f\left(P \cdot r_f^i(Q) \cdot f^i\right) \;\middle|\; 0 \leq i \leq \log_f(Q) \right\}\right) - \log_f(Q)$$

Using Proposition 9.1.3.15 (4).

$$\begin{aligned}
&\leq \max\left(\left\{ \log_f\left(P \cdot r_f^i(Q)\right) + i \;\middle|\; 0 \leq i \leq \log_f(Q) \right\}\right) - \log_f(Q) \\
&\leq \max\left(\left\{ \log_f\left(P \cdot r_f^i(Q)\right) \;\middle|\; 0 \leq i \leq \log_f(Q) \right\}\right) + \log_f(Q) - \log_f(Q) \\
&= \max\left(\left\{ \log_f\left(P \cdot r_f^i(Q)\right) \;\middle|\; 0 \leq i \leq \log_f(Q) \right\}\right)
\end{aligned}$$

We can thus take R to be the f-reduced polynomial $r_f^i(Q)$, where $0 \leq i \leq \log_f(Q)$ is chosen to maximize $\log_f\left(P \cdot r_f^i(Q)\right)$. $\square$

Proposition 9.4.2.4. *Assume MonOrdMonicPoly, and assume furthermore that the degree of f satisfies $\deg_{\preceq}(f) \geq (1, \dots, 1)$ and that $f_{\vec{i}} = 0$ for any $\vec{i} \in \mathbb{Z}_{\geq 0}^n$ such that $\vec{i} \not\leq \deg_{\preceq}(f)$, i. e. every variable divides the leading monomial of f and every monomial appearing in f divides the leading monomial.*

Let $P \in k[x_1, \dots, x_n]$ be an f-reduced polynomial such that $P_{\vec{i}} = 0$ for every $\vec{i} \in \mathbb{Z}_{\geq 0}^n$ such that $\vec{i} \not\leq \deg_{\preceq}(f)$, i. e. every monomial in P divides the lead monomial of f.

Then $\operatorname{logdim}_f(P) \leq 1$. ♡

Proof. By Proposition 9.4.2.3, it suffices to show that for any f-reduced polynomial Q the inequality $\log_f(P \cdot Q) \leq 1$ holds. Using Proposition 9.4.1.4 we can furthermore reduce to the case $P = x^{\vec{j}}$ with $\vec{j} < \deg_{\preceq}(f)$ and $Q = x^{\vec{i}}$ with $\vec{i} \not\geq \deg_{\preceq}(f)$.

By Proposition 9.1.3.12 we can write the product $P \cdot Q = x^{\vec{j}+\vec{i}}$ as

$$x^{\vec{j}+\vec{i}} = R_2 f^2 + R_1 f + R_0 \tag{$*$}$$

such that R_1 and R_0 are f-reduced polynomials, and R_2 is any polynomial. What we have to show is then that $R_2 = 0$. We prove this by contradiction and assume that $R_2 \neq 0$. It then follows from Proposition 9.1.2.3 (4) that

$$\left(R_2 f^2\right)_{\deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)} \neq 0$$

so that it suffices to show that

$$\left(x^{\vec{j}+\vec{i}}\right)_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)} = (R_1 f)_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)} = (R_0)_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)} = 0$$

in contradiction to ($*$).

We start with $(x^{\vec{j}+\vec{i}})_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)}$, which could only be nonzero if the following equation would hold.

$$\vec{j} + \vec{i} = \deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)$$

However, as $\vec{j} < \deg_{\preceq}(f)$ we would then obtain

$$\begin{aligned} \vec{i} &= \left(\vec{j} + \vec{i}\right) - \vec{j} \\ &> \left(\deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)\right) - \deg_{\preceq}(f) \\ &\geq \deg_{\preceq}(f) \end{aligned}$$

which would contradict $\vec{i} \not\geq \deg_{\preceq}(f)$. Thus $(x^{\vec{j}+\vec{i}})_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)} = 0$ must hold.

Next, if $(R_1 f)_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)}$ were nonzero, then there would exist $\vec{a}, \vec{b} \in \mathbb{Z}_{\geq 0}^n$ such that $(R_1)_{\vec{a}} \neq 0$ and $f_{\vec{b}} \neq 0$ and such that the equation

$$\vec{a} + \vec{b} = \deg_{\preceq}(R_2) + 2\deg_{\preceq}(f)$$

holds. Using that, by assumption on f, the inequality $\vec{b} \leq \deg_{\preceq}(f)$ must hold, we obtain completely like in the previous case, with $\vec{b}$ taking the place of $\vec{j}$, that

$$\vec{a} \geq \deg_{\preceq}(f)$$

which contradicts the assumption that R_2 is f-reduced.

Finally, that

$$(R_0)_{\deg_{\preceq}(R_2)+2\deg_{\preceq}(f)} = 0$$

follows directly from R_0 being f-reduced. □

Proposition 9.4.2.5. *Assume MonOrdMonicPoly, and let $P \in k[x_1, \ldots, x_n]$ be an f-reduced polynomial. Assume that for every $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ such that $f_{\vec{i}} \neq 0$ or $P_{\vec{i}} \neq 0$ the following property holds: If $1 \leq j \leq n$ and $\deg_{\preceq}(f)_j \neq 0$, then $\vec{i}_j \leq \deg_{\preceq}(f)_j$. In other words, we require that every monomial appearing in f or P divides the leading monomial of f after setting those variables that do not appear in the leading monomial of f to 1.*

Then $\operatorname{logdim}_f(P) \leq 1$. ♡

Proof. Let $\{i'_1, \ldots, i'_r\}$ be the subset of $\{1, \ldots, n\}$ of elements for which $\deg_{\preceq}(f)_{i'_j} = 0$, let $\{i_1, \ldots, i_l\}$ be the complement, and let $\varphi\colon \{i_1, \ldots, i_l\} \to \{1, \ldots, n\}$ be the inclusion. Note that $\deg_{\preceq}(f)$ is then in the image of ψ from Construction 9.1.1.11. Denote by f' and P' the elements of $(k[x_{i'_1}, \ldots, x_{i'_r}])[x_{i_1}, \ldots, x_{i_l}]$ corresponding to f and P under the isomorphism of Construction 9.1.1.11. Note that by Proposition 9.1.3.5, f' is monic and $\deg_{\preceq}(f) = \psi\Big(\deg_{\preceq}(f')\Big)$ by Proposition 9.1.2.4. Then the assumptions on f and P then translate to f' and P' satisfying the assumptions required in Proposition 9.4.2.4. We can thus conclude that $\operatorname{logdim}_{f'}(P') \leq 1$. As by Remark 9.4.2.2 we also have $\operatorname{logdim}_f(P) = \operatorname{logdim}_{f'}(P')$, we are done. □

Corollary 9.4.2.6. *Assume MonOrdMonicPoly, and assume that f satisfies the property required in Proposition 9.4.2.5.*

Then for every $1 \leq i \leq n$ the partial derivative $\frac{\partial f}{\partial x_j}$ satisfies the property required of P in Proposition 9.4.2.5, and so $\operatorname{logdim}_f\left(\frac{\partial f}{\partial x_j}\right) \leq 1$. In particular, $q_f^2(\mathrm{d}\, f \cdot P) = 0$ for every f-reduced polynomial P. ♡

Proof. Every monomial in $\frac{\partial f}{\partial x_j}$ divides a monomial in f. □

Notation 9.4.2.7. Assume MonOrdMonicPoly. Then we define $\operatorname{logdim}_f(\mathrm{d}\, f)$ as follows.

$$\operatorname{logdim}_f(\mathrm{d}\, f) := \max\left(\left\{ \operatorname{logdim}_f\left(\frac{\partial f}{\partial x_i}\right) \,\middle|\, 1 \leq i \leq n \right\}\right)$$

In particular, the conclusion of Corollary 9.4.2.6 can be phrased as $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$, and $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$ implies $q_f^2(\mathrm{d}\, f \cdot P) = 0$ for every f-reduced polynomial P. ♢

9.5. A smaller strict model for the mixed complex

Assume MonOrdMonicPoly and that Conjecture D holds for f. As was already discussed in the introduction of Section 9.4, we would like to show that the strict mixed structure on X_f from Construction 9.2.2.1 restricts to the subcomplex $X^e_{f,0}$ that we constructed in Section 9.3.1, which would allow us to conclude that $X^e_{f,0}$ even represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1,\ldots,x_n]/f)$ as an object of $\mathcal{M}$ixed.

The work of Section 9.4 now allows us to concisely state a condition on f that implies that the strict mixed structure restricts like that, namely the condition $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$. We show that this indeed implies that the strict mixed structure of X_f restricts to $X^e_{f,0}$ in the short section Section 9.5.1.

In continuation to Section 9.3.2, in which we gave a different (independent from X_f) description of the chain complex $X^e_{f,0}$ by constructing an isomorphism between $X^e_{f,0}$ and a chain complex with underlying graded k-module

$$k[x_1,\ldots,x_n]/f \otimes \Lambda(\mathrm{d}\, x_1,\ldots,\mathrm{d}\, x_n) \otimes \Gamma(t) \tag{9.6}$$

we will upgrade that isomorphism to an isomorphism of strict mixed complexes in Section 9.5.2.

9.5.1. Restricting the strict mixed structure

Proposition 9.5.1.1. *Assume MonOrdMonicPoly and that* $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$.

Then the strict mixed structure of X_f from Construction 9.2.2.1 restricts to the subcomplex[17] $X^e_{f,0}$. *Thus the inclusion $X^e_{f,0} \to X_f$ is a quasiisomorphism of strict mixed complexes.* ♡

Proof. That the inclusion $X^e_{f,0} \to X_f$ is a quasiisomorphism was already shown in Proposition 9.3.1.3, so it suffices to show that $X^e_{f,0}$ is closed under d. Unpacking the definition of $X^e_{f,0}$ and using the formula for d obtained in Proposition 9.2.3.4 this means that we need to show that for $\vec{i} \not\geq \deg_{\preceq}(f)$, $\vec{\epsilon} \in \{0,1\}^n$ and $m \geq 0$ the element

$$\begin{aligned}\mathrm{d}\left(e_{\vec{i},0,\vec{\epsilon},m}\right) = E^{[m]}\left(\mathrm{d}\left(x^{\vec{i}}\right)\mathrm{d}\, x^{\vec{\epsilon}} + m q^1_f\left(\mathrm{d}\, f \cdot x^{\vec{i}}\right)\mathrm{d}\, x^{\vec{\epsilon}}\right)\\ + (m-1)C^{[m-1]}\left(q^2_f\left(\mathrm{d}\, f \cdot r^0_f\left(\mathrm{d}\, f \cdot x^{\vec{i}}\,\mathrm{d}\, x^{\vec{\epsilon}}\right)\right)\right)\end{aligned}$$

is again in $X^e_{f,0}$. For this it suffices to show the following.

(1) $\mathrm{d}\left(x^{\vec{i}}\right)$ is f-reduced.

(2) $q^1_f\left(\mathrm{d}\, f \cdot x^{\vec{i}}\right)$ is f-reduced.

[17] See Definition 9.3.1.1 for the definition and Proposition 9.3.1.2 for being a subcomplex.

(3) $q^2_f(\mathrm{d}\, f \cdot R) = 0$ if R is f-reduced.

(1) follows immediately from Proposition 9.2.3.1 (1), (2) follows from $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$ with Proposition 9.1.3.15 (6), and (3) follows from $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$. □

9.5.2. An alternative description of the smaller strict mixed model

We can now transfer the strict mixed structure on $X^e_{f,0}$ via the isomorphism of chain complexes φ from Construction 9.3.2.1 and Proposition 9.3.2.2. We first describe the resulting d, and then show that φ is compatible with it.

Construction 9.5.2.1. Assume MonOrdMonicPoly and that $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$.
Recall the commutative differential graded k-algebra

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t) \tag{9.7}$$

as well as the morphisms p defined in Construction 9.3.2.1.
We will define a k-linear operator[18] d that increases degree by 1 on (9.7) by

$$\mathrm{d}\Big(p(P)\,\mathrm{d}\, x^{\vec{\epsilon}} t^{[m]}\Big) \coloneqq \bigg(p\Big(\mathrm{d}\Big(r^0_f(P)\Big)\Big) + mp\Big(q^1_f\Big(\mathrm{d}\, f \cdot r^0_f(P)\Big)\Big)\bigg)\,\mathrm{d}\, x^{\vec{\epsilon}} t^{[m]} \tag{9.8}$$

for $P \in k[x_1, \ldots, x_n]$, $\vec{\epsilon} \in \{0,1\}^n$, and $m \geq 0$. Note that r^0_f is zero on the ideal generated by f, so d as defined above is well-defined. ♢

Proposition 9.5.2.2. *Assume MonOrdMonicPoly and that* $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$. *Then the isomorphism*

$$\varphi\colon X^e_{f,0} \to k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

of chain complexes from Construction 9.3.2.1 and Proposition 9.3.2.2 is compatible with the operators d *defined on either side. In particular,* d *as defined in Construction 9.5.2.1 on the codomain defines a strict mixed complex structure on*

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

and this strict mixed complex is isomorphic as a mixed complex to $X^e_{f,0}$. ♡

Proof. Using the description for d on $X^e_{f,0}$ obtained in the proof of Proposition 9.5.1.1, we obtain for $\vec{i} \not\geq \deg_{\preceq}(f)$, $\vec{\epsilon} \in \{0,1\}^n$ and $m \geq 0$ the following calculation.

$$\varphi\Big(\mathrm{d}\Big(e_{\vec{i},0,\vec{\epsilon},m}\Big)\Big) = \varphi\bigg(E^{[m]}\Big(\mathrm{d}\Big(x^{\vec{i}}\Big)\,\mathrm{d}\, x^{\vec{\epsilon}} + mq^1_f\Big(\mathrm{d}\, f \cdot x^{\vec{i}}\Big)\,\mathrm{d}\, x^{\vec{\epsilon}}\Big)\bigg)$$

[18] We will later show that under the isomorphism φ this operator agrees with the d that is part of the strict mixed complex structure on $X^e_{f,0}$, so that the operator d defined here defines a strict mixed complex structure will then be automatic.

$$
\begin{aligned}
&= \left(p\left(\mathrm{d}\left(x^{\vec{i}}\right)\right) + mp\left(q_f^1\left(\mathrm{d}\, f \cdot x^{\vec{i}}\right)\right)\right) \mathrm{d}\, x^{\vec{\epsilon}} t^{[m]} \\
&= \left(p\left(\mathrm{d}\left(r_f^0\left(x^{\vec{i}}\right)\right)\right) + mp\left(q_f^1\left(\mathrm{d}\, f \cdot r_f^0\left(x^{\vec{i}}\right)\right)\right)\right) \mathrm{d}\, x^{\vec{\epsilon}} t^{[m]} \\
&= \mathrm{d}\left(p\left(x^{\vec{i}}\right) \mathrm{d}\, x^{\vec{\epsilon}} t^{[m]}\right) \\
&= \mathrm{d}\left(\varphi\left(e_{\vec{i},0,\vec{\epsilon},m}\right)\right)
\end{aligned}
$$
□

We can now put everything together to obtain the main result.

Proposition 9.5.2.3. *Assume MonOrdMonicPoly, that Conjecture D*[19] *holds for f, and that* $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$[20].

Then there is an equivalence

$$\mathrm{HH}_{\mathsf{Mixed}}\big(k[x_1, \ldots, x_n]/f\big) \simeq \gamma_{\mathsf{Mixed}}\big(k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)\big)$$

in $\mathcal{M}$ixed*, where*

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

is the mixed complex described in Construction 9.3.2.1, Construction 9.5.2.1, and Proposition 9.5.2.2. ♡

Proof. Combine Proposition 9.2.2.2 with Proposition 9.5.1.1 and Proposition 9.5.2.2. □

Proposition 9.5.2.3 is the last missing piece to prove Theorem A that was stated in the introduction.

Proof of Theorem A. Combine Proposition 9.5.2.3 with Proposition 7.5.3.1 and Corollary 9.4.2.6. □

9.6. On the quasiisomorphisms constructed by the Buenos Aires Cyclic Homology Group

Assume MonOrdMonicPoly and let $A := k[x_1, \ldots, x_n]/f$. In [BACH], an $A \otimes A$-free resolution $R_s(A)$ of A is constructed, together with morphisms of $A \otimes A$-chain complexes

$$h\colon R_s(A) \to \overline{\mathrm{C}}^{\mathrm{Bar}}(A) \qquad \text{and} \qquad g\colon \overline{\mathrm{C}}^{\mathrm{Bar}}(A) \to R_s(A)$$

where $\overline{\mathrm{C}}^{\mathrm{Bar}}(A)$ refers to the normalized bar construction that relates to the bar construction defined in Construction 6.3.2.1 as the normalized standard Hochschild complex

[19] Note that Conjecture D holds if $n = 1$ or $n = 2$ and 2 is invertible in k by Proposition 7.5.3.1.

[20] Recall from Corollary 9.4.2.6 and Proposition 9.4.2.5 that this holds in particular if for every $\vec{i} \in \mathbb{Z}^n_{\geq 0}$ such that $f_{\vec{i}} \neq 0$ the following property holds: If $1 \leq j \leq n$ and $\deg_{\preceq}(f)_j \neq 0$, then $\vec{i}_j \leq \deg_{\preceq}(f)_j$.

relates to the standard Hochschild complex; in chain degree $n \geq 0$ the complex $\overline{\mathrm{C}}^{\mathrm{Bar}}(A)$ is given by $A \otimes (A/k \cdot \{1\})^{\otimes n} \otimes A$. It it shown in [BACH, 2.5.11] that g and h are mutual homotopy inverses. Tensoring over $A \otimes A$ from the left with A one then obtains quasiisomorphisms[21]

$$\overline{h}\colon \overline{R_s}(A) \to \overline{\mathrm{C}}(A) \qquad \text{and} \qquad \overline{g}\colon \overline{\mathrm{C}}(A) \to \overline{R_s}(A)$$

so that $\gamma(\overline{R_s}(A)) \simeq \gamma(\overline{\mathrm{C}}(A))$ in $\mathcal{D}(k)$. By Propositions 6.3.1.10 and 6.3.4.1 the chain complex $\overline{R_s}(A)$ is thus a strict model for $\mathrm{HH}(A)$ as an object of $\mathcal{D}(k)$. As was remarked in Section 9.3.2, the chain complex $\overline{R_s}(A)$ is isomorphic to the chain complex $k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$ described in Construction 9.3.2.1. Corollary 9.3.2.3 could thus also be deduced directly from the results of [BACH].

The question now arises whether one could similarly give an alternative proof of Proposition 9.5.2.3 and Theorem A, perhaps even without requiring the assumption that Conjecture D holds for f and that $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$, by showing that $\overline{g}$ or $\overline{h}$ can be lifted to a morphism of strict mixed complexes, and using that the normalized standard Hochschild complex $\overline{\mathrm{C}}(A)$ represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(A)$ even as an object in $\mathcal{M}\mathrm{ixed}$ by Propositions 6.3.1.10 and 6.3.4.1.

The following two propositions show that this is in general not possible; there is in general no strict mixed complex structure on $\overline{R_s}(A)$ that makes $\overline{g}$ or $\overline{h}$ into a morphism of strict mixed complexes. The counterexamples we use are $f = x_1x_2x_3$ for g and $f = x_1x_2$ for h. Note that both of these polynomials satisfy $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$ by Corollary 9.4.2.6.

This leaves open the question of whether it is possible to prove that $\overline{g}$ or $\overline{h}$ can be upgraded to a strongly homotopy linear morphism of strict mixed complexes (see Section 4.2.3). This is what the author tried originally for $f = x_1x_2x_3$, but without succeeding. The amount of data required for the higher homotopies combined with the complicated definitions of $\overline{g}$ and $\overline{h}$ may make this infeasible as n gets large.

In the rest of this section we will assume that the reader is familiar with the definitions and notation from [BACH]. We will however deviate from the notation from [BACH] when we have already established notation for the same thing. In particular, if P is an element of $k[x_1, \ldots, x_n]$, then we will write $q_f^1(P)$ rather than $\overline{P}$ used in [BACH, 2.2.1], and we denote by $\overline{P}$ the residue class of P in $A/k \cdot \{1\}$, as in Proposition 6.3.1.10. We will denote by $\overline{\varphi}$ the morphism $A \otimes_{A \otimes A} \varphi$, with φ as in [BACH, 2.5.1].

Proposition 9.6.0.1. *Let $f = x_1x_2x_3$ and $A := k[x_1, x_2, x_3]/f$. Then there is no strict mixed structure on $\overline{R_s}(A)$ such that $\overline{g}$ is a morphism of strict mixed complexes.* ♡

Proof. If $\overline{g}$ were a morphism of strict mixed complexes, then the following equation would need to hold.

$$\mathrm{d}\big(\overline{g}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2})\big) = \overline{g}\big(\mathrm{d}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2})\big)$$

However, we will show that this is not possible no matter what the strict mixed complex structure on $\overline{R_s}(A)$ is, as $\overline{g}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2})$ is already zero, making the left hand side zero, while the right hand side is nonzero.

[21] Compare with Proposition 6.3.2.4 for the identification $\overline{\mathrm{C}}(A) \cong A \otimes_{A \otimes A} \overline{\mathrm{C}}^{\mathrm{Bar}}(A)$.

We begin by showing that $\overline{g}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2}) = 0$. We begin with the definition of g from [BACH, 2.5.4].

$$\begin{aligned}
&\overline{g}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2}) \\
&= x_2x_3\overline{g}_2(1 \otimes \overline{x_1} \otimes \overline{x_2}) \\
&= x_2x_3 \cdot \left(-q_f^1(x_1x_2)\overline{g}_0(1) \blacksquare t + (-1)^1 \sum_{i_1<i_2} \overline{\varphi}_{i_2i_1}(1 \otimes \overline{x_1} \otimes \overline{x_2})e_{i_1i_2} \right)
\end{aligned}$$

$q_f^1(x_1x_2) = 0$, so the first summand vanishes. We plug in the definition of $\overline{\varphi}$ from [BACH, 2.5.1].

$$\begin{aligned}
&= x_2x_3 \cdot \left(- \sum_{i_1<i_2} \left(\overline{\varphi}^0_{i_2i_1}(1 \otimes \overline{x_1} \otimes \overline{x_2}) + \overline{\varphi}^1_{i_2i_1}(1 \otimes \overline{x_1} \otimes \overline{x_2})\right)e_{i_1i_2} \right) \\
&= x_2x_3 \cdot \left(- \sum_{i_1<i_2} \left(\frac{\partial x_1}{\partial x_{i_2}} \cdot \frac{\partial x_2}{\partial x_{i_1}} + \overline{\varphi}^1_{i_2i_1}(1 \otimes \overline{x_1} \otimes \overline{x_2}) \right) e_{i_1i_2} \right)
\end{aligned}$$

The first summand can only be nonzero if both $i_2 = 1$ and $i_1 = 2$, but this does not actually occur as $i_1 < i_2$.

$$\begin{aligned}
&= x_2x_3 \cdot \left(- \sum_{i_1<i_2} \left(\overline{\varphi}^1_{i_2i_1}(1 \otimes \overline{x_1} \otimes \overline{x_2})\right)e_{i_1i_2} \right) \\
&= x_2x_3 \cdot \left(- \sum_{i_1<i_2} \left(-1 \cdot \frac{\partial q_f^1(x_1x_2)}{\partial x_{i_2}} \cdot \overline{\varphi}^0_{i_1}(1 \otimes f) \right) e_{i_1i_2} \right)
\end{aligned}$$

This is zero as $q_f^1(x_1x_2) = 0$.

It remains to show that $\overline{g}\big(\mathrm{d}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2})\big)$ is not zero. We begin by evaluating $\mathrm{d}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2})$ using Proposition 6.3.1.10.

$$\begin{aligned}
&\overline{g}_3\big(\mathrm{d}(x_2x_3 \otimes \overline{x_1} \otimes \overline{x_2})\big) \\
&= \overline{g}_3(1 \otimes \overline{x_2x_3} \otimes \overline{x_1} \otimes \overline{x_2} + 1 \otimes \overline{x_1} \otimes \overline{x_2} \otimes \overline{x_2x_3} + 1 \otimes \overline{x_2} \otimes \overline{x_2x_3} \otimes \overline{x_1}) \\
&= -\Big(q_f^1(x_1x_2x_3)\overline{g}_1(1 \otimes \overline{x_2}) + q_f^1(x_1x_2)\overline{g}_1(1 \otimes \overline{x_2x_3}) + q_f^1\big(x_2^2x_3\big)\overline{g}_1(1 \otimes \overline{x_1}) \Big) \blacksquare t \\
&\quad + \big(\overline{\varphi}_{321}(1 \otimes \overline{x_2x_3} \otimes \overline{x_1} \otimes \overline{x_2}) + \overline{\varphi}_{321}(1 \otimes \overline{x_1} \otimes \overline{x_2} \otimes \overline{x_2x_3}) \\
&\qquad + \overline{\varphi}_{321}(1 \otimes \overline{x_2} \otimes \overline{x_2x_3} \otimes \overline{x_1})\big) \cdot e_{123}
\end{aligned}$$

We have three elements to which $\overline{\varphi}_{321} = \overline{\varphi}^0_{321} + \overline{\varphi}^1_{321}$ is applied. The $\overline{\varphi}^0_{321}$ component is zero for all three terms; for the first one because $\frac{\partial x_2}{\partial x_1} = 0$, for the second one because $\frac{\partial x_1}{\partial x_3} = 0$, and for the last one because $\frac{\partial x_2}{\partial x_3} = 0$.

$$\begin{aligned}
&= -\overline{g}_1(1 \otimes \overline{x_2}) \blacksquare t \\
&\quad + \overline{\varphi}^1_{321}(1 \otimes \overline{x_2x_3} \otimes \overline{x_1} \otimes \overline{x_2}) \cdot e_{123} \\
&\quad + \overline{\varphi}^1_{321}(1 \otimes \overline{x_1} \otimes \overline{x_2} \otimes \overline{x_2x_3}) \cdot e_{123}
\end{aligned}$$

$$+\,\overline{\varphi}^1_{321}(1\otimes\overline{x_2}\otimes\overline{x_2x_3}\otimes\overline{x_1})\cdot e_{123}$$

The definition of $\overline{\varphi}^1_{321}$ has a factor that is a partial derivative of q^1_f of the product of two neighboring tensor factors. q^1_f of such a product can only possibly be nonzero if we multiply the first two tensor factors in $1\otimes\overline{x_2x_3}\otimes\overline{x_1}\otimes\overline{x_2}$ or the last two in $1\otimes\overline{x_2}\otimes\overline{x_2x_3}\otimes\overline{x_1}$. In both cases the product is $x_1x_2x_3$, so that the value of q^1_f will be 1. Forming any partial derivative then yields zero.

$$\begin{aligned}
&= -\overline{g}_1(1\otimes\overline{x_2})\blacksquare\, t\\
&= \big(\overline{\varphi}_1(1\otimes\overline{x_2})\cdot e_1+\overline{\varphi}_2(1\otimes\overline{x_2})\cdot e_2+\overline{\varphi}_3(1\otimes\overline{x_2})\cdot e_3\big)\blacksquare\, t\\
&= \left(\frac{\partial x_2}{\partial x_1}\cdot e_1+\frac{\partial x_2}{\partial x_2}\cdot e_2+\frac{\partial x_2}{\partial x_3}\cdot e_3\right)\blacksquare\, t\\
&= e_2t
\end{aligned}$$

□

Proposition 9.6.0.2. *Let $f=x_1x_2$ and $A:=k[x_1,x_2]/f$. Then there is no strict mixed structure on $\overline{R_s}(A)$ such that $\overline{h}$ is a morphism of strict mixed complexes.* ♡

Proof. If $\overline{h}$ were a morphism of strict mixed complexes, then the following equation would need to hold.

$$\overline{h}\big(\mathrm{d}(x_1t)\big)=\mathrm{d}\Big(\overline{h}(x_1t)\Big)$$

However, we will show that this is not possible no matter what the strict mixed complex structure on $\overline{R_s}(A)$ is, as $\mathrm{d}\Big(\overline{h}(x_1t)\Big)$ does not lie in the image of $\overline{h}$.

We begin by calculating $h(t)$, for which we have the following by [BACH, After 2.4.5, 2.2.4 (g), and 1.1].

$$\begin{aligned}
&h(t)\\
&=\epsilon_0\left(-\frac{T_1(x_1x_2)}{T(x_1)}(1\otimes\overline{x_1}\otimes 1)-\frac{T_2(x_1x_2)}{T(x_2)}(1\otimes\overline{x_2}\otimes 1)\right)\\
&=\epsilon_0\big(-(1\otimes x_2)(1\otimes\overline{x_1}\otimes 1)-(x_1\otimes 1)(1\otimes\overline{x_2}\otimes 1)\big)\\
&=\epsilon_0\big(-(1\otimes\overline{x_1}\otimes x_2)-(x_1\otimes\overline{x_2}\otimes 1)\big)\\
&=-\big(1\otimes\overline{1}\otimes\overline{x_1}\otimes x_2\big)-(1\otimes\overline{x_1}\otimes\overline{x_2}\otimes 1)\\
&=-(1\otimes\overline{x_1}\otimes\overline{x_2}\otimes 1)
\end{aligned}$$

We can thus conclude the following for $\overline{h}(t)$.

$$\overline{h}(t)=-1\otimes\overline{x_1}\otimes\overline{x_2}$$

We can now evaluate $\mathrm{d}\Big(\overline{h}(x_1t)\Big)$ as follows, using Proposition 6.3.1.10.

$$\begin{aligned}
&\mathrm{d}\Big(\overline{h}(x_1t)\Big)\\
&=-\,\mathrm{d}(x_1\otimes\overline{x_1}\otimes\overline{x_2})\\
&=-1\otimes\overline{x_1}\otimes\overline{x_1}\otimes\overline{x_2}-1\otimes\overline{x_1}\otimes\overline{x_2}\otimes\overline{x_1}-1\otimes\overline{x_2}\otimes\overline{x_1}\otimes\overline{x_1}
\end{aligned}$$

Note that $\overline{\mathrm{C}}_3(A)$ is a free k-module that has a basis that is given by elements of the following form.

$$x^{\vec{i}} \otimes \overline{x^{\vec{j_1}}} \otimes \overline{x^{\vec{j_2}}} \otimes \overline{x^{\vec{j_3}}} \text{ for } \vec{i}, \vec{j_1}, \vec{j_2}, \vec{j_3} \in \mathbb{Z}^2_{\geq 0} \text{ such that } \vec{i}, \vec{j_1}, \vec{j_2}, \vec{j_3} \ngeq (1,1) \text{ and } \vec{j_1}, \vec{j_2}, \vec{j_3} \neq \vec{0}$$

We can define a submodule J spanned by the basis elements of the above form such that there exist $1 \leq a < b \leq 3$ such that $\vec{j_a} = (1,0)$ and $\vec{j_b} = (0,1)$. In other words, J is spanned elements in which two of the last three tensor factors are x_1 and x_2, and appearing in that order. Note that $\mathrm{d}(\overline{h}(x_1 t))$ is a linear combination of three basis elements of $\overline{\mathrm{C}}_3(A)$, and while the first two lie in J, this is not the case for $1 \otimes \overline{x_2} \otimes \overline{x_1} \otimes \overline{x_1}$. This implies that $\mathrm{d}(\overline{h}(x_1 t))$ does not lie in J, so it suffices to show that the image of $\overline{h}_3$ is a submodule of J.

$\overline{R_s}(A)_3$ is generated by elements of the form $x^{\vec{i}} e_j t$ with $\vec{i} \in \mathbb{Z}^2_{\geq 0}$ and $j \in \{1, 2\}$. The image of $\overline{h}_3$ is thus generated by elements of the following form, using Propositions 6.3.2.10 and 6.3.2.11.

$$\begin{aligned}
&\overline{h}_3\left(x^{\vec{i}} e_j t\right) \\
&= x^{\vec{i}} \cdot \left(-1 \otimes \overline{x_j}\right) \cdot \left(-1 \otimes \overline{x_1} \otimes \overline{x_2}\right) \\
&= x^{\vec{i}} \otimes \overline{x_j} \otimes \overline{x_1} \otimes \overline{x_2} - x^{\vec{i}} \otimes \overline{x_1} \otimes \overline{x_j} \otimes \overline{x_2} + x^{\vec{i}} \otimes \overline{x_1} \otimes \overline{x_2} \otimes \overline{x_j}
\end{aligned}$$

This shows that the image of $\overline{h}_3$ is contained in J. □

9.7. On a question of Larsen

Let n be a positive integer and f an element of $k[x_1, \ldots, x_n]$ that is monic and of positive degree when considered as a polynomial in the single variable x_1 with coefficients in $k[x_2, \ldots, x_n]$. Then Larsen constructs in [Lar95, 2.11] a strict mixed complex and asks the question whether it gives the cyclic homology of $k[x_1, \ldots, x_n]/f$, having answered this question in the affirmative for $n = 2$ in [Lar95, 2.10].

In the $n = 2$ case, what Larsen actually shows is that there is a strongly homotopy linear[22] quasiisomorphism from the strict mixed complex Larsen constructs to the normalized standard Hochschild complex. As the normalized standard Hochschild complex as well as the strict mixed complex Larsen constructs are bounded below, it follows from [Kas87, 2.3] using the argument of the proof of [Kas87, 2.6] that this strongly homotopy linear quasiisomorphism induces an isomorphism of cyclic homology groups.

By Remark 4.4.4.2, the strongly homotopy linear quasiisomorphism constructed by Larsen induces an equivalence in $\mathcal{M}\mathrm{ixed}$, and as the normalized standard Hochschild complex represents Hochschild homology as a mixed complex by Propositions 6.3.4.1 and 6.3.1.10, this implies that Larsen's strict mixed complex represents the Hochschild homology $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, x_2]/f)$ as an object of $\mathcal{M}\mathrm{ixed}$. Applying [Hoy18, 2.1, 2.2, and

[22]See Definition 4.2.3.1 for a definition. The definition stated in [Lar95, 1.4.1] differs slightly, likely due to a mistake, see a discussion in Remark 9.7.0.1.

2.3] this in turn also implies the statement regarding cyclic homology groups, without invoking [Kas87, 2.3].

Using Corollary 9.4.2.6 it is easy to see that the conditions stated at the start of this section for f imply that $\operatorname{logdim}_f(\mathrm{d}\, f) \leq 1$. If we assume that Conjecture D holds for f, then Proposition 9.5.2.3 will thus provide a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, \ldots, x_n]/f)$ as an object of $\mathcal{M}\mathrm{ixed}$.

We claim that the strict mixed complex

$$k[x_1, \ldots, x_n]/f \otimes \Lambda(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n) \otimes \Gamma(t)$$

used in Proposition 9.5.2.3 and described in Construction 9.3.2.1, Construction 9.5.2.1, and Proposition 9.5.2.2 is in fact isomorphic as a strict mixed complex to the strict mixed complex constructed by Larsen in [Lar95, 2.11], so that proving Conjecture D will result in an affirmative answer to Larsen's question. This is essentially clear if one understands both definitions, but due to the very different notations used, we say some words about this.

That the underlying commutative graded k-algebras are isomorphic via an isomorphism that maps our x_i, $\mathrm{d}\, x_i$, and $t^{[m]}$ to Larsen's x_i, dx_i, and $(-1)^m z^{[m]}$ is clear by looking at [Lar95, 2.11]. Comparing the formulas for the boundary operator (denoted by b in [Lar95]) given in Construction 9.3.2.1 and [Lar95, 2.11], it is also clear that this isomorphism is compatible with the boundary operators.

The differential d is denoted by B in [Lar95], and defined in [Lar95, 2.11] by the following formula.

$$B(\alpha) \coloneqq \mathrm{d}\, \alpha + \left[df, z\frac{\partial \alpha}{\partial z} \right] \tag{9.9}$$

Let $\alpha = p(P)dx^{\vec{\epsilon}} z^{[m]}$ for $P \in k[x_1, \ldots, x_n]$, $\vec{\epsilon} \in \{0,1\}^n$, and $m \geq 0$. The summand $\mathrm{d}\, \alpha$ is then notation for $p(\mathrm{d}(r_f^0(P)))dx^{\vec{\epsilon}} z^{[m]}$, so corresponds to the first summand in the formula (9.8) in Construction 9.5.2.1.

The term $z\frac{\partial \alpha}{\partial z}$ is given by[23]

$$z \cdot \frac{\partial p(P)dx^{\vec{\epsilon}} z^{[m]}}{\partial z} = z \cdot p(P)dx^{\vec{\epsilon}} z^{[m-1]} = m \cdot p(P)dx^{\vec{\epsilon}} z^{[m]}$$

so that we are left to consider the term $\left[df, m \cdot p(P)dx^{\vec{\epsilon}} z^{[m]} \right]$.

The notation $[-,-]$ is defined in [Lar95, 2.1.1], and in our notation

$$\left[df, m \cdot p(P)dx^{\vec{\epsilon}} z^{[m]} \right]$$

corresponds to[24]

$$q_f^1\left(\mathrm{d}\, f \cdot r_f^0(m \cdot P)dx^{\vec{\epsilon}} z^{[m]} \right)$$

so that the second summand in (9.9) corresponds to the second summand in (9.8) in Construction 9.5.2.1.

[23] Recall that $z^{[m]}$ is $\frac{1}{m!}z^m$.

[24] We use that $\mathrm{d}\, f$ is f-reduced.

Remark 9.7.0.1. A definition of what we call strongly homotopy linear morphisms of strict mixed complexes is given around [Lar95, 1.4.1], which however differs in signs from the one we gave in Definition 4.2.3.1, with a plus sign on the left hand side. It is noted just after [Lar95, 1.4.1] that the sign conventions differ from those of [Kas87]. However, this changed sign does not seem to be a matter of convention but rather a mistake, with the definition of [Lar95] leading to a different notion, making the results of [Kas87] inapplicable. Luckily the inductive method to construct $i^{(2k+2)}$ in [Lar95, Display between (1.4.1) and (1.4.2)] works with the correct definition (4.15), while the first step of the induction actually fails when using [Lar95, 1.4.1]. Thus the results of [Lar95] should hold with the corrected definition.

In the following we will construct a morphism of chain complexes $f\colon X \to Y$ between strict mixed complexes that can be extended to a strongly homotopy linear morphism using the definition we gave in Definition 4.2.3.1 and that is also used in [Kas87, 2.2] and [Lod98, 2.5.14], but that can not be extended using the definition of [Lar95, 1.4.1], thereby showing that the sign difference is not just a matter of conventions.

Let X be the strict mixed complex whose underlying $\mathbb{Z}$-graded k-module is free with generator x in degree 0 and y in degree 1, with $\mathrm{d} = 0$ and $\partial(y) = x$. As the underlying chain complex is cofibrant and acyclic, we should expect that every chain morphism out of it can be extended to a strongly homotopy linear morphism. Indeed, this is the case with the definition we give here. Let $f\colon X \to Y$ be a morphism of chain complexes to any other strict mixed complex Y. Then setting

$$\begin{aligned} f^{(1)}(x) &\coloneqq \mathrm{d}\big(f(y)\big) \\ f^{(1)}(y) &\coloneqq 0 \\ f^{(n)} &\coloneqq 0 \qquad \text{for } n > 1 \end{aligned}$$

and extending k-linearly defines the necessary data to extend f to a strongly homotopy linear morphism, as it is easy to check that (4.15) is satisfied.

Let us now consider the strict mixed complex Y whose underlying $\mathbb{Z}$-graded k-module is free on a in degree 0, on c and e in degree 1, and on g in degree 2, with d and ∂ defined by extending k-linearly from the following definitions.

$$\begin{array}{cccc} \partial(a) \coloneqq 0 & \partial(c) \coloneqq a & \partial(e) \coloneqq 0 & \partial(g) \coloneqq -e \\ \mathrm{d}(a) \coloneqq e & \mathrm{d}(c) \coloneqq g & \mathrm{d}(e) \coloneqq 0 & \mathrm{d}(g) \coloneqq 0 \end{array}$$

The following diagram depicts the strict mixed complex Y using the conventions from Convention 4.2.1.7.

$$\begin{tikzcd} & g \arrow[dr, "-1"] & \\ c \arrow[ur, squiggly, "1"] \arrow[dr, "1"'] & & e \\ & a \arrow[ur, squiggly, "1"'] & \end{tikzcd}$$

Now define a morphism of chain complexes $f\colon X \to Y$ by k-linearly extending $f(x) \coloneqq a$ and $f(y) \coloneqq c$. Assume that $f^{(1)}$ were a morphism of $\mathbb{Z}$-graded k-modules from X to Y increasing degree by 2 and satisfying the following equation.

$$f^{(1)} \circ \partial + \partial \circ f^{(1)} = f \circ \mathrm{d} - \mathrm{d} \circ f$$

Then we obtain

$$\partial\Big(f^{(1)}(x)\Big) = f\big(\mathrm{d}(x)\big) - \mathrm{d}\big(f(x)\big) - f^{(1)}\big(\partial(x)\big) = f(0) - \mathrm{d}(a) - f^{(1)}(0) = -e$$

which implies that $f^{(1)}(x) = g$. We then need

$$\partial\Big(f^{(1)}(y)\Big) = f\big(\mathrm{d}(y)\big) - \mathrm{d}\big(f(y)\big) - f^{(1)}\big(\partial(y)\big) = f(0) - \mathrm{d}(c) - f^{(1)}(x) = -g - g = -2g$$

to hold. However, if $2 \neq 0$ in k, then this is impossible, as $2g$ is then not a boundary in Y. This shows that the notion defined by [Lar95, 1.4.1] is genuinely different to the notion of strongly homotopy linear morphisms as defined in (4.15) as well as [Kas87, 2.2] and [Lod98, 2.5.14]. $\diamondsuit$

Remark 9.7.0.2. In [HN20, Theorem 1], a description is given of an object of $\mathcal{D}(\mathbb{Z})^{\mathrm{B}\mathbb{T}}$ related to $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, x_2]/f)$ for $f = x_1^a - x_2^b$ for $a, b \geq 2$ relatively prime integers. It is stated that this description follows from the results of [Lar95], but as so far there was no proof in the literature that strongly homotopy linear quasiisomorphisms induce equivalences in $\mathcal{M}$ixed, this constituted a gap in [HN20], which is filled by Sections 4.2.3 and 4.4.4 and in particular Remark 4.4.4.2.[25]

If 2 is in invertible in k then one can now also use Proposition 9.5.2.3 in combination with Proposition 7.5.3.1, which gives a new proof of the statement that the strict mixed complex constructed by Larsen represents $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(k[x_1, x_2]/f)$ in $\mathcal{M}$ixed. However to use this for [HN20, Theorem 1] slightly more work would be needed to also identify the decomposition – see Section 1.6 (3). $\diamondsuit$

[25] However the construction of the higher homotopies of the strongly homotopy linear map constructed in [Lar95] ultimately depends on the choice of a contracting homotopy K^t in [Lar95, Lemma 1.3]. It is unclear to the author which choice should be used as the canonical one to obtain a canonical equivalence in [HN20, Theorem 1] as claimed.

Chapter 10.

Example: $x_1^2 - x_2x_3$

Just like Proposition 8.3.0.1 was a stepping stone for Theorem A, we also view Theorem A as a stepping stone; for any particular polynomial f of interest one will most likely want to further simplify the strict mixed complex provided by Theorem A before using it as input for further calculations.

In this chapter we thus go through one relatively simple but nontrivial example in detail: Conditional on Conjecture D[1] holding for for f we describe $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}[x_1, x_2, x_3]/f)$, where f is the polynomial $f = x_1^2 - x_2x_3$ that geometrically defines a cone. We will describe the process step by step in the order one might proceed in when first working out the example.

10.1. Applying Theorem A

In order to be able to apply Theorem A, f needs to be in particular monic with respect to a chosen monomial order. While f is monic with respect to any monomial order, which one we choose matters with regards to what the degree of f will be – either x_2x_3 or x_1^2 could be chosen as the leading term.

We choose $\preceq$ to be the lexicographic monomial ordering on three variables so that x_1^2 is the leading term. We then have $\deg_{\preceq}(f) = (2, 0, 0)$, and for $\vec{i} \in \mathbb{Z}^3_{\geq 0}$ the monomial $x^{\vec{i}}$ is f-reduced if and only if $i_1 \leq 1$. We can now apply Theorem A to obtain a strict mixed complex representing $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1, x_2, x_3]/(x_1^2 - x_2x_3)\big)$, conditional on Conjecture D holding for for f.

Proposition 10.1.0.1. *Let $f = x_1^2 - x_2x_3$ as an element of $\mathbb{Z}[x_1, x_2, x_3]$, and assume that Conjecture D holds for f. Then there is an equivalence*

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\big(\mathbb{Z}[x_1, x_2, x_3]/f\big) \simeq \gamma_{\mathsf{Mixed}}\big(\mathbb{Z}[x_1, x_2, x_3]/f \otimes \Lambda(\mathrm{d}\,x_1, \mathrm{d}\,x_2, \mathrm{d}\,x_3) \otimes \Gamma(t)\big)$$

in $\mathcal{M}\mathrm{ixed}$, *where*

$$Y \coloneqq \mathbb{Z}[x_1, x_2, x_3]/f \otimes \Lambda(\mathrm{d}\,x_1, \mathrm{d}\,x_2, \mathrm{d}\,x_3) \otimes \Gamma(t)$$

[1] That the discussion of the example in this chapter is conditional on a conjecture is of course slightly unsatisfactory, but allows us to discuss an illustrative example with nontrivial features.

is the strict mixed complex with underlying graded abelian group as indicated, with x_i of degree 0, $\mathrm{d}\,x_i$ of degree 1 and t of degree 2, and with boundary operator and differential given by the following formulas[2]*, for $a, b \geq 0$, $\vec{\epsilon} \in \{0,1\}^3$, and $m \geq 0$.*

$$\begin{aligned}
&\partial\Big(p\Big(x_2^a x_3^b\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} t^{[m]}\Big)\\
&= \Big(-2 \cdot p\Big(x_1 x_2^a x_3^b\Big)\,\mathrm{d}\,x_1 + p\Big(x_2^a x_3^{b+1}\Big)\,\mathrm{d}\,x_2 + p\Big(x_2^{a+1} x_3^b\Big)\,\mathrm{d}\,x_3\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} t^{[m-1]}\\
&\partial\Big(p\Big(x_1 x_2^a x_3^b\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} t^{[m]}\Big)\\
&= \Big(-2 \cdot p\Big(x_2^{a+1} x_3^{b+1}\Big)\,\mathrm{d}\,x_1 + p\Big(x_1 x_2^a x_3^{b+1}\Big)\,\mathrm{d}\,x_2 + p\Big(x_1 x_2^{a+1} x_3^b\Big)\,\mathrm{d}\,x_3\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} t^{[m-1]}\\
&\mathrm{d}\Big(p\Big(x_2^a x_3^b\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} t^{[m]}\Big)\\
&= \Big(a \cdot p\Big(x_2^{a-1} x_3^b\Big)\,\mathrm{d}\,x_2 + b \cdot p\Big(x_2^a x_3^{b-1}\Big)\,\mathrm{d}\,x_3\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} t^{[m]}\\
&\mathrm{d}\Big(p\Big(x_1 x_2^a x_3^b\Big)\,\mathrm{d}\,x^{\vec{\epsilon}} t^{[m]}\Big)\\
&= \Big((1+2m) \cdot p\Big(x_2^a x_3^b\Big)\,\mathrm{d}\,x_1 + a \cdot p\Big(x_1 x_2^{a-1} x_3^b\Big)\,\mathrm{d}\,x_2 + b \cdot p\Big(x_1 x_2^a x_3^{b-1}\Big)\,\mathrm{d}\,x_3\Big)\,\mathrm{d}\,x^{\epsilon} t^{[m]}
\end{aligned}$$

In the formulas above, terms involving negative exponents of a variable are to be interpreted as 0. ♡

Proof. As x_1 is the only variable occurring in the leading term of f and the exponent of x_1 in the other term x_2x_3 is 0, the assumptions of Theorem A are satisfied, so that it suffices to check that the formulas for ∂ and d from Theorem A specialize to the ones given in the statement above. We have

$$\mathrm{d}\,f = 2x_1\,\mathrm{d}\,x_1 - x_3\,\mathrm{d}\,x_2 - x_2\,\mathrm{d}\,x_3$$

so the two formulas for ∂ follow directly from their description in Theorem A, where in the second formula we need only note that $p(x_1^2 x_2^a x_3^b) = p(x_2^{a+1} x_3^{b+1})$.

The formula for d from Theorem A is as follows, for $\eta \in \{0,1\}$.

$$\mathrm{d}\Big(p\Big(x_1^\eta x_2^a x_3^b\Big)\,\mathrm{d}\,x^{\epsilon} t^{[m]}\Big) = \Big(p\Big(\mathrm{d}\Big(x_1^\eta x_2^a x_3^b\Big)\Big) + m \cdot p\Big(q_f^1\Big(\mathrm{d}\,f \cdot x_1^\eta x_2^a x_3^b\Big)\Big)\Big)\,\mathrm{d}\,x^{\epsilon} t^{[m]}$$

As the maximum exponent of x_1 occurring in $x_2^a x_3^b$ and $\mathrm{d}\,f$ is 0 and 1, respectively, $\mathrm{d}\,f \cdot x_2^a x_3^b$ is f-reduced and thus $q_f^1(\mathrm{d}\,f \cdot x_2^a x_3^b) = 0$, so that the first formula for d follows.

[2] We use p as notation for the quotient morphism $\mathbb{Z}[x_1, x_2, x_3] \to \mathbb{Z}[x_1, x_2, x_3]/(x_1^2 - x_2x_3)$, like in Construction 9.3.2.1

For the second formula for d, we first note that

$$\begin{aligned}
\mathrm{d}\, f \cdot x_1 x_2^a x_3^b &= 2x_1^2 x_2^a x_3^b \,\mathrm{d}\, x_1 - x_1 x_2^a x_3^{b+1} \,\mathrm{d}\, x_2 - x_1 x_2^{a+1} x_3^b \,\mathrm{d}\, x_3 \\
&= \left(2x_2^a x_3^b \,\mathrm{d}\, x_1\right) \cdot f + 2x_2^{a+1} x_3^{b+1} \,\mathrm{d}\, x_1 - x_1 x_2^a x_3^{b+1} \,\mathrm{d}\, x_2 - x_1 x_2^{a+1} x_3^b \,\mathrm{d}\, x_3
\end{aligned}$$

which implies that

$$q_f^1\left(\mathrm{d}\, f \cdot x_1 x_2^a x_3^b\right) = 2x_2^a x_3^b \,\mathrm{d}\, x_1$$

The following calculation then shows the second formula for d from the statement.

$$\begin{aligned}
&\mathrm{d}\left(p\left(x_1 x_2^a x_3^b\right) \mathrm{d}\, x^\epsilon t^{[m]}\right) \\
&= \left(p\left(\mathrm{d}\left(x_1 x_2^a x_3^b\right)\right) + m \cdot p\left(2x_2^a x_3^b \,\mathrm{d}\, x_1\right)\right) \mathrm{d}\, x^\epsilon t^{[m]} \\
&= \left(p\left(x_2^a x_3^b\right) \mathrm{d}\, x_1 + a \cdot p\left(x_1 x_2^{a-1} x_3^b\right) \mathrm{d}\, x_2 + b \cdot p\left(x_1 x_2^a x_3^{b-1}\right) \mathrm{d}\, x_3 \right.\\
&\quad \left. +2m \cdot p\left(x_2^a x_3^b\right) \mathrm{d}\, x_1\right) \mathrm{d}\, x^\epsilon t^{[m]} \\
&= \left((1+2m) \cdot p\left(x_2^a x_3^b\right) \mathrm{d}\, x_1 + a \cdot p\left(x_1 x_2^{a-1} x_3^b\right) \mathrm{d}\, x_2 + b \cdot p\left(x_1 x_2^a x_3^{b-1}\right) \mathrm{d}\, x_3\right) \mathrm{d}\, x^\epsilon t^{[m]} \square
\end{aligned}$$

10.2. Comparison with the mixed complex of de Rham forms

To describe Y it will be useful to compare it to the mixed complex of de Rham forms. We first note the following about $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$.

Remark 10.2.0.1. It follows from [Wei94, 9.2.7] that the identification

$$\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]\,/\,\mathbb{Z}} \cong \mathbb{Z}[x_1,x_2,x_3] \otimes \Lambda(\mathrm{d}\, x_1, \mathrm{d}\, x_2, \mathrm{d}\, x_3)$$

from Section 7.1 induces an isomorphism

$$\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}} \cong \left(\mathbb{Z}[x_1,x_2,x_3]/f \otimes \Lambda(\mathrm{d}\, x_1, \mathrm{d}\, x_2, \mathrm{d}\, x_3)\right)/\,\mathrm{d}\, f$$

of strict mixed complexes[3]. ♢

We next define a morphism of strict mixed complexes from Y to $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$.

Definition 10.2.0.2. Consider the following morphism of graded abelian groups.

$$\varphi\colon Y \to \Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$$

[3]The boundary operators are zero, and the differential d maps x_i to $\mathrm{d}\, x_i$ and satisfies the Leibniz rule.

$$p\left(x^{\vec{i}}\right) \mathrm{d}\, x^{\vec{\epsilon}} t^{[m]} \mapsto \begin{cases} 0 & m > 0 \\ p(x^{\vec{i}}) \mathrm{d}\, x^{\vec{\epsilon}} & m = 0 \end{cases} \qquad \text{for } \vec{i} \in \mathbb{Z}^3_{\geq 0}, \vec{\epsilon} \in \{0,1\}^3, m \geq 0$$

It is clear from Proposition 10.1.0.1 that φ is compatible with the chain complex and mixed structure so that φ is a morphism of strict mixed complexes.

We furthermore define the morphism of strict mixed complexes

$$\psi \colon K \to Y$$

to be the kernel of φ. ♢

10.3. Grading

To make it easier to discuss K and Y, we equip them with a $\mathbb{Z}^2_{\geq 0}$-grading.

Construction 10.3.0.1. We upgrade $\mathbb{Z}[x_1, x_2, x_3]$ to a $\mathbb{Z}^2_{\geq 0}$-graded ring by declaring $\deg_{\mathrm{gr}}(x_1) = (1,1)$, $\deg_{\mathrm{gr}}(x_2) = (2,0)$, and $\deg_{\mathrm{gr}}(x_3) = (0,2)$. This makes f into a homogeneous polynomial of grading $\deg_{\mathrm{gr}}(f) = (2,2)$, so $\mathbb{Z}[x_1, x_2, x_3]/(f)$ inherits a grading where $\deg_{\mathrm{gr}}(p(x^{\vec{i}})) = \deg_{\mathrm{gr}}(r_f^0(x^{\vec{i}}))$ (note that f being homogeneous ensures that $r_f^0(x^{\vec{i}})$ is homogeneous). Declaring $\deg_{\mathrm{gr}}(\mathrm{d}\, x_i) = \deg_{\mathrm{gr}}(x_i)$ and $\deg_{\mathrm{gr}}(t^{[m]}) = m \cdot (2,2)$ makes Y and $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$ into $\mathbb{Z}^2_{\geq 0}$-graded strict mixed complexes, as one can easily see by inspecting the formulas for ∂ and d in Proposition 10.1.0.1. Furthermore, $\varphi \colon Y \to \Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$ clearly respects the grading, so the kernel K obtains an induced grading, making $\psi \colon K \to Y$ into a morphism of $\mathbb{Z}^2_{\geq 0}$-graded strict mixed complexes as well.

Let us denote the sub-mixed-complex of Y (of $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$, of K) of homogeneous elements of grading $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ by $Y(\vec{j})$ (by $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}(\vec{j})$, by $K(\vec{j})$), so that we obtain a sum decomposition as a strict mixed complex

$$Y \cong \bigoplus_{\vec{j} \in \mathbb{Z}^2_{\geq 0}} Y\left(\vec{j}\right)$$

and similarly for $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$ and K. ♢

Remark 10.3.0.2. Note that the additive submonoid of $\mathbb{Z}^2_{\geq 0}$ generated by $(1,1)$, $(2,0)$, and $(0,2)$ is not equal to all of $\mathbb{Z}^2_{\geq 0}$; it contains precisely those elements (a,b) for which the sum $a+b$ is even[4]. It follows that

$$Y(\vec{j}) \cong \Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}(\vec{j}) \cong K(\vec{j}) \cong 0$$

if $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ such that $j_1 + j_2$ is odd.

[4]This is obviously an additive condition, so as it holds for the three generators it holds for the full submonoid. On the other hand, if $(a,b) \in \mathbb{Z}^2_{\geq 0}$ with $a+b = 2c$ even, and without loss of generality say $b > a$, then $(a,b) = a \cdot (1,1) + (c-a) \cdot (0,2)$.

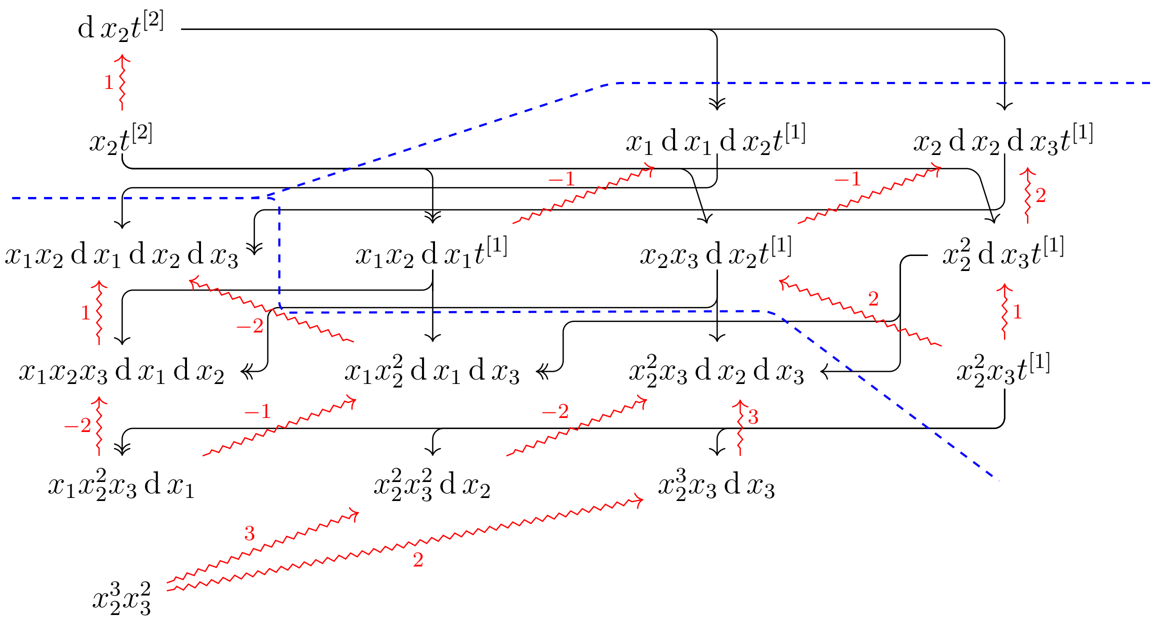
Note that the mixed complexes $Y(\vec{j})$ for $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ such that $j_1 + j_2$ is even might look different depending on the parity of j_1; In the even case, x_1 and $\mathrm{d}\,x_1$ must always "occur together", while in the odd case they never do. Indeed, one consequence is that the summand $(1+2m) \cdot p\big(x_2^a x_3^b\big)\,\mathrm{d}\,x_1$ in the second formula for d in Proposition 10.1.0.1 vanishes in the even case, as $\mathrm{d}\,x_1 \cdot \mathrm{d}\,x_1 = 0$. ♢

10.4. Non-diagonal pieces

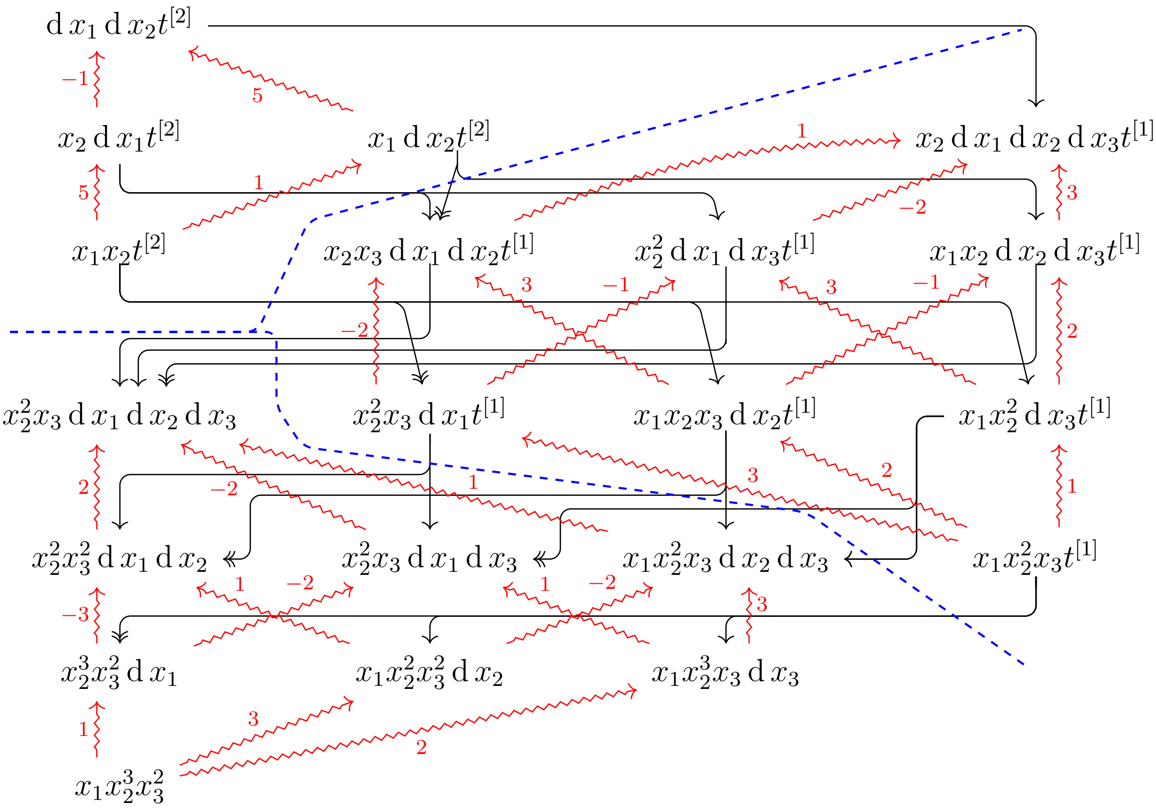
10.4.1. A first look at $Y((6,4))$ and $Y((7,5))$

We will next look at two illustrative examples to understand the mixed complexes $Y(\vec{j})$ better, one where j_1 is even and one where it is odd. We will depict the strict mixed complexes diagrammatically in the manner introduced in Convention 4.2.1.7, with respect to the basis given by elements of the form $p(x^{\vec{i}})\,\mathrm{d}\,x^{\vec{\epsilon}} t^{[m]}$. In this basis, the components of ∂ all have absolute value 0, 1, or 2. To make the diagram more readable, we omit the labels to the respective arrows and instead use a normal arrowhead to indicate an absolute value of 1, and a double arrowhead to indicate an absolute value of 2, while not indicating the sign to avoid overloading the diagram. We also omit p from the notation and write e. g. $x_2^3 x_3^2$ instead of $p(x_2^3 x_3^2)$.

We first consider $Y((6,4))$.



Next, the following diagram depicts $Y((7,5))$ as representative of the odd case.



Looking at these diagrams we can see that in both cases we can split of a large acyclic subcomplex (ignoring the mixed structure for now). Let us discuss the first case $Y((6,4))$. Starting from the top, we can first replace the basis element $p(x_2)\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[1]}$ with $\partial(\mathrm{d}\,x_2 t^{[2]}) = -p(x_2)\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[1]} - 2\cdot p(x_1)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[1]}$. Then $\mathrm{d}\,x_2 t^{[2]}$ and the new basis element generate a subcomplex that splits off as an acyclic summand. Continuing downward, we can replace $p(x_2^2)\,\mathrm{d}\,x_3 t^{[1]}$ with $\partial(p(x_2)t^{[2]})$, and so on. In the end, the only basis elements that "survive" are $p(x_2^3x_3^2)$, $p(x_1x_2^2x_3)\,\mathrm{d}\,x_1$, $p(x_2^2x_3^2)\,\mathrm{d}\,x_2$, and $p(x_1x_2x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2$.

10.4.2. A new basis

In general, we would like to do the following. For $a, \epsilon_1, \epsilon_2 \in \{0,1\}$ and $b, c, m \geq 0$, we would like to replace the basis element $p(x_1^a x_2^b x_3^c)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2}\,\mathrm{d}\,x_3 t^{[m]}$ of $Y(\vec{j})$ by the element $\partial(p(x_1^a x_2^{b-1} x_3^c)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} t^{[m+1]})$. Roughly, we divide by $x_2\,\mathrm{d}\,x_3$, increase the divided power of t by one, and then take the boundary. This is of course not possible if $b = 0$. So when could $b = 0$ happen? If $p(x_1^a x_3^c)\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2}\,\mathrm{d}\,x_3 t^{[m]}$ is in $Y(\vec{j})$, then we have $j_1 = 2m + a + \epsilon_1 + 2\epsilon_2$ and $j_2 = 2m + a + 2c + \epsilon_1 + 2$. As $\epsilon_2 \leq 1$ this implies that such an element can only occur in $Y(\vec{j})$ if $j_1 \leq j_2$.

So we are lead to distinguish three cases: For $Y(\vec{j})$ with $j_1 > j_2$, we can "eliminate" basis elements divisible by $\mathrm{d}\,x_3$, and for $Y(\vec{j})$ with $j_1 < j_2$, we can analogously "eliminate" basis elements divisible by $\mathrm{d}\,x_2$, leaving the case of $Y(\vec{j})$ with $j_1 = j_2$ to still be analyzed (and which will indeed turn out to be more interesting).

We will now carry out the idea we just sketched and first construct the indicated new basis for $Y(\vec{j})$ for $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ with $j_1 \neq j_2$. We will then be able to use this to show that $K(\vec{j})$ is acyclic.

Definition 10.4.2.1. To ease notation in the following, we make the following definitions for $\vec{j} \in \mathbb{Z}^2_{\geq 0}$.

$$\begin{aligned}
V &= \{0,1\} \times \mathbb{Z}^2_{\geq 0} \times \{0,1\}^3 \times \mathbb{Z}_{\geq 0} \\
V' &= \{0,1\} \times \mathbb{Z}^2_{\geq 0} \times \{0,1\}^2 \times \mathbb{Z}_{\geq 0} \\
V\left(\vec{j}\right) &= \left\{ (a,b,c,\epsilon_1,\epsilon_2,\epsilon_3,m) \in V \;\middle|\; \deg_{\mathrm{gr}}\left(p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} \,\mathrm{d}\, x_3^{\epsilon_3} t^{[m]}\right) = \vec{j} \right\} \\
V_2\left(\vec{j}\right) &= \left\{ (a,b,c,\epsilon_1,\epsilon_2,m) \in V' \;\middle|\; \deg_{\mathrm{gr}}\left(p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\right) = \vec{j} \right\} \\
V_3\left(\vec{j}\right) &= \left\{ (a,b,c,\epsilon_1,\epsilon_3,m) \in V' \;\middle|\; \deg_{\mathrm{gr}}\left(p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_3^{\epsilon_3} t^{[m]}\right) = \vec{j} \right\}
\end{aligned}$$

◇

Proposition 10.4.2.2. *Let $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ with $j_1 > j_2$. Then the set*

$$\begin{aligned}
\mathcal{B}_2(\vec{j}) = &\left\{ p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}) \right\} \\
&\cup \left\{ \partial\left(p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\right) \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}), m > 0 \right\}
\end{aligned}$$

forms a basis of $Y(\vec{j})$. Analogously, let $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ with $j_1 < j_2$. Then the set

$$\begin{aligned}
\mathcal{B}_3(\vec{j}) = &\left\{ p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_3^{\epsilon_3} t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_3,m) \in V_3(\vec{j}) \right\} \\
&\cup \left\{ \partial\left(p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_3^{\epsilon_3} t^{[m]}\right) \;\middle|\; (a,b,c,\epsilon_1,\epsilon_3,m) \in V_3(\vec{j}), m > 0 \right\}
\end{aligned}$$

forms a basis of $Y(\vec{j})$. ♡

Proof. We only discuss the statement for $j_1 > j_2$, the other is completely analogous. We will refer to the basis given by elements of the form $p(x^{\vec{i}}) \,\mathrm{d}\, x^{\vec{\epsilon}} t^{[m]}$ used up to now as the *monomial basis*. We wrote $\mathcal{B}_2(\vec{j})$ as a union, and will call elements of the first set elements of the first type and elements of the second set elements of the second type.

Note that the monomial basis can be written as follows, following the discussion before Definition 10.4.2.1 showing that any element of the monomial basis divisible by $\mathrm{d}\, x_3$ must have x_2 as a factor as well.

$$\left\{ p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}) \right\}$$

$$\cup \left\{ p\left(x_1^a x_2^{b+1} x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} \,\mathrm{d}\, x_3 t^{[m-1]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}), m > 0 \right\}$$

In this subdivision of the basis elements of the monomial basis, the first subset is exactly equal to the elements of $\mathcal{B}_2(\vec{j})$ of the first type.

For the elements of the second type we note that for $(a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j})$ with $m > 0$, they have the following form.

$$\begin{aligned} &\partial\left(p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\right) \\ &= (-1)^{\epsilon_1+\epsilon_2} p\left(x_1^a x_2^{b+1} x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} \,\mathrm{d}\, x_3 t^{[m-1]} \\ &\quad (-1)^{\epsilon_1} p\left(x_1^a x_2^b x_3^{c+1}\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2+1} t^{[m-1]} - 2p\left(x_1^{a+1} x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1+1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m-1]} \end{aligned}$$

Note that the first summand is always the negative of the corresponding (also indexed by $(a,b,c,\epsilon_1,\epsilon_2,m)$) basis element of second type in the monomial basis, while the other two summands are multiples of elements of the first type. This shows the claim. □

10.4.3. Non-diagonal pieces of K are acyclic

Proposition 10.4.3.1. *Let $\vec{j} \in \mathbb{Z}^2_{\geq 0}$ with $j_1 \neq j_2$. Then $K(\vec{j})$ is acyclic.* ♡

Proof. We again only discuss the case $j_1 > j_2$, as the other case is completely analogous.

Using Remark 10.2.0.1 and the same kind of argument as in the proof of Proposition 10.4.2.2 shows that

$$\left\{ p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,0) \in V_2(\vec{j}) \right\}$$

is a basis of $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}(\vec{j})$. It thus follows immediately from Proposition 10.4.2.2 that

$$\begin{aligned} &\left\{ p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}), m > 0 \right\} \\ \cup &\left\{ \partial\left(p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\right) \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}), m > 0 \right\} \end{aligned}$$

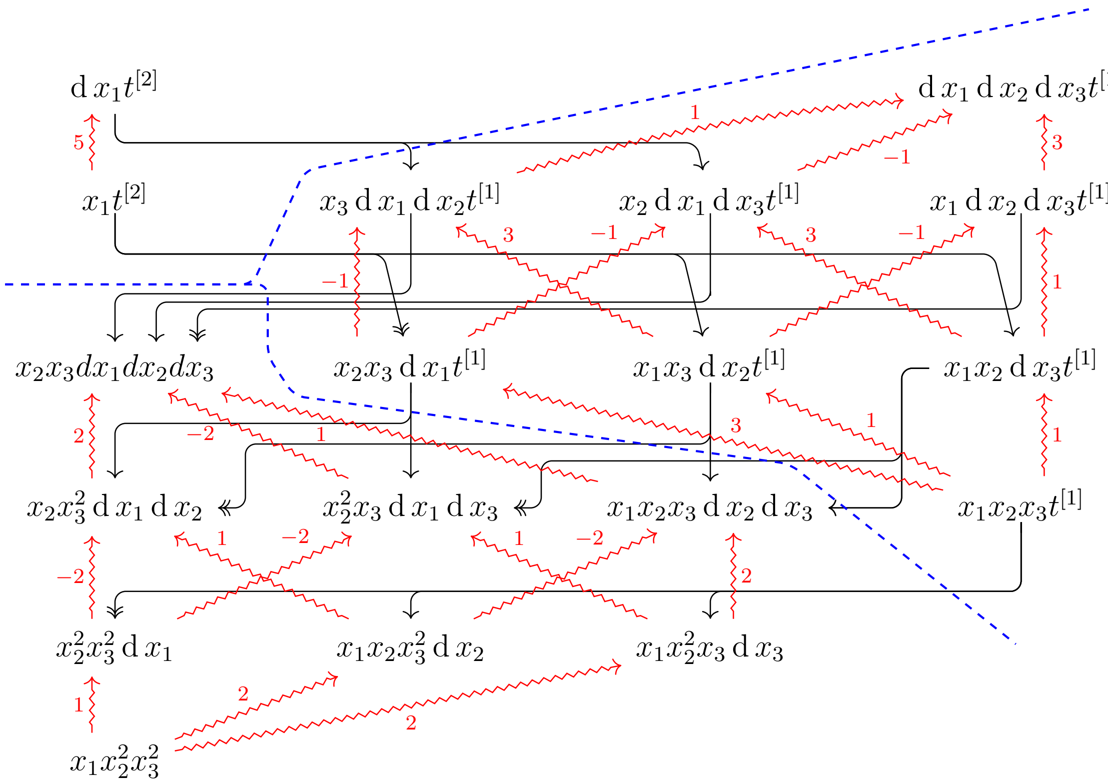
is a basis for $K(\vec{j})$. We can thus easily define a contracting homotopy h of $K(\vec{j})$ as follows, where $(a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j})$ with $m > 0$.

$$\begin{aligned} h\left(p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\right) &:= 0 \\ h\left(\partial\left(p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\right)\right) &:= p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \end{aligned}$$ □

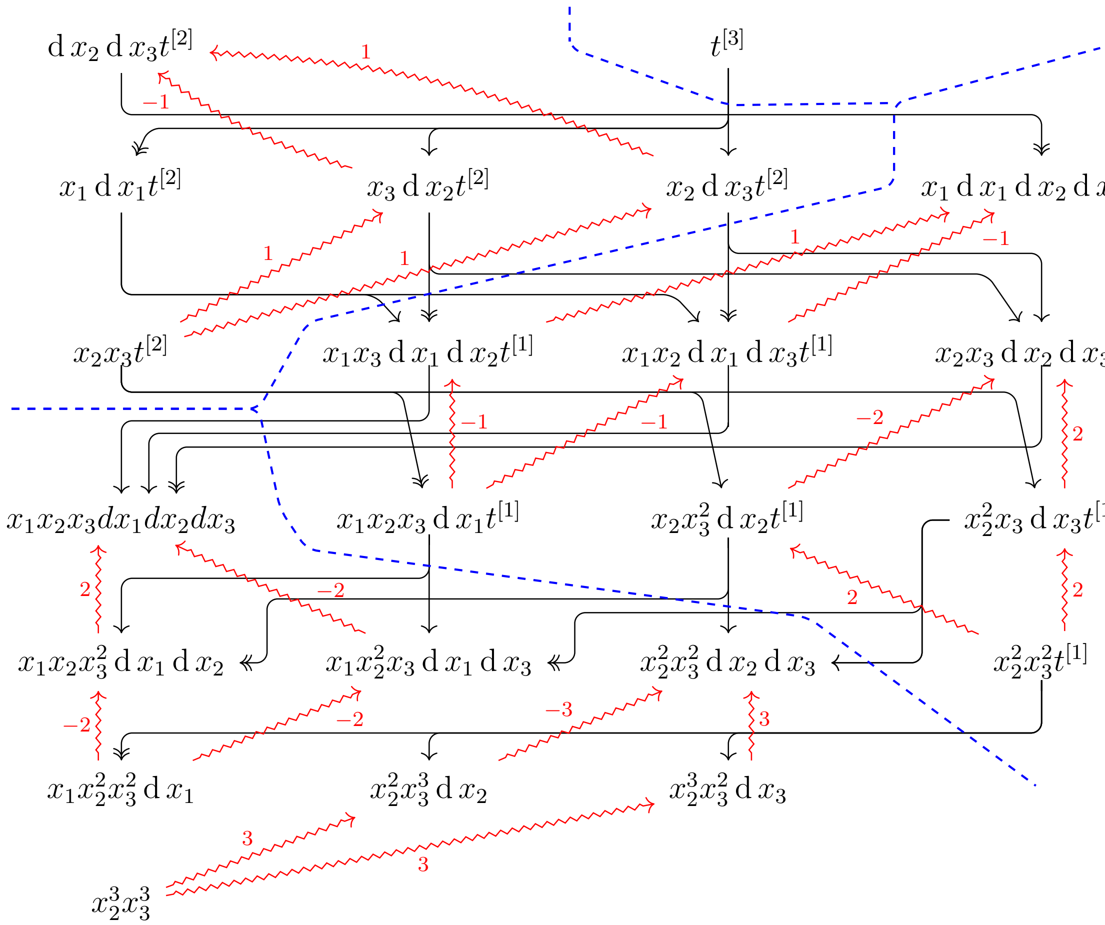
10.5. Diagonal pieces

10.5.1. A first look at $Y((5,5))$ and $Y((6,6))$

Let us now look at what happens when when $j_1 = j_2$. The following is the diagram for $Y((5,5))$. We use the same conventions as we did for $Y((6,4))$ and $Y((7,5))$ above.



As mentioned in Remark 10.3.0.2, $Y((j,j))$ may differ in character depending on the parity of j, so let us also look at $Y((6,6))$.



We can already see the difference between these two cases as well as $Y(\vec{j})$ with $j_1 \neq j_2$ in these two examples. Indeed, note how in the diagrams for both $Y((5,5))$ and $Y((6,6))$ the upper element in the rightmost column represents a nonzero element in the homology of K, showing that $K(\vec{j})$ is in general not acyclic for $j_1 = j_2$, in contrast to the case $j_1 \neq j_2$ (see Proposition 10.4.3.1). In $Y((6,6))$ this element in homology is of order 2, in contrast to $Y((5,5))$, where it is of infinite order.

10.5.2. A new basis

To simplify $Y(\vec{j})$ for $j_1 = j_2$ we make a similar base change as we did for $j_1 \neq j_2$. We again try to eliminate replace basis elements from the monomial basis that are divisible by $\mathrm{d}\,x_3$, as in Proposition 10.4.2.2. This time, we will not be able to write all of the relevant elements as boundaries, however the formulas themselves still make sense.

Notation 10.5.2.1. Let $j \geq 0$ be an integer and $(a, b, c, \epsilon_1, \epsilon_2, 1, m) \in V((j, j))$. We will define an element $b_{(a,b,c,\epsilon_1,\epsilon_2,m)}$ of $Y((j, j))$, by distinguishing three cases. If $b > 0$, then we define $b_{(a,b,c,\epsilon_1,\epsilon_2,m)}$ as follows.

$$\begin{aligned} b_{(a,b,c,\epsilon_1,\epsilon_2,m)} &\coloneqq \partial\Big(p\Big(x_1^a x_2^{b-1} x_3^c\Big)\, \mathrm{d}\, x_1^{\epsilon_1}\, \mathrm{d}\, x_2^{\epsilon_2} t^{[m+1]}\Big) \\ &= (-1)^{\epsilon_1+\epsilon_2} p\Big(x_1^a x_2^b x_3^c\Big)\, \mathrm{d}\, x_1^{\epsilon_1}\, \mathrm{d}\, x_2^{\epsilon_2}\, \mathrm{d}\, x_3 t^{[m]} \\ &\quad + (-1)^{\epsilon_1} p\Big(x_1^a x_2^{b-1} x_3^{c+1}\Big)\, \mathrm{d}\, x_1^{\epsilon_1}\, \mathrm{d}\, x_2^{1+\epsilon_2} t^{[m]} \\ &\quad - 2p\Big(x_1^{a+1} x_2^{b-1} x_3^c\Big)\, \mathrm{d}\, x_1^{1+\epsilon_1}\, \mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \end{aligned}$$

If instead $b = 0$, then note that this implies $\epsilon_2 = 1$ and $c = 0$. We then make the following definitions.

$$\begin{aligned} b_{(0,0,0,\epsilon_1,1,m)} &\coloneqq (-1)^{\epsilon_1+1}\, \mathrm{d}\, x_1^{\epsilon_1}\, \mathrm{d}\, x_2\, \mathrm{d}\, x_3 t^{[m]} \\ b_{(1,0,0,\epsilon_1,1,m)} &\coloneqq (-1)^{\epsilon_1+1} p(x_1)\, \mathrm{d}\, x_1^{\epsilon_1}\, \mathrm{d}\, x_2\, \mathrm{d}\, x_3 t^{[m]} - 2p(x_3)\, \mathrm{d}\, x_1^{1+\epsilon_1}\, \mathrm{d}\, x_2 t^{[m]} \end{aligned} \qquad \diamondsuit$$

Proposition 10.5.2.2. *Let $j \geq 0$. Then the following form a basis for $Y((j,j))$.*

$$\begin{aligned} \mathcal{B}((j,j)) = &\left\{ p\Big(x_1^a x_2^b x_3^c\Big)\, \mathrm{d}\, x_1^{\epsilon_1}\, \mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2((j,j)) \right\} \\ &\cup \left\{ b_{(a,b,c,\epsilon_1,\epsilon_2,m)} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,1,m) \in V((j,j)) \right\} \end{aligned} \qquad \heartsuit$$

Proof. The proof is very similar to Proposition 10.4.2.2. The monomial basis can be written as follows.

$$\begin{aligned} &\left\{ p\Big(x_1^a x_2^b x_3^c\Big)\, \mathrm{d}\, x_1^{\epsilon_1}\, \mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2(\vec{j}) \right\} \\ \cup &\left\{ p\Big(x_1^a x_2^b x_3^c\Big)\, \mathrm{d}\, x_1^{\epsilon_1}\, \mathrm{d}\, x_2^{\epsilon_2}\, \mathrm{d}\, x_3 t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,1,m) \in V((j,j)) \right\} \end{aligned}$$

Again the elements of $\mathcal{B}((j,j))$ of the first type correspond to elements of the monomial basis of the first type, and the element of the second type indexed by $(a,b,c,\epsilon_1,\epsilon_2,1,m)$ is – up to sign – the sum of the corresponding element of the second type indexed by $(a,b,c,\epsilon_1,\epsilon_2,1,m)$ of the monomial basis and a linear combination of elements of the first type. $\square$

We can record the following behavior of the new basis with respect to the boundary operator.

Proposition 10.5.2.3. *Let $j \geq 0$. Then the following holds in $Y((j,j))$ for elements $(a,b,c,\epsilon_1,\epsilon_2,m)$ of $V_2((j,j))$.*

$$\partial\Big(p\Big(x_1^a x_2^b x_3^c\Big)\, \mathrm{d}\, x_1^{\epsilon_1}\, \mathrm{d}\, x_2^{\epsilon_2} t^{[m]}\Big) = \begin{cases} 0 & \text{if } m = 0 \\ b_{(a,b+1,c,\epsilon_1,\epsilon_2,m-1)} & \text{if } m > 0 \end{cases}$$

Furthermore, the following holds for $(a,b,c,\epsilon_1,\epsilon_2,1,m) \in V((j,j))$.

$$\partial\big(b_{(a,b,c,\epsilon_1,\epsilon_2,m)}\big) = \begin{cases} 2 \cdot b_{(1,0,0,1,1,m-1)} & \text{if } (a,b,\epsilon_1) = (0,0,0) \text{ and } m > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \heartsuit$$

Proof. The first formula follows immediately from the definitions in Notation 10.5.2.1. The second formula follows from $\partial^2 = 0$ if $b > 0$ and from Proposition 10.1.0.1 if $m = 0$. So we can assume that $b = 0$ and $m > 0$. We distinguish three cases: first $\epsilon_1 = 1$, then $(a, \epsilon_1) = (0, 0)$, and finally $(a, \epsilon_1) = (1, 0)$. In each case the formula follows by writing out the elements and using Proposition 10.1.0.1[5].

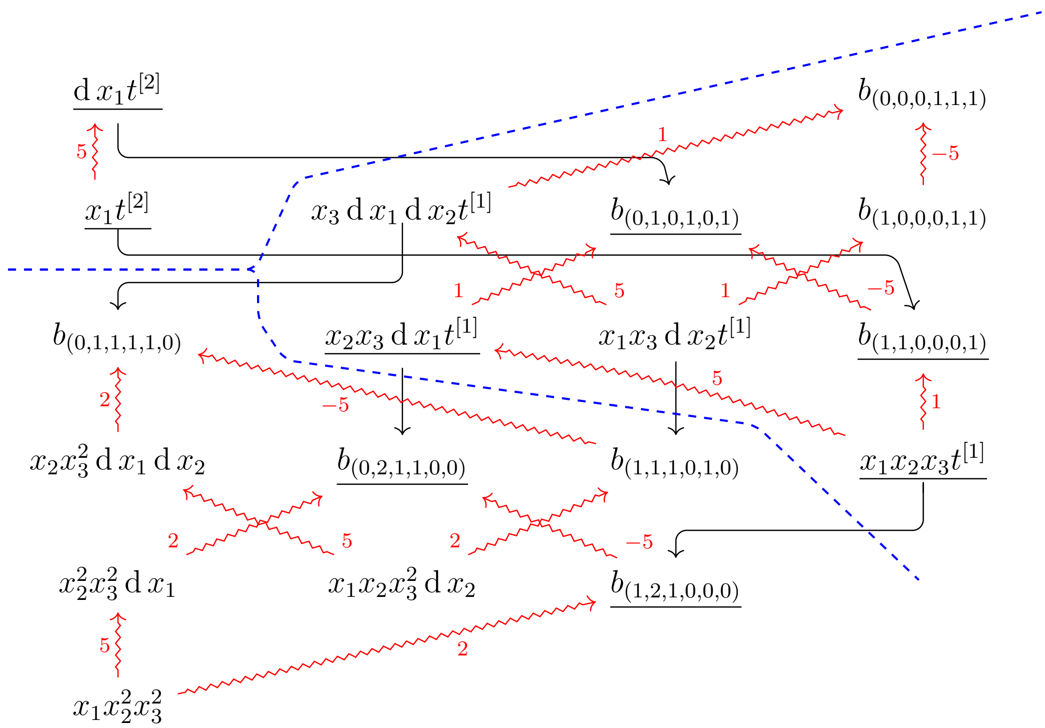
$$\begin{aligned}
\partial\big(b_{(a,0,0,1,1,m)}\big) &= \partial\Big(p(x_1^a)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]}\Big) = 0\\
\partial\big(b_{(0,0,0,0,1,m)}\big) &= \partial\Big(-\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]}\Big)\\
&= 2p(x_1)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m-1]} = 2\cdot b_{(1,0,0,1,1,m-1)}\\
\partial\big(b_{(1,0,0,0,1,m)}\big) &= \partial\Big(-p(x_1)\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]} - 2p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[m]}\Big)\\
&= 2p(x_2x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m-1]} - 2p(x_2x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m-1]}\\
&= 0
\end{aligned}$$

□

Note that we can see a distinction between the cases of $Y((j, j))$ with j odd and even in Proposition 10.5.2.3; the first (non-zero) case in the formula for $\partial\big(b_{(a,b,c,\epsilon_1,\epsilon_2,m)}\big)$ only occurs if j is even.

10.5.3. Another look at $Y((5, 5))$

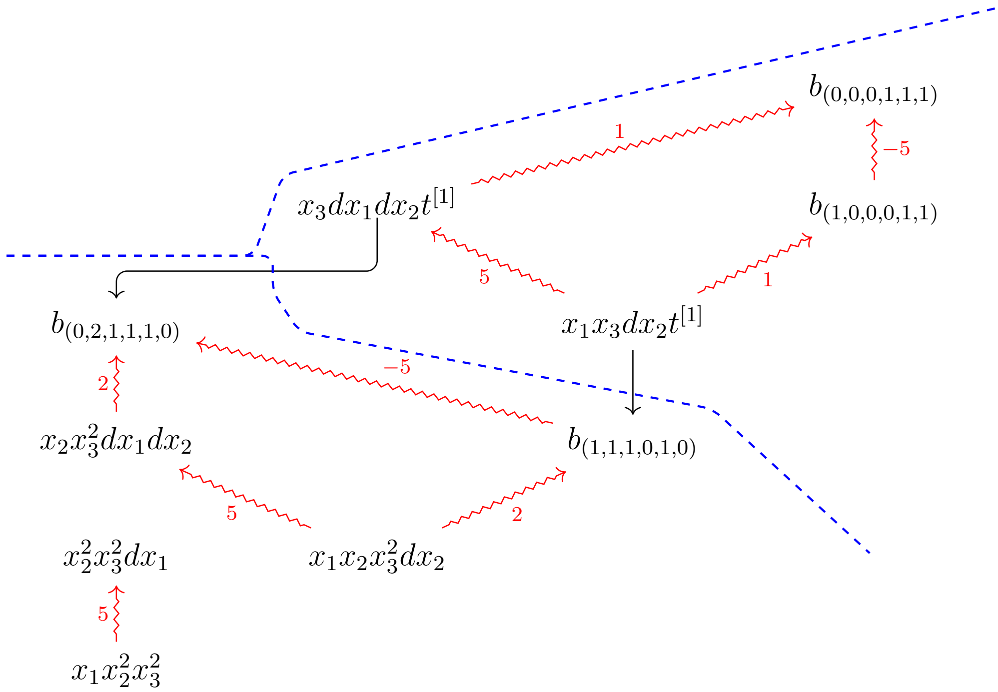
We can now look at $Y((5, 5))$ again, but in this new basis.



(10.1)

[5]Note that $b = 0$ implies $\epsilon_2 = 1$ and $c = 0$.

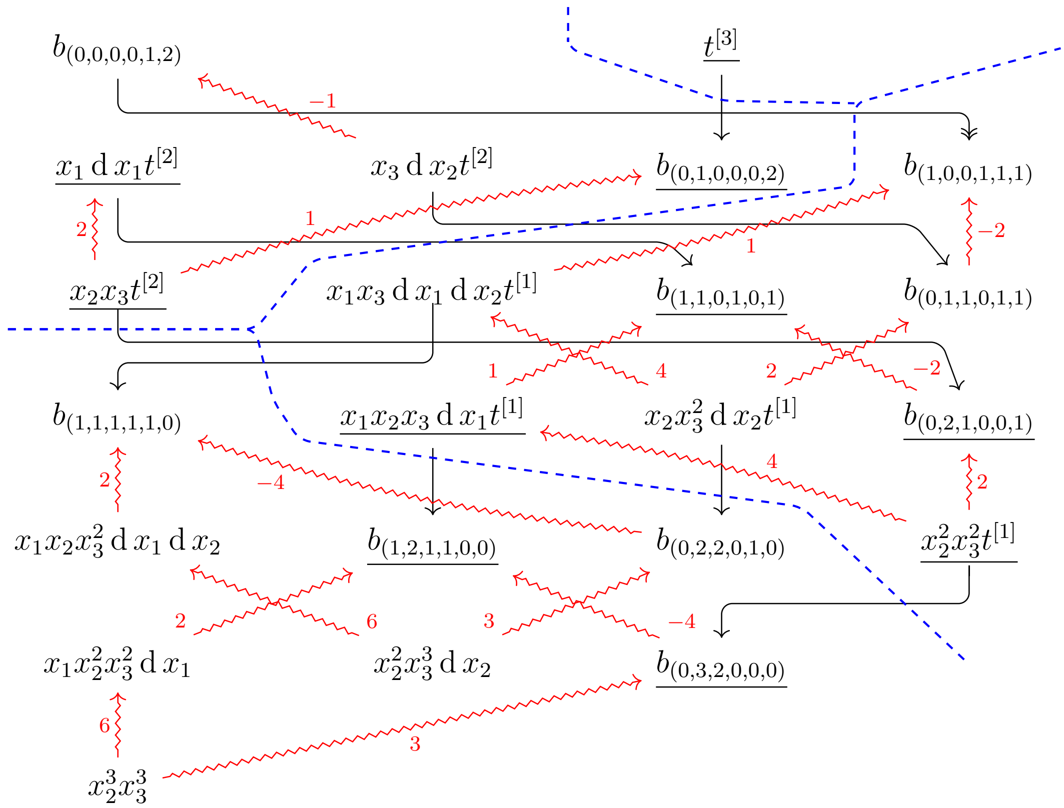
Note that the $\mathbb{Z}$-graded-abelian group generated by the underlined basis elements is closed under both boundary operator and differential. It is also acyclic, so the the quotient map from $Y((5,5))$ obtained by dividing out this sub-mixed-complex is a quasi-isomorphism. The following diagram depicts the resulting strict mixed complex.



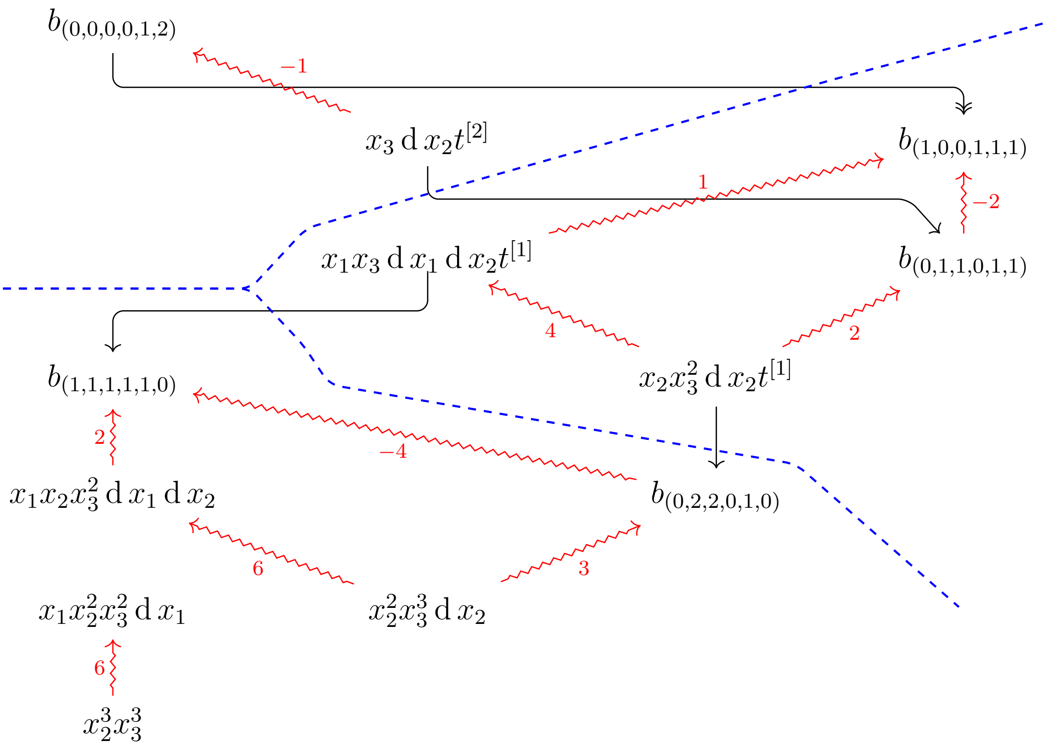
From this we can read off that $K((5,5))$ will not be acyclic, but rather equivalent to the strict mixed subcomplex generated by $b_{(1,0,0,0,1,1)}$ and $b_{(0,0,0,1,1,1)}$, which is isomorphic to $\mathrm{D}_{-5}[4]$, where we use notation from Definition 4.2.1.5.

10.5.4. Another look at $Y((6,6))$

Let us now consider the even case. The following diagram depicts $Y((6,6))$ in the basis from Proposition 10.5.2.2.



We again underlined basis elements that generate an acyclic sub-mixed-complex that we can divide out, obtaining the strict mixed complex depicted in the diagram below.



This time we see that $K((6,6))$ will be equivalent to $\mathbb{Z}/2[5]$, generated by $b_{(1,0,0,1,1,1)}$.

10.5.5. A basis for $K((j,j))$

We will now show how the description above of $K((j,j))$ generalizes to $j \geq 5$ other than 5 and 6, whereas $K((j,j))$ for $j < 5$ is acyclic. We start by describing a basis of $K((j,j))$.

Proposition 10.5.5.1. *Let $j \geq 5$. Then a basis of $K((j,j))$ is given by the following set.*

$$\left\{ p\Big(x_1^a x_2^b x_3^c\Big) \,\mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2((j,j)), m > 0 \right\}$$
$$\cup \left\{ b_{(a,b,c,\epsilon_1,\epsilon_2,m)} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,1,m) \in V((j,j)) \right\}$$

Furthermore, $K((0,0)) \cong 0$, $K((1,1)) \cong 0$, a basis of $K((2,2))$ is given by

$$\left\{ b_{(0,1,0,0,0,0)}, t^{[1]} \right\}$$

a basis of $K((3,3))$ is given by

$$\left\{ b_{(1,1,0,0,0,0)}, b_{(0,1,0,1,0,0)}, x_1 t^{[1]}, \mathrm{d}\, x_1 t^{[1]} \right\}$$

and a basis of $K((4,4))$ *is given by the following set.*

$$\begin{aligned} &\left\{ b_{(0,2,1,0,0,0)}, b_{(1,1,0,1,0,0)}, b_{(0,1,1,0,1,0)}, 2 \cdot b_{(1,0,0,1,1,0)} \right\} \\ \cup &\left\{ p(x_2x_3)t^{[1]}, p(x_1)\,\mathrm{d}\,x_1t^{[1]}, p(x_3)\,\mathrm{d}\,x_2t^{[1]}, b_{(0,1,0,0,0,1)}, b_{(0,0,0,0,1,1)}, t^{[2]} \right\} \end{aligned}$$ ♡

Proof. We first consider the case $j \geq 5$. This assumption implies that if $(a, b, c, \epsilon_1, \epsilon_2, \epsilon_3, 0)$ is an element of $V((j,j))$, then $b > 0$. In other words, every element of grading (j,j) of the monomial basis of $\mathbb{Z}[x_1, x_2, x_3]/f \otimes \Lambda(\mathrm{d}\,x_1, \mathrm{d}\,x_2, \mathrm{d}\,x_3)$ is divisible by x_2. Like in Proposition 10.4.3.1 we can thus conclude that

$$\left\{ p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} \;\middle|\; (a, b, c, \epsilon_1, \epsilon_2, 0) \in V_2((j,j)) \right\}$$

is a basis of $\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}((j,j))$.

By Proposition 10.5.2.2 the set

$$\begin{aligned} &\left\{ p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} \;\middle|\; (a, b, c, \epsilon_1, \epsilon_2, 0) \in V_2((j,j)) \right\} \\ \cup &\left\{ p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a, b, c, \epsilon_1, \epsilon_2, m) \in V_2((j,j)), m > 0 \right\} \\ \cup &\left\{ b_{(a,b,c,\epsilon_1,\epsilon_2,m)} \;\middle|\; (a, b, c, \epsilon_1, \epsilon_2, 1, m) \in V((j,j)) \right\} \end{aligned}$$

is a basis of $Y((j,j))$, and elements of the first type (of this decomposition into three subsets) are mapped by φ to the corresponding element of $\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$. It thus suffices to show that elements of the second and third type are mapped to 0 by φ. If $m > 0$ in either of the two types of elements, then this is clear. So it remains to consider elements of the form $b_{(a,b,c,\epsilon_1,\epsilon_2,0)}$ for $(a, b, c, \epsilon_1, \epsilon_2, 1, 0) \in V((j,j))$. As mentioned at the start, this implies $b > 0$. It thus follows from Proposition 10.5.2.3 that

$$b_{(a,b,c,\epsilon_1,\epsilon_2,0)} = \partial\left(p(x_1^a x_2^{b-1} x_3^c\,\mathrm{d}\,x_1^{\epsilon_1}\,\mathrm{d}\,x_2^{\epsilon_2} t^{[m+1]})\right)$$

from which $\varphi(b_{(a,b,c,\epsilon_1,\epsilon_2,0)}) = 0$ follows from the $m > 0$ case as φ is a morphism of chain complexes.

The cases of $0 \leq j \leq 4$ can be done by inspecting each case individually. The difference to the case $j \geq 5$ is that terms that are divisible by $\mathrm{d}\,x_3$ but not by t need not automatically be divisible by x_2 as well. This means that for example $b_{(1,0,0,1,1,0)}$ is not in the kernel of φ (but $2 \cdot b_{(1,0,0,1,1,0)}$ is), whereas the analogous element of $Y((6,6))$, namely $b_{(1,1,1,1,1,0)}$, *does* lie in the kernel. □

10.5.6. $K((j,j))$ for $j < 5$

We can now already finish the case of $j < 5$.

Proposition 10.5.6.1. *Let* $0 \leq j < 5$. *Then* $K((j,j))$ *is acyclic.* ♡

Proof. This follows immediately from Proposition 10.5.5.1 in combination with Proposition 10.5.2.3. □

10.5.7. Splitting an acyclic summand off of $K((j,j))$ for $j > 5$

We now turn back to $K((j,j))$ for $j \geq 5$. We start by splitting off an acyclic summand.

Proposition 10.5.7.1. *Let $j \geq 5$. Then define $K_{\mathrm{acyc}}((j,j))$ to be the sub-$\mathbb{Z}$-graded-abelian-group of $K((j,j))$ with basis the following set (compare Proposition 10.5.5.1).*

$$\left\{ p\left(x_1^a x_2^b x_3^c\right) \mathrm{d}\, x_1^{\epsilon_1} \,\mathrm{d}\, x_2^{\epsilon_2} t^{[m]} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,m) \in V_2((j,j)), m > 0 \right\}$$
$$\cup \left\{ b_{(a,b,c,\epsilon_1,\epsilon_2,m)} \;\middle|\; (a,b,c,\epsilon_1,\epsilon_2,1,m) \in V((j,j)), b > 0 \right\}$$

Furthermore, define $K'((j,j))$ to be the sub-$\mathbb{Z}$-graded-abelian-group of $K((j,j))$ with basis the following set.

$$\left\{ b_{(a,0,0,\epsilon_1,1,m)} \;\middle|\; (a,0,0,\epsilon_1,1,1,m) \in V((j,j)) \right\}$$

Then the following hold.

(1) $K_{\mathrm{acyc}}((j,j))$ is a subcomplex of $K((j,j))$.

(2) $K_{\mathrm{acyc}}((j,j))$ is acyclic.

(3) $K'((j,j))$ a subcomplex of $K((j,j))$.

(4) $K((j,j))$ is the sum of $K_{\mathrm{acyc}}((j,j))$ and $K'((j,j))$ as chain complexes.

(5) The inclusion of $K'((j,j))$ into $K((j,j))$ is a quasiisomorphism. ♡

Proof. Proof of claims (1), (2) and (3): Follows immediately from Proposition 10.5.2.3.

Proof of claim (4): If $(a,0,c,\epsilon_1,\epsilon_2,1,m)$ is an element of $V((j,j))$, then this implies that $c = 0$ and $\epsilon_2 = 1$. The claim now follows from Proposition 10.5.5.1.

Proof of claim (5): Immediate consequence of the preceding claims. □

10.5.8. Description of the strict mixed structure

We next need to understand how d acts on $K'((j,j))$.

Proposition 10.5.8.1. *Let $j \geq 5$. Then a basis of $K'((j,j))$ is given by the following set.*

$$\begin{aligned} &\left\{ b_{(0,0,0,0,1,\frac{j-2}{2})}, b_{(1,0,0,1,1,\frac{j-4}{2})} \right\} && \text{if } 2 \mid j \\ &\left\{ b_{(0,0,0,1,1,\frac{j-3}{2})}, b_{(1,0,0,0,1,\frac{j-3}{2})} \right\} && \text{if } 2 \nmid j \end{aligned}$$

Furthermore, the following holds for $m \geq 0$.

$$\begin{aligned} \mathrm{d}\left(b_{(0,0,0,0,1,m)}\right) &= 0 \\ \mathrm{d}\left(b_{(1,0,0,1,1,m)}\right) &= 0 \\ \mathrm{d}\left(b_{(0,0,0,1,1,m)}\right) &= 0 \\ \mathrm{d}\left(b_{(1,0,0,0,1,m)}\right) &= -(2m+3) \cdot b_{(0,0,0,1,1,m)} \end{aligned}$$

♡

Proof. The claim about the basis follows directly from Proposition 10.5.7.1, it merely involves spelling out what a, ϵ_1, and m can be such that $(a, 0, 0, \epsilon_1, 1, 1, m) \in V((j, j))$.

For the formulas for d, we use the definition from Notation 10.5.2.1 and then apply Proposition 10.1.0.1.

$$\begin{aligned}
\mathrm{d}\big(b_{(0,0,0,0,1,m)}\big) &= \mathrm{d}\Big(-\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]}\Big) = 0\\
\mathrm{d}\big(b_{(1,0,0,1,1,m)}\big) &= \mathrm{d}\Big(p(x_1)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]}\Big) = 0\\
\mathrm{d}\big(b_{(0,0,0,1,1,m)}\big) &= \mathrm{d}\Big(\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]}\Big) = 0\\
\mathrm{d}\big(b_{(1,0,0,0,1,m)}\big) &= \mathrm{d}\Big(-p(x_1)\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]} - 2\cdot p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[m]}\Big)\\
&= -(1+2m)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]} - 2\cdot \mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3 t^{[m]}\\
&= -(2m+3) b_{(0,0,0,1,1,m)}
\end{aligned}$$

□

10.5.9. A smaller model for $K((j,j))$ for $j > 5$

Proposition 10.5.8.1 implies that $K'((j,j))$ is equivalent as a strict mixed complex to $K((j,j))$ for $j \geq 5$, as we record next.

Proposition 10.5.9.1. *Let $j \geq 5$. Then the strict mixed structure of $K((j,j))$ restricts to $K'((j,j))$ and the inclusion $K'((j,j)) \to K((j,j))$ is a weak equivalence of strict mixed complexes.*

Furthermore, if j is even, then $K'((j,j))$ is isomorphic to the mapping cone of $\mathbb{Z}[j-1] \xrightarrow{2\cdot -} \mathbb{Z}[j-1]$. If j is odd, then $K'((j,j))$ is isomorphic to $\mathrm{D}_j[j-1]$ (see Definition 4.2.1.5 for the notation). ♡

Proof. That the strict mixed structure of $K((j,j))$ restricts to $K'((j,j))$ follows directly from Proposition 10.5.8.1, and that the inclusion is a weak equivalence of strict mixed complexes then follows from Proposition 10.5.7.1 (5).

The identification of $K'((j,j))$ up to isomorphism follows from Proposition 10.5.2.3 and Proposition 10.5.8.1. For the isomorphism to $\mathrm{D}_j[j-1]$, note that $\mathrm{D}_j \cong \mathrm{D}_{-j}$. □

10.6. $\mathrm{HH}_{\mathrm{Mixed}}\big(\mathbb{Z}[x_1, x_2, x_3]/(x_1^2 - x_2x_3)\big)$ as a non-split extension

We can now sum up all the results by coming back to Hochschild homology.

Proposition 10.6.0.1. *Assume that Conjecture D holds for the polynomial $f = x_1^2 - x_2x_3$ in $\mathbb{Z}[x_1, x_2, x_3]$. Then there is a cofiber sequence*

$$\left(\bigoplus_{j\geq 5, 2\mid j} \mathbb{Z}/2[j-1]\right) \oplus \left(\bigoplus_{j\geq 5, 2\nmid j} \gamma_{\mathsf{Mixed}}\big(\mathrm{D}_j[j-1]\big)\right)$$

$$\to \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2-x_2x_3)) \to \gamma_{\mathsf{Mixed}}\left(\left(\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right)$$

in $\mathcal{M}$ixed *that does not split.* ♡

Proof. By definition of K we have a pullback square

$$\begin{tikzcd} K \arrow[r, "\psi"] \arrow[d] & Y \arrow[d, "\varphi"] \\ 0 \arrow[r] & \Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}} \end{tikzcd}$$

in Mixed. It is clear from Definition 10.2.0.2 and Remark 10.2.0.1 that φ is levelwise surjective and hence a fibration in Mixed. As every object in Mixed is fibrant, it follows from [HTT, A.2.4.4] that the above square is also a homotopy pullback square.

We can apply $\gamma_{\mathsf{Mixed}}(-^{\mathrm{cof}})$ (where $-^{\mathrm{cof}}$ is the cofibrant replacement functor for Mixed) to this diagram to obtain a commutative square in $\mathcal{M}$ixed that is a pullback square by [HA, 1.3.4.23][6] By Proposition 4.4.3.1 $\mathcal{M}$ixed is stable, so said square is also a pushout square, so we have shown existence of a cofiber sequence as follows.

$$\gamma_{\mathsf{Mixed}}\left(K^{\mathrm{cof}}\right) \to \gamma_{\mathsf{Mixed}}\left(Y^{\mathrm{cof}}\right) \to \gamma_{\mathsf{Mixed}}\left(\left(\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right)$$

We can identify $\gamma_{\mathsf{Mixed}}(Y^{\mathrm{cof}})$ with $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}(\mathbb{Z}[x_1,x_2,x_3]/f)$ along the equivalence from Proposition 10.1.0.1, and for $\gamma_{\mathsf{Mixed}}(K^{\mathrm{cof}})$ we obtain a sequence of equivalences

$$\begin{aligned} &\gamma_{\mathsf{Mixed}}\left(K^{\mathrm{cof}}\right) \\ \simeq\ &\gamma_{\mathsf{Mixed}}\left(\bigoplus_{\vec{j}\in\mathbb{Z}^2_{\geq 0}} K(\vec{j})^{\mathrm{cof}}\right) \\ \simeq\ &\bigoplus_{\vec{j}\in\mathbb{Z}^2_{\geq 0}} \gamma_{\mathsf{Mixed}}\left(K(\vec{j})^{\mathrm{cof}}\right) \\ \simeq\ &\left(\bigoplus_{j\geq 5, 2\mid j} \mathrm{cofib}\left(\mathbb{Z}[j-1] \xrightarrow{2\cdot -} \mathbb{Z}[j-1]\right)\right) \oplus \left(\bigoplus_{j\geq 5, 2\nmid j} \gamma_{\mathsf{Mixed}}(\mathrm{D}_j[j-1])\right) \end{aligned}$$

where in the first equivalence we used the decomposition from Construction 10.3.0.1 and that coproducts of quasiisomorphisms are quasiisomorphisms, in the second we used that coproducts of cofibrant objects are homotopy coproducts and [HA, 1.3.4.24], and in the third we used Propositions 10.4.3.1, 10.5.6.1 and 10.5.9.1[7]. This shows existence of a cofiber sequence as claimed.

[6]See Propositions 4.4.1.7 and 4.4.2.2 for $\mathcal{M}$ixed being the underlying ∞-category of the model category Mixed.

[7]Note that D_n has cofibrant underlying chain complex.

It remains to show that this cofiber sequence does not split. So suppose that there is a morphism

$$\gamma_{\mathsf{Mixed}}\left(\left(\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right) \to \mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)\right)$$

in $\mathcal{M}$ixed such that postcomposition with the morphism induced by φ is homotopic to the identity on $\gamma_{\mathsf{Mixed}}((\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}})^{\mathrm{cof}})$. By Propositions 4.4.1.7 and 4.4.2.2 and [Hov99, 1.2.10 (ii)] we can then lift this section to a triangle

$$\begin{tikzcd}
 & \left(\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}} \arrow[dl, "s"'] \arrow[d, "i"] \\
Y \arrow[r, "\varphi"'] & \Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}
\end{tikzcd}$$

in Mixed that commutes up to homotopy, with i a quasiisomorphism. We will denote $(\Omega^{\bullet}_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}})^{\mathrm{cof}}$ by C below. The following argument will use $Y((5,5))$, and it will likely be helpful to follow along with diagram (10.1). We will in particular read off ∂ and d from that diagram; to verify those formulas one uses the formulas in Proposition 10.1.0.1 and the definition of the basis elements in Notation 10.5.2.1. The diagram below provides an overview over the argument; The left column depicts elements of C and the right column of $Y((5,5))$ In both columns we depict where the elements are mapped by ∂ and d using the conventions of Convention 4.2.1.7, and the horizontal arrows correspond to application of s followed by the projection $Y \to Y((5,5))$ associated with the decomposition from Construction 10.3.0.1.

$$\begin{tikzcd}
\delta \arrow[r, mapsto] \arrow[d] & ? \arrow[d, dashed] \\
\mathrm{d}\,\beta \arrow[r, mapsto] & (2-5d)\cdot b_{(0,0,0,1,1,1)} \\
\beta \arrow[r, mapsto] \arrow[u, squiggly] \arrow[d] & 2\cdot p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[1]} + c\cdot b_{(0,1,0,1,0,1)} + d\cdot b_{(1,0,0,0,1,1)} \arrow[u, squiggly] \arrow[d] \\
\mathrm{d}\,\alpha \arrow[r, mapsto] & 2\cdot b_{(0,1,1,1,1,0)} \\
\alpha \arrow[r, mapsto] \arrow[u, squiggly] & p(x_2x_3^2)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 \arrow[u, squiggly]
\end{tikzcd}$$

As the homology of the fiber of φ is concentrated in degrees above 3 by the already obtained cofiber sequence, $\mathrm{H}_2(\varphi)$ is an isomorphism. From diagram (10.1) we can read off that $p(x_2x_3^2)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2$ is a cycle in Y_2 that represents a nontrivial homology class. There must thus be a cycle $\alpha \in C_2$ such that $s(\alpha) = p(x_2x_3^2)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 + \partial y$, with $y \in Y_3$.

As α is a cycle, we have
$$\partial(\mathrm{d}\,\alpha) = -\,\mathrm{d}(\partial\alpha) = 0$$
so $\mathrm{d}\,\alpha$ is a cycle. We furthermore obtain
$$\begin{aligned} s(\mathrm{d}\,\alpha) &= \mathrm{d}\big(s(\alpha)\big) = \mathrm{d}\big(p(x_2x_3^2)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 + \partial y\big)\\ &= 2\cdot b_{(0,1,1,1,1,0)} + \mathrm{d}\,\partial y = \partial\Big(p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[1]} - \mathrm{d}\,y\Big)\end{aligned}$$
so that $s(\mathrm{d}\,\alpha)$ is a boundary. As $\mathrm{H}_3(s)$ has to be injective, this implies that $\mathrm{d}\,\alpha$ must be a boundary. So let $\beta\in C_4$ be such that $\partial\beta = \mathrm{d}\,\alpha$.

Using the description of a basis for $Y_4((5,5))$ from Proposition 10.5.2.2 we can write $s(\beta)$ as
$$s(\beta) = a\cdot p(x_1)t^{[2]} + b\cdot p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[1]} + c\cdot b_{(0,1,0,1,0,1)} + d\cdot b_{(1,0,0,0,1,1)} + z$$
with $a,b,c,d\in\mathbb{Z}$, and $z\in\bigoplus_{\vec{j}\in\mathbb{Z}^2_{\geq 0},\vec{j}\neq(5,5)} Y_4(\vec{j})$. It follows that
$$\begin{aligned} 2\cdot b_{(0,1,1,1,1,0)} + \mathrm{d}\,\partial y &= s(\mathrm{d}\,\alpha) = s(\partial\beta)\\ &= \partial\big(s(\beta)\big)\\ &= a\cdot b_{(1,1,0,0,0,1)} + b\cdot b_{(0,1,1,1,1,0)} + c\cdot 0 + d\cdot 0 + \partial z\end{aligned}$$
where both $\mathrm{d}\,\partial y$ and ∂z are elements of $\bigoplus_{\vec{j}\in\mathbb{Z}^2_{\geq 0},\vec{j}\neq(5,5)} Y_3(\vec{j})$, so we can conclude that $a=0$ and $b=2$.

We have
$$\partial(\mathrm{d}\,\beta) = -\,\mathrm{d}(\partial\beta) = -\,\mathrm{d}(\mathrm{d}\,\alpha) = 0$$
so $\mathrm{d}\,\beta$ is a cycle. As $\mathrm{H}_5(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}})\cong 0$, it thus follows that $\mathrm{d}\,\beta$ must be of the form $\partial\delta$ for some element $\delta\in C_6$, and hence $\mathrm{d}\,s(\beta) = s(\mathrm{d}\,\beta)$ must be a cycle in Y_5 that is also a boundary. But we can calculate
$$\begin{aligned} \mathrm{d}\,s(\beta) &= \mathrm{d}\Big(2\cdot p(x_3)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2 t^{[1]} + c\cdot b_{(0,1,0,1,0,1)} + d\cdot b_{(1,0,0,0,1,1)} + z\Big)\\ &= 2\cdot b_{(0,0,0,1,1,1)} + 0 - 5d\cdot b_{(0,0,0,1,1,1)} + \mathrm{d}\,z\\ &= (2-5d)\cdot b_{(0,0,0,1,1,1)} + \mathrm{d}\,z\end{aligned}$$
which, as z lies in $\bigoplus_{\vec{j}\in\mathbb{Z}^2_{\geq 0},\vec{j}\neq(5,5)} Y_4(\vec{j})$ and $(2-5d)\cdot b_{(0,0,0,1,1,1)}$ is a cycle representing a nontrivial homology class, is in contradiction to $\mathrm{d}\,s(\beta)$ being a boundary. □

10.7. Non-formality of $\mathrm{HH}_{\mathrm{Mixed}}\big(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2-x_2x_3)\big)$

Let M be a strict mixed complex. Then the relation $\mathrm{d}\circ\partial + \partial\circ\mathrm{d} = 0$ ensures that $\mathrm{d}\colon M_* \to M_{*+1}$ maps cycles to cycles, and thus induces an operator increasing degree by 1 on homology. Equipping $\mathrm{H}_\bullet(X)$ with the zero boundary operator we can then consider $\mathrm{H}_\bullet(M)$ again as an object of Mixed.

Now let M be a mixed complex, i.e. an object in the ∞-category $\mathcal{M}\mathrm{ixed}$. Then we can make a similar construction using the functors $\mathrm{H}_*\colon \mathcal{D}(k) \to \mathrm{LMod}_k(\mathsf{Ab})$ defined in Definition 4.3.3.1. From the element d in $\mathrm{H}_1(\mathrm{D})$ we obtain with Proposition 4.3.2.1 (5) a morphism $k[1] \to \mathrm{D}$ in $\mathcal{D}(k)$ which induces a morphism

$$M[1] \simeq k[1] \otimes M \to \mathrm{D} \otimes M \to M$$

in $\mathcal{D}(k)$, where the second morphism is the action of D on M. This morphism induces an operator increasing degree by 1 in H_*, and $\mathrm{d}^2 = 0$ in $\mathrm{H}_*(\mathrm{D})$ implies that this operator squares to 0. Equipping $\mathrm{H}_\bullet(M)$ with this operator as d and the zero boundary operator we again obtain a strict mixed complex. Proposition 4.3.3.2 ensures that the just discussed two constructions agree, i.e. if M is a strict mixed complex with cofibrant underlying chain complex, then the strict mixed complexes $\mathrm{H}_\bullet(\gamma_{\mathsf{Mixed}}(M))$ and $\mathrm{H}_\bullet(M)$ are naturally isomorphic.

Given an object M of $\mathcal{M}\mathrm{ixed}$, we can now ask whether M is *formal*, i.e. whether there is an equivalence

$$M \simeq \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(M)^{\mathrm{cof}}\right)$$

in $\mathcal{M}\mathrm{ixed}$. In the next proposition we show that, still assuming that Conjecture D holds for the polynomial $x_1^2 - x_2x_3$ in $\mathbb{Z}[x_1, x_2, x_3]$, that $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)\right)$ is *not* formal. Note that

$$\mathrm{H}_\bullet\left(\gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(M)^{\mathrm{cof}}\right)\right) \cong \mathrm{H}_\bullet(M)$$

for every mixed complex M. This implies (under the assumption of Conjecture D) that there are at least two mixed complexes whose homology, as a strict mixed complex, is isomorphic to

$$\mathrm{H}_\bullet\left(\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)\right)\right)$$

so the mixed complex $\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)\right)$ can not be recovered from its homology (even including the action of d) alone.

Proposition 10.7.0.1. *Assume that Conjecture D holds for the polynomial $f = x_1^2 - x_2x_3$ in $\mathbb{Z}[x_1,x_2,x_3]$. Then there is no equivalence*

$$\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)\right)$$

and

$$\gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet\left(\mathrm{HH}_{\mathcal{M}\mathrm{ixed}}\left(\mathbb{Z}[x_1,x_2,x_3]/(x_1^2 - x_2x_3)\right)\right)^{\mathrm{cof}}\right)$$

in $\mathcal{M}\mathrm{ixed}$. ♡

Proof. We will make use of the cofiber sequence constructed in Proposition 10.6.0.1, for which we will use the following notation.

$$F \to Z \xrightarrow{\Phi} R$$

We show the claim by contradiction and assume that there is an equivalence

$$\Theta\colon \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(Z)^{\mathrm{cof}}\right) \xrightarrow{\simeq} Z$$

in $\mathcal{M}\mathrm{ixed}$.

Note that F has homology concentrated in degrees ≥ 4, so Φ induces an isomorphism in homology on degrees ≤ 3. As R has homology concentrated in degrees ≤ 3, it follows that there is a unique morphism of underlying chain complexes $s\colon \mathrm{H}_\bullet(R) \to \mathrm{H}_\bullet(Z)$ such that $\mathrm{H}_\bullet(\Phi)\circ s$ is the identity.

We claim that s is in fact also compatible with d and thus a morphism in Mixed. As $\mathrm{H}_\bullet(\Phi)$ is an isomorphism in degrees ≤ 3, it automatically follows that $\mathrm{d}\circ s = s\circ \mathrm{d}$ on elements of degree ≤ 2. What remains to show is that d applied to every element of $\mathrm{H}_3(Z)$ is zero. From Proposition 10.6.0.1 and the previous discussion in this chapter we know that the elements of

$$\mathrm{H}_4(Z)\cong \mathrm{H}_4(Y)$$

are precisely represented by the integer multiples of the element $b_{(1,0,0,0,1,1)}$ of $Y((5,5))$ (see in particular Propositions 10.5.8.1 and 10.5.9.1). From the sum decomposition of Y it follows that it suffices to show that there is no cycle in $Y((5,5))$ that is mapped by d to a linear combination of basis elements of $Y((5,5))$ with respect to the basis from Proposition 10.5.2.2, in which $b_{(1,0,0,0,1,1)}$ has nonzero coefficient. But this follows from Proposition 10.5.2.3 and can be read off of the first diagram in Section 10.5.3.

Note that

$$R = \gamma_{\mathsf{Mixed}}\left(\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right)$$

and that are isomorphisms as follows in Mixed;

$$\mathrm{H}_\bullet\left(\gamma_{\mathsf{Mixed}}\left(\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right)\right) \cong \mathrm{H}_\bullet\left(\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right)$$

as discussed before this proposition,

$$\mathrm{H}_\bullet\left(\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)^{\mathrm{cof}}\right) \cong \mathrm{H}_\bullet\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right)$$

induced by the cofibrant replacement, and

$$\mathrm{H}_\bullet\left(\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}\right) \cong \Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$$

as $\Omega^\bullet_{\mathbb{Z}[x_1,x_2,x_3]/f\,/\,\mathbb{Z}}$ has zero boundary operator.

Combining these isomorphisms and applying $\gamma_{\mathsf{Mixed}}(-^{\mathrm{cof}})$ we obtain an equivalence

$$\alpha\colon R \xrightarrow{\simeq} \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(R)^{\mathrm{cof}}\right)$$

in $\mathcal{M}\mathrm{ixed}$.

We can now consider the composition

$$\lambda\colon R \xrightarrow{\alpha} \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(R)^{\mathrm{cof}}\right) \xrightarrow{\gamma_{\mathsf{Mixed}}(s^{\mathrm{cof}})} \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(Z)^{\mathrm{cof}}\right) \xrightarrow{\Theta} Z \xrightarrow{\Phi} R$$

in $\mathcal{M}\mathrm{ixed}$. As α and Θ are equivalences, they induce isomorphisms on homology. The morphism s induces an isomorphism in homology in degrees ≤ 3, so $\gamma_{\mathsf{Mixed}}(s^{\mathrm{cof}})$ does so too, and we already used above that Φ induces an isomorphism in homology in degrees ≤ 3. It follows that λ induces an isomorphism in degrees ≤ 3. As R has homology concentrated in those degrees, it follows that λ actually induces an isomorphism on homology in all degrees and is thus an equivalence.

Now define ϱ to be the composition

$$R \xrightarrow{\lambda^{-1}} R \xrightarrow{\alpha} \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(R)^{\mathrm{cof}}\right) \xrightarrow{\gamma_{\mathsf{Mixed}}(s^{\mathrm{cof}})} \gamma_{\mathsf{Mixed}}\left(\mathrm{H}_\bullet(Z)^{\mathrm{cof}}\right) \xrightarrow{\Theta} Z$$

in $\mathcal{M}\mathrm{ixed}$. Then it follows that

$$\Phi \circ \varrho \simeq \lambda \circ \lambda^{-1} \simeq \mathrm{id}_Z$$

so ϱ is a section of Φ. This contradicts the fact that the cofiber sequence from Proposition 10.6.0.1 does not split. $\square$

Appendix A.

∞-category theory

This is the first of two appendices in which we collect a number of small results on various basic staples of ∞-category theory, the second one being Appendix D[1].

In Section A.1 we will see that the homotopy category of the underlying ∞-category of a model category is canonically equivalent to the homotopy category of the model category. We will then discuss mapping spaces in ∞-categories in Section A.2, and collect some results relating to the $(\infty, 2)$-category of ∞-categories $\mathcal{C}\mathrm{at}_\infty$ in Section A.3.

A.1. Homotopy categories of model categories

Given a model category C with a class of weak equivalences W, we can form its homotopy category $\mathrm{Ho}_W(\mathsf{C})$ of C, as discussed for example in [Hov99, Section 1.2]. There is also another way to produce a 1-category called "homotopy category" from C: We can first pass to the underlying ∞-category $\mathsf{C}[W^{-1}]$ of C (see [HA, 1.3.4.1]), and then take the homotopy category $\mathrm{Ho}(\mathsf{C}[W^{-1}])$ of this ∞-category as explained in [HTT, 1.2.3]. The following proposition shows that these two notions of "homotopy category" are consistent with each other, i. e. they are canonically equivalent.

Proposition A.1.0.1. *Let C be a model category with class of weak equivalences W. Then there exists an equivalence $\mathrm{Ho}_W \mathsf{C} \simeq \mathrm{Ho}\big(\mathsf{C}[W^{-1}]\big)$ fitting into a commutative diagram as follows*

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \mathsf{C}[W^{-1}] \arrow[d, "\beta"] \\
\mathrm{Ho}_W \mathsf{C} \arrow[r, dash, "\simeq"'] & \mathrm{Ho}\big(\mathsf{C}[W^{-1}]\big)
\end{tikzcd}$$

where $\mathrm{Ho}_W \mathsf{C}$ is the homotopy category of the model category C (see [Hov99, 1.2]), $\mathrm{Ho}\big(\mathsf{C}[W^{-1}]\big)$ is the homotopy category of the ∞-category $\mathsf{C}[W^{-1}]$ (see [HTT, 1.2.3]), and the functors α, β, and γ are the canonical ones. ♡

Proof. The functor α sends morphisms in W to equivalences[2], and β sends all equivalences to isomorphisms as $\mathrm{Ho}\big(\mathsf{C}[W^{-1}]\big)$ is a 1-category. The universal property of $\mathrm{Ho}_W \mathsf{C}$

[1] Some parts of Appendix D depend on Appendices B and C.

[2] See [HA, 1.3.4.1] for a definition of $\mathsf{C}[W^{-1}]$.

(see [Hov99, 1.2.2]) furnishes us with a functor Φ making the following diagram commute.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \mathsf{C}[W^{-1}] \arrow[d, "\beta"] \\
\mathrm{Ho}_W \mathsf{C} \arrow[r, "\Phi"] & \mathrm{Ho}\big(\mathsf{C}[W^{-1}]\big)
\end{tikzcd}$$

As isomorphisms are in particular equivalences, the universal property of $\mathsf{C}[W^{-1}]$ (see [HA, 1.3.4.1]) provides us with a functor $\psi\colon \mathsf{C}[W^{-1}] \to \mathrm{Ho}_W \mathsf{C}$ satisfying $\psi \circ \alpha \simeq \gamma$. Applying Ho we obtain a commuting diagram as follows.

$$\begin{tikzcd}
 & \mathsf{C} \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\
\mathsf{C}[W^{-1}] \arrow[rr, "\psi"] \arrow[d, "\beta"'] & & \mathrm{Ho}_W \mathsf{C} \arrow[d, "\cong"] \\
\mathrm{Ho}\,\mathsf{C}[W^{-1}] \arrow[rr] \arrow[urr, dashed, "\Psi"] & & \mathrm{Ho}(\mathrm{Ho}_W \mathsf{C})
\end{tikzcd}$$

As $\mathrm{Ho}_W \mathsf{C}$ already is a 1-category, we can identify $\mathrm{Ho}(\mathrm{Ho}_W \mathsf{C})$ with $\mathrm{Ho}_W \mathsf{C}$. Call the resulting functor $\Psi\colon\ \mathrm{Ho}\,\mathsf{C}[W^{-1}] \to \mathrm{Ho}_W \mathsf{C}$.

Using the uniqueness part of the universal properties of α, β, and γ one concludes that the compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are naturally isomorphic to the respective identity functors. □

A.2. Mapping spaces

In this section we discuss two results relating to mapping spaces of ∞-categories. In Proposition A.2.0.1 we show that mapping spaces can be calculated as certain pullbacks in $\mathfrak{Cat}_\infty$. We will then apply this result in Proposition A.2.0.2 to show that a pullback diagram in $\mathfrak{Cat}_\infty$ induces pullback diagrams of the respective mapping spaces.

Proposition A.2.0.1. *Let $\mathcal{C}$ be an ∞-category and X and Y two objects of $\mathcal{C}$. Then there is a natural (in $\mathcal{C}$) pullback square in $\mathfrak{Cat}_\infty$*

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d] & \mathcal{C}^{[1]} \arrow[d] \\
\{(X,Y)\} \arrow[r] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

where the right vertical functor sends a morphism $f\colon A \to B$ to (A, B). ♡

Proof. We give a proof for this claim in the setting of quasicategories. The discussion in [HTT, Discussion after 1.2.2.5 and 4.2.1.8] exhibits the mapping space as a pullback of quasicategories, so we need to argue why this is a homotopy pullback in the Joyal model structure, and then identify the resulting (iterated) homotopy pullback with the pullback square we claimed. So let $\mathtt{C}$ be a quasicategory modeling the ∞-category $\mathcal{C}$. In [HTT, 4.2.1.8], a model for $\operatorname{Map}_{\mathcal{C}}(X,Y)$ is identified with the pullback in simplicial sets of the following diagram.

$$\mathtt{C}^{\{X\}/} \to \mathtt{C} \leftarrow \{Y\}$$

Applying [HTT, 4.2.1.6][3] to $X = \mathtt{C}$, $S = \{Y\}$, $K = \{X\}$, and $K_0 = \emptyset$, we obtain that

$$\mathtt{C}^{\{X\}/} \to \mathtt{C}^{\emptyset/} \times_{\{Y\}^{\emptyset/}} \{Y\}^{\{X\}/} \cong \mathtt{C} \times_{\{Y\}} \{Y\} \cong \mathtt{C}$$

is a left fibration. By [HTT, 2.4.2.4 and 3.3.1.4] this implies that the pullback of

$$\mathtt{C}^{\{X\}/} \to \mathtt{C} \leftarrow \{Y\}$$

is already a homotopy pullback in the Joyal model structure.

Unpacking the definition of $\mathtt{C}^{\{X\}/}$ (see [HTT, after 4.2.1.4]) one can write $\mathtt{C}^{\{X\}/}$ as the pullback in simplicial sets of the following diagram.

$$\{X\} \to \mathtt{C}^{\{0\}} \leftarrow \mathtt{C}^{\Delta^1}$$

It follows from [HTT, 2.4.7.12] (applied to $\mathrm{id}_{\mathtt{C}}$) that $\mathtt{C}^{\Delta^1} \to \mathtt{C}^{\{0\}}$ is a cartesian fibration, so again by [HTT, 3.3.1.4] the pullback in simplicial sets is already a homotopy pullback in the Joyal model structure. Together this implies that the ∞-groupoid $\operatorname{Map}_{\mathcal{C}}(X,Y)$ is naturally equivalent to the iterated pullback

$$\left(\{X\} \times_{\mathcal{C}^{\{0\}}} \mathcal{C}^{[1]}\right) \times_{\mathcal{C}^{\{1\}}} \{Y\}$$

in $\mathcal{C}\mathrm{at}_\infty$. Using [HTT, 4.4.2.2] one can rewrite this iterated pullback into the form that was stated in the claim. □

Proposition A.2.0.2. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[d, "H"] \\ \mathcal{E} \arrow[r] & \mathcal{F} \end{tikzcd} \tag{$*$}$$

be a pullback square in $\mathcal{C}\mathrm{at}_\infty$, and X, Y two objects in $\mathcal{C}$. Then the commutative diagram in $\mathcal{S}$ induced by $(*)$ on mapping spaces

$$\begin{tikzcd} \operatorname{Map}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d] & \operatorname{Map}_{\mathcal{D}}\big(F(X),F(Y)\big) \arrow[d] \\ \operatorname{Map}_{\mathcal{E}}\big(G(X),G(Y)\big) \arrow[r] & \operatorname{Map}_{\mathcal{F}}\big(H(F(X)),H(F(Y))\big) \end{tikzcd}$$

is a pullback diagram. ♡

[3] $\mathtt{C} \to \{Y\}$ is a categorical fibration by [HTT, 2.4.6.1].

Proof. As $\mathcal{C}$ is given as a pullback in $\mathcal{C}\mathrm{at}_\infty$ and products as well as $\mathrm{Fun}([1], -)$ preserve limits, we can write $\mathrm{Map}_{\mathcal{C}}(X, Y)$ as a pullback of pullbacks by applying Proposition A.2.0.1: The ∞-groupoid $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is the pullback of the following diagram.

$$\begin{tikzcd}
\Big\{\big(G(X), G(Y)\big)\Big\} \times_{\{(H(F(X)), H(F(Y)))\}} \Big\{\big(F(X), F(Y)\big)\Big\} \arrow[d] \\
(\mathcal{E} \times \mathcal{E}) \times_{(\mathcal{F} \times \mathcal{F})} (\mathcal{D} \times \mathcal{D}) \\
\mathcal{E}^{[1]} \times_{\mathcal{F}^{[1]}} \mathcal{D}^{[1]} \arrow[u]
\end{tikzcd}$$

Commuting the two limits [HTT, 5.5.2.3] and applying Proposition A.2.0.1 again we can conclude that the commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}\big(F(X), F(Y)\big) \arrow[d] \\
\mathrm{Map}_{\mathcal{E}}\big(G(X), G(Y)\big) \arrow[r] & \mathrm{Map}_{\mathcal{F}}\big(H(F(X)), H(F(Y))\big)
\end{tikzcd}$$

induced by $(*)$ is a pullback diagram in $\mathcal{C}\mathrm{at}_\infty$, and hence a pullback diagram in $\mathcal{S}$ by [HTT, 1.2.13.7]. $\square$

A.3. The $(\infty, 2)$-category of ∞-categories

In this section we discuss some results concerning the $(\infty, 2)$-category of ∞-categories. We will characterize pullbacks in the underlying ∞-category $\mathcal{C}\mathrm{at}_\infty$ in Section A.3.1, and show that checking that a natural transformation is an equivalence can be done pointwise in Section A.3.2.

A.3.1. Pullbacks in $\mathcal{C}\mathrm{at}_\infty$

Proposition A.3.1.1. *Let*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[d, "H"] \\
\mathcal{E} \arrow[r] & \mathcal{F}
\end{tikzcd} \tag{A.1}$$

be a commutative diagram in $\mathcal{C}\mathrm{at}_\infty$. Then diagram (A.1) is a pullback diagram if and only if the induced diagram on ∞-groupoid cores

$$\begin{tikzcd}
\mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{D}^{\simeq} \arrow[d] \\
\mathcal{E}^{\simeq} \arrow[r] & \mathcal{F}^{\simeq}
\end{tikzcd} \tag{A.2}$$

as well as the induced diagram on mapping spaces

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}\big(F(X),F(Y)\big) \arrow[d] \\
\mathrm{Map}_{\mathcal{E}}\big(G(X),G(Y)\big) \arrow[r] & \mathrm{Map}_{\mathcal{F}}\Big(H\big(F(X)\big),H\big(F(Y)\big)\Big)
\end{tikzcd} \tag{A.3}$$

for every pair of objects X and Y in $\mathcal{C}$ are pullback diagrams in $\mathcal{S}$. ♡

Proof. The functor $(-)^{\simeq}\colon \mathcal{C}\mathrm{at}_{\infty} \to \mathcal{S}$ is right adjoint to the inclusion (see [HTT, 1.2.5]) and thus preserves pullbacks, which together with Proposition A.2.0.2 shows the "only if"-direction.

For the "if"-direction, consider the following commutative diagram in $\mathcal{C}\mathrm{at}_{\infty}$ induced by (A.1), where the small square is to be a pullback diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, dashed, "\Phi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
& \mathcal{E} \arrow[r] & \mathcal{F}
\end{tikzcd}$$

It suffices to show that Φ is an equivalence. The already proven "only if"-direction and the assumption that (A.2) is a pullback diagram imply that $\Phi^{\simeq}$ is an equivalence of spaces, which implies that Φ is essentially surjective (see [HTT, 1.2.10.1]). Analogously we deduce from (A.3) that Φ is fully faithful (see [HTT, 1.2.10.1] and Definition B.2.0.1 below). Thus Φ is an equivalence. □

Remark A.3.1.2. In Proposition A.3.1.1, if diagrams (A.3) are pullback diagrams, then it follows immediately from the proof that we can replace the condition that diagram (A.2) is a pullback diagram with the a-priori weaker claim that the map $\Phi^{\simeq}$ constructed in the proof is induces a surjection on π_0. As $(-)^{\simeq}$ preserves pullbacks we can identify $\Phi^{\simeq}$ with the induced functor $\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq} \times_{\mathcal{E}^{\simeq}} \mathcal{F}^{\simeq}$. ♢

A.3.2. Natural transformations

Proposition A.3.2.1 ([Lur21, Theorem 01DK])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, F and G two functors $\mathcal{C} \to \mathcal{D}$, and $\Phi\colon F \to G$ a natural transformation. Then Φ is an equivalence in* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *if and only if $\Phi_X\colon F(X) \to F(X)$ is an equivalence in $\mathcal{D}$ for every object X of $\mathcal{C}$.* ♡

Proof. Equivalences can be described via colimits; A morphism f in some ∞-category $\mathcal{E}$ is an equivalence if and only if the corresponding functor $[0]^{\triangleright} \simeq [1] \to \mathcal{E}$ is a colimit diagram, see [HTT, 4.4.1 and 1.2.4.1]. The claim now follows from the fact that colimits in functor categories can be detected pointwise by [HTT, 5.1.2.3 (2)]. □

Appendix B.

(Fully) faithful functors and monomorphisms in $\mathcal{C}\mathrm{at}_\infty$

In this appendix we discuss three important classes of functors of ∞-categories that are all in some sense analogues to the notion of injections of sets. These are the *faithful* functors, *fully faithful* functors, as well as *monomorphisms* in $\mathcal{C}\mathrm{at}_\infty$.

The notion of monomorphism can be defined in any ∞-category, not just $\mathcal{C}\mathrm{at}_\infty$, so we begin by discussing monomorphisms in this greater generality in Section B.1. We then define (fully) faithful functors Section B.2 and discuss some immediate consequences of the definitions. Before discussing these three classes of functors of ∞-categories further, we will need to show an intermediate result in Section B.3, stability of (fully) faithful functors under $\mathrm{Fun}(\mathcal{I}, -)$ for an ∞-category $\mathcal{I}$. We will then be ready to discuss monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ in detail in Section B.4. In Section B.5 we will cover a number of stability results, including descriptions of replete images, for (fully) faithful functors and monomorphisms in $\mathcal{C}\mathrm{at}_\infty$, under $\mathrm{Fun}(\mathcal{I}, -)$, pullbacks along another functor, and pullbacks. We will end this section with Section B.6, in which we will discuss the correspondence between monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ with codomain a fixed ∞-category $\mathcal{C}$ and replete subcategories of $\mathrm{Ho}\,\mathcal{C}$.

B.1. Monomorphisms

Let $\mathcal{C}$ be an ∞-category and $f\colon X \to Y$ a morphism in $\mathcal{C}$. Then f is called a *monomorphism*[1] if the morphism

$$f_*\colon \mathrm{Map}_{\mathcal{C}}(Z, X) \to \mathrm{Map}_{\mathcal{C}}(Z, Y)$$

is a monomorphism in $\mathcal{S}$ for every object Z of $\mathcal{C}$.

In Section B.1.1 we will give a number of equivalent characterizations for monomorphisms in $\mathcal{S}$, before discussing the interaction of monomorphisms with compositions in Section B.1.2 and with limits in Section B.1.3.

B.1.1. Monomorphisms in the ∞-category $\mathcal{S}$

The following proposition recalls the notion of *monomorphisms* in the ∞-category $\mathcal{S}$.

[1] See the definition given in [HTT, Between 5.5.6.13 and 5.5.6.14] as well as [HTT, 5.5.6.8].

Proposition B.1.1.1. *Let $f\colon X \to Y$ be a morphism in $\mathcal{S}$. Then the following are equivalent.*

(1) f is a monomorphism in the sense of [HTT, Directly after 5.5.6.13], i. e. if f is (-1)-truncated in the sense of [HTT, 5.5.6.8].

(2) For every point y in Y the fiber of f over y is (-1)-truncated, i. e. empty or contractible.

(3) For every point x in X the fiber of f over $f(x)$ is (-2)-truncated, i. e. contractible.

(4) For every point x in X the morphism induced by f

$$\pi_n(X, x) \to \pi_n\big(Y, f(x)\big) \tag{B.1}$$

is a bijection for $n > 0$ and an injection for $n = 0$.

(5) The induced morphism on path components $\pi_0(f)$ is injective and the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y) \end{tikzcd}$$

is a pullback diagram in $\mathcal{S}$.

(6) Considering f as a functor of ∞-categories (via the inclusion of ∞-groupoids into $\mathcal{C}\mathrm{at}_\infty$) the induced map on mapping spaces[2]

$$\mathrm{Map}_X\big(x, x'\big) \to \mathrm{Map}_Y\big(f(x), f(x')\big) \tag{B.2}$$

is an equivalence for every pair of points x and x' in X.

♡

Proof. Proof that (1) is equivalent to (2): This is [HTT, 5.5.6.9].

Proof that (2) is equivalent to (3): Follows from the fact that points in Y are equivalent to $f(x)$ for a point x in X if and only if the fiber of f over y is not empty.

Proof that (5) implies (1): As any injective morphism of discrete spaces satisfies (3) and hence (1), and monomorphisms are stable under taking pullbacks by [HTT, 5.5.6.12], (5) implies (1).

Proof that (3) is equivalent to (4): Follows immediately from the long exact sequence of homotopy groups.

[2] These are the path spaces if we think of X and Y as spaces.

Proof that (3) implies (5): That $\pi_0(f)$ is injective is part of (4). Now consider the following diagram, where the small square is a pullback square.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\
& P \arrow[r, "\psi"] \arrow[d] & Y \arrow[d] \\
& \pi_0(X) \arrow[r, "\pi_0(f)"'] & \pi_0(Y)
\end{tikzcd}$$

It suffices to show that φ is an equivalence. By the long exact sequence of homotopy groups, it suffices for this to show that $\pi_0(\varphi)$ is surjective and the fiber of φ over every point in P is contractible. As $Y \to \pi_0(Y)$ is 1-connective (see [HTT, 6.5.1.10] for a definition) we obtain that $P \to \pi_0(X)$ is 1-connective by [HTT, 6.5.1.16 (6)], and as $X \to \pi_0(X)$ is 1-connective as well it follows that $\pi_0(\varphi)$ must be an isomorphism.

Now let p be a point in P. Consider the following diagram of pullback squares.

$$\begin{tikzcd}
\mathrm{fib}_p(\varphi) \arrow[r] \arrow[d] & \{p\} \arrow[d] & \\
\mathrm{fib}_{\psi(p)}(f) \arrow[r] \arrow[d] & \mathrm{fib}_{\psi(p)}(\psi) \arrow[r] \arrow[d] & \{\psi(p)\} \arrow[d] \\
X \arrow[r, "\varphi"] & P \arrow[r, "\psi"] \arrow[d] & Y \arrow[d] \\
& \pi_0(X) \arrow[r] & \pi_0(Y)
\end{tikzcd}$$

As $\pi_0(\varphi)$ is surjective, $\psi(p)$ is equivalent to $f(x)$ for some point x in X, so it follows from the assumption that $\mathrm{fib}_{\psi(p)}(f)$ is contractible. Furthermore, as $\mathrm{fib}_{\psi(p)}(\psi)$ can be identified as a fiber of a map of discrete spaces, it is discrete as well. It follows, using the long exact sequence of homotopy groups associated to the fiber sequence

$$\mathrm{fib}_p(\varphi) \to \mathrm{fib}_{\psi(p)}(f) \to \mathrm{fib}_{\psi(p)}(\psi)$$

that $\mathrm{fib}_p(\varphi)$ is contractible.

Proof that (6) is equivalent to (4): Let x and x' be points of X. We distinguish two cases. If x and x' are not in the same path component, then $\mathrm{Map}_X(x, x')$ is empty, and so (B.2) is an equivalence if and only if $\mathrm{Map}_Y(f(x), f(x'))$ is empty. That this is the case for all points x and x' in different path components of X is equivalent to $\pi_0(f)$ being injective.

If x and x' are two points of X that lie in the same path component, then the map (B.2) can be identified with the induced map on loop spaces.

$$\Omega_x(X) \xrightarrow{\Omega_x(f)} \Omega_{f(x)}(Y)$$

As $\pi_n(\Omega_x(f)) \cong \pi_{n+1}(f)$ (where on the left we use the constant loop at x as the basepoint, and at the right the point x) we can conclude that (B.2) being an equivalence for all x and x' in the same path component of X is equivalent to (B.1) being an isomorphism for $n > 0$. $\square$

B.1.2. Monomorphisms and composition

Proposition B.1.2.1. *Let $\mathcal{C}$ be an ∞-category and $f\colon X \to Y$ and $g\colon Y \to Z$ two morphisms in $\mathcal{C}$ such that g is a monomorphism. Then $g \circ f$ is a monomorphism if and only if f is a monomorphism.* $\heartsuit$

Proof. The statement for $\mathcal{C} = \mathcal{S}$ follows immediately from criterion Proposition B.1.1.1 (4). The claim for general $\mathcal{C}$ now follows immediately from the definition. $\square$

B.1.3. Monomorphisms and limits

Proposition B.1.3.1. *Let $\mathcal{I}$ and $\mathcal{C}$ be ∞-categories, $A, B\colon \mathcal{I} \to \mathcal{C}$ two functors, and F a natural transformation from A to B. Assume that for every object X of $\mathcal{I}$ the morphism $F(X)\colon A(X) \to B(X)$ in $\mathcal{C}$ is a monomorphism. Then the morphism $\lim_{\mathcal{I}} A \xrightarrow{\lim_{\mathcal{I}} F} \lim_{\mathcal{I}} B$ in $\mathcal{C}$ is a monomorphism as well.* $\heartsuit$

Proof. We first prove the claim for $\mathcal{C} = \mathcal{S}$. Let y be a point in $\lim_{\mathcal{I}} B$. We have to show that $\mathrm{fib}_y(\lim_{\mathcal{I}} F)$ is (-1)-truncated. But as limits commute with limits, we have an equivalence

$$\mathrm{fib}_y\left(\lim_{\bullet\in\mathcal{I}} F(\bullet)\right) \simeq \lim_{\bullet\in\mathcal{I}}\big(\mathrm{fib}_{\mathrm{pr}_\bullet(y)} F(\bullet)\big)$$

so that the claim follows from the closure of (-1)-truncated objects under limits, see [HTT, 5.5.6.5].

The case of general $\mathcal{C}$ now follows from this special case using that for every object X of $\mathcal{C}$ the functor

$$\mathrm{Map}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathcal{S}$$

preserves limits. $\square$

B.2. (Fully) faithful functors

In this section we define the notions of *(fully) faithful* functors of ∞-categories[3] and record some direct consequences of the definition.

Definition B.2.0.1. Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a functor of ∞-categories. We say that ι is *(fully) faithful* if for every pair of objects X and Y of $\mathcal{C}'$ the morphism in $\mathcal{S}$ induced by ι

$$\mathrm{Map}_{\mathcal{C}'}(X, Y) \to \mathrm{Map}_{\mathcal{C}}\big(F(X), F(Y)\big)$$

is a monomorphism (is an equivalence). $\diamondsuit$

[3]Fully faithful functors are defined in [HTT, 1.2.10.1].

Remark B.2.0.2. It is clear from the definition, that the notions of (fully) faithfulness agree with the classical definitions on 1-categories. Furthermore, as $\pi_0 \colon \mathcal{S} \to \mathsf{Set}$ sends equivalences to isomorphisms and monomorphisms to monomorphisms (see Proposition B.1.1.1), if a functor ι of ∞-categories is (fully) faithful, then the same is true for the functor $\operatorname{Ho}\iota$ of ordinary categories. ♢

Proposition B.2.0.3. *Let* $\iota \colon \mathcal{C}' \to \mathcal{C}$ *be a faithful functor of* ∞*-categories. Then the commutative diagram*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\iota"] \arrow[d] & \mathcal{C} \arrow[d] \\ \operatorname{Ho}(\mathcal{C}') \arrow[r, "\operatorname{Ho}\iota"'] & \operatorname{Ho}(\mathcal{C}) \end{tikzcd} \tag{B.3}$$

is a pullback diagram in $\mathcal{C}\mathrm{at}_\infty$*.* ♡

Proof. We use Proposition A.3.1.1 and Remark A.3.1.2. Let X and Y be two object of $\mathcal{C}'$. Diagram (B.3) induces the following diagram of mapping spaces.

$$\begin{tikzcd} \operatorname{Map}_{\mathcal{C}'}(X,Y) \arrow[r] \arrow[d] & \operatorname{Map}_{\mathcal{C}}(\iota X, \iota Y) \arrow[d] \\ \pi_0\big(\operatorname{Map}_{\mathcal{C}'}(X,Y)\big) \arrow[r] & \pi_0\big(\operatorname{Map}_{\mathcal{C}}(\iota X, \iota Y)\big) \end{tikzcd}$$

That this square is a pullback square in $\mathcal{S}$ follows now from Proposition B.1.1.1.

It now remains to show that

$$\mathcal{C}'^{\simeq} \to \mathcal{C}^{\simeq} \times_{\operatorname{Ho}(\mathcal{C})^{\simeq}} \operatorname{Ho}\big(\mathcal{C}'\big)^{\simeq}$$

induces a surjection on π_0. The map[4]

$$\mathcal{C}'^{\simeq} \to \operatorname{Ho}(\mathcal{C}')^{\simeq} \simeq \operatorname{Ho}(\mathcal{C}'^{\simeq}) \simeq \tau_{\le 1}(\mathcal{C}'^{\simeq})$$

is 2-connective. Similarly, $\mathcal{C}^{\simeq} \to \operatorname{Ho}(\mathcal{C})^{\simeq}$ is 2-connective, so by [HTT, 6.5.1.16 (6)] the projection $\mathrm{pr}_2 \colon \mathcal{C}^{\simeq} \times_{\operatorname{Ho}(\mathcal{C})^{\simeq}} \operatorname{Ho}(\mathcal{C}')^{\simeq} \to \operatorname{Ho}(\mathcal{C}')^{\simeq}$ is 2-connective as well. We thus have a

[4] That $\operatorname{Ho}(\mathcal{C})^{\simeq} \simeq \operatorname{Ho}(\mathcal{C}^{\simeq})$ can be seen directly using the definitions, it boils down to the subspace of $\operatorname{Map}_{\mathcal{C}'}(X,Y)$ spanned by the equivalences consisting exactly of the path components that as elements of $\pi_0(\operatorname{Map}_{\mathcal{C}'}(X,Y)) = \operatorname{Mor}_{\operatorname{Ho}\mathcal{C}'}(X,Y)$ correspond to isomorphisms in $\operatorname{Ho}\mathcal{C}'$. That $\operatorname{Ho}(\mathcal{C}'^{\simeq}) \simeq \tau_{\le 1}(\mathcal{C}'^{\simeq})$ amounts to the fact that the diagram of inclusions

$$\begin{tikzcd} \mathcal{S}_{\le 1} \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\ \mathsf{Cat} \arrow[r] & \mathcal{C}\mathrm{at}_\infty \end{tikzcd}$$

is left adjointable in the sense of [HTT, 7.3.1.1]. However, in this situation this follows from the horizontal functors being fully faithful.

commuting triangle

$$
\begin{tikzcd}
\mathcal{C}'^{\simeq} \arrow[rr] \arrow[dr] & & \mathcal{C}^{\simeq} \times_{\mathrm{Ho}(\mathcal{C})^{\simeq}} \mathrm{Ho}(\mathcal{C}')^{\simeq} \arrow[dl, "\mathrm{pr}_2"] \\
& \mathrm{Ho}(\mathcal{C}')^{\simeq} &
\end{tikzcd}
$$

where the two non-horizontal maps are 2-connective, so the horizontal map must in particular induce a surjection on π_0. □

Proposition B.2.0.4. *Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a faithful functor. Then for any objects X and Y of $\mathcal{C}'$, the induced map*

$$\mathrm{Map}_{\mathcal{C}'^{\simeq}}(X, Y) \to \mathrm{Map}_{\mathcal{C}^{\simeq}}(\iota X, \iota Y) \tag{B.4}$$

is a monomorphism in $\mathcal{S}$. ♡

Proof. The map in question is by definition the induced vertical map by taking limits of the horizontal diagrams in the following commutative diagram

$$
\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}'}(X,Y) \arrow[r] \arrow[d] & \pi_0\big(\mathrm{Map}_{\mathcal{C}'}(X,Y)\big) \arrow[d] & \pi_0\big(\mathrm{Map}_{\mathcal{C}'^{\simeq}}(X,Y)\big) \arrow[l] \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[r] & \pi_0\big(\mathrm{Map}_{\mathcal{C}}(X,Y)\big) & \pi_0\big(\mathrm{Map}_{\mathcal{C}^{\simeq}}(X,Y)\big) \arrow[l]
\end{tikzcd}
$$

where the vertical maps are induced by ι, the horizontal maps from the left to the middle are the canonical ones, and the horizontal maps from the right to the middle are the inclusions of the path components spanned by invertible morphisms.

As all vertical maps are monomorphisms, it follows from Proposition B.1.3.1 that (B.4) is a monomorphism as well. □

B.3. (Fully) Faithful functors and Fun

When we discuss monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ in Section B.4, we will need to use a first stability result for (fully) faithful functors that we prove in this section; for an ∞-category $\mathcal{I}$, the functor $\mathrm{Fun}(\mathcal{I}, -)\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ preserves (fully) faithful functors.

Proposition B.3.0.1. *Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a (fully) faithful functor of ∞-categories and let $\mathcal{I}$ be some ∞-category. Then the induced functor*

$$\iota_*\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}') \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$$

is (fully) faithful as well. ♡

Proof. Let F and G be two objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{D}')$. Mapping spaces in functor categories can be written as ends, see [Gla16, 2.3]. Concretely, the map induced by ι_* on mapping spaces

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D}')}(F, G) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(\iota \circ F, \iota \circ G)$$

can be identified with the following map of ends, induced by the maps induced by ι on mapping spaces $\mathrm{Map}_{\mathcal{D}'}(\bullet, \bullet) \to \mathrm{Map}_{\mathcal{D}}(\iota(\bullet), \iota(\bullet))$.

$$\int_{\bullet \in \mathcal{C}} \mathrm{Map}_{\mathcal{D}'}\big(F(\bullet), G(\bullet)\big) \longrightarrow \int_{\bullet \in \mathcal{C}} \mathrm{Map}_{\mathcal{D}}\big(\iota(F(\bullet), \iota(G(\bullet))\big)$$

If ι is fully faithful, then this is an equivalence as ends, like other limits, are invariant under equivalences, so ι_* is fully faithful as well.

If ι is faithful, then we can use that limits commute with limits, so for $\varphi\colon F \to G$ a morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D}')$ we obtain

$$\begin{aligned}
&\mathrm{fib}_\varphi\Big(\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D}')}(F, G) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(\iota \circ F, \iota \circ G)\Big) \\
\simeq &\int_{\bullet \in \mathcal{C}} \mathrm{fib}_{\varphi_\bullet}\Big(\mathrm{Map}_{\mathcal{D}'}\big(F(\bullet), G(\bullet)\big) \to \mathrm{Map}_{\mathcal{D}}\big(\iota(F(\bullet), \iota(G(\bullet))\big)\Big) \\
\simeq &\int_{\bullet \in \mathcal{C}} * \simeq *
\end{aligned}$$

where in the second-to-last step we use that ι is faithful in combination with criterion (3) of Proposition B.1.1.1, and in the last step we use that a limit of a diagram that is pointwise a terminal object (which is a limit over the empty diagram) is the terminal object (as limits commute with limits). Thus ι_* is again faithful. $\square$

B.4. Monomorphisms in $\mathcal{C}\mathrm{at}_\infty$

In this section we discuss monomorphisms in $\mathcal{C}\mathrm{at}_\infty$. We start in Section B.4.1 by giving several equivalent characterizations of monomorphisms in $\mathcal{C}\mathrm{at}_\infty$, that will be crucial for later results. In Section B.4.2 we will then discuss the analogue of monomorphism in $\mathcal{C}\mathrm{at}_\infty$ for 1-categories, the notion of *pseudomonic* functors, as well as the relation between monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ and pseudomonic functors in Cat. Section B.4.3 will provide the important criterion for lifting along monomorphisms in $\mathcal{C}\mathrm{at}_\infty$. Finally, we end this section with Section B.4.4, where we show that faithful functors are monomorphisms.

B.4.1. Equivalent characterizations of monomorphisms in $\mathcal{C}\mathrm{at}_\infty$

In this section we provide a number of equivalent characterizations of monomorphisms in $\mathcal{C}\mathrm{at}_\infty$. We also show that monomorphisms in $\mathcal{C}\mathrm{at}_\infty$ are conservative functors, i. e. reflect equivalences.

Proposition B.4.1.1. *Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a functor of ∞-categories. Then the following are equivalent.*

(1) *ι is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$ in the sense of [HTT, After 5.5.6.13].*

(2) *For every ∞-category $\mathcal{I}$, the induced map*

$$(\iota_*)^{\simeq}\colon \operatorname{Fun}\big(\mathcal{I},\mathcal{C}'\big)^{\simeq} \to \operatorname{Fun}(\mathcal{I},\mathcal{C})^{\simeq}$$

is a monomorphism in $\mathcal{S}$.

(3) *ι is faithful and the induced functor on ∞-groupoid cores $\iota^{\simeq}\colon \mathcal{C}'^{\simeq} \to \mathcal{C}^{\simeq}$ is a monomorphism in $\mathcal{S}$.*

(4) *ι is faithful and for every two objects X and Y in $\mathcal{C}'$ and equivalence $f\colon \iota X \to \iota Y$ there is an equivalence $f'\colon X \to Y$ such that $\iota f'$ is homotopic to f.*

♡

Proof. Proof that (1) is equivalent to (2): This follows immediately by unpacking the definition of monomorphisms and using that $\operatorname{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{I},-) \simeq \operatorname{Fun}(\mathcal{I},-)^{\simeq}$ by definition [HTT, 3.0.0.1].

Proof that (2) implies (3): Applying the assumption to $\mathcal{I} = [0]$, we deduce immediately that $\iota^{\simeq}$ is a monomorphism in $\mathcal{S}$. Let X and Y be objects of $\mathcal{C}'$. Using that $(-)^{\simeq}$ preserves pullbacks as a right adjoint [HTT, 1.2.5] we obtain from Proposition A.2.0.1 that the map induced by ι

$$\operatorname{Map}_{\mathcal{C}'}(X,Y) \to \operatorname{Map}_{\mathcal{C}}(\iota X,\iota Y) \tag{$*$}$$

is the map induced on limits of the horizontal diagrams in the following commutative diagram.

$$\begin{tikzcd}
\operatorname{Fun}\big([1],\mathcal{C}'\big)^{\simeq} \arrow[r] \arrow[d] & \operatorname{Fun}\big(\{0,1\},\mathcal{C}'\big)^{\simeq} \arrow[d] & \{(X,Y)\} \arrow[l] \arrow[d] \\
\operatorname{Fun}\big([1],\mathcal{C}\big)^{\simeq} \arrow[r] & \operatorname{Fun}\big(\{0,1\},\mathcal{C}\big)^{\simeq} & \{(\iota X,\iota Y)\} \arrow[l]
\end{tikzcd}$$

where the vertical maps are induced by ι, the horizontal maps from the left to the middle are induced by precomposition with the inclusion of $\{0,1\}$ into $[1]$ and the horizontal maps from the right to the middle are the inclusions of the functors sending 0 to the first component of the tuple and 1 to the second component. The vertical map on the right is an equivalence and thus a monomorphism, and the other two vertical maps are monomorphisms by assumption. It follows from Proposition B.1.3.1 that $(*)$ is a monomorphism as well.

Proof that (3) implies (4): Follows immediately from description Proposition B.1.1.1 (6) of monomorphisms in $\mathcal{S}$ applied to $\iota^{\simeq}$.

Proof that (4) implies (2): Let $\mathcal{I}$ be an ∞-category. As $\operatorname{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{I},-) \simeq \operatorname{Fun}(\mathcal{I},-)^{\simeq}$ preserves limits, we obtain from Proposition B.2.0.3 a pullback diagram of spaces as follows

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{I},\mathcal{C}')^{\simeq} \arrow[r, "(\iota_*)^{\simeq}"] \arrow[d] & \operatorname{Fun}(\mathcal{I},\mathcal{C})^{\simeq} \arrow[d] \\
\operatorname{Fun}(\mathcal{I},\operatorname{Ho}\mathcal{C}')^{\simeq} \arrow[r, "\big((\operatorname{Ho}\iota)_*\big)^{\simeq}"'] & \operatorname{Fun}(\mathcal{I},\operatorname{Ho}\mathcal{C})^{\simeq}
\end{tikzcd}$$

where the vertical maps are induced by postcomposition with the canonical functors. We have to show that the top map is a monomorphism, so as monomorphisms are stable under pullback by [HTT, 5.5.6.12], it suffices to show that $((\mathrm{Ho}\,\iota)_*)^{\simeq}$ is a monomorphism in $\mathcal{S}$. Note that as $\mathrm{Ho}\,\iota$ is a functor of 1-categories, we can identify $((\mathrm{Ho}\,\iota)_*)^{\simeq}$ with the following functor.

$$\big((\mathrm{Ho}\,\iota)_*\big)^{\simeq}\colon\ \mathrm{Fun}\big(\mathrm{Ho}\,\mathcal{I},\mathrm{Ho}\,\mathcal{C}'\big)^{\simeq}\to\mathrm{Fun}(\mathrm{Ho}\,\mathcal{I},\mathrm{Ho}\,\mathcal{C})^{\simeq}$$

Let F and G be functors from $\mathrm{Ho}\,\mathcal{I}$ to $\mathrm{Ho}\,\mathcal{C}'$, considered as objects of $\mathrm{Fun}(\mathrm{Ho}\,\mathcal{I},\mathrm{Ho}\,\mathcal{C}')^{\simeq}$. By criterion Proposition B.1.1.1 (6) it suffices to show that postcomposition with $\mathrm{Ho}\,\iota$ induces an equivalence on mapping spaces as follows.

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Ho}\,\mathcal{I},\mathrm{Ho}\,\mathcal{C}')^{\simeq}}(X,Y)\to\mathrm{Map}_{\mathrm{Fun}(\mathrm{Ho}\,\mathcal{I},\mathrm{Ho}\,\mathcal{C})^{\simeq}}(\iota\circ X,\iota\circ Y)\tag{$**$}$$

By Remark B.2.0.2 together with Proposition B.3.0.1 the functor $(\mathrm{Ho}\,\iota)_*$ is faithful, so by Proposition B.2.0.4, the map $(**)$ is already a monomorphism, so that it suffices to show that it induces a surjection on π_0. So let $\Phi\colon\iota\circ F\to\iota\circ G$ be a natural isomorphism of functors from $\mathrm{Ho}\,\mathcal{I}$ to $\mathrm{Ho}\,\mathcal{C}$. We have to show that we can lift Φ to a natural transformation from F to G. Let X be an object of $\mathrm{Ho}\,\mathcal{I}$. Then we can apply the assumption on ι and lift the isomorphism $\Phi_X\colon\iota(F(X))\to\iota(G(X))$ in $\mathrm{Ho}\,\mathcal{C}'$ to an isomorphism $\Phi'_X\colon F(X)\to G(X)$ in $\mathrm{Ho}\,\mathcal{C}$ such that $\iota(\Phi'_X)=\Phi_X$. It remains to check that Φ' defines a natural transformation from F to G. As F and G are functors of 1-categories, this is a property, not data, and it suffices to check that for every morphism $f\colon X\to Y$ in $\mathcal{I}$, the diagram

$$\begin{tikzcd}
F(X) \arrow[r, "\Phi'_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\Phi'_Y"'] & G(Y)
\end{tikzcd}$$

commutes. But as $\mathrm{Ho}\,\iota$ is faithful, it suffices to check that ι applied to this square yields a commutative square, which is the case as Φ is a natural transformation. □

Proposition B.4.1.2. *Let $\iota\colon\mathcal{C}'\to\mathcal{C}$ be a monomorphism in Cat_∞. Then ι is conservative, i. e. reflect equivalences.* ♡

Proof. Let $f\colon X\to Y$ be a morphism in $\mathcal{C}'$ such that $\iota(f)$ is an equivalence. By Proposition B.4.1.1 (4) we can lift $\iota(f)$ to an equivalence $f'\colon X\to Y$ in $\mathcal{C}'$. But faithfulness of ι implies that $\pi_0(\mathrm{Map}_{\mathcal{C}'}(X,Y))\to\pi_0(\mathrm{Map}_{\mathcal{C}}(\iota X,\iota Y))$ is injective, hence f and f' must be homotopic, so f is an equivalence as well. □

B.4.2. Pseudomonic functors and replete images

The notion of monomorphisms in Cat_∞ corresponds to the notion of *pseudomonic* functors of 1-categories, as we discuss in this section. Like injective maps of sets, pseudomonic functors of 1-categories are, up to equivalence, determined by their image. In the case of pseudomonic functors we will usually consider a more invariant notion of image, the *replete image*, which we also introduce below.

Remark B.4.2.1. Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a monomorphism in $\mathfrak{Cat}_\infty$. Then if follows immediately from Proposition B.4.1.1 (4) and Remark B.2.0.2 that $\operatorname{Ho}\iota\colon \operatorname{Ho}\mathcal{C}' \to \operatorname{Ho}\mathcal{C}$ is a pseudomonic functor, i. e. $\operatorname{Ho}\iota$ satisfies the following two conditions.

(1) $\operatorname{Ho}\iota$ is faithful.

(2) If X and Y are two objects of $\operatorname{Ho}\mathcal{C}'$ and $f\colon (\operatorname{Ho}\iota)(X) \to (\operatorname{Ho}\iota)(Y)$ is an isomorphism in $\operatorname{Ho}\mathcal{C}$, then f lifts to an isomorphism $f'\colon X \to Y$ in $\operatorname{Ho}\mathcal{C}'$ such that $(\operatorname{Ho}\iota)(f') = f$.

If $\iota\colon \mathsf{C}' \to \mathsf{C}$ is a pseudomonic functor of 1-categories, then it follows similarly that ι is a monomorphism in $\mathfrak{Cat}_\infty$. ♢

Definition B.4.2.2. Let C' be a subcategory of the 1-category C. We say that C' is a *replete* subcategory of C if the collection of morphisms in C' is closed under isomorphisms in the arrow category $\operatorname{Fun}([1], \mathsf{C})$.

If $F\colon \mathsf{C} \to \mathsf{D}$ is a functor of 1-categories, then the *replete image* $\operatorname{Im} F$ of F is the replete subcategory of D generated by the image of F, i. e. it consists of those objects isomorphic to an object of the form $F(X)$ for X in C, and those morphisms isomorphic in the arrow category of D to a morphism of the form $F(f)$ for f a morphism in C. ♢

Remark B.4.2.3. Let $\iota\colon \mathsf{C}' \to \mathsf{C}$ be a pseudomonic functor of 1-categories. Then it follows directly from the definitions that the induced functor $\iota'\colon \mathsf{C}' \to \operatorname{Im}\iota$ is essentially surjective as well as fully faithful and thus an equivalence. ♢

B.4.3. Lifting along monomorphisms

We now show that monomorphisms in $\mathfrak{Cat}_\infty$ have the expected property: We can check whether two functors into the domain of a monomorphism ι are homotopic by checking their compositions with ι, and any functor into the target of ι can be lifted as long as its image is contained in the image of ι.

Proposition B.4.3.1. *Let $\iota\colon \mathcal{D}' \to \mathcal{D}$ be a monomorphism in $\mathfrak{Cat}_\infty$ and $F\colon \mathcal{C} \to \mathcal{D}$ a functor of ∞-categories.*

Then F can be lifted along ι, i. e. there exists a commutative diagram as follows

$$\begin{tikzcd} & \mathcal{D}' \arrow[d, "\iota"] \\ \mathcal{C} \arrow[ur, dashed, "F'"] \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

if and only if $\operatorname{Im}(\operatorname{Ho}F)$ is contained in $\operatorname{Im}(\operatorname{Ho}\iota)$. If this is the case, then the lift is essentially unique in the sense that the fiber over F of the map

$$(\iota_*)^{\simeq}\colon \operatorname{Fun}\big(\mathcal{C}, \mathcal{D}'\big)^{\simeq} \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$$

is contractible. ♡

Proof. Proof of the "only if"-direction: Clear.

Proof of the "if"-direction: By Proposition B.4.1.1 ι is faithful and so the right square in the following commutative diagram is a pullback square by Proposition B.2.0.3.

$$\begin{tikzcd} & \mathcal{D}' \arrow[r] \arrow[d, "\iota"] & \operatorname{Ho}\mathcal{D}' \arrow[d, "\operatorname{Ho}\iota"] \\ \mathcal{C} \arrow[r, "F"'] \arrow[ur, dashed, "F'"] & \mathcal{D} \arrow[r] & \operatorname{Ho}\mathcal{D} \end{tikzcd}$$

It thus suffices to show that the composition $\widetilde{F}$ of F with the canonical functor $\mathcal{D} \to \operatorname{Ho}\mathcal{D}$ can be lifted along $\operatorname{Ho}\iota$. But $\operatorname{Ho}\iota$ factors by Remark B.4.2.3 as an equivalence composed with the inclusion $\operatorname{Im}(\operatorname{Ho}\iota) \to \operatorname{Ho}\mathcal{C}$, and by assumption $\widetilde{F}$ factors over this inclusion.

Proof that the lift is essentially unique if it exists: As we assume a lift exists, the fiber can not be empty. That it is then contractible follows from Proposition B.4.1.1 (2). □

B.4.4. (Fully) faithful functors are monomorphisms

In this short section we show that (fully) faithful functors are monomorphisms.

Proposition B.4.4.1. *Fully faithful functors of ∞-categories are monomorphisms in $\mathfrak{Cat}_\infty$.* ♡

Proof. Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a fully faithful functor. We will use criterion Proposition B.4.1.1 (4). That ι is faithful is clear. Let X and Y be objects of $\mathcal{C}'$ and $f\colon \iota X \to \iota Y$ an equivalence in $\mathcal{C}$. Let f^{-1} be an inverse of f. As ι is fully faithful, we can lift f to a morphism $f'\colon X \to Y$ and f^{-1} to a morphism $f''\colon Y \to X$. But as ι also induces an equivalence $\operatorname{Map}_{\mathcal{C}'}(X, X) \to \operatorname{Map}_{\mathcal{C}}(\iota X, \iota X)$, we can also lift the homotopy $f^{-1} \circ f \simeq \mathrm{id}_{\iota X}$ to a homotopy $f'' \circ f' \simeq \mathrm{id}_X$, and similarly $f' \circ f'' \simeq \mathrm{id}_Y$, so $f'\colon X \to Y$ is an equivalence with $\iota f' \simeq f$. □

B.5. Stability properties of (fully) faithful functors and monomorphisms in $\mathfrak{Cat}_\infty$

In this section we show that monomorphism in $\mathfrak{Cat}_\infty$ as well as (fully) faithful functors are stable under various constructions. In Section B.5.1 we handle the case of induced functors on functor ∞-categories, and in Sections B.5.2 and B.5.3 we discuss two types of stability under taking pullbacks.

Section B.5.2 concerns taking the pullback along an arbitrary other functor, i. e. we show that if $\iota_{\mathcal{D}}$ is faithful (fully faithful, a monomorphism), then the functor $\iota_{\mathcal{C}}$, defined via a pullback diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\iota_{\mathcal{C}}"] \arrow[d, "F'"'] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r, "\iota_{\mathcal{D}}"'] & \mathcal{D} \end{tikzcd}$$

in $\mathfrak{Cat}_\infty$, with F any functor, is so as well.

In Section B.5.3 we instead consider stability under taking pullbacks in the arrow ∞-category; in Proposition B.1.3.1 we already showed that a natural transformation between two diagrams that is pointwise a monomorphism induces a monomorphism between the two limits. Section B.5.3 specializes this to the case of pullbacks in $\mathfrak{Cat}_\infty$, and adds additional information regarding the replete image of the induced functor.

B.5.1. Functor ∞-categories

Proposition B.5.1.1. *Let* $\iota\colon \mathcal{C}' \to \mathcal{C}$ *be a monomorphism in* $\mathfrak{Cat}_\infty$ *and* $\mathcal{I}$ *an* ∞*-category. Then the induced functor*

$$\iota_*\colon \operatorname{Fun}(\mathcal{I}, \mathcal{C}') \to \operatorname{Fun}(\mathcal{I}, \mathcal{C})$$

is a monomorphism in $\mathfrak{Cat}_\infty$ *as well.*

Let J *be defined to be the replete subcategory of* $\operatorname{Ho}\operatorname{Fun}(\mathcal{I}, \mathcal{C})$ *where*

- *a functor* $F\colon \mathcal{I} \to \mathcal{C}$ *considered as an object of* $\operatorname{Ho}\operatorname{Fun}(\mathcal{I}, \mathcal{C})$ *is in* J *if and only if* $\operatorname{Im}(\operatorname{Ho} F)$ *is contained in* $\operatorname{Im}(\operatorname{Ho}\iota)$*.*
- *a natural transformation* $\Phi\colon F \to G$ *of functors* $\mathcal{I} \to \mathcal{C}$*, considered as a morphism from* F *to* G *in* $\operatorname{Ho}\operatorname{Fun}(\mathcal{I}, \mathcal{C})$*, is in* J *if and only if* F *and* G *are objects of* J *and* Φ_X *is in* $\operatorname{Im}(\operatorname{Ho}\iota)$ *for every object* X *of* $\mathcal{I}$*.*

Then the replete image $\operatorname{Im}(\operatorname{Ho}\iota_*)$ *of the functor*

$$\operatorname{Ho}(\iota_*)\colon \operatorname{Ho}\operatorname{Fun}(\mathcal{I}, \mathcal{C}') \to \operatorname{Ho}\operatorname{Fun}(\mathcal{I}, \mathcal{C})$$

is equal to J*.* ♡

Proof. Proof that ι_* *is a monomorphism*: Follows from description Proposition B.4.1.1 (2) using that for any ∞-category $\mathcal{J}$ there is a natural equivalence as follows.

$$\operatorname{Fun}\big(\mathcal{J}, \operatorname{Fun}(\mathcal{I}, -)\big) \simeq \operatorname{Fun}(\mathcal{J} \times \mathcal{I}, -)$$

Proof that $\operatorname{Im}(\operatorname{Ho}\iota_*)$ *is contained in* J: Clear

Proof that J *is contained in* $\operatorname{Im}(\operatorname{Ho}\iota_*)$: It suffices to show an inclusion of the respective collection of morphisms, as the case of objects is covered by the identity morphisms. So let $\Phi\colon F \to G$ be a natural transformation of functors $\mathcal{I} \to \mathcal{C}$, considered as a morphism from F to G in $\operatorname{Ho}\operatorname{Fun}(\mathcal{I}, \mathcal{C})$, and assume that Φ lies in J. What we have to show is that Φ can be lifted along ι, i. e. that there is a natural transformation Φ' of functors $\mathcal{I} \to \mathcal{C}'$ such that $\iota \circ \Phi' \simeq \Phi$. But we can encode Φ as a functor $\widetilde{\Phi}\colon [1] \times \mathcal{I} \to \mathcal{C}$, and the assumptions mean precisely that $\operatorname{Im}(\operatorname{Ho}\widetilde{\Phi})$ is contained $\operatorname{Im}(\operatorname{Ho}\iota)$. That we can lift Φ along ι now follows from Proposition B.4.3.1. □

Remark B.5.1.2. Let $\iota\colon \mathcal{C}' \to \mathcal{C}$ be a fully faithful functor. By B.4.4.1 ι is also a monomorphism in $\mathfrak{Cat}_\infty$, so we can apply Proposition B.5.1.1. In this case, the replete subcategory J of $\operatorname{Ho}\operatorname{Fun}(\mathcal{I}, \mathcal{C})$ appearing in the statement of Proposition B.5.1.1 has a simpler description, using that $\operatorname{Ho}\iota$ is full: J is the full subcategory of $\operatorname{Ho}\operatorname{Fun}(\mathcal{I}, \mathcal{C})$ spanned by those functors $F\colon \mathcal{I} \to \mathcal{C}$ for which $F(X)$ is in the essential image of $\operatorname{Ho}\iota$ for every object X of $\mathcal{I}$. ◇

B.5.2. Pullbacks along another functor

Proposition B.5.2.1. *Let*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\iota_\mathcal{C}"] \arrow[d, "F'"'] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r, "\iota_\mathcal{D}"'] & \mathcal{D} \end{tikzcd} \tag{B.5}$$

be a pullback square in $\mathcal{C}\mathrm{at}_\infty$ and assume that $\iota_\mathcal{D}$ is faithful (fully faithful, a monomorphism). Then $\iota_\mathcal{C}$ is faithful (fully faithful, a monomorphism) as well.

Furthermore, if $\iota_\mathcal{D}$ is a monomorphism[5]*, then the induced diagram on homotopy categories*

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{C}') \arrow[r, "\mathrm{Ho}\,\iota_\mathcal{C}"] \arrow[d, "\mathrm{Ho}\,F'"'] & \mathrm{Ho}(\mathcal{C}) \arrow[d, "\mathrm{Ho}\,F"] \\ \mathrm{Ho}(\mathcal{D}') \arrow[r, "\mathrm{Ho}\,\iota_\mathcal{D}"'] & \mathrm{Ho}(\mathcal{D}) \end{tikzcd} \tag{B.6}$$

is a pullback[6]. ♡

Proof. That $\iota_\mathcal{C}$ is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$ if $\iota_\mathcal{D}$ is follows immediately from pullbacks of monomorphisms being pullbacks again, see [HTT, 5.5.6.12].

We next show the first statement for (fully) faithful functors. Let X and Y be objects of $\mathcal{C}'$. We have to show that the map in $\mathcal{S}$

$$\mathrm{Map}_{\mathcal{C}'}(X, Y) \to \mathrm{Map}_\mathcal{C}(\iota_\mathcal{C}(X), \iota_\mathcal{C}(Y))$$

induced by $\iota_\mathcal{C}$ is a monomorphism (is an equivalence). By Proposition A.2.0.2 the commutative square induced by (B.5)

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}'}(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}_\mathcal{C}\big(\iota_\mathcal{C}(X), \iota_\mathcal{C}(Y)\big) \arrow[d] \\ \mathrm{Map}_{\mathcal{D}'}\big(F'(X), F'(Y)\big) \arrow[r] & \mathrm{Map}_\mathcal{D}\big(\iota_\mathcal{D}(F'(X)), \iota_\mathcal{D}(F'(Y))\big) \end{tikzcd}$$

is a pullback diagram in $\mathcal{S}$. As $\iota_\mathcal{D}$ is (fully) faithful the lower horizontal map is a monomorphism (equivalence), and hence so is the upper horizontal map (see [HTT, 5.5.6.12] for monomorphisms being preserved by pullbacks) This shows that $\iota_\mathcal{C}$ is (fully) faithful.

Finally it remains to show that diagram (B.6) is a pullback diagram if $\iota_\mathcal{D}$ is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$. By Remark B.4.2.1, the functors $\mathrm{Ho}\,\iota_\mathcal{D}$ and $\mathrm{Ho}\,\iota_\mathcal{C}$ are pseudomonic, so this boils down to showing that the replete image of $\mathrm{Ho}\,\iota_\mathcal{C}$ is equal to the $\mathrm{Ho}\,F$-preimage of the replete image of $\mathrm{Ho}\,\iota_\mathcal{D}$. It is clear that $\mathrm{Ho}\,F$ maps the replete image of $\mathrm{Ho}\,\iota_\mathcal{C}$ to the replete image of $\mathrm{Ho}\,\iota_\mathcal{D}$. On the other hand, if f is a morphism in $\mathcal{C}$ such that $\mathrm{Ho}\,F(f)$ is in the replete image of $\mathrm{Ho}\,\iota_\mathcal{D}$, then there must exist a morphism g' in $\mathcal{D}$ and an equivalence

[5] Recall that by Proposition B.4.4.1 fully faithful functors are automatically monomorphisms in $\mathcal{C}\mathrm{at}_\infty$.
[6] We take the pullback in the ∞-category of 1-categories.

$\iota_{\mathcal{D}}(g') \simeq F(f)$ in $\operatorname{Fun}([1], \mathcal{D})$. We can interpret the situation as a commuting square as depicted as the outer square in the following diagram.

$$\begin{tikzcd}
{[1]} \arrow[dr, dashed] \arrow[drr, bend left, "\mathrm{const}_f"] \arrow[ddr, bend right, "\mathrm{const}_{g'}"'] & & \\
& \mathcal{C}' \arrow[r, "\iota_{\mathcal{C}}"] \arrow[d, "F'"'] & \mathcal{C} \arrow[d, "F"] \\
& \mathcal{D}' \arrow[r, "\iota_{\mathcal{D}}"'] & \mathcal{D}
\end{tikzcd}$$

As the small square is a pullback square we obtain the dashed functor, which we can interpret as a morphism in $\mathcal{C}'$ that is mapped by $\operatorname{Ho}\iota_{\mathcal{C}}$ to a morphism isomorphic to C. That the objects of the two replete subcategories we are to compare agree can be proven analogously, or deduced from this by considering identity morphisms. □

B.5.3. Pullbacks

Proposition B.5.3.1. *Let*

$$\begin{tikzcd}
& \mathcal{C}' \arrow[rr] \arrow[dl, "F_{\mathcal{C}}"'] \arrow[dd] & & \mathcal{D}' \arrow[dl, "F_{\mathcal{D}}"'] \arrow[dd, "R'"] \\
\mathcal{C} \arrow[rr, "P"] \arrow[dd, "Q"'] & & \mathcal{D} \arrow[dd, "R"] & \\
& \mathcal{E}' \arrow[rr, "S'"] \arrow[dl, "F_{\mathcal{E}}"'] & & \mathcal{F}' \arrow[dl, "F_{\mathcal{F}}"'] \\
\mathcal{E} \arrow[rr, "S"'] & & \mathcal{F} &
\end{tikzcd} \tag{B.7}$$

be a commuting cube of ∞*-categories such that* $F_{\mathcal{D}}$, $F_{\mathcal{E}}$, *and* $F_{\mathcal{F}}$ *are faithful (fully faithful, monomorphisms) and the front and back squares are pullback squares in* $\mathfrak{Cat}_\infty$. *Then the functor* $F_{\mathcal{C}}$ *is faithful (fully faithful, a monomorphism) as well.*

Furthermore, if $F_{\mathcal{F}}$ *is a monomorphism*[7] *in* $\mathfrak{Cat}_\infty$, *then an object (morphism) in* $\operatorname{Ho}\mathcal{C}$ *is in* $\operatorname{Im}(\operatorname{Ho}F_{\mathcal{C}})$ *if and only if it is mapped by* $\operatorname{Ho}P$ *and* $\operatorname{Ho}Q$ *to an object (morphism) in* $\operatorname{Im}(\operatorname{Ho}F_{\mathcal{D}})$ *and* $\operatorname{Im}(\operatorname{Ho}F_{\mathcal{E}})$, *respectively.* ♡

Proof. To show that $F_{\mathcal{C}}$ is again faithful or fully faithful we apply Proposition A.2.0.2 and use Proposition B.1.3.1 and that the formation of pullbacks is invariant under equivalences. The case of monomorphisms in $\mathfrak{Cat}_\infty$ is even simpler, as it follows directly from Proposition B.1.3.1.

It remains to show the statement concerning replete images. The "only if"-direction is clear. We show that a morphism in $\operatorname{Ho}\mathcal{C}$ satisfying the assumption lies in $\operatorname{Im}(\operatorname{Ho}F_{\mathcal{C}})$, the

[7] By Proposition B.4.4.1, fully faithful functors are monomorphisms as well.

statement for objects follows from this by considering identity morphisms. As the front of (B.7) is a pullback diagram, a morphism in $\mathcal{C}$ satisfying the assumptions corresponds to a commutative square

$$\begin{tikzcd}
{[1]} \arrow[r, "\Phi_{\mathcal{D}}"] \arrow[d, "\Phi_{\mathcal{E}}"'] & \mathcal{D} \arrow[d, "R"] \\
\mathcal{E} \arrow[r, "S"'] & \mathcal{F}
\end{tikzcd}$$

such that $\operatorname{Im}(\operatorname{Ho}\Phi_{\mathcal{D}})$ is contained in $\operatorname{Im}(F_{\mathcal{D}})$ and $\operatorname{Im}(\operatorname{Ho}\Phi_{\mathcal{E}})$ is contained in $\operatorname{Im}(F_{\mathcal{E}})$. What we have to show is that we can extend this square to a commutative cube as follows.

$$\begin{tikzcd}
 & {[1]} \arrow[dl, "\mathrm{id}_{[1]}"'] \arrow[rr, dashed, "\Phi_{\mathcal{D}'}"] \arrow[dd, dashed, "\Phi_{\mathcal{E}'}"] & & \mathcal{D}' \arrow[dl, "F_{\mathcal{D}}"'] \arrow[dd, "R'"] \\
{[1]} \arrow[rr, "\Phi_{\mathcal{D}}"] \arrow[dd, "\Phi_{\mathcal{E}}"'] & & \mathcal{D} \arrow[dd, "R"] & \\
 & \mathcal{E}' \arrow[dl, "F_{\mathcal{E}}"'] \arrow[rr, "S'"] & & \mathcal{F}' \arrow[dl, "F_{\mathcal{F}}"] \\
\mathcal{E} \arrow[rr, "S"'] & & \mathcal{F} &
\end{tikzcd} \tag{$*$}$$

The assumptions on $\operatorname{Im}(\operatorname{Ho}\Phi_{\mathcal{D}})$ and $\operatorname{Im}(\operatorname{Ho}\Phi_{\mathcal{E}})$ imply that we can fill the dashed arrows together with the top and left squares by Proposition B.4.3.1, as $F_{\mathcal{D}}$ and $F_{\mathcal{D}}$ are monomorphisms. We are left to find a filler for the back square and the cube. But this amounts to lifting the homotopy between $F_{\mathcal{F}} \circ R' \circ \Phi_{\mathcal{D}'}$ and $F_{\mathcal{F}} \circ S' \circ \Phi_{\mathcal{E}'}$ encoded by the other sides to a homotopy between $R' \circ \Phi_{\mathcal{D}'}$ and $S' \circ \Phi_{\mathcal{E}'}$. This is possible as the following map induced by $F_{\mathcal{F}}$

$$\operatorname{Map}_{(\operatorname{Fun}([1],\mathcal{F}')^{\simeq})}\big(R' \circ \Phi_{\mathcal{D}'}, S' \circ \Phi_{\mathcal{E}'}\big) \to \operatorname{Map}_{(\operatorname{Fun}([1],\mathcal{F})^{\simeq})}\big(F_{\mathcal{F}} \circ R' \circ \Phi_{\mathcal{D}'}, F_{\mathcal{F}} \circ S' \circ \Phi_{\mathcal{E}'}\big)$$

is an equivalence by Proposition B.4.1.1 (2) and Proposition B.1.1.1 (6). □

B.6. Subcategories

In this short section we briefly discuss how monomorphisms into a fixed ∞-category $\mathcal{C}$ correspond to replete subcategories of $\operatorname{Ho}\mathcal{C}$.

Remark B.6.0.1. Let $\mathcal{C}$ be an ∞-category and $(\operatorname{Ho}\mathcal{C})'$ a replete subcategory of $\operatorname{Ho}\mathcal{C}$. Then define $\iota\colon \mathcal{C}' \to \mathcal{C}$ as in the following pullback diagram

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "\iota"] \arrow[d] & \mathcal{C} \arrow[d] \\
(\operatorname{Ho}\mathcal{C})' \arrow[r, "\iota'"'] & \operatorname{Ho}\mathcal{C}
\end{tikzcd}$$

where the right vertical functor is the canonical one. As the inclusion of a replete subcategory of a 1-category is a pseudomonic functor of 1-categories, it follows from Remark B.4.2.1 that ι' is a monomorphism in $\mathfrak{Cat}_\infty$. By Proposition B.5.2.1 ι is also a monomorphism, and furthermore the induced functor $\mathrm{Ho}(\mathcal{C}') \to \mathrm{Ho}(\mathcal{C})'$ is an equivalence[8], so $\mathrm{Im}(\mathrm{Ho}\,\iota) = (\mathrm{Ho}\,\mathcal{C})'$.

By Proposition B.4.3.1, two monomorphisms $\iota' : \mathcal{C}' \to \mathcal{C}$ and $\iota'' : \mathcal{C}'' \to \mathcal{C}$ are equivalent as functors to $\mathcal{C}$ in the sense that there is a commutative triangle

$$
\begin{tikzcd}
\mathcal{C}' \arrow[dr, "\iota'"] \arrow[dd, dash, "\simeq"'] & \\
& \mathcal{C} \\
\mathcal{C}'' \arrow[ur, "\iota''"'] &
\end{tikzcd}
$$

if and only if $\mathrm{Im}(\mathrm{Ho}\,\iota') = \mathrm{Im}(\mathrm{Ho}\,\iota'')$. This implies that all monomorphisms arise up to equivalence from the above construction, and that there is a bijection between equivalence classes of monomorphisms with target $\mathcal{C}$ and replete subcategories of $\mathrm{Ho}\,\mathcal{C}$. ◊

[8] As $\mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}(\mathrm{Ho}\,\mathcal{C})$ is.

Appendix C.

(Co)Cartesian Fibrations

For many technical parts of this thesis, (co)cartesian fibrations play a crucial role. For a very readable model-independent introduction [Maz19a] can be recommended. For a full introduction to (co)cartesian fibrations and their properties in the setting of quasicategories see [HTT, 2.4]. We will follow [Maz19a] in deviating somewhat from Lurie's terminology by using a more model-independent definition: For us, a cartesian fibration is a morphism in $\mathcal{C}\mathrm{at}_\infty$ that can be represented by a morphism of quasicategories that is a cartesian fibration in Lurie's sense (see [HTT, 2.4.2.1]). Equivalently, those are the functors of ∞-categories which satisfy condition [HTT, 2.4.1.1 (2)], with the pullback in the definition of cartesian morphisms in [HTT, 2.4.1.1] replaced by the homotopy pullback in $\mathcal{C}\mathrm{at}_\infty$. For a definition along these lines, see [Maz19a, 3]. It is shown in [Maz19a, 4.3 and 4.4] that these two descriptions coincide, and we can thus use the latter model-independent definition while still making use of all the properties of (co)cartesian fibrations proved in [HTT].

In this appendix we collect some statements relating to (co)cartesian fibrations that we need; in Section C.1 we will show a number of stability statements, and in Section C.2 we will discuss compatibility of cocartesian fibrations with products.

C.1. Stability properties of (co)cartesian fibrations

In this section we discuss stability of (co)cartesian fibrations under some constructions. Concretely, in Section C.1.1 we consider pullbacks of cartesian fibrations along any other functor, in Section C.1.2 we discuss a condition under which restrictions of cartesian fibrations along fully faithful functors are again cartesian fibrations, and in Section C.1.3 we show that if $p\colon \mathcal{C} \to \mathcal{D}$ and $q\colon \mathcal{D} \to \mathcal{E}$ are cartesian fibrations, then p is also a morphism of cartesian fibrations from qp to q, i. e. maps qp-cartesian morphisms to q-cartesian morphisms.

Remark C.1.0.1. The definitions of cocartesian and cartesian fibrations are dual to each other: $p\colon \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration if and only if $p^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a cartesian fibration [HTT, 2.4.2.1]. Because of this it suffices to prove many statements for only one of the two (usually cartesian fibrations), the other case following by passing to opposite ∞-categories. To avoid overly long statements we will not state the dual versions in the propositions below, but use them without further comment. $\diamondsuit$

C.1.1. Pullbacks

We record the following fact, that is clear from [HTT, 2.4.1 and 2.4.2], but not stated like this.

Proposition C.1.1.1. *Let*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "F"] \arrow[d, "p'"'] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D}' \arrow[r] & \mathcal{D} \end{tikzcd}$$

be a pullback diagram of ∞-categories where p is a cartesian fibration.

Then p' is also a cartesian fibration and a morphism $\varphi\colon X \to Y$ in $\mathcal{C}'$ is p'-cartesian if and only if $F(\varphi)$ is p-cartesian. ♡

Proof. That p' is also a cartesian fibration is [HTT, 2.4.2.3 (2)], which follows from [HTT, 2.4.1.3 (2)], which also covers the "if"-direction. For the "only if"-direction, let $\varphi\colon X \to Y$ be a p'-cartesian morphism in $\mathcal{C}'$. Then φ is in particular locally p'-cartesian[1], so we can apply [HTT, 2.4.1.12] to conclude that $F(\varphi)$ is locally p-cartesian. As p is a cartesian fibration we can then apply [HTT, 2.4.2.13] to show that $F(\varphi)$ is in fact p-cartesian. □

C.1.2. Restriction along fully faithful functors

Proposition C.1.2.1. *Let $p'\colon \mathcal{C}' \to \mathcal{D}$ be a cartesian fibration of ∞-categories and $\iota\colon \mathcal{C} \to \mathcal{C}'$ a fully faithful functor. Assume that for every object Y in $\mathcal{C}$ and every p'-cartesian morphism $f'\colon X' \to \iota(Y)$ in $\mathcal{C}'$ there is an object X in $\mathcal{C}$ with $\iota(X) \simeq X'$.*

Let $p = p'\iota$. Then p is also a cartesian fibration, and a morphism f in $\mathcal{C}$ is p-cartesian if and only if $\iota(f)$ is p'-cartesian. ♡

Proof. We start by noting that the "if"-direction, i. e. the criterion for checking when a morphism of $\mathcal{C}$ is p-cartesian, follows immediately from [HTT, 2.4.4.3].

We can now use this criterion to show that p has a sufficient supply of cartesian lifts to be a cartesian fibration. So let Y be an object in $\mathcal{C}$ and $g\colon X \to p(Y)$ a morphism in $\mathcal{D}$. Then there exists a p'-cartesian lift $\overline{g}'\colon \overline{X}' \to \iota(Y)$ in $\mathcal{C}'$, as p' is a cartesian fibration. By the assumption on ι, there exists an object $\overline{X}$ of $\mathcal{C}$ such that $\iota(\overline{X}) \simeq \overline{X}'$. As ι is also fully faithful, there exists a morphism $\overline{g}\colon \overline{X} \to Y$ in $\mathcal{C}$ such that $\iota(\overline{g}) \simeq \overline{g}'$ and hence $p(\overline{g}) \simeq g$. We can now use the already proven criterion to deduce that $\overline{g}$ is p-cartesian from $\overline{g}'$ being p'-cartesian. This finishes the proof that p is a cartesian fibration.

Finally, let $f\colon X \to Z$ be a p-cartesian morphism in $\mathcal{C}$. We want to show that $\iota(f)$ is p'-cartesian. In $\mathcal{C}'$ we can factor $\iota(f)$ as $\iota(f) = \psi' \circ \varphi'$, where ψ' is p'-cartesian and φ' is

[1]This follows from the already proved "if"-direction. See [HTT, 2.4.1.11] for a definition of locally p'-cartesian morphisms.

a morphism in $\mathcal{C}'_{p(X)}$, as depicted in the following commutative diagram

$$\begin{tikzcd} \iota(X) \arrow[r, "\varphi'"] \arrow[dr, "\iota(f)"'] & Y' \arrow[d, "\psi'"] \\ & \iota(Z) \end{tikzcd}$$

lying over the following commutative diagram in $\mathcal{D}$.

$$\begin{tikzcd} p(X) \arrow[r, "\mathrm{id}_{p(X)}"] \arrow[dr, "p(f)"'] & p(X) \arrow[d, "p(f)"] \\ & p(Z) \end{tikzcd}$$

Using the assumptions on ι, we can find an object Y in $\mathcal{C}$ together with an equivalence $\vartheta\colon Y' \xrightarrow{\simeq} \iota(Y)$, as well as a commuting diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & Y \arrow[dl, "\psi"] \\ & Z & \end{tikzcd}$$

in $\mathcal{C}$ which maps to the following composite commutative diagram in $\mathcal{C}'$.

$$\begin{tikzcd} \iota(X) \arrow[rr, "\iota(\varphi)"] \arrow[dr, dashed, "\varphi'"] \arrow[ddr, "\iota(f)"'] & & \iota(Y) \arrow[ddl, "\iota(\psi)"] \\ & Y' \arrow[ur, dashed, "\vartheta"'] \arrow[d, dashed, "\psi'"] & \\ & \iota(Z) & \end{tikzcd}$$

As ϑ is an equivalence and ψ' is p'-cartesian, also $\iota(\psi)$ is p'-cartesian, so that we can conclude that ψ is p-cartesian by the already proven “if”-direction. If follows from [HTT, 2.4.1.7] that φ is also p-cartesian. Furthermore, $p(\varphi)$ is an equivalence as the composition of the two equivalences $\mathrm{id}_{p(X)}$ and $p'(\vartheta)$, so by [HTT, 2.4.1.5] φ itself is an equivalence. Thus $\iota(\varphi)$ is an equivalence and hence by [HTT, 2.4.1.5] p'-cartesian, and so $\iota(f)$ is p'-cartesian by [HTT, 2.4.1.7]. □

C.1.3. Morphisms of cartesian fibrations

Proposition C.1.3.1. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "p"] \arrow[dr, "s"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{E} & \end{tikzcd}$$

be a commutative diagram of ∞-categories such that p, q, and s are cartesian fibrations.

Then p is a morphism of cartesian fibrations over $\mathcal{E}$, i. e. maps s-cartesian morphisms to q-cartesian morphisms. ♡

Proof. Let $f\colon X \to Y$ be an s-cartesian morphism in $\mathcal{C}$. As q is a cartesian fibration, there exists a q-cartesian lift $g\colon Z \to p(Y)$ in $\mathcal{D}$ of $s(f)$. As p is a cartesian fibration, we can further lift g to a p-cartesian morphism $f'\colon X' \to Y$ in $\mathcal{C}$. By [HTT, 2.4.1.3 (3)] f' is even s-cartesian, so by uniqueness of cartesian lifts (see [HTT, 2.4.1.9]) f' and f are equivalent as morphisms in $\mathcal{C}$ and hence $p(f) \simeq p(f') \simeq g$ is q-cartesian because g is. □

C.2. Cocartesian fibrations and products

Let $\mathcal{D}$ be an ∞-category and $F\colon \mathcal{D} \to \mathfrak{Cat}_\infty$ a functor. Let $\mathcal{O}$ be an ∞-operad. By [HA, 2.4.2.4] the ∞-category $\mathrm{Mon}_{\mathcal{O}}(\mathfrak{Cat}_\infty)$ of $\mathcal{O}$-monoids in $\mathfrak{Cat}_\infty$ can be identified with the ∞-category of $\mathcal{O}$-monoidal ∞-categories. If F preserves products, then we obtain an induced functor on $\mathcal{O}$-monoids, which we can thus interpret as functorially producing $\mathcal{O}$-monoidal ∞-categories out of $\mathcal{O}$-monoids in $\mathcal{D}$. We will be very interested in this situation in this thesis, in particular in Chapter 3. However, it will usually be easier to construct and work with the cocartesian fibration $p\colon \mathcal{C} \to \mathcal{D}$ associated to F rather than with F directly. For this reason we will start this section by describing the property of F preserving products in terms of the cocartesian fibration p (see Definition C.2.0.1), and will then prove some consequences of this property, as well as one result (see Proposition C.2.0.4) that can help deduce that a cocartesian fibration has this property.

Definition C.2.0.1. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration. We say that p has *fibers compatible with products* if $\mathcal{D}$ admits all products and for any set I and collection of objects Y_i in $\mathcal{D}$ for $i \in I$, the functor

$$\mathcal{C}_{\prod_{i\in I} Y_i} \xrightarrow{\prod_{i\in I}(\mathrm{pr}_i)_!} \prod_{i\in I} \mathcal{C}_{Y_i}$$ ◇

is an equivalence of ∞-categories, where $\mathrm{pr}_j\colon \prod_{i\in I} Y_i \to Y_j$ is the projection and $(\mathrm{pr}_j)_!$ is the functor induced by pr_j on fibers.

Remark C.2.0.2. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration that is classified by a functor $F : \mathcal{D} \to \mathfrak{Cat}_\infty$. Then p has fibers compatible with products if and only if $\mathcal{D}$ admits all products and F preserves products. ◇

If $p\colon \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration whose fibers are compatible with products, then we will see in the next proposition that $\mathcal{C}$ admits all products as well, and p preserves them. In fact we can say more and also describe concretely how to construct products in $\mathcal{C}$.

Proposition C.2.0.3. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration whose fibers are compatible with products in the sense of Definition C.2.0.1.*

Let I be a set and $(X_i)_{i\in I}$ a collection of objects in $\mathcal{C}$. As fibers of p are compatible with products, we obtain the following equivalence.

$$\mathcal{C}_{\prod_{i\in I} p(X_i)} \xrightarrow{\prod_{i\in I}(\mathrm{pr}_i)_!} \prod_{i\in I} \mathcal{C}_{p(X_i)}$$

There thus exists an object X in $\mathcal{C}$ lying over $\prod_{i\in I} p(X_i)$ together with p-cocartesian morphisms $\overline{\mathrm{pr}_i}\colon X \to X_i$ lying over the projections $\mathrm{pr}_i\colon \prod_{i\in I} p(X_i) \to p(X_i)$.

Then the morphisms $\overline{\mathrm{pr}_i}$ exhibit X as a product of the collection of objects X_i for $i \in I$ in $\mathcal{C}$. In particular, $\mathcal{C}$ admits all products and p preserves products. ♡

Proof. We use notation as in the statement. By [HTT, 4.4.1] we need to prove for every object Z of $\mathcal{C}$ that the map

$$\mathrm{Map}_{\mathcal{C}}(Z, X) \xrightarrow{\prod_{i\in I}(\overline{\mathrm{pr}_i}\circ -)} \prod_{i\in I} \mathrm{Map}_{\mathcal{C}}(Z, X_i)$$

is an equivalence. This map fits into the following commutative square as the left vertical map, with the horizontal maps being induced by p.

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(Z, X) \arrow[r] \arrow[d, "\prod_{i\in I}(\overline{\mathrm{pr}_i}\circ -)"'] & \mathrm{Map}_{\mathcal{D}}\left(p(Z), \prod_{i\in I} p(X_i)\right) \arrow[d, "\prod_{i\in I}(\mathrm{pr}_i\circ -)"] \\
\prod_{i\in I} \mathrm{Map}_{\mathcal{C}}(Z, X_i) \arrow[r] & \prod_{i\in I} \mathrm{Map}_{\mathcal{D}}\big(p(Z), p(X_i)\big)
\end{tikzcd} \tag{$*$}$$

As by definition the projections pr_i exhibit $\prod_{i\in I} p(X_i)$ as a product of $(X_i)_{i\in I}$, it follows by [HTT, 4.4.1] that the right vertical map is an equivalence. Let $f\colon p(Z) \to \prod_{i\in I} p(X_i)$ be a morphism. We can extend diagram ($*$) to a morphism of fiber sequences by taking the fiber of the top horizontal map over f and of the lower horizontal map over $(\mathrm{pr}_i \circ f)_{i\in I}$. By the five lemma it will then suffice to show that for every such f the induced map on fibers is an equivalence.

To identify this induced map on fibers, we let $\overline{f}\colon Z \to f_!Z$ be a p-cocartesian lift of f, let $\overline{\mathrm{pr}_j}'\colon f_!Z \to \mathrm{pr}_{j!}(f_!Z)$ be a p-cocartesian lift of $\mathrm{pr}_j\colon \prod_{i\in I} p(X_i) \to p(X_j)$, and ponder the following diagram.

$$\begin{tikzcd}
\operatorname{Map}_{\mathcal{C}_{\prod_{i\in I} p(X_i)}}(f_! Z, X) \arrow[r, "-\circ \overline{f}"] \arrow[d, "pr_!"] & \operatorname{Map}_{\mathcal{C}}(Z, X) \arrow[r, "p"] \arrow[d, "\overline{\mathrm{pr}_j}\circ -"] & \operatorname{Map}_{\mathcal{D}}\left(p(Z), \prod_{i\in I} p(X_i)\right) \arrow[d, "\mathrm{pr}_j \circ -"] \\
\operatorname{Map}_{\mathcal{C}_{p(X_j)}}\left(\mathrm{pr}_{j!}(f_! Z), X_j\right) \arrow[r, "-\circ\left(\overline{\mathrm{pr}_j}' \circ \overline{f}\right)"] & \operatorname{Map}_{\mathcal{C}}(Z, X_j) \arrow[r, "p"] & \operatorname{Map}_{\mathcal{D}}\big(p(Z), p(X_j)\big)
\end{tikzcd} \tag{$**$}$$

The top and bottom rows come with homotopies of the composition to const_f and $\mathrm{const}_{\mathrm{pr}_j \circ f}$, respectively. For the top horizontal sequence this homotopy is indicated in the following diagram, the case for the lower horizontal diagram is analogous.

$$\begin{tikzcd}
& \operatorname{Map}_{\mathcal{C}_{\prod_{i\in I} p(X_i)}}(f_! Z, X) \arrow[dl, "-\circ \overline{f}"'] \arrow[d] \arrow[dd, bend left=60, "\mathrm{const}_{\mathrm{id}}"] \arrow[ddd, bend left=80, "\mathrm{const}_f"] \\
\operatorname{Map}_{\mathcal{C}}(Z, X) \arrow[ddr, bend right, "p"'] & \operatorname{Map}_{\mathcal{C}}(f_! Z, X) \arrow[l, "-\circ \overline{f}"] \arrow[d, "p"] \\
& \operatorname{Map}_{\mathcal{D}}\left(\prod_{i\in I} p(X_i), \prod_{i\in I} p(X_i)\right) \arrow[d, "-\circ f"] \\
& \operatorname{Map}_{\mathcal{D}}\left(p(Z), \prod_{i\in I} p(X_i)\right)
\end{tikzcd}$$

By [HTT, 2.4.4.2 and the discussion preceding it], this homotopy upgrades the top row of diagram ($**$) into a fiber sequence, and analogously for the bottom row.

Unpacking the various definitions we can also upgrade the vertical morphisms in diagram ($**$) into a morphism of fiber sequences. For example commutativity of the left square essentially boils down to the functor $\mathrm{pr}_j\colon \mathcal{C}_{\prod p(X_i)} \to \mathcal{C}_{X_j}$ by definition sending a morphism $g\colon f_! Z \to X$ to the essentially unique morphism $\mathrm{pr}_{i!} g$ that fits in a commutative diagram

$$\begin{tikzcd}
f_! Z \arrow[r] \arrow[d, "g"'] & \mathrm{pr}_{i!} f_! Z \arrow[d, "\mathrm{pr}_{i!} g"] \\
X \arrow[r] & \mathrm{pr}_{i!} X
\end{tikzcd}$$

where the horizontal morphisms are p-cocartesian lifts of pr_i, see [HTT, 5.2.1].

We have thus shown that the induced morphism on fibers (which we have to show is an equivalence) can be identified with the morphism

$$\prod_{i\in I} (\mathrm{pr}_{i!})\colon \operatorname{Map}_{\mathcal{C}_{\prod_{i\in I} p(X_i)}}(f_! Z, X) \to \prod_{i\in I} \operatorname{Map}_{\mathcal{C}_{p(X_j)}}\left(\mathrm{pr}_{j!}(f_! Z), X_j\right)$$

But that this is an equivalence follows immediately from

$$\prod_{i\in I}(\mathrm{pr}_{i!})\colon \mathcal{C}_{\prod_{i\in I}p(X_i)} \to \prod_{i\in I}\mathcal{C}_{p(X_j)}$$

being an equivalence and mapping spaces in products of ∞-categories being equivalent to the respective product of mapping spaces. □

The following proposition will be key to show that some cocartesian fibrations we are interested in have fibers that are compatible with products.

Proposition C.2.0.4. *Let*

$$\begin{tikzcd} \mathcal{C}' \arrow[rr, "F"] \arrow[dr, "p'"'] & & \mathcal{C} \arrow[dl, "p"] \\ & \mathcal{D} & \end{tikzcd}$$

be a morphism of cocartesian fibrations over $\mathcal{D}$ and assume that p' and p have fibers that are compatible with products in the sense of Definition C.2.0.1.

If F is also a cocartesian fibration, then its fibers are also compatible with products. ♡

Proof. Let I be a set and $(X_i)_{i\in I}$ a collection of objects in $\mathcal{C}$. Proposition C.2.0.3 provides us with an object X in $\mathcal{C}_{\prod_{i\in I}p(X_i)}$ together with, for every element j of I, p-cocartesian lifts $\overline{\mathrm{pr}}_i\colon X \to X_j$ of the projections $\mathrm{pr}_j\colon \prod_{i\in I}p(X_i) \to p(X_j)$, such that the collection of morphisms $(\overline{\mathrm{pr}_i})_{i\in I}$ exhibits X as the product of $(X_i)_{i\in I}$ in $\mathcal{C}$.

As F is a morphism of cocartesian fibrations, we obtain a commutative square as depicted as the right hand square in the following diagram.

$$\begin{tikzcd} \mathcal{C}'_X \arrow[r] \arrow[d] & \mathcal{C}'_{\prod_{i\in I}p(X_i)} \arrow[r, "F_{\prod_{i\in I}p(X_i)}"] \arrow[d, "\prod_{i\in I}\left((\mathrm{pr}_i)_!^{p'}\right)"'] & \mathcal{C}_{\prod_{i\in I}p(X_i)} \arrow[d, "\prod_{i\in I}\left((\mathrm{pr}_i)_!^{p}\right)"] \\ \prod_{i\in I}\mathcal{C}'_{X_i} \arrow[r] & \prod_{i\in I}\mathcal{C}'_{p(X_i)} \arrow[r, "\prod_{i\in I}F_{p(X_i)}"] & \prod_{i\in I}\mathcal{C}_{p(X_i)} \end{tikzcd} \tag{$*$}$$

Taking fibers in the horizontal direction, over X in the top line, and over $(X_i)_{i\in I}$ in the bottom line, we obtain the induced commutative square depicted on the left. As by assumption both p' and p have fibers that are compatible with products, the middle and right vertical functors are equivalences, and hence so is the induced left vertical functor. We are not quite done however, as a priori this functor is the induced functor constructed from p'-cocartesian morphisms, whereas we need to show that the functor

$$\prod_{i\in I}(\overline{\mathrm{pr}_i})_!^F : \mathcal{C}'_X \to \prod_{i\in I}\mathcal{C}'_{X_i} \tag{$**$}$$

is an equivalence, which is constructed from F-cocartesian morphisms.

So let Y be an object in $\mathcal{C}'_X$ and let $\overline{\mathrm{pr}_i}'\colon X \to \overline{\mathrm{pr}_i}_!(X)$ be an F-cocartesian lift of $\overline{\mathrm{pr}_i}$. As $\overline{\mathrm{pr}_i}'$ maps under F to the p-cocartesian morphism $\overline{\mathrm{pr}_i}$, we can conclude by [HTT, 2.4.1.3 (3)] that $\overline{\mathrm{pr}_i}'$ is in fact also an p'-cocartesian lift of pr_i. We can thus identify the functor (∗∗) with the left vertical functor in diagram (∗). □

If $p\colon \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration whose fibers are compatible with products, then by Proposition C.2.0.3 $\mathcal{C}$ admits products and p preserves products, so we obtain an induced symmetric monoidal functor $p^\times\colon \mathcal{C}^\times \to \mathcal{D}^\times$ with respect to the cartesian symmetric monoidal structures, see [HA, 2.4.1.8]. It will be useful for us to know that $p^\times$ is again a cocartesian fibration, so we will show this as Proposition C.2.0.6 below, after the following technical prerequisite.

Proposition C.2.0.5. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration with fibers compatible with products in the sense of Definition C.2.0.1. Then products of p-cocartesian morphisms are again p-cocartesian.* ♡

Proof. Let I be a set and let $f_i\colon C_i \to C'_i$ be a p-cocartesian morphism in $\mathcal{C}$ for every element i of I. We have to show that the product $f \coloneqq \prod_{i\in I} f_i\colon \prod_{i\in I} C_i \to \prod_{i\in I} C'_i$ is p-cocartesian. By Proposition C.2.0.3, p preserves products, so f lies over the morphism $\prod_{i\in I} p(f_i)$. We can then factor f as indicated in the following diagram

$$\begin{tikzcd} & \varphi_!\left(\prod_{i\in I} C_i\right) \arrow[d, dashed, "\psi"] \\ \prod\limits_{i\in I} C_i \arrow[ur, "\varphi"] \arrow[r, "f"'] & \prod\limits_{i\in I} C'_i \end{tikzcd}$$

where φ is a p-cocartesian lift of $\prod_{i\in I} p(f_i)$ and ψ lies over $\mathrm{id}_{\prod_{i\in I} p(C'_i)}$. It then suffices to show that ψ is an equivalence.

Let i be an element of I, and let $\overline{\mathrm{pr}_i}\colon \varphi_!(\prod_{i\in I} C_i) \to C''_i$ be a p-cocartesian lift of $\mathrm{pr}_i\colon \prod_{i\in I} p(C'_i) \to p(C'_i)$. It then follows from Proposition C.2.0.3 that the collection $(\overline{\mathrm{pr}_i})_{i\in I}$ exhibits $\varphi_!(\prod_{i\in I} C_i)$ as a product $\prod_{i\in I} C''_i$. Furthermore, ψ induces morphisms $\psi_j\colon C''_j \to C'_j$ for every element j of I as in the following diagram, and ψ can thus be identified with the product $\prod_{i\in I} \psi_i$.

$$\begin{tikzcd} \prod\limits_{i\in I} C''_i \arrow[r, "\psi"] \arrow[d, "\mathrm{pr}_j"] & \prod\limits_{i\in I} C'_i \arrow[d, "\mathrm{pr}_j"] \\ C''_j \arrow[r, dashed, "\psi_j"] & C'_j \end{tikzcd}$$

The following commuting diagram depicts the situation:

$$\begin{tikzcd}
\prod_{i\in I} C_i \arrow[r, "f"] \arrow[d, "\mathrm{pr}_j"] \arrow[rr, bend left, "\varphi"] & \prod_{i\in I} C_i' \arrow[d, "\mathrm{pr}_j"] & \prod_{i\in I} C_i'' \arrow[l, "\prod_{i\in I}\psi_i"'] \arrow[d, "\mathrm{pr}_j"] \\
C_j \arrow[r, "f_j"'] & C_j' & C_j'' \arrow[l, "\psi_j"]
\end{tikzcd}$$

In the outer commuting diagram, all morphisms except possibly ψ_j are p-cocartesian, so by [HTT, 2.4.1.7] also ψ_j is p-cocartesian. It then follows from [HTT, 2.4.1.5] and $p(\psi_j) = \mathrm{id}_{p(C_j)}$ that ψ_j is even an equivalence. Hence $\psi = \prod_{i\in I}\psi_i$ is an equivalence, and thus f is p-cocartesian as it is equivalent to the p-cocartesian morphism φ. □

Proposition C.2.0.6. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration whose fibers are compatible with products in the sense of Definition C.2.0.1. Let $p^\times\colon \mathcal{C}^\times \to \mathcal{D}^\times$ be the induced symmetric monoidal functor between the respective cartesian symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{D}$ as in [HA, 2.4.1.8] (using that $\mathcal{C}$ has all products and p preserves products by Proposition C.2.0.3).*

Then $p^\times$ is also a cocartesian fibration. ♡

Proof. We will apply [GHN15, 9.6][2] to the commutative triangle

$$\begin{tikzcd}
\mathcal{C}^\times \arrow[rr, "p^\times"] \arrow[dr, "q"'] & & \mathcal{D}^\times \arrow[dl, "r"] \\
& \mathsf{Fin}_* &
\end{tikzcd}$$

where q and r are the cocartesian fibrations that are part of the structure of a symmetric monoidal ∞-category. In this situation (the dual version of) [GHN15, 9.6] states that $p^\times$ is a cocartesian fibration if the following hold:

(a) q and r are cocartesian fibrations.

(b) $p^\times$ sends q-cocartesian morphisms to r-cocartesian morphisms.

(c) For each object $\langle n\rangle$ in Fin_*, the induced functor on fibers $p^\times_{\langle n\rangle}\colon \mathcal{C}^\times_{\langle n\rangle} \to \mathcal{D}^\times_{\langle n\rangle}$ is a cocartesian fibration.

(d) Let $n, m \geq 0$, let $f_1, \ldots, f_n, g_1, \ldots, g_m$ be morphisms in $\mathcal{C}$ (with $f_i\colon X_i \to X_i'$ and $g_i\colon Y_i \to Y_i'$), and let φ and ψ be morphisms in $\mathcal{C}^\times$ such that the following square

[2][GHN17] is the published version of [GHN15], but does not contain [GHN15, 9.6].

in $\mathcal{C}^\times$ commutes[3]

$$\begin{tikzcd}
X_1 \oplus \cdots \oplus X_n \arrow[r, "\varphi"] \arrow[d, "f_1 \oplus \cdots \oplus f_n"'] & Y_1 \oplus \cdots \oplus Y_m \arrow[d, "g_1 \oplus \cdots \oplus g_m"] \\
X_1' \oplus \cdots \oplus X_n' \arrow[r, "\psi"'] & Y_1' \oplus \cdots \oplus Y_m'
\end{tikzcd} \tag{$*$}$$

and lies over a commuting square of the following form in Fin_*, with $\alpha\colon \langle n\rangle \to \langle m\rangle$ some morphism.

$$\begin{tikzcd}
\langle n\rangle \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & \langle m\rangle \arrow[d, "\mathrm{id}"] \\
\langle n\rangle \arrow[r, "\alpha"'] & \langle m\rangle
\end{tikzcd}$$

Assume that φ and ψ are q-cocartesian and $f_1 \oplus \cdots \oplus f_n$ is $\left(p^\times\right)_{\langle n\rangle}$-cocartesian. Then $g_1 \oplus \cdots \oplus g_m$ is $\left(p^\times\right)_{\langle m\rangle}$-cocartesian.

Condition (a) holds by definition, and (b) holds as $p^\times$ is a symmetric monoidal functor from $\mathcal{C}^\times$ to $\mathcal{D}^\times$ (see the definition in [HA, 2.1.3.7]). The functor $p^\times_{\langle n\rangle}$ can be identified with $p^{\times n}\colon \mathcal{C}^{\times n} \to \mathcal{D}^{\times n}$, so (c) follows from the fact that products of cocartesian fibrations are again cocartesian fibrations (which follows from [HTT, 2.4.2.3]).

So now suppose we are in the situation of condition (d). We have to show that $g_1 \oplus \cdots \oplus g_m$ is $p^\times_{\langle m\rangle}$-cocartesian. Unpacking the data of the commutative square ($*$) we see that it corresponds to the data of a commutative square

$$\begin{tikzcd}
\prod\limits_{\alpha(i)=j} X_i \arrow[r, "\varphi_j"] \arrow[d, "\prod_{\alpha(i)=j} f_i"'] & Y_j \arrow[d, "g_j"] \\
\prod\limits_{\alpha(i)=j} X_i' \arrow[r, "\psi_j"'] & Y_j'
\end{tikzcd}$$

in $\mathcal{C}$ for every $1 \le j \le m$. That φ and ψ are q-cocartesian implies that φ_j and ψ_j are equivalences, so we can conclude that g_j is equivalent to $\prod_{\alpha(i)=j} f_i$ in $\mathcal{C}$. As $f_1 \oplus \cdots \oplus f_n$ is $p^\times_{\langle n\rangle}$-cocartesian, it follows from the identification $p^\times_{\langle n\rangle} \simeq p^{\times n}$ in combination with [HTT, 3.1.2.1] that f_i is p-cocartesian for each $1 \le i \le n$. Applying Proposition C.2.0.5 we can then conclude that $\prod_{\alpha(i)=j} f_i$ is also p-cocartesian, so g_j is equivalent to a p-cocartesian morphism and thus p-cocartesian as well. Applying the equivalence $p^\times_{\langle m\rangle} \simeq p^{\times m}$ and [HTT, 3.1.2.1] again we conclude that $g_1 \oplus \cdots \oplus g_m$ is $p^\times_{\langle m\rangle}$-cocartesian. □

[3] We are using the notation from [HA, 2.1.1.15]: For $f_1, \ldots, f_n\colon \mathcal{C} \to \mathcal{C}$ we denote by $f_1 \oplus \cdots \oplus f_n$ the morphism in $\mathcal{C}_{\langle n\rangle}$ which under the equivalence $\mathcal{C}_{\langle n\rangle} \simeq \mathcal{C}^n$ corresponds to the tuple $(f_1, \ldots, f_n)$.

Appendix D.

More ∞-category theory

This appendix is really a continuation of Appendix A and collects some facts about more basic concepts of ∞-category theory: Undercategories in Section D.1 and adjunctions in Section D.2.

D.1. Undercategories

In this section we discuss undercategories. [HTT, 1.2.9.5] gives a definition in terms of quasicategories, so we start in Section D.1.1 by providing a model independent construction that can be carried out in $\mathfrak{Cat}_\infty$. We then show in Section D.1.2 that the property of a functor being (fully) faithful or a monomorphism is preserved by passing to induced functors on undercategories. Finally, in Section D.1.3 we describe mapping spaces in an overcategory $\mathcal{C}_{X/}$ as pullbacks of mapping spaces in $\mathcal{C}$.

D.1.1. Model independent construction

Proposition D.1.1.1. *Let $\mathcal{C}$ be an ∞-category and X an object of $\mathcal{C}$. Let $\mathtt{C}$ be a quasicategory representing $\mathcal{C}$ and $\mathtt{X}$ an object of $\mathtt{C}$ representing X.*

Then the undercategory $\mathtt{C}_{\mathtt{X}/}$ defined as in [HTT, 1.2.9.5], together with its projection functor $\mathtt{C}_{\mathtt{X}/} \to \mathtt{C}$ represent the functor

$$\mathrm{ev}_1 \circ \mathrm{pr}_1 \colon \operatorname{Fun}\big([1], \mathcal{C}\big) \times_{\mathcal{C}} \{X\} \to \mathcal{C}$$

in $\mathfrak{Cat}_\infty$, where the pullback is taken with respect to the functor ev_0 and the inclusion of $\{X\}$ into $\mathcal{C}$. ♡

Proof. The inclusion of $\{0\}$ into $[1]$ is a cofibration of simplicial sets, so the functor

$$\mathrm{ev}_0 \colon \mathsf{sSet}\big([1], \mathtt{C}\big) \to \mathtt{C}$$

is a Kan fibration by [Hov99, 4.2.8 and 4.2.2]. In particular, using [HTT, 3.3.1.4 and 2.4.2.4], the pullback (along morphisms like in the statement) $\mathsf{sSet}\big([1], \mathtt{C}\big) \times_{\mathtt{C}} \{\mathtt{X}\}$ is a homotopy pullback in the Joyal model structure, and thus represents $\operatorname{Fun}\big([1], \mathcal{C}\big) \times_{\mathcal{C}} \{X\}$.

The claim now follows from checking that $\mathsf{sSet}\big([1], \mathtt{C}\big)$ satisfied the defining universal property of $\mathtt{C}_{\mathtt{X}/}$ (see [HTT, 1.2.9.5 and 1.2.9.2]). □

D.1.2. Undercategories and (fully) faithful functors, monomorphisms

Proposition D.1.2.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a monomorphism (faithful functor, fully faithful functor) in $\mathcal{C}\mathrm{at}_\infty$ and X an object of $\mathcal{C}$. Then the induced functor on undercategories $\mathcal{C}_{X/} \to \mathcal{D}_{F(X)/}$ is a monomorphism (faithful functor, fully faithful functor) as well.* ♡

Proof. Using the description of undercategories from Proposition D.1.1.1, this follows immediately from Proposition B.5.1.1, Proposition B.3.0.1, and Proposition B.5.3.1. □

D.1.3. Mapping spaces in undercategories

In this section we show that mapping spaces in undercategories can be calculated through the expected pullback diagram. Before we can show this, we need the following small result on how initial objects interact with functors which are retractions.

Proposition D.1.3.1. *Let $\iota\colon \mathcal{C} \to \mathcal{D}$ and $r\colon \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories and assume that $r \circ \iota$ is homotopic to the identity functor.*

Let X be an initial object of $\mathcal{D}$. As X is initial, there is an essentially unique morphism $f\colon X \to \iota r X$ in $\mathcal{D}$. Assume that $rf\colon rX \to r\iota r X$ is an equivalence. Then rX is an initial object of $\mathcal{C}$. ♡

Proof. Let Y be an object of $\mathcal{C}$ and consider the following commutative diagram of mapping spaces.

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(rX, Y) \arrow[d, "\iota"'] & \\
\mathrm{Map}_{\mathcal{D}}(\iota r X, \iota Y) \arrow[r, "f^*"] \arrow[d, "r"'] & \mathrm{Map}_{\mathcal{D}}(X, \iota Y) \arrow[d, "r"] \\
\mathrm{Map}_{\mathcal{C}}(r\iota r X, r\iota Y) \arrow[r, "r(f)^*"'] & \mathrm{Map}_{\mathcal{C}}(rX, r\iota Y)
\end{tikzcd}$$

The left vertical composite is homotopic to the identity by the assumption that $r\iota \simeq \mathrm{id}_{\mathcal{C}}$ and the bottom horizontal functor is an equivalence as $r(f)$ is an equivalence by assumption. As the mapping space in the middle right is contractible by the assumption that X is initial, it thus follows that the top left mapping space $\mathrm{Map}_{\mathcal{C}}(rX, Y)$ is also contractible[1], which is what we need to show. □

Proposition D.1.3.2. *Let $\mathcal{C}$ be an ∞-category, X an object of $\mathcal{C}$, and $f\colon X \to Y$ and $g\colon X \to Z$ morphisms in $\mathcal{C}$. Let $p\colon \mathcal{C}_{X/} \to \mathcal{C}$ be the projection functor.*

Then the commutative diagram in $\mathcal{S}$

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}_{X/}}(f, g) \arrow[r] \arrow[d, "p"'] & \{g\} \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(Y, Z) \arrow[r, "f^*"'] & \mathrm{Map}_{\mathcal{C}}(X, Z)
\end{tikzcd}$$

[1] As a retract of a contractible space.

is a pullback diagram. ♡

Proof. Note that there is a degenerate commutative triangle

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "f"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

in $\mathcal{C}$ that we interpret as a morphism $\mathrm{id}_X \to f$ in $\mathcal{C}_{X/}$, which we will call f'.

By [HTT, 2.1.2.2], $p\colon \mathcal{C}_{X/} \to \mathcal{C}$ is a left fibration, and hence by (the dual of) [HTT, 2.4.2.4] a cocartesian fibration such that every morphism of $\mathcal{C}_{X/}$ is p-cocartesian. Applying (the dual of) [HTT, 2.4.4.3] to the p-cocartesian morphism $f'\colon \mathrm{id}_X \to f$, we obtain the following pullback diagram in $\mathcal{S}$.

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}_{X/}}(f,g) \arrow[r, "f'^*"] \arrow[d, "p"'] & \mathrm{Map}_{\mathcal{C}_{X/}}(\mathrm{id}_X,g) \arrow[d, "p"] \\ \mathrm{Map}_{\mathcal{C}}(Y,Z) \arrow[r, "f^*"'] & \mathrm{Map}_{\mathcal{C}}(X,Z) \end{tikzcd}$$

Note that

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "g"] & \\ X \arrow[rr, "g"'] & & Z \end{tikzcd}$$

is a point in $\mathrm{Map}_{\mathcal{C}_{X/}}(\mathrm{id}_X, g)$ that maps to g under p, so it suffices to show that the mapping space $\mathrm{Map}_{\mathcal{C}_{X/}}(\mathrm{id}_X, g)$ is contractible, i. e. that id_X is an initial object in $\mathcal{C}_{X/}$.

We provide a quick proof for this fact here in the setting of quasicategories. So let $\mathtt{C}$ be a quasicategory and X an object of $\mathtt{C}$. To show that id_X is an initial object of $\mathtt{C}_{X/}$ it suffices by Proposition D.1.3.1[2] to provide a retraction r of the inclusion $\mathtt{C}_{X/} \to \{i\} \star \mathtt{C}_{X/}$ that sends the unique morphism $i \to \mathrm{id}_X$ in $\{i\} \star \mathtt{C}_{X/}$ to an equivalence.

Using the universal property of $\mathtt{C}_{X/}$ (see [HTT, 1.2.9.2]) it suffices for this to give a morphism[3]

$$\varphi\colon \big(\{x\} \star \{i\}\big) \star \mathtt{C}_{X/} \to \mathtt{C}$$

such that the restriction of φ to $\{x\} \star \mathtt{C}_{X/} \to \mathtt{C}$ is adjoint to the identity of $\mathtt{C}_{X/}$ (this corresponds to r being a retraction of the inclusion) and such that the unique 2-simplex

$$\begin{tikzcd} x \arrow[dr] \arrow[dd] & \\ & \mathrm{id}_X \\ i \arrow[ur] & \end{tikzcd}$$

[2] The idea for this argument is from the proof of [HTT, 1.2.12.5].
[3] We are using associativity of the join operation $\star$, see [HTT, 1.2.8].

in $\bigl(\{x\} \star \{i\}\bigr) \star \mathtt{C}_{X/}$ is mapped by φ to the degenerate 2-simplex

$$\begin{tikzcd} X \arrow[dd, "\mathrm{id}_X"'] \arrow[dr, "\mathrm{id}_X"] & \\ & X \\ X \arrow[ur, "\mathrm{id}_X"'] & \end{tikzcd}$$

which covers the condition of the unique morphism $i \to \mathrm{id}_X$ being sent to an equivalence.

We can define such a morphism as follows: Let $q\colon \{x\} \star \{i\} \to \{x\}$ be the unique morphism. Then we take the composite

$$\bigl(\{x\} \star \{i\}\bigr) \star \mathtt{C}_{X/} \xrightarrow{q \star \mathrm{id}_{\mathtt{C}_{X/}}} \{x\} \star \mathtt{C}_{X/} \longrightarrow \mathtt{C}$$

where the second morphism is adjoint to $\mathrm{id}_{\mathtt{C}_{X/}}$. □

D.2. Adjunctions

In this section we discuss adjunctions of ∞-categories. In Section D.2.1 we briefly recall the two equivalent descriptions of adjunctions that are explicitly given in [HTT] and prove that they are equivalent to a third characterization. In Section D.2.2 we discuss the interaction of adjunctions with $\mathrm{Fun}(\mathcal{C}, -)$ for some ∞-category $\mathcal{C}$.

D.2.1. Equivalent characterizations of adjoints

There are several ways to define adjunctions of ∞-categories. The definition used in [HTT] describes adjunctions as cocartesian and cartesian fibrations over $[1]$ (see [HTT, 5.2.2.1]). Lurie also shows that adjunctions are equivalently given by pairs of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ together with a unit transformation $u\colon \mathrm{id}_{\mathcal{C}} \to G \circ F$ satisfying the usual property for mapping spaces (see [HTT, 5.2.2.7 and 5.2.2.8]). We will use both descriptions and refer to [HTT, 5.2.2] for full definitions and how to translate between the two descriptions. We will also need a related third description, which we prove in the next proposition.

Proposition D.2.1.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories, and $\eta\colon \mathrm{id}_{\mathcal{C}} \to G \circ F$ a natural transformation. Then the following are equivalent.*

(1) There exists a natural transformation $\epsilon\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ and the composite natural transformations

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$$

and

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$$

are homotopic to id_F and id_G.

(2) η is a unit transformation for (F, G) in the sense of [HTT, 5.2.2.7]. ♡

Proof. Let us first assume (2). The proof of (1) is really an extension of what is shown in the proof of [HTT, 5.2.2.8], so we will assume the reader is familiar with that proof and sketch the additions that need to be made.

In [HTT, 5.2.2.8], assuming (2), an adjunction $q\colon \mathcal{M} \to [1]$ in the sense of [HTT, 5.2.2.1] associated to F and G is constructed from η. Let $\Phi\colon [1] \times \mathcal{C} \to \mathcal{M}$ be the pointwise (in $\mathcal{C}$) q-cocartesian natural transformation from the inclusion[4] of $\mathcal{C}$ into $\mathcal{M}$ to F exhibiting F as associated to q and similarly Ψ for G.

It is clear from unpacking the definitions, that the unit transformation extracted from q in the other direction of [HTT, 5.2.2.8] can be identified with η. One can extract a natural transformation $\epsilon\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ in a completely analogous manner, as we will also explain in more detail now.

Both natural transformations η and ϵ are obtained are by combining [HTT, 3.1.2.1][5] and [HTT, 2.4.1.4] to lift find fillers in certain diagrams of natural transformations. For example, for ϵ we consider the following diagram of functors $\mathcal{D} \to \mathcal{M}$

$$\begin{tikzcd}
 & \mathrm{id}_{\mathcal{D}} \\
G \arrow[ur, "\Psi"] \arrow[dr, "\Phi G"'] & \\
 & FG \arrow[uu, dashed, "\epsilon"']
\end{tikzcd}$$

where a filler for the dashed arrow and the triangle can be found as the bottom left arrow is cocartesian.

To show that

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$$

is homotopic to the identity, we can now ponder the following diagram of functors $\mathcal{C} \to \mathcal{M}$.

$$\begin{tikzcd}
 & F \\
GF \arrow[ur, "\Psi F"] \arrow[r, "\Phi GF"] \arrow[dr, "\Psi F"] & FGF \arrow[u, dashed, "\epsilon F"'] \\
\mathrm{id}_{\mathcal{C}} \arrow[u, dashed, "\eta"] \arrow[r, "\Phi"'] & F \arrow[u, dashed, "F\eta"']
\end{tikzcd}$$

[4] We identify $\mathcal{C}$ with $\mathcal{M}_0$ and $\mathcal{D}$ with $\mathcal{M}_1$.

[5] That induced functors $q_*\colon \mathrm{Fun}(\mathcal{J}, \mathcal{M}) \to \mathrm{Fun}(\mathcal{J}, [1])$ are again (co)cartesian fibrations and natural transformations are q_*-(co)cartesian if and only if they are pointwise q-(co)cartesian.

The dashed arrow on the left comes with a filler for the triangle at the bottom left and uses that ΨF is q_*-cartesian. The dashed arrow on the bottom right then comes with a filler for the lower square and uses that Φ is q_*-cocartesian. The dashed arrow on the upper right comes with a filler for the upper triangle and uses that ΦGF is q_*-cocartesian. We can thus conclude that $\epsilon F \circ F\eta$ is a filler in the following diagram.

$$\begin{tikzcd} GF \arrow[r, "\Psi F"] & F \\ \mathrm{id}_{\mathcal{C}} \arrow[u, "\eta"] \arrow[r, "\Phi"'] & F \arrow[u, dashed, "\epsilon F \circ F\eta"'] \end{tikzcd}$$

But by definition of η (see the lower left triangle in the previous diagram), one such filler is id_F, so it follows that $\epsilon F \circ F\eta \simeq \mathrm{id}_F$. The other case is completely analogous. This shows (1).

We now assume (1) and show that η is a unit transformation for (F, G). For this we have to show that for every object C in $\mathcal{C}$ and object D in $\mathcal{D}$, the composition

$$\mathrm{Map}_{\mathcal{D}}\big(F(C), D\big) \xrightarrow{G} \mathrm{Map}_{\mathcal{C}}\big(GF(C), G(D)\big) \xrightarrow{(\eta_C)^*} \mathrm{Map}_{\mathcal{C}}\big(C, G(D)\big)$$

is an equivalence. Using ϵ we can define a map in the opposite direction as

$$\mathrm{Map}_{\mathcal{C}}\big(C, G(D)\big) \xrightarrow{F} \mathrm{Map}_{\mathcal{D}}\big(F(C), FG(D)\big) \xrightarrow{(\epsilon_D)_*} \mathrm{Map}_{\mathcal{D}}\big(F(C), D\big)$$

and it follows immediately from (1) that these two maps are inverse equivalences. □

D.2.2. Adjunctions and Fun

In this short section we show that for $\mathcal{C}$ an ∞-category, the functor $\mathrm{Fun}(\mathcal{C}, -)$ preserves adjunctions in a manner made precise in the next proposition.

Proposition D.2.2.1. *Let $p\colon \mathcal{M} \to [1]$ be a cartesian and cocartesian functor, and $F\colon \mathcal{M}_0 \to \mathcal{M}_1$ the corresponding left adjoint, $G\colon \mathcal{M}_1 \to \mathcal{M}_0$ the corresponding right adjoint, and $u\colon \mathrm{id}_{\mathcal{M}_0} \to G \circ F$ the corresponding unit transformation.*

Let $\mathcal{C}$ be an ∞-category. Then the functor $p'\colon \mathcal{M}' \to [1]$ that is defined by the following pullback diagram

$$\begin{tikzcd} \mathcal{M}' \arrow[r] \arrow[d, "p'"'] & \mathrm{Fun}(\mathcal{C}, \mathcal{M}) \arrow[d, "p_*"] \\ {[1]} \arrow[r, "\mathrm{const}"'] & \mathrm{Fun}\big(\mathcal{C}, [1]\big) \end{tikzcd}$$

is also a cartesian and cocartesian fibration and hence defines an adjunction. Furthermore, the fibers $\mathcal{M}'_0$ and $\mathcal{M}'_1$ can be identified with $\mathrm{Fun}(\mathcal{C}, \mathcal{M}_0)$ and $\mathrm{Fun}(\mathcal{C}, \mathcal{M}_1)$, and under this identification the encoded left adjoint can be identified with F_*, the encoded right adjoint with G_*, and the corresponding unit transformation with u_*. ♡

Proof. That p' is again a cartesian and cocartesian fibration follows from [HTT, 3.1.2.1] and Proposition C.1.1.1. Using composability of pullback diagrams and $\mathrm{Fun}(\mathcal{C}, -)$ preserving pullbacks we obtain the following chain of equivalences with which we can identify $\mathcal{M}'_i$ as stated.

$$\begin{aligned}
\mathcal{M}'_i &\simeq \mathrm{Fun}(\mathcal{C}, \mathcal{M}) \times_{\mathrm{Fun}(\mathcal{C},[1])} \{\mathrm{const}_i\} \\
&\simeq \mathrm{Fun}(\mathcal{C}, \mathcal{M}) \times_{\mathrm{Fun}(\mathcal{C},[1])} \mathrm{Fun}\big(\mathcal{C}, \{i\}\big) \\
&\simeq \mathrm{Fun}\big(\mathcal{C}, \mathcal{M} \times_{[1]} \{i\}\big) \\
&\simeq \mathrm{Fun}(\mathcal{C}, \mathcal{M}_i)
\end{aligned}$$

Let the commuting diagram

$$\begin{tikzcd}
\mathcal{M}_0 \times [1] \arrow[rr, "F'"] \arrow[dr, "\mathrm{pr}_2"'] & & \mathcal{M} \arrow[dl, "p"] \\
& {[1]} &
\end{tikzcd}$$

exhibit F as the left adjoint to p (see [HA, 5.2.1.1 and 5.2.2.1]). We can then construct a diagram exhibiting F_* as the left adjoint to p' as indicated in the following diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}, \mathcal{M}_0) \times [1] \arrow[rr] \arrow[dr, dashed, "(F_*)'"] \arrow[ddr, "\mathrm{pr}_2"'] & & \mathrm{Fun}\big(\mathcal{C}, \mathcal{M}_0 \times [1]\big) \arrow[d, "F'_*"] \arrow[dd, bend left=60, "(\mathrm{pr}_2)_*"] \\
& \mathcal{M}' \arrow[r] \arrow[d, "p'"] & \mathrm{Fun}(\mathcal{C}, \mathcal{M}) \arrow[d, "p_*"] \\
& {[1]} \arrow[r, "\mathrm{const}"'] & \mathrm{Fun}\big(\mathcal{C}, [1]\big)
\end{tikzcd}$$

where the top horizontal functor is the composition

$$\mathrm{Fun}(\mathcal{C}, \mathcal{M}_0) \times [1] \xrightarrow{\mathrm{id} \times \mathrm{const}} \mathrm{Fun}(\mathcal{C}, \mathcal{M}_0) \times \mathrm{Fun}\big(\mathcal{C}, [1]\big) \xrightarrow{\simeq} \mathrm{Fun}\big(\mathcal{C}, \mathcal{M}_0 \times [1]\big)$$

That $(F_*)'$ as constructed in the above diagram indeed exhibits F_* as the left adjoint associated to p' follows from the description of cocartesian morphisms in [HTT, 3.1.2.1] and Proposition C.1.1.1.

The statements regarding G_* and u_* can be proven analogously. □

Appendix E.

∞-operads and algebras

This appendix collects various results concerning ∞-operads and their ∞-categories of algebras.

We begin in Section E.1 with generic facts on (morphisms of) ∞-operads. For most of the remaining sections we then turn towards ∞-categories of algebras. In Section E.2 we will look into the relationship between ∞-categories of algebras and base changes of ∞-operads, and in Section E.3 we show that passing from morphisms of ∞-operads to functors between the respective ∞-categories of algebras preserves various properties.

If $\mathcal{O}$ is an ∞-operad and $\mathcal{C}$ is a symmetric monoidal ∞-category, then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ inherits an induced symmetric monoidal structure, which will be discussed in Section E.4. If $\mathcal{O}'$ is another ∞-operad, then the symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ allows us to take $\mathcal{O}'$-algebras in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. In Section E.5 we will show that there is another way to describe $\mathcal{O}'$-algebras in $\mathcal{O}$-algebras in $\mathcal{C}$, namely as $\mathcal{O} \otimes \mathcal{O}'$-algebras in $\mathcal{C}$. In Section E.6 we then discuss the commutative ∞-operad Comm and show that the tensor product of ∞-operads of any ∞-operad $\mathcal{O}$ with Comm is equivalent to Comm again.

In Section E.7 we discuss colimits of algebras as well as free algebras, and in particular when they are preserved by induced functors on algebra ∞-categories. Finally, in Section E.8 we discuss relative tensor products and when monoidal functors preserve them. We also show that pushouts of commutative algebras are given by relative tensor products.

E.1. ∞-operads

In this section we collect three statements relating to properties of morphisms of ∞-operads or helpful for showing that a functor is a morphism of ∞-operads or a symmetric monoidal functor. Concretely, Section E.1.1 helps showing that a morphism of ∞-operads between symmetric monoidal ∞-categories is symmetric monoidal, Section E.1.2 is about consequences of a morphism of ∞-operads being conservative, and Section E.1.3 discusses functors that are pullbacks of a morphism of ∞-operads along a cocartesian fibration of ∞-operads and vice versa.

E.1.1. Symmetric monoidal functors

By definition[1], a morphism of ∞-operads between symmetric monoidal ∞-categories is symmetric monoidal if it is a morphism of cocartesian fibrations, so preserves all cocartesian morphisms[2]. In the following proposition, we show that it suffices to check cocartesian lifts of two select morphisms in Fin_*: The multiplication $\mu\colon \langle 2\rangle \to \langle 1\rangle$ and unit $\epsilon\colon \langle 0\rangle \to \langle 1\rangle$. This is an analogue of [HA, 2.1.2.9] which similarly reduces the amount of inert morphisms that need to be checked to verify a functor over Fin_* is a morphism of ∞-operads.

Proposition E.1.1.1. *Let*

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "p_{\mathcal{C}}"'] & & \mathcal{D}^{\otimes} \arrow[dl, "p_{\mathcal{D}}"] \\ & \mathsf{Fin}_* & \end{tikzcd}$$

be a commutative diagram of morphisms of ∞-operads, and assume that $p_{\mathcal{C}}$ and $p_{\mathcal{D}}$ exhibit $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ as symmetric monoidal ∞-categories. Then the following two conditions are equivalent.

(1) $F^{\otimes}$ is symmetric monoidal, i. e. maps $p_{\mathcal{C}}$-cocartesian morphisms to $p_{\mathcal{D}}$-cocartesian morphisms.

(2) $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian lifts of the active morphism[3] $\mu\colon \langle 2\rangle \to \langle 1\rangle$ and $p_{\mathcal{C}}$-cocartesian lifts of the unique morphism $\epsilon\colon \langle 0\rangle \to \langle 1\rangle$ to $p_{\mathcal{D}}$-cocartesian morphisms. ♡

Proof. It is clear that (1) implies (2), so it remains to show the converse direction. Morphisms in Fin_* are generated (by composition) by morphisms of the following forms (compare [HA, 2.1.2.2]).

(A) Inert morphisms[4].

(B) For every $n \geq 1$ the morphism $\mu_n\colon \langle n+1\rangle \to \langle n\rangle$ that sends an element i of $\langle n+1\rangle^{\circ}$ to i if $i \leq n$, and to n otherwise[5].

(C) For every $n \geq 0$ the inclusion $\epsilon_n\colon \langle n\rangle \to \langle n+1\rangle$ (i. e. sending i to i).

As the collection of cocartesian morphisms is closed under composition [HTT, 2.4.1.7] and cocartesian lifts with fixed source object are unique up to equivalence [HTT, 2.4.1.9], it suffices to prove that $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian lifts of morphisms of type (A), (B),

[1] See [HA, 2.1.3.7].
[2] With respect to the respective canonical cocartesian fibrations of ∞-operads to Fin_*.
[3] So this is the morphism that sends 1 and 2 to 1.
[4] Note that in particular all isomorphisms are inert.
[5] So n is the unique element of the target that has two preimages, n and $n+1$.

and (C) to $p_{\mathcal{D}}$-cocartesian morphisms. By assumption we already know that $F^{\otimes}$ is a morphism of ∞-operads and hence preserves inert morphisms, so this covers type (A).

We now show that $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian lifts of morphisms of type (B) to $p_{\mathcal{D}}$-cocartesian morphisms. So let $n \geq 1$, let μ_n be the morphism of Fin_* defined in (B), and let $f\colon X \to Y$ be a $p_{\mathcal{C}}$-cocartesian lift of μ_n. As $p_{\mathcal{D}}$ is a cocartesian fibration, we can lift μ_n to a $p_{\mathcal{D}}$-cocartesian morphism $\overline{f}\colon F^{\otimes}(X) \to (\mu_n)_!(F^{\otimes}(X))$, and obtain an induced morphism g lying over $\mathrm{id}_{\langle n\rangle}$, such that there is a commutative diagram as follows.

$$\begin{tikzcd}
 & (\mu_n)_!\big(F^{\otimes}(X)\big) \arrow[d, dashed, "g"] \\
F^{\otimes}(X) \arrow[ur, "\overline{f}"] \arrow[r, "F^{\otimes}(f)"'] & F^{\otimes}(Y)
\end{tikzcd}$$

By [HTT, 2.4.1.7 and 2.4.1.5], $F^{\otimes}(f)$ is $p_{\mathcal{D}}$-cocartesian if and only if g is an equivalence, so we prove the latter.

Let us first consider $\rho^j_!(g)$ for $1 \leq j < n$. This is the induced morphism indicated in the following diagram, where $\overline{r}$ and r are $p_{\mathcal{D}}$-cocartesian lifts of ρ^j.

$$\begin{tikzcd}
 & (\mu_n)_!\big(F^{\otimes}(X)\big) \arrow[r, "\overline{r}"] \arrow[d, dashed, "g"] & (\rho^j \circ \mu_n)_!\big(F^{\otimes}(X)\big) \arrow[d, dashed, "\rho^j_!(g)"] \\
F^{\otimes}(X) \arrow[ur, "\overline{f}"] \arrow[r, "F^{\otimes}(f)"'] & F^{\otimes}(Y) \arrow[r, "r"'] & \rho^j_!\big(F^{\otimes}(Y)\big)
\end{tikzcd}$$

But note that for $1 \leq j < n$ the composition $\rho^j \circ \mu_n$ is equal to ρ^j. The morphism $\rho^j_!(g)$ is thus also equivalent to the morphism $g_j\colon \rho^j_!(F^{\otimes}(X)) \to \rho^j_!\big(F^{\otimes}(Y)\big)$ induced by $r \circ F^{\otimes}(f)$. Now let

$$\begin{tikzcd}
 & Y \arrow[r, "s"] \arrow[d, dashed, "\mathrm{id}"] & \rho^j_!(Y) \arrow[d, dashed, "\rho^j_!(\mathrm{id})"] \\
X \arrow[ur, "f"] \arrow[r, "f"'] & Y \arrow[r, "s"'] & \rho^j_!(Y)
\end{tikzcd}$$

be the diagram constructed completely analogously from f in $\mathcal{C}^{\otimes}$, with s a $p_{\mathcal{C}}$-cocartesian lift of ρ^j. In this case we can use f itself as a $p_{\mathcal{C}}$-cocartesian lift of μ_n, and the identity morphism can play the role of g. In particular, the morphism $f_j\colon \rho^j_!(Y) \to \rho^j_!(Y)$ induced by $s \circ f$ is an equivalence. As $F^{\otimes}$ preserves inert morphisms $F^{\otimes}(s)$ can be identified with r, and $F^{\otimes}(s \circ f)$ with $\overline{r} \circ \overline{f}$. This implies that $F^{\otimes}(f_j) \simeq g_j$, and as $F^{\otimes}$ preserves equivalences, g_j must be an equivalence.

Let us now consider $\rho^n(g)$. In this case $\rho^n \circ \mu_n$ is not ρ^n, but $\mu \circ \rho^{n,n+1}$, where $\rho^{n,n+1}\colon \langle n+1\rangle \to \langle 2\rangle$ maps i to $*$ if $i < n$, maps n to 1, and maps $n+1$ to 2. We can argue completely analogously to the previous case, but have to additionally use that $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian lifts of μ to $p_{\mathcal{D}}$-cocartesian morphisms, which is the case by assumption (2).

As the functor

$$\mathcal{D}_{\langle n\rangle} \xrightarrow{\prod_{1\leq j\leq n}\rho^j_!} \mathcal{D}^{\times n}_{\langle 1\rangle}$$

is an equivalence and we showed that $\rho^j_!(g)$ is an equivalence for every $1 \leq j \leq n$, we can conclude that g is an equivalence. Thus we have shown that $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian lifts of morphisms of type (B) to $p_{\mathcal{D}}$-cocartesian morphisms.

The case of morphisms of type (C) is similar, in this case we will need to use the assumption regarding ϵ. □

E.1.2. Conservative morphisms of ∞-operads

In the following proposition we record a very useful consequence of a morphism of ∞-operads being conservative.

Proposition E.1.2.1. *Let*

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "p_{\mathcal{C}}"'] & & \mathcal{D}^{\otimes} \arrow[dl, "p_{\mathcal{D}}"] \\ & \mathsf{Fin}_* & \end{tikzcd}$$

be a commutative diagram of morphisms of ∞-operads, and assume that $F^{\otimes}$ is a conservative functor, i. e. reflects equivalences. Then the following hold.

(1) A morphism f in $\mathcal{C}^{\otimes}$ is inert if and only if $F^{\otimes}(f)$ is inert.

(2) Assume that $p_{\mathcal{C}}$ and $p_{\mathcal{D}}$ exhibit $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ as symmetric monoidal ∞-categories, and that $F^{\otimes}$ is symmetric monoidal. Then a morphism f in $\mathcal{C}^{\otimes}$ is $p_{\mathcal{C}}$-cocartesian if and only if $F^{\otimes}(f)$ is $p_{\mathcal{D}}$-cocartesian. ♡

Proof. In both cases the “only if”-direction is handled directly by the assumption that $F^{\otimes}$ is a morphism of ∞-operads, and that $F^{\otimes}$ is even symmetric monoidal in the case of (2).

We will prove the “if”-direction of both (1) and (2) at the same time. So let $f\colon X \to Y$ be a morphism in $\mathcal{C}^{\otimes}$ that lies over a morphism φ in Fin_* and is mapped by $F^{\otimes}$ to a $p_{\mathcal{D}}$-cocartesian morphism in $\mathcal{D}^{\otimes}$. For (1) assume additionally that φ is inert. We have to show that f is $p_{\mathcal{D}}$-cocartesian.

We can factor f as indicated in the following commutative diagram in $\mathcal{C}^{\otimes}$

$$\begin{tikzcd} & \varphi_! X \arrow[d, dashed, "f''"] \\ X \arrow[ur, "f'"] \arrow[r, "f"'] & Y \end{tikzcd}$$

such that f' is $p_{\mathcal{C}}$-cocartesian and f'' lies over an identity morphism in Fin_*. Both $F^{\otimes}(f')$ and $F^{\otimes}(f)$ are $p_{\mathcal{D}}$-cocartesian morphisms, so by [HTT, 2.4.1.7 and 2.4.1.5] $F^{\otimes}(f'')$ is an equivalence. As $F^{\otimes}$ is conservative, it follows that f'' is also an equivalence, which by [HTT, 2.4.1.7 and 2.4.1.5] implies that f is $p_{\mathcal{C}}$-cocartesian. □

E.1.3. Base changes of cocartesian fibrations of ∞-operads

By Proposition C.1.1.1 a pullback of a cocartesian fibration along any functor is again a cocartesian fibration. The next proposition can be considered an upgrade of this statement to the situation in which both functors are morphisms of ∞-operads.

Proposition E.1.3.1. *Let*

$$\begin{tikzcd} \mathcal{C}'^{\otimes} \arrow[r, "q"] \arrow[d, "p'"'] & \mathcal{C}^{\otimes} \arrow[d, "p"] & \\ \mathcal{O}'^{\otimes} \arrow[r, "r"'] & \mathcal{O}^{\otimes} \arrow[r, "p_{\mathcal{O}}"'] & \mathsf{Fin}_* \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}\mathrm{at}_\infty$ such that the square is a pullback square, $p_{\mathcal{O}}$ and r are morphisms of ∞-operads, and p is a cocartesian fibration of ∞-operads.

Then p' is a cocartesian fibration of ∞-operads and q is a morphism of ∞-operads. Furthermore, a morphism f in $\mathcal{C}'^{\otimes}$ is inert if and only if $q(f)$ and $p'(f)$ are inert. ♡

Proof. By Proposition C.1.1.1 p' is a cocartesian fibration, and the description of p'-cocartesian morphisms also implies that if $n \geq 0$ and X_i are objects in $\mathcal{O}'$ for $1 \leq i \leq n$, and $f_i \colon X_1 \oplus \cdots \oplus X_n \to X_i$ are the canonical inert morphisms in $\mathcal{O}'^{\otimes}$, then the induced functor on fibers

$$\mathcal{C}'^{\otimes}_{X_1 \oplus \cdots \oplus X_n} \xrightarrow{\prod_{1 \leq i \leq n} f_{i!}} \mathcal{C}'^{\otimes}_{X_i} \tag{$*$}$$

can be identified with the following functor that is induced on the fibers of p.

$$\mathcal{C}^{\otimes}_{r(X_1 \oplus \cdots \oplus X_n)} \xrightarrow{\prod_{1 \leq i \leq n} r(f_i)_!} \mathcal{C}^{\otimes}_{r(X_i)}$$

As r is a morphism of ∞-operads we can for each $1 \leq i \leq n$ identify $r(f_i)$ with the inert morphism $r(X_1) \oplus \cdots \oplus r(X_n) \to r(X_i)$. As p is a cocartesian fibration of ∞-operads, it thus follows that $(*)$ is an equivalence, so p' is a cocartesian fibration of ∞-operads[6].

Let f be a morphism in $\mathcal{C}'^{\otimes}$. It remains to show that f is inert if and only if $q(f)$ and $p'(f)$ are inert. Denote the compositions from the four ∞-categories in the square to Fin_* by p with subscript the name of the underlying ∞-category.

Assume that f is inert. Then f is by definition $p_{\mathcal{C}'}$-cocartesian, and as p' preserves inert morphisms, $p'(f)$ is inert, so $p_{\mathcal{O}'}$-cocartesian. It follows from [HTT, 2.4.1.3 (3)] that f is p'-cocartesian. By Proposition C.1.1.1 we then obtain that $q(f)$ is p-cocartesian. Furthermore, $p(q(f)) = r(p'(f))$ is inert, i. e. $p_{\mathcal{O}}$-cocartesian, as r is a morphism of ∞-operads. We can again use [HTT, 2.4.1.3 (3)] to conclude that $q(f)$ is $p_{\mathcal{C}}$-cocartesian, so inert.

Now assume that $q(f)$ and $p'(f)$ are inert. Again, as r is a morphism of ∞-operads, $p(q(f)) = r(p'(f))$ is inert, so by [HTT, 2.4.1.3 (3)] $q(f)$ is p-cocartesian, which by Proposition C.1.1.1 implies that f is p'-cocartesian, from which we can deduce with another application of [HTT, 2.4.1.3 (3)] that f is $p_{\mathcal{C}'}$-cocartesian, so inert. □

[6] See [HA, 2.1.2.13 and 2.1.2.12].

E.2. Alg and base change

This section concerns the interaction of Alg with base changes, with the upshot being the following. Given a commutative diagram

$$\begin{tikzcd} & \mathcal{C}'^{\otimes} \arrow[r, "F^{\otimes}"] \arrow[d, "p'"'] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ \mathcal{O}''^{\otimes} \arrow[r, "\alpha"'] & \mathcal{O}'^{\otimes} \arrow[r, "\beta"'] & \mathcal{O}^{\otimes} \end{tikzcd}$$

of ∞-operads such that the square is a pullback diagram in $\mathfrak{Cat}_\infty$, we will obtain an induced pullback diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{C}') \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}''}(p)"] \\ \{\beta \circ \alpha\} \arrow[r] & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{O}) \end{tikzcd}$$

in $\mathfrak{Cat}_\infty$ of ∞-categories of algebras.

Construction E.2.0.1. Let

$$\begin{tikzcd} & \mathcal{C}'^{\otimes} \arrow[r, "F^{\otimes}"] \arrow[d, "p'"'] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ \mathcal{O}''^{\otimes} \arrow[r, "\alpha"'] & \mathcal{O}'^{\otimes} \arrow[r, "\beta"'] & \mathcal{O}^{\otimes} \end{tikzcd}$$

be a commutative diagram of ∞-operads such that the square is a pullback diagram in $\mathfrak{Cat}_\infty$.

Applying $\mathrm{Fun}(\mathcal{O}''^{\otimes}, -)$ to the pullback square we obtain the pullback on the right in the following diagram, with the left square a pullback square as well, by definition.

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{O}'^{\otimes}}\big(\mathcal{O}''^{\otimes}, \mathcal{C}'^{\otimes}\big) \arrow[r] \arrow[d] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{C}'^{\otimes}\big) \arrow[r, "F^{\otimes}_*"] \arrow[d, "p'_*"'] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[d, "p_*"] \\ \{\alpha\} \arrow[r] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{O}'^{\otimes}\big) \arrow[r, "\beta_*"'] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{O}^{\otimes}\big) \end{tikzcd}$$

Comparing the combined outer pullback square [HTT, 4.4.2.1] to the pullback square

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[r] \arrow[d] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[d, "p_*"] \\ \{\beta \circ \alpha\} \arrow[r] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{O}^{\otimes}\big) \end{tikzcd}$$

we obtain a canonical equivalence

$$\mathrm{Fun}_{\mathcal{O}'^{\otimes}}\big(\mathcal{O}''^{\otimes}, \mathcal{C}'^{\otimes}\big) \simeq \mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}\big)$$

of ∞-categories. ◊

Proposition E.2.0.2. *In the situation of Construction E.2.0.1 the equivalence*

$$\mathrm{Fun}_{\mathcal{O}'^{\otimes}}(\mathcal{O}''^{\otimes}, \mathcal{C}'^{\otimes}) \simeq \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes})$$

restricts to an equivalence on the full subcategories of algebras as follows.

$$\mathrm{Alg}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{C}') \simeq \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}}(\mathcal{C}) \qquad \heartsuit$$

Proof. Unpacking the definitions the statement boils down to the following: Let

$$\begin{tikzcd} \mathcal{O}''^{\otimes} \arrow[rr, "A"] \arrow[dr, "\alpha"'] & & \mathcal{C}'^{\otimes} \arrow[dl, "p'"] \\ & \mathcal{O}'^{\otimes} & \end{tikzcd}$$

be a commuting diagram and let f be an inert morphism in $\mathcal{O}''^{\otimes}$. Denote by $q\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ the unique morphism of ∞-operads. We have to show that $A(f)$ is $q\beta p'$-cocartesian if and only if $F^{\otimes}(A(f))$ is qp-cocartesian.

As α is a morphism of ∞-operads, it preserves inert morphisms, so $\alpha(f) = p'(A(f))$ is $q\beta$-cocartesian. Then [HTT, 2.4.1.3 (3)] implies that $A(f)$ is $q\beta p'$-cocartesian if and only if $A(f)$ is p'-cocartesian. By Proposition C.1.1.1 $A(f)$ is p'-cocartesian if and only if $F^{\otimes}(A(f))$ is p-cocartesian. But as $\beta \circ \alpha$ preserves inert morphisms, $\beta(\alpha(f))$ is q-cocartesian, so again by [HTT, 2.4.1.3 (3)] $F^{\otimes}(A(f))$ is p-cocartesian if and only if $F^{\otimes}(A(f))$ is qp-cocartesian. $\square$

Proposition E.2.0.3 ([HA, 2.1.3.1])**.** *Let $\gamma\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ and $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be morphisms of ∞-operads. Then the pullback diagram of ∞-categories*

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[d, "p_*"] \\ \{\gamma\} \arrow[r] & \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}) \end{tikzcd}$$

induces on full subcategories a pullback diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}'}(p)"] \\ \{\gamma\} \arrow[r] & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}) \end{tikzcd}$$

of ∞-categories[7]*.* $\heartsuit$

[7] We are using the the definition given in [HA, 2.1.3.1] for $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ as a full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$. The alternative description as the pullback given in this statement is also mentioned in [HA, 2.1.3.1].

Proof. There is a commutative cube in $\mathfrak{Cat}_\infty$

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \arrow[dl] \arrow[dd] \\
\mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes) \arrow[rr, crossing over] \arrow[dd] & & \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes) & \\
& \{\gamma\} \arrow[rr] \arrow[dl] & & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}) \arrow[dl] \\
\{\gamma\} \arrow[rr] & & \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{O}^\otimes) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

with all functors from the back to the front inclusions of full subcategories. One can use Proposition B.5.2.1 to show that the top and bottom squares are pullback squares as follows: For the top square, consider the induced diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[dr, dashed, "\theta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{D} \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \arrow[d] \\
& \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes) \arrow[r] & \mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)
\end{tikzcd}$$

where $\mathcal{D}$ is constructed as a pullback of the square. The right vertical functor is fully faithful, so by Proposition B.5.2.1 the left vertical functor is fully faithful as well. As we also know that the functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$ is also fully faithful, it follows that the induced functor θ is fully faithful too. To show that θ is an equivalence it thus suffices to show essential surjectivity [HTT, 1.2.10]. As $\mathcal{D} \to \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$ is fully faithful, an object in $\mathcal{D}$ can be thought of as an object in $\mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$, i. e. a commutative triangle

$$\begin{tikzcd}
\mathcal{O}'^\otimes \arrow[rr, "A"] \arrow[dr, "\gamma"'] & & \mathcal{C}^\otimes \arrow[dl, "p"] \\
& \mathcal{O}^\otimes &
\end{tikzcd} \tag{E.1}$$

such that the corresponding object in $\mathrm{Fun}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$, i. e. A, lies in $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$, i. e. A must be a morphism of ∞-operads. But this is precisely the condition for an object of $\mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$ to lie in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$, so θ is essentially surjective and hence an equivalence, which implies that the top square of the cube is a pullback diagram. That the bottom square is a pullback diagram can be proven analogously.

As the top and front of the cube are pullback diagrams, the composite of those two squares is a pullback diagram as well by [HTT, 4.4.2.1]. This composite is equivalent to the composite formed by the back and bottom squares, so using that the bottom square is a pullback and the other direction of [HTT, 4.4.2.1] we can conclude that the back square is a pullback as well. $\square$

Remark E.2.0.4. Combining Proposition E.2.0.2 and Proposition E.2.0.3 in the situation of Construction E.2.0.1, we obtain the following pullback diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}''/\mathcal{O}'}(\mathcal{C}') \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}''}(p)"] \\ \{\beta \circ \alpha\} \arrow[r] & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{O}) \end{tikzcd}$$

in Cat_∞. Tracing through the definitions is is not difficult to see that this square is also natural in $\mathcal{C}$ (with $\mathcal{O}$,$\mathcal{O}'$, and $\mathcal{O}''$ staying fixed and $\mathcal{C}'$ changing with $\mathcal{C}$ as a pullback). $\diamondsuit$

E.3. Properties preserved by Alg

In this section we show that passing to ∞-categories of algebras preserves several properties of functors. Specifically, we discuss pullbacks in Section E.3.1, cocartesian fibrations in Section E.3.2, adjoints in Section E.3.3, the property of a functor being conservative in Section E.3.4, and fully faithfulness in Section E.3.5.

E.3.1. Pullbacks

Proposition E.3.1.1. *Let*

$$\begin{tikzcd} \mathcal{C}^\otimes \arrow[r, "F^\otimes"] \arrow[d, "G^\otimes"'] & \mathcal{D}^\otimes \arrow[d, "H^\otimes"] & \mathcal{O}'^\otimes \arrow[d, "\alpha^\otimes"] \\ \mathcal{E}^\otimes \arrow[r, "K^\otimes"'] & \mathcal{F}^\otimes \arrow[r, "p_{\mathcal{F}}"'] & \mathcal{O}^\otimes \end{tikzcd}$$

be a commutative diagram of ∞-operads such that the square is a pullback diagram in Cat_∞. Assume furthermore that a morphism f in $\mathcal{C}^\otimes$ is inert if and only if $F^\otimes(f)$ and $G^\otimes(f)$ are inert.

Then the induced commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(F)"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(G)"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(H)"] \\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{E}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(K)"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{F}) \end{tikzcd}$$

is a pullback diagram in Cat_∞. $\heartsuit$

Proof. As $\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, -\big)$ preserves pullbacks and limits commute with each other, we first obtain an induced pullback square as follows.

$$\begin{tikzcd}
\mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[r, "F_*^{\otimes}"] \arrow[d, "G_*^{\otimes}"'] & \mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \arrow[d, "H_*^{\otimes}"] \\
\mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{E}^{\otimes}\big) \arrow[r, "K_*^{\otimes}"'] & \mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{F}^{\otimes}\big)
\end{tikzcd}$$

Let $\mathcal{P}$ be defined to be the pullback in the square in the following diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[drrr, bend left, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(F)"] \arrow[drr, dashed, "\varphi"] \arrow[dd, bend right, "\iota"'] \arrow[ddrr, bend right, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(G)"] & & & \\
& & \mathcal{P} \arrow[r] \arrow[d] \arrow[dll, "\psi"] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(H)"] \\
\mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) & & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{E}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(K)"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{F})
\end{tikzcd}$$

where φ and ψ are the induced functors, and ι is the canonical fully faithful inclusion. It suffices to show that φ is an equivalence. By Proposition B.5.3.1, ψ is fully faithful with essential image spanned by those functors $A\colon \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ over $\mathcal{O}^{\otimes}$ whose compositions with $F^{\otimes}$ and $G^{\otimes}$ send inert morphisms to inert morphisms. But by the assumptions on inert morphisms in $\mathcal{C}^{\otimes}$, this means that the essential image of ψ is exactly the essential image of ι. It now follows from Proposition B.4.4.1 and Proposition B.4.3.1 that φ is an equivalence. □

E.3.2. Cocartesian fibrations

Proposition E.3.2.1. *Let*

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q"] \\
& \mathcal{O}^{\otimes} &
\end{tikzcd}$$

be a commuting diagram of maps of ∞-operads, and let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be another map of ∞-operads.

If $F^{\otimes}$ is a cocartesian fibration, then the induced functor

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(F)\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$$

is a cocartesian fibration as well. ♡

Proof. Consider the following commutative diagram induced by $F^{\otimes}$

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[dr, "\varphi"] \arrow[drr, bend left, "\iota_{\mathcal{C}}"] \arrow[ddr, bend right, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(F)"'] & & & \\
& \mathcal{E} \arrow[r, "\iota_{\mathcal{E}}"] \arrow[d, "p"] & \mathrm{Fun}_{\mathcal{O}^{\otimes}}\left(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\right) \arrow[r] \arrow[d, "F^{\otimes}_{*}"] & \mathrm{Fun}\left(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\right) \arrow[d, "F^{\otimes}_{*}"] \\
& \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \arrow[r, "\iota_{\mathcal{D}}"'] & \mathrm{Fun}_{\mathcal{O}^{\otimes}}\left(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\right) \arrow[r] & \mathrm{Fun}\left(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\right)
\end{tikzcd}$$

where $\mathcal{E}$ is defined to be the pullback of the middle square, ι_C, ι_D, and the two right horizontal functors are the canonical ones, and φ is the induced functor into the pullback.

By [HTT, 3.1.2.1], the right vertical morphism $F^{\otimes}_{*} = \mathrm{Fun}\left(\mathrm{id}_{\mathcal{O}'^{\otimes}}, F^{\otimes}\right)$ is a cocartesian fibration, so as both squares are pullback squares we can apply Proposition C.1.1.1 to conclude that p is also a cocartesian fibration. As cocartesian fibrations are closed under composition [HTT, 2.4.2.3 (3)], it thus suffices to show that φ is a cocartesian fibration.

By definition, $\iota_{\mathcal{D}}$ and $\iota_{\mathcal{C}}$ are inclusions of full subcategories and hence fully faithful, and as a pullback of $\iota_{\mathcal{D}}$, Proposition B.5.2.1 implies that $\iota_{\mathcal{E}}$ is fully faithful as well. It follows that φ is also fully faithful, so by Proposition C.1.2.1 it suffices to show that for any object A in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ and p-cocartesian morphism $\theta\colon \varphi(A) \to B'$ in $\mathcal{E}$ there exists an object B in $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ such that $\varphi(B)$ is equivalent to B'.

Unpacking definitions, this means the following. Assume we have given a morphism $\theta\colon A \to B$ in $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$, which we can think of as a natural transformation between two commuting triangles as in the following diagram.

$$\begin{tikzcd}
\mathcal{O}'^{\otimes} \arrow[rr, bend left, "A"] \arrow[rr, bend right, "B"'] \arrow[rr, phantom, "\Downarrow\theta"] \arrow[dr, "\alpha"'] & & \mathcal{C}^{\otimes} \arrow[dl, "p"] \\
& \mathcal{O}^{\otimes} &
\end{tikzcd}$$

We furthermore assume that:

(a) A preserves inert morphisms. This corresponds to A lying in the full subcategory $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$.

(b) $F^{\otimes} \circ B\colon \mathcal{O}'^{\otimes} \to \mathcal{D}^{\otimes}$ preserves inert morphisms. This corresponds to B lying in the full subcategory $\mathcal{E}$ of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$.

(c) For every object O in $\mathcal{O}'^{\otimes}$, the morphism $\theta_O\colon A(O) \to B(O)$ in $\mathcal{C}$ is $F^{\otimes}$-cocartesian. This corresponds to θ (considered as a morphism in $\mathcal{E}$) being p-cocartesian.

We then have to show that B preserves inert morphisms.

In the following we let $q_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_{*}$ and $q_{\mathcal{D}}\colon \mathcal{D}^{\otimes} \to \mathsf{Fin}_{*}$ be the canonical maps of ∞-operads. Let $f\colon U \to V$ be an inert morphism in $\mathcal{O}^{\otimes}$. We have to show that $B(f)$ is

$q_{\mathcal{C}}$-cocartesian. The natural transformation θ induces the following commuting square in $\mathcal{C}^{\otimes}$.

$$\begin{tikzcd} A(U) \arrow[r, "\theta_U"] \arrow[d, "A(f)"'] & B(U) \arrow[d, "B(f)"] \\ A(V) \arrow[r, "\theta_V"'] & B(V) \end{tikzcd}$$

By (a), A preserves inert morphisms, so $A(f)$ is inert, hence $q_{\mathcal{C}}$-cocartesian. As $F^{\otimes}$ is a map of ∞-operads it also preserves inert morphism, and thus $F^{\otimes}(A(f))$ is $q_{\mathcal{D}}$-cocartesian. It then follows from [HTT, 2.4.1.3 (3)] that $A(f)$ is $F^{\otimes}$-cocartesian. By (c) both θ_U and θ_V are also $F^{\otimes}$-cocartesian, so it follows from [HTT, 2.4.1.7] that $B(f)$ is $F^{\otimes}$-cocartesian as well. Finally, $F^{\otimes}(B(f))$ is $q_{\mathcal{D}}$-cocartesian by (b), so by applying [HTT, 2.4.1.3 (3)] in the other direction we can conclude that $B(f)$ is $q_{\mathcal{C}}$-cocartesian. □

E.3.3. Adjoints

Proposition E.3.3.1. *Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, $p_{\mathcal{D}}\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$, and $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be maps of ∞-operads and let $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ and $G^{\otimes}\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ be maps of ∞-operads over $\mathcal{O}^{\otimes}$. Let $u\colon \mathrm{id}_{\mathcal{C}^{\otimes}} \to G^{\otimes} \circ F^{\otimes}$ be a natural transformation exhibiting $F^{\otimes}$ as left adjoint to $G^{\otimes}$ and assume that $p_{\mathcal{C}}$ maps u to the identity natural transformation of $p_{\mathcal{C}}$ (in other words, u is a unit for an adjunction between $F^{\otimes}$ and $G^{\otimes}$ relative to $\mathcal{O}^{\otimes}$ in the sense of [HA, 7.3.2.3]).*

Then the induced natural transformation $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(u)\colon \mathrm{id}_{\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})} \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(G \circ F)$ exhibits $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(F)$ as left adjoint to $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(G)$. ♡

Proof. Applying $\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, -\big)$ we obtain two commuting triangles as indicated in the following diagram

$$\begin{tikzcd} \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[rr, shift left, "F_*^{\otimes}"] \arrow[dr, "p_{\mathcal{C}*}"'] & & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \arrow[ll, shift left, "G_*^{\otimes}"] \arrow[dl, "p_{\mathcal{D}*}"] \\ & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}\big) & \end{tikzcd}$$

as well as a natural transformation $u_*\colon \mathrm{id}_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})} \to G_*^{\otimes} \circ F_*^{\otimes}$. By Proposition D.2.2.1, u_* exhibits $F_*^{\otimes}$ as left adjoint to $G_*^{\otimes}$. As $p_{\mathcal{C}*}$ maps u_* to the identity natural transformation of $p_{\mathcal{C}*}$, this makes u_* into the unit for an adjunction between $F_*^{\otimes}$ and $G_*^{\otimes}$ relative to $\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}\big)$ in the sense of [HA, 7.3.2.3]. Taking the pullback of this adjunction along $\{\alpha\} \to \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}\big)$ and applying [HA, 7.3.2.5] yields an induced adjunction between $\mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)$ and $\mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big)$. The claim now follows by restricting the relevant functors and natural transformation to the full subcategories of ∞-operad maps [HA, 2.1.2.7]. □

E.3.4. Reflecting equivalences

Proposition E.3.4.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories and $F\colon \mathcal{C} \to \mathcal{D}$ a symmetric monoidal functor. Let $\mathcal{O}$ be an ∞-operad.*

Assume that F is conservative, i. e. reflects equivalences. Then $\mathrm{Alg}_{\mathcal{O}}(F)$ is conservative as well. ♡

Proof. There is a commutative diagram as follows for every object X in $\mathcal{O}$.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(F)"] \arrow[d, "\mathrm{ev}_X"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) \arrow[d, "\mathrm{ev}_X"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

Now suppose that φ is a morphism in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ such that $\mathrm{Alg}_{\mathcal{O}}(F)(\varphi)$ is an equivalence. By [HA, 3.2.2.6] ev_X preserves equivalences, so the morphism

$$\mathrm{ev}_X\big(\mathrm{Alg}_{\mathcal{O}}(F)(\varphi)\big) = F(\mathrm{ev}_X(\varphi))$$

is an equivalence for every object X of $\mathcal{O}$. As F is conservative, this implies that $\mathrm{ev}_X(\varphi)$ is an equivalence for every object X of $\mathcal{O}$, which by another application of [HA, 3.2.2.6] implies that φ is an equivalence. □

E.3.5. Fully faithfulness

Proposition E.3.5.1. *Let*

$$\begin{tikzcd}
 & \mathcal{C}'^{\otimes} \arrow[rr, "\iota^{\otimes}"] \arrow[dr, "p_{\mathcal{C}'}"'] & & \mathcal{C}^{\otimes} \arrow[dl, "p_{\mathcal{C}}"] \\
\mathcal{O}'^{\otimes} \arrow[rr, "\alpha^{\otimes}"'] & & \mathcal{O}^{\otimes} &
\end{tikzcd}$$

be a commutative diagram of ∞-operads and assume that ι is fully faithful.

Then the functor

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\iota)\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}\big(\mathcal{C}'\big) \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$$

is fully faithful. Furthermore, an object A of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ lies in the essential image of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\iota)$ if and only if for every object X of $\mathcal{O}'$ the evaluation $\mathrm{ev}_X(A)$ of A at X lies in the essential image of ι. ♡

Proof. Combining Proposition B.3.0.1, Proposition B.5.1.1, and Proposition B.5.3.1 we obtain that

$$\iota^{\otimes}_{*}\colon\ \mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{C}'^{\otimes}\big) \to \mathrm{Fun}_{\mathcal{O}^{\otimes}}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)$$

is fully faithful with essential image spanned by those functors $F^{\otimes}\colon \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ over $\mathcal{O}^{\otimes}$ for which $F^{\otimes}(X)$ lies in the essential image of $\iota^{\otimes}$ for every object X of $\mathcal{O}'^{\otimes}$. There is a commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}'^\otimes) \arrow[r, "\iota_*^\otimes"] & \operatorname{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes) \\
\operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \arrow[u] \arrow[r, "\operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\iota)"'] & \operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}') \arrow[u]
\end{tikzcd}$$

where the the vertical functors are the canonical inclusions and thus by definition fully faithful, so it follows that $\operatorname{Alg}_{\mathcal{O}'/\mathcal{O}}(\iota)$ is also fully faithful, with essential image spanned by those algebras whose associated functors $F^\otimes\colon \mathcal{O}'^\otimes \to \mathcal{C}^\otimes$ are such that $F^\otimes(X)$ lies in the essential image of $\iota^\otimes$ for every object X of $\mathcal{O}'^\otimes$.

As $F^\otimes$ and $\alpha^\otimes$ are morphisms of ∞-operads, we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{O}'^\otimes_{\langle n\rangle} \arrow[r, "F^\otimes_{\langle n\rangle}"] \arrow[d, "\simeq"'] & \mathcal{C}^\otimes_{\langle n\rangle} \arrow[d, "\simeq"] & \mathcal{C}'^\otimes_{\langle n\rangle} \arrow[l, "\alpha^\otimes_{\langle n\rangle}"'] \arrow[d, "\simeq"] \\
\mathcal{O}'^{\times n} \arrow[r, "F^{\times n}"'] & \mathcal{C}^{\times n} & \mathcal{C}'^{\times n} \arrow[l, "\alpha^{\times n}"]
\end{tikzcd}$$

for every $n \geq 0$ that shows that $F^\otimes(X)$ lying in the essential image of $\iota^\otimes$ for every object X of $\mathcal{O}'^\otimes$ is equivalent to $F(X)$ lying in the essential image of ι for every object X of $\mathcal{O}$. □

E.4. Induced ∞-operad structures on Alg

Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $\mathcal{O}$ an ∞-operad. Then the tensor product on $\mathcal{C}$ induces a symmetric monoidal structure on $\operatorname{Alg}_{\mathcal{O}}(\mathcal{C})$ such that the forgetful functor $\operatorname{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ can be upgraded to a symmetric monoidal functor. In the setting of quasicategories, this structure is constructed in [HA, 3.2.4.1, 3.2.4.2, and 3.2.4.3]. However, it is not immediately obvious from the definition that this construction does not depend on the choice of representatives (or in other words, whether it is invariant under categorical equivalences). In Section E.4.1 we will give a description of the construction that can be performed entirely in $\mathfrak{Cat}_\infty$, i. e. without the help of models like quasicategories, and show that it agrees with the one given by Lurie. Apart from the aesthetic gain from being able to work as model independently as possible, the reformulated description will also be helpful in some results we will prove later.

In Section E.4.2 we will then collect a number of properties that the induced ∞-operad structure has, deducing most of them from the results of [HA, 3.2.4]. It would also be possible to prove these statements without referring back to the quasicategorical model. However, we need to show agreement of the two approaches anyway, as throughout the text we will need to make use of several other results from [HA] using the induced ∞-operad structure on algebras, so giving an independent, more model-independent proof of the statements discussed in Section E.4.2 would not save us from having to go through the comparison in Section E.4.1.

E.4.1. The quasicategorical model

In this section we discuss Lurie's quasicategorical model for induced ∞-operad structures on ∞-categories of algebras, and compare it to a more model-independent definition.

We will make use of the following convention during our discussion.

Convention E.4.1.1. In contrast with the rest of the text, wherever we explicitly invoke this convention every notion should be taken to refer to the respective quasicategorical notion as defined in [HTT] and [HA]. So for example the claim that a diagram of quasicategories commutes means that it is a strictly commuting diagram of simplicial sets, and an ∞-operad is a map of simplicial sets $\mathtt{O}^{\otimes} \to \mathsf{Fin}_*$ where $\mathtt{O}$ is a quasicategory and such that the map satisfies some properties, rather than a morphism $\mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ in $\mathcal{C}\mathrm{at}_\infty$ satisfying some properties. ♢

We start by reviewing the construction given in [HA, 3.2.4.1].

Definition E.4.1.2 ([HA, 3.2.4.1])**.** We make use of Convention E.4.1.1 in this construction. Let $\mathtt{p}_{\mathtt{O}}\colon \mathtt{O}^{\otimes} \to \mathsf{Fin}_*$, $\mathtt{p}_{\mathtt{O}'}\colon \mathtt{O}'^{\otimes} \to \mathsf{Fin}_*$, and $\mathtt{p}_{\mathtt{O}''}\colon \mathtt{O}''^{\otimes} \to \mathsf{Fin}_*$ be ∞-operads, and let $\mathtt{q}\colon \mathtt{C}^{\otimes} \to \mathtt{O}''^{\otimes}$ be a fibration of ∞-operads, i. e. a map of ∞-operads where $\mathtt{q}$ is also a categorical fibration of quasicategories (see [HA, 2.1.2.10]). Let $\mathtt{f}\colon \mathtt{O}^{\otimes} \times \mathtt{O}'^{\otimes} \to \mathtt{O}''^{\otimes}$ be a bifunctor of ∞-operads, i. e. a functor of quasicategories such that the diagram

$$\begin{tikzcd} \mathtt{O}^{\otimes} \times \mathtt{O}'^{\otimes} \arrow[r, "\mathtt{f}"] \arrow[d, "\mathtt{p}_{\mathtt{O}} \times \mathtt{p}_{\mathtt{O}'}"'] & \mathtt{O}''^{\otimes} \arrow[d, "\mathtt{p}_{\mathtt{O}''}"] \\ \mathsf{Fin}_* \times \mathsf{Fin}_* \arrow[r, "\wedge"'] & \mathsf{Fin}_* \end{tikzcd}$$

commutes and such that $\mathtt{f}$ sends pairs of inert morphisms to inert morphisms, see [HA, 2.2.5.3].

Define $\widetilde{\Phi}$ to be the functor $\mathsf{sSet}_{/\mathtt{O}^{\otimes}} \to \mathsf{Set}$ that sends $\mathtt{g}\colon \mathtt{K} \to \mathtt{O}^{\otimes}$ to the set of commutative diagrams as indicated below.

$$\begin{tikzcd} \mathtt{K} \times \mathtt{O}'^{\otimes} \arrow[r, "\mathtt{F}"] \arrow[d, "\mathtt{g} \times \mathrm{id}_{\mathtt{O}'^{\otimes}}"'] & \mathtt{C}^{\otimes} \arrow[d, "\mathtt{r}"] \\ \mathtt{O}^{\otimes} \times \mathtt{O}'^{\otimes} \arrow[r, "\mathtt{f}"'] & \mathtt{O}''^{\otimes} \end{tikzcd} \tag{E.2}$$

Furthermore, define $\Phi\colon \mathsf{sSet}_{/\mathtt{O}^{\otimes}} \to \mathsf{Set}$ to be the functor which sends $\mathtt{g}\colon \mathtt{K} \to \mathtt{O}^{\otimes}$ to the subset of $\widetilde{\Phi}(\mathtt{g})$ of commutative diagrams (E.2) which have the property that $\mathtt{F}(\mathrm{id}_{\mathtt{k}}, \alpha)$ is inert for every vertex $\mathtt{k}$ of $\mathtt{K}$ and every inert morphism α in $\mathtt{O}''^{\otimes}$.

We say that an object $\mathtt{r}$ in $\mathsf{sSet}_{/\mathtt{O}^{\otimes}}$ is *a quasicategorical model (a quasicategorical pre-model) for the ∞-operad structure on algebras* with respect to $\mathtt{f}$, $\mathtt{q}$, etc. as introduced above, if there exists a natural bijection of functors $\mathsf{sSet}_{/\mathtt{O}^{\otimes}} \to \mathsf{Set}$ between $\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^{\otimes}}}(-, \mathtt{r})$ and Φ (between $\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^{\otimes}}}(-, \mathtt{r})$ and $\widetilde{\Phi}$).

Note that the Yoneda lemma implies that if a quasicategorical (pre-)model for the ∞-operad structure on algebras exists, then it is unique up to isomorphism in $\mathsf{sSet}_{/\mathtt{O}^{\otimes}}$. We will give a more concrete construction of a quasicategorical (pre-)model for the ∞-operad structure on algebras below. ♢

Remark E.4.1.3. In this remark we make use of Convention E.4.1.1, and assume that we are in the situation of Definition E.4.1.2. Let $\widetilde{\mathtt{r}}\colon \widetilde{\mathrm{Alg}}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes} \to \mathtt{O}^{\otimes}$ be a quasicategorical pre-model for the ∞-operad structure on algebras, and let ϕ be a natural bijection $\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^{\otimes}}}(-, \widetilde{\mathtt{r}}) \cong \widetilde{\Phi}$. We then define a sub-simplicial set $\mathrm{Alg}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes}$ of $\widetilde{\mathrm{Alg}}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes}$ as the sub-simplicial set spanned by those vertices $\mathtt{A}$ which correspond under ϕ to maps

$$\mathtt{O}'^{\otimes} \cong \{\mathtt{A}\} \times \mathtt{O}'^{\otimes} \xrightarrow{\mathtt{F}} \mathtt{C}^{\otimes}$$

that preserve inert morphisms.

Let $\mathtt{r}\colon \mathrm{Alg}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes} \to \mathtt{O}^{\otimes}$ be the restriction of $\widetilde{\mathtt{r}}$ to $\mathrm{Alg}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes}$. As the condition defining the natural subset Φ of $\widetilde{\Phi}$ can be checked vertex-wise (in $\mathtt{K}$, where we use the notation from (E.2)), it is clear that ϕ restricts to a bijection between $\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^{\otimes}}}(-, \mathtt{r})$ and Φ. We conclude that $\mathtt{r}$ is a quasicategorical model for the ∞-operad structure on algebras. ♢

Proposition E.4.1.4. *In this proposition Convention E.4.1.1 applies. Assume we are in the situation of Definition E.4.1.2. Let $\widetilde{\mathtt{r}}$ be the functor*

$$\widetilde{\mathtt{r}}\colon\ \mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes}\right)} \mathtt{O}^{\otimes} \xrightarrow{\mathrm{pr}_2} \mathtt{O}^{\otimes}$$

where the functor $\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big) \to \mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes}\big)$ that is part of the pullback is q_*, and the functor $\mathtt{O}^{\otimes} \to \mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes}\big)$ is the adjoint functor to $\mathtt{f}$. Let $\mathtt{r}$ be the restriction of $\widetilde{\mathtt{r}}$ to $\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big)' \times_{\mathrm{Fun}\left(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes}\right)} \mathtt{O}^{\otimes}$, where $\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big)'$ is the sub-simplicial set of $\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big)$ spanned by the vertices which are functors $\mathtt{O}'^{\otimes} \to \mathtt{C}^{\otimes}$ that preserve inert morphisms.

Then $\widetilde{\mathtt{r}}$ is a quasicategorical pre-model for the ∞-operad structure on algebras and $\mathtt{r}$ is a quasicategorical model for the ∞-operad structure on algebras. ♡

Proof. Let $\mathtt{g}\colon \mathtt{K} \to \mathtt{O}^{\otimes}$ be an object in $\mathsf{sSet}_{/\mathtt{O}^{\otimes}}$. There is a chain of bijections which are natural in $\mathtt{g}$ as follows.

$$\begin{aligned}
&\mathrm{Mor}_{\mathsf{sSet}_{/\mathtt{O}^{\otimes}}}(\mathtt{g}, \widetilde{\mathtt{r}})\\
&\cong \mathrm{Mor}_{\mathsf{sSet}}\left(\mathtt{K}, \mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes}\right)} \mathtt{O}^{\otimes}\right) \times_{\mathrm{Mor}_{\mathsf{sSet}}\left(\mathtt{K}, \mathtt{O}^{\otimes}\right)} \{\mathtt{g}\}\\
&\cong \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{K} \times \mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big) \times_{\mathrm{Mor}_{\mathsf{sSet}}\left(\mathtt{K} \times \mathtt{O}'^{\times}, \mathtt{O}''^{\otimes}\right)} \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{K}, \mathtt{O}^{\otimes}\big) \times_{\mathrm{Mor}_{\mathsf{sSet}}\left(\mathtt{K}, \mathtt{O}^{\otimes}\right)} \{\mathtt{g}\}\\
&\cong \mathrm{Mor}_{\mathsf{sSet}}\big(\mathtt{K} \times \mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big) \times_{\mathrm{Mor}_{\mathsf{sSet}}\left(\mathtt{K} \times \mathtt{O}'^{\times}, \mathtt{O}''^{\otimes}\right)} \{\mathtt{f} \circ (\mathtt{g} \times \mathrm{id}_{\mathtt{O}'^{\otimes}})\}\\
&\cong \widetilde{\Phi}(\mathtt{g})
\end{aligned}$$

This shows the claim about $\widetilde{\mathtt{r}}$. The claim for $\mathtt{r}$ follows using Remark E.4.1.3 after noting that for a vertex $\mathtt{A}$ of $\operatorname{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}) \times_{\operatorname{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes})} \mathtt{O}^{\otimes}$ considered as a functor

$$\mathtt{a}\colon \{\mathtt{A}\} \to \operatorname{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}) \times_{\operatorname{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes})} \mathtt{O}^{\otimes}$$

the composition of the chain of bijections above with the projection to

$$\operatorname{Mor}_{\mathsf{sSet}}(\{\mathtt{A}\} \times \mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}) \cong \operatorname{Mor}_{\mathsf{sSet}}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes})$$

sends $\mathtt{a}$ to $\operatorname{pr}_1(\mathtt{A})$. □

We can now state the construction of the induced ∞-operad structure on ∞-categories of algebras without referring to quasicategories.

Proposition E.4.1.5. *Let $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$, $p_{\mathcal{O}'}\colon \mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$, and $p_{\mathcal{O}''}\colon \mathcal{O}''^{\otimes} \to \mathsf{Fin}_*$ be ∞-operads and let $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}''^{\otimes}$ be a morphism of ∞-operads. Let $f\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads.*

Let $\mathtt{p}_{\mathtt{O}}\colon \mathtt{O}^{\otimes} \to \mathsf{Fin}_*$, $\mathtt{p}_{\mathtt{O}'}\colon \mathtt{O}'^{\otimes} \to \mathsf{Fin}_*$, and $\mathtt{p}_{\mathtt{O}''}\colon \mathtt{O}''^{\otimes} \to \mathsf{Fin}_*$ be functors of quasicategories which represent $p_{\mathcal{O}}$, $p_{\mathcal{O}'}$, and $p_{\mathcal{O}''}$, respectively. Let $\mathtt{q}\colon \mathtt{C}^{\otimes} \to \mathtt{O}''^{\otimes}$ be a categorical fibration of quasicategories representing q and let $\mathtt{f}\colon \mathtt{O}^{\otimes} \times \mathtt{O}'^{\otimes} \to \mathtt{O}''^{\otimes}$ be a functor of quasicategories representing f.

Define $\operatorname{Alg}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})$, $\widetilde{\operatorname{Alg}}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})$, $\widetilde{\mathtt{r}}$, $\mathtt{r}$, $\mathtt{s}'$, and $\mathtt{s}$ via the following diagram, where the two squares are to be pullback diagrams, $\widehat{\mathtt{f}}$ is adjoint to $\mathtt{f}$, and $\mathtt{i}_{\mathrm{Fun}}$ is the inclusion of the full sub-simplicial set spanned by those vertices which correspond to functors that preserve inert morphisms.

$$\begin{tikzcd}
\operatorname{Alg}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes} \arrow[r, "\mathtt{i}_{\mathrm{Alg}}"] \arrow[d, "\mathtt{s}"'] \arrow[rr, bend left, "\mathtt{r}"] & \widetilde{\operatorname{Alg}}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes} \arrow[r, "\widetilde{\mathtt{r}}"] \arrow[d, "\mathtt{s}'"'] & \mathtt{O}^{\otimes} \arrow[d, "\widehat{\mathtt{f}}"] \\
\operatorname{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes})' \arrow[r, "\mathtt{i}_{\mathrm{Fun}}"'] & \operatorname{Fun}(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}) \arrow[r, "\mathtt{q}_*"'] & \operatorname{Fun}(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes})
\end{tikzcd} \tag{E.3}$$

Then the above diagram represents the following diagram in $\mathcal{C}\mathrm{at}_{\infty}$, were both squares are pullback diagrams as well, $\widehat{f}$ is adjoint to f, and ι_{Fun} is the inclusion of the full subcategory spanned by those functors that preserve inert morphisms.

$$\begin{tikzcd}
\operatorname{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\iota_{\mathrm{Alg}}"] \arrow[d, "s"'] \arrow[rr, bend left, "r"] & \widetilde{\operatorname{Alg}}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\widetilde{r}"] \arrow[d, "s'"'] & \mathcal{O}^{\otimes} \arrow[d, "\widehat{f}"] \\
\operatorname{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})' \arrow[r, "\iota_{\mathrm{Fun}}"'] & \operatorname{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r, "q_*"'] & \operatorname{Fun}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})
\end{tikzcd}$$

♡

Proof. What we have to show is that both squares in diagram (E.3) are homotopy pullback diagrams with respect to the Joyal model structure. We begin by showing that $\mathtt{q}_*$ and $\mathtt{i}_{\mathrm{Fun}}$ are categorical fibrations.

By assumption $\mathtt{q}\colon \mathtt{C}^{\otimes} \to \mathtt{O}''^{\otimes}$ is a categorical fibration of quasicategories. A map of simplicial sets is a categorical fibration if and only if it has the right lifting property with respect to maps of simplicial sets which are monomorphisms as well as categorical equivalences (see [HTT, 2.2.5.1]). By adjoining the lifting problems we need to solve to show that $\mathtt{q}_*$ is a categorical fibration we are reduced to showing that if $\mathtt{j}$ is a map of simplicial sets which is a monomorphism as well as a categorical equivalence, then $\mathtt{j} \times \mathrm{id}_{\mathtt{O}'^{\otimes}}$ is so as well. That $\mathtt{j} \times \mathrm{id}_{\mathtt{O}'^{\otimes}}$ is again a monomorphism is clear, and that it is also a categorical equivalence is [HTT, 2.2.5.4].

We next argue that $\mathtt{i}_{\mathrm{Fun}}$ is also a categorical fibration. As $\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big)$ is a quasicategory by [HTT, 1.2.7.3 (1)], we can apply [HTT, 2.4.6.5] so that it suffices to show that $\mathtt{i}_{\mathrm{Fun}}$ is an inner fibration and that for any natural equivalence $\varphi\colon \mathtt{g} \to \mathtt{g}'$ of functors $\mathtt{O}'^{\otimes} \to \mathtt{C}^{\otimes}$ such that $\mathtt{g}$ preserves inert morphisms it follows that $\mathtt{g}'$ preserves inert morphisms as well. The latter property follows immediately from the fact that cocartesian morphisms are closed under equivalences. It remains to show that $\mathtt{i}_{\mathrm{Fun}}$ is an inner fibration. But note that every horn inclusion $\Lambda^n_i \subseteq \Delta^n$ for $0 < i < n$ is an isomorphism on 0-simplices, and as $\mathtt{i}_{\mathrm{Fun}}$ is the inclusion of a full sub-simplicial set lifting positive dimensional simplices is always possible, so $\mathtt{i}_{\mathrm{Fun}}$ is an inner fibration.

We have now shown that $\mathtt{q}_*$ and $\mathtt{i}_{\mathrm{Fun}}$ are both categorical fibrations. By assumption $\mathtt{O}^{\otimes}$ is a quasicategory and $\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{O}''^{\otimes}\big)$ is a quasicategory by [HTT, 1.2.7.3 (1)], so it follows from [HTT, A.2.4.4, variant (i) and A.2.4.5] that the right square in diagram (E.3) is a homotopy pullback square with respect to the Joyal model structure. As a pullback of the categorical fibration $\mathtt{q}_*$ is the functor $\widetilde{\mathtt{r}}$ a categorical fibration as well, so as $\mathtt{O}^{\otimes}$ is a quasicategory, $\widetilde{\mathrm{Alg}}_{\mathtt{O}'/\mathtt{O}''}(\mathtt{C})^{\otimes}$ is also a quasicategory [HTT, 2.4.6.1]. It was already mentioned that $\mathrm{Fun}\big(\mathtt{O}'^{\otimes}, \mathtt{C}^{\otimes}\big)$ is a quasicategory, so we can apply [HTT, A.2.4.4, variant (i) and A.2.4.5] again to conclude that the left square in diagram (E.3) is also a homotopy pullback square with respect to the Joyal model structure. □

E.4.2. Properties of the induced ∞-operad structure

In Proposition E.4.1.5 we gave a construction of the induced ∞-operad structure on ∞-categories of algebras that could be formulated without referring back to quasicategorical models. In this section we collect the properties of this construction.

Remark E.4.2.1. In the situation of Proposition E.4.1.5, Proposition B.5.2.1 shows that as ι_{Fun} is a fully faithful functor, so is ι_{Alg}. We can thus identify

$$\iota_{\mathrm{Alg}}\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes}\right)} \mathcal{O}^{\otimes}$$

with the inclusion of the full subcategory spanned by those objects whose projection to the first factor is a functor $\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ that preserves inert morphisms. ◇

Remark E.4.2.2. Let $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{O}''$ be ∞-operads, let

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "q_{\mathcal{C}}"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q_{\mathcal{D}}"] \\ & \mathcal{O}''^{\otimes} & \end{tikzcd}$$

be a commutative diagram of ∞-operads, and let $f\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads. Then the functor indicated as the right vertical functor in the following diagram induces a functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}$ on algebras that makes the diagram commute

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\iota^{\mathcal{C}}_{\mathrm{Alg}}"] \arrow[d, dashed, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}"'] & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes}\right)} \mathcal{O}^{\otimes} \arrow[d, "\left(F^{\otimes}\right)_{*} \times_{\mathrm{id}} \mathrm{id}"] \\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D})^{\otimes} \arrow[r, "\iota^{\mathcal{D}}_{\mathrm{Alg}}"'] & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes}\right)} \mathcal{O}^{\otimes} \end{tikzcd}$$

where $\iota^{\mathcal{C}}_{\mathrm{Alg}}$ and $\iota^{\mathcal{D}}_{\mathrm{Alg}}$ are as in Remark E.4.2.1. This follows immediately from the description in Remark E.4.2.1, as F preserves inert morphisms as a morphism of ∞-operads. From the definition it is also clear that $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}$ is compatible with the projections to $\mathcal{O}^{\otimes}$. ♢

In light of Proposition E.4.1.4 and Proposition E.4.1.5, all the properties listed in [HA, 3.2.4.2 and 3.2.4.3] apply to $r\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$. We re-state them as the proposition below for easier reference.

Proposition E.4.2.3 ([HA, 3.2.4.2 and 3.2.4.3])**.** *Let $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{O}''$ be ∞-operads, let $q_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}''^{\otimes}$ be a morphism of ∞-operads, and let $f\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads.*

Let

$$\iota^{\mathcal{C}}_{\mathrm{Alg}}\colon\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes}\right)} \mathcal{O}^{\otimes}$$

be as in Proposition E.4.1.5 and Remark E.4.2.1 and denote by $r_{\mathcal{C}}$ the composition $\mathrm{pr}_2 \circ \iota^{\mathcal{C}}_{\mathrm{Alg}}$. Then the following hold:

(0) Let X be an object of $\mathcal{O}$. Then $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}_X$ can be identified with $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})$, where the latter ∞-category of algebras is taken with respect to the following morphism of ∞-operads.

$$f_X\colon \mathcal{O}'^{\otimes} \simeq \{X\} \times \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \xrightarrow{f} \mathcal{O}''^{\otimes}$$

This identification is compatible with the respective inclusions into the following ∞-categories.

$$\left(\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes}\right)} \mathcal{O}^{\otimes}\right) \times_{\mathcal{O}^{\otimes}} \{X\} \simeq \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes}\right)} \{f_X\}$$

(1) *The functor* $r_{\mathcal{C}}$ *is a morphism of* ∞*-operads.*

(2) *A morphism* α *in* $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}$ *lying over an inert morphism in* $\mathcal{O}^{\otimes}$ *is inert if and only if for every object* X *of* $\mathcal{O}'$*, the morphism* $\mathrm{ev}_X\Big(\mathrm{pr}_1(\iota^{\mathcal{C}}_{\mathrm{Alg}}(\alpha))\Big)$ *in* $\mathcal{C}^{\otimes}$ *is inert.*

(3) *If* $q_{\mathcal{C}}$ *is a cocartesian fibration of* ∞*-operads, then so is* $r_{\mathcal{C}}$*.*

(4) *Assume that* $q_{\mathcal{C}}$ *is a cocartesian fibration of* ∞*-operads. Then a morphism* α *in* $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}$ *is* $r_{\mathcal{C}}$*-cocartesian if and only if for every object* X *of* $\mathcal{O}'$*, the morphism obtained by evaluating at* X*, i. e.* $\mathrm{ev}_X\Big(\mathrm{pr}_1(\iota^{\mathcal{C}}_{\mathrm{Alg}}(\alpha))\Big)$*, is* $q_{\mathcal{C}}$*-cocartesian.*

(5) *Let* X *be an object of* $\mathcal{O}'$*. Then the functor* $\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota^{\mathcal{C}}_{\mathrm{Alg}}$ *is a morphism of* ∞*-operads and fits into a commutative diagram*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\mathrm{ev}_X\circ\mathrm{pr}_1\circ\iota^{\mathcal{C}}_{\mathrm{Alg}}"] \arrow[d, "r_{\mathcal{C}}"'] & \mathcal{C}^{\otimes} \arrow[d, "q_{\mathcal{C}}"] \\ \mathcal{O}^{\otimes} \arrow[r] & \mathcal{O}''^{\otimes} \end{tikzcd}$$

where the bottom horizontal functor is the following composition.

$$\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \times \{X\} \xrightarrow{f} \mathcal{O}''^{\otimes}$$

Furthermore, if $q_{\mathcal{C}}$ *is a cocartesian fibration of* ∞*-operads, then* $\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota^{\mathcal{C}}_{\mathrm{Alg}}$ *sends* $r_{\mathcal{C}}$*-cocartesian morphisms to* $q_{\mathcal{C}}$*-cocartesian morphisms.*

We can also consider how the above properties behave under induced functors as in Remark E.4.2.2. So let

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "q_{\mathcal{C}}"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q_{\mathcal{D}}"] \\ & \mathcal{O}''^{\otimes} & \end{tikzcd}$$

be a commutative diagram of ∞*-categories, and let* $\iota^{\mathcal{D}}_{\mathrm{Alg}}$ *and* $r_{\mathcal{D}}$ *be defined analogously to* $\iota^{\mathcal{C}}_{\mathrm{Alg}}$ *and* $r_{\mathcal{C}}$*. Then the following hold.*

(6) *Let* X *be an object of* $\mathcal{O}$*. Then there is a commutative diagram*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}_X \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}_X"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)"] \\ \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D})^{\otimes}_X \arrow[r, dash, "\simeq"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D}) \end{tikzcd}$$

where the horizontal functors are the equivalences from (0).

(7) *The functor*

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D})^{\otimes}$$

is a morphism of ∞*-operads.*

(8) *If* $q_{\mathcal{C}}$ *and* $q_{\mathcal{D}}$ *are cocartesian fibrations of* ∞*-operads, and* F *is an* $\mathcal{O}''$*-monoidal functor, then the functor*

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}\colon \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D})^{\otimes}$$

is $\mathcal{O}$*-monoidal.*

(9) *Let* X *be an object of* $\mathcal{O}'$*. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \arrow[r, "\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(F)^{\otimes}"] \arrow[d, "\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota^{\mathcal{C}}_{\mathrm{Alg}}"'] & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{D})^{\otimes} \arrow[d, "\mathrm{ev}_X \circ \mathrm{pr}_1 \circ \iota^{\mathcal{D}}_{\mathrm{Alg}}"] \\
\mathcal{C}^{\otimes} \arrow[r, "F^{\otimes}"'] & \mathcal{D}^{\otimes}
\end{tikzcd}$$

of ∞*-operads.* ♡

Proof. Claims (0) to (4) are just restatements of [HA, 3.2.4.2 and 3.2.4.3] which applies in this form due to Proposition E.4.1.4 and Proposition E.4.1.5.

Claim (5) follows directly from (2) and (4). Claim (6) follows immediately from Remark E.4.2.2 and (0). Combining that F is a morphism of ∞-operads with the description of inert morphisms in (2) implies (7), and if F is a $\mathcal{O}''$-monoidal, then combining this with (4) implies (8). Finally, (9) is immediate from the definitions. □

Remark E.4.2.4. Let $\mathcal{O}_L$, $\mathcal{O}'_L$, $\mathcal{O}_R$, and $\mathcal{O}'$ be ∞-operads, let $q_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}'^{\otimes}$ be a morphism of ∞-operads, let

$$f\colon \mathcal{O}_L^{\otimes} \times \mathcal{O}_R^{\otimes} \to \mathcal{O}'^{\otimes}$$

be a bifunctor of ∞-operads, and let $\alpha^{\otimes}\colon \mathcal{O}'_L \to \mathcal{O}_L$ be a morphism of ∞-operads.

We obtain another bifunctor of ∞-categories f' as the following composition.

$$f'\colon \mathcal{O}'^{\otimes}_L \times \mathcal{O}^{\otimes}_R \xrightarrow{\alpha^{\otimes}\times\mathrm{id}} \mathcal{O}^{\otimes}_L \times \mathcal{O}^{\otimes}_R \xrightarrow{f} \mathcal{O}'^{\otimes}$$

We obtain a pullback diagram as follows

$$\begin{tikzcd}
\mathrm{Fun}\big(\mathcal{O}_R^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathcal{O}_R^{\otimes}, \mathcal{O}'^{\otimes}\right)} \mathcal{O}'^{\otimes}_L \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{id}\times_{\mathrm{id}}\alpha^{\otimes}"'] & \mathcal{O}'^{\otimes}_L \arrow[d, "\alpha^{\otimes}"] \\
\mathrm{Fun}\big(\mathcal{O}_R^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}\left(\mathcal{O}_R^{\otimes}, \mathcal{O}'^{\otimes}\right)} \mathcal{O}^{\otimes}_L \arrow[r, "\mathrm{pr}_2"'] & \mathcal{O}^{\otimes}_L
\end{tikzcd}$$

where the pullbacks on the left are take with respect to the morphisms as in Proposition E.4.1.5, on the top with respect to f' and the bottom with respect to f.

It is clear from the definition of ι_{Alg} (see Remark E.4.2.1) that an object lies in the essential image of the functor ι_{Alg} associated to the bifunctor f' if and only if $\mathrm{id} \times_{\mathrm{id}} \alpha^{\otimes}$ maps that object to the essential image of the functor ι_{Alg} associated to the bifunctor f.

It thus follows from Proposition B.5.3.1, Proposition B.4.4.1, and Proposition B.4.3.1 that the above pullback diagram induces another pullback diagram as follows, where the $\mathrm{Alg}_{\mathcal{O}_R/\mathcal{O}'}(\mathcal{C})^{\otimes}$ at the top left is the one with respect to the bifunctor f' and the one at the bottom left is with respect to the bifunctor f.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}_R/\mathcal{O}'}(\mathcal{C})^{\otimes} \arrow[r, "\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}"] \arrow[d] & \mathcal{O}'^{\otimes}_L \arrow[d, "\alpha^{\otimes}"] \\
\mathrm{Alg}_{\mathcal{O}_R/\mathcal{O}'}(\mathcal{C})^{\otimes} \arrow[r, "\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}}"'] & \mathcal{O}^{\otimes}_L
\end{tikzcd}$$

By Proposition E.4.2.3 (7), the horizontal functors are morphisms of ∞-operads, $\alpha^{\otimes}$ is by assumption a morphism of ∞-operads, and it then follows from Proposition E.4.2.3 (2) that the left vertical functor is also a morphism of ∞-operads. ♢

E.5. Iterating Alg

Proposition E.4.2.3 allows us to "iterate" passing to the ∞-category of algebras. In this section we show that there is an alternative description of algebras of algebras: There is an equivalence of ∞-categories between the ∞-category of $\mathcal{O}$-algebras in $\mathcal{O}'$-algebras $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}))$ and the ∞-category of $\mathcal{O} \otimes \mathcal{O}'$-algebras $\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C})$. This equivalence goes through an intermediate step, the ∞-category $\mathrm{BiFunc}(\mathcal{O}, \mathcal{O}', \mathcal{C})$ of bifunctors of ∞-operads.

Proposition E.5.0.1. *Let $p_{\mathcal{O}} \colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$, $p'_{\mathcal{O}} \colon \mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$, and $p_{\mathcal{C}} \colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be ∞-operads.*

Then there is a commutative diagram as follows[8]

$$\begin{tikzcd}
\mathrm{BiFunc}(\mathcal{O}, \mathcal{O}'; \mathcal{C}) \arrow[r] \arrow[d, "\Phi_2"', "\simeq"] & \mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[d, "\widehat{(-)}"] \\
\mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})\big) \arrow[r] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)\Big)
\end{tikzcd}$$

where $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ carries the ∞-operad structure from Proposition E.4.2.3, see [HA, 3.2.4.4] [9]*, the horizontal functors are the canonical ones, and the functor $\widehat{(-)}$ sends a functor G to its adjoint $\widehat{G}$. The functor Φ_2 is an equivalence.* ♡

[8]See [HA, 2.2.5.3] for a definition of BiFunc.

[9]There is a bifunctor of ∞-operads $\mathsf{Fin}_* \times \mathcal{O}'^{\otimes} \xrightarrow{\mathrm{id} \times p_{\mathcal{O}'}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{-\wedge-} \mathsf{Fin}_*$ and it is with respect to this bifunctor that we apply Proposition E.4.2.3.

Proof. We consider the following diagram, in which the outer square corresponds to the square from the statement. We will explain the individual functors in the text below.

$$\begin{tikzcd}
\mathrm{BiFunc}(\mathcal{O}, \mathcal{O}'; \mathcal{C}) \arrow[r, dashed, "\Phi_2", "\simeq"'] \arrow[ddd, "j"'] & \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})\big) \arrow[d, "i"] \\
& \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}^{\otimes}, \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}\big) \arrow[d, "(\iota_{\mathrm{Alg}})_*"] \\
& \mathcal{E} \coloneqq \mathrm{Fun}_{\mathsf{Fin}_*}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_*\Big) \arrow[dd, bend left=60, "P_3"] \\
\mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[r, "V", "\simeq"'] \arrow[d, "P_1"'] & \mathrm{Fun}_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*)}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)\Big) \arrow[u, "W"', "\simeq"] \arrow[d, "P_2"] \\
\mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[r, shift left, "\widehat{(-)}", "\simeq"'] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)\Big) \arrow[l, shift left, "\widecheck{(-)}"]
\end{tikzcd}$$

Functors P_1, P_2, and P_3 are constructed from the relevant projection and forgetful functors: P_1 forgets that the functor was over Fin_*, and similarly for P_2. The functor P_3 additionally postcomposes with the projection to the first factor. Functors $\widecheck{(-)}$ and $\widehat{(-)}$ send functors to their respective adjoints, both are equivalences.

We use notation from Proposition E.4.2.3, so ι_{Alg} is the inclusion of the full subcategory $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ of $\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_*$ of those objects whose projection to the first factor is a functor $\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ preserving inert morphisms. It follows from Proposition B.3.0.1 that $(\iota_{\mathrm{Alg}})_*$ is also fully faithful, and applying Proposition B.5.3.1 and Remark B.5.1.2 we can further conclude that the functor ι_{Alg_*} in the diagram is fully faithful, with essential image spanned by precisely those objects of $\mathcal{E}$ which are mapped by P_3 to functors

$$\mathcal{O}^{\otimes} \to \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)$$

which evaluated at every object of $\mathcal{O}^{\otimes}$ yield a functor $\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ that preserves inert morphisms.

The functor i is the canonical inclusion of the full subcategory of those functors $\mathcal{O}^{\otimes} \to \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ over Fin_* which send inert morphisms to inert morphisms. Using Proposition C.1.2.1, Proposition C.1.1.1, and [HTT, 3.1.2.1] we can reformulate this condition: i is the inclusion of the full subcategory of objects who are mapped by $P_3 \circ (\iota_{\mathrm{Alg}})_*$ to functors

$$\mathcal{O}^{\otimes} \to \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)$$

which send an inert morphism in $\mathcal{O}^{\otimes}$ to a natural transformation for which every component is an inert morphism in $\mathcal{C}^{\otimes}$.

We can summarize the above discussion as follows: The composition $(\iota_{\mathrm{Alg}})_* \circ i$ is fully faithful, and an object E of $\mathcal{E}$ is in the essential image of $(\iota_{\mathrm{Alg}})_* \circ i$ precisely when $\widetilde{P_3(E)}$ is a functor $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ that preserves inert morphisms separately in each variable. As identity morphism in $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ are inert [HTT, 2.4.1.5] and cocartesian morphisms are closed under composition [HTT, 2.4.1.7], this condition is equivalent to the functor sending pairs of inert morphisms to inert morphisms in $\mathcal{C}^{\otimes}$.

The functor W is an equivalence and constructed using compatibility of Fun with pullbacks, the $\times-$Fun-adjunction, as well as the pasting law for pullbacks [HTT, 4.4.2.1]; It is the following composition.

$$
\begin{aligned}
&\mathrm{Fun}_{\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathsf{Fin}_*)}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})\Big)\\
\simeq\ &\mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})\Big) \times_{\mathrm{Fun}\left(\mathcal{O}^{\otimes},\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathsf{Fin}_*)\right)} \{(\widehat{p_{\mathcal{O}} \wedge p_{\mathcal{O}'}})\}\\
\simeq\ &\mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})\Big) \times_{\mathrm{Fun}\left(\mathcal{O}^{\otimes},\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathsf{Fin}_*)\right)} \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathsf{Fin}_*)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times_{\mathrm{Fun}(\mathcal{O}^{\otimes},\mathsf{Fin}_*)} \{(\widehat{p_{\mathcal{O}} \wedge p_{\mathcal{O}'}})\}\\
\simeq\ &\mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathsf{Fin}_*)} \mathsf{Fin}_*\Big) \times_{\mathrm{Fun}(\mathcal{O}^{\otimes},\mathsf{Fin}_*)} \{(\widehat{p_{\mathcal{O}} \wedge p_{\mathcal{O}'}})\}\\
\simeq\ &\mathrm{Fun}_{\mathsf{Fin}_*}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathsf{Fin}_*)} \mathsf{Fin}_*\Big)
\end{aligned}
$$

It is clear that W defined like this satisfies $P_3 \circ W \simeq P_2$.

The equivalence V is defined using quite similar manipulations, as indicated below.

$$
\begin{aligned}
&\mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})\\
\simeq\ &\mathrm{Fun}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{O}^{\otimes}\times\mathcal{O}'^{\otimes},\mathsf{Fin}_*)} \{p_{\mathcal{O}} \wedge p_{\mathcal{O}'}\}\\
\simeq\ &\mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})\Big) \times_{\mathrm{Fun}\left(\mathcal{O}^{\otimes},\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathsf{Fin}_*)\right)} \{\widetilde{p_{\mathcal{O}} \wedge p_{\mathcal{O}'}}\}\\
\simeq\ &\mathrm{Fun}_{\mathrm{Fun}(\mathcal{O}'^{\otimes},\mathsf{Fin}_*)}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})\Big)
\end{aligned}
$$

It is clear that then $P_2 \circ V \simeq \widehat{(-)} \circ P_1$.

The description obtained above of the essential image of the fully faithful functor $(\iota_{\mathrm{Alg}})_* \circ i$ now implies that the composition $V^{-1} \circ W^{-1} \circ (\iota_{\mathrm{Alg}})_* \circ i$ is fully faithful with essential image spanned by those functors $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ that map pairs of inert morphisms to inert morphisms. But this is by definition [HA, 2.2.5.3] precisely the essential image of the fully faithful functor j. This shows that an induced functor Φ_2 making the diagram commute exists and that Φ_2 is an equivalence. □

Proposition E.5.0.2. *Let $p_{\mathcal{O}} \colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$, $p'_{\mathcal{O}} \colon \mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$, and $p_{\mathcal{C}} \colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be ∞-operads, and let $F \colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads (see [HA, 2.2.5.3]).*

Then there exists a commutative diagram as follows

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \arrow[r] \arrow[d, "\Phi_1"'] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[d, "F^*"] \\ \mathrm{BiFunc}(\mathcal{O}, \mathcal{O}'; \mathcal{C}) \arrow[r] & \mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \end{tikzcd}$$

where the horizontal functors are the canonical ones.

By definition [HA, 2.2.5.3] F exhibits $\mathcal{O}''^{\otimes}$ as a tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ if and only if Φ_1 is an equivalence for every ∞-operad $\mathcal{C}$. ♡

Proof. The existence of the induced dashed functor Φ_1 on full subcategories in the following diagram follows immediately from the fact that F maps pairs of inert morphisms to inert morphisms.

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \arrow[r, dashed, "\Phi_1", "\simeq"'] \arrow[d] & \mathrm{BiFunc}(\mathcal{O}, \mathcal{O}'; \mathcal{O}'') \arrow[d] \\ \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[r, "F^*"'] & \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes}\big) \end{tikzcd}$$

□

E.6. The commutative ∞-operad

Let $\mathcal{O}$ be an ∞-operad. In the next proposition we show that the ∞-operad Comm has the property that the tensor product of $\mathcal{O}$ and Comm is given by Comm again.

Proposition E.6.0.1. *Let $p_{\mathcal{O}} \colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be a reduced*[10] *∞-operad and denote the essentially unique object in $\mathcal{O}$ by $\mathfrak{o}$.*

Then the bifunctor of ∞-operads[11]

$$\alpha \colon \mathcal{O}^{\otimes} \times \mathsf{Comm}^{\otimes} \xrightarrow{p_{\mathcal{O}} \times \mathrm{id}} \mathsf{Comm}^{\otimes} \times \mathsf{Comm}^{\otimes} \xrightarrow{- \wedge -} \mathsf{Comm}^{\otimes}$$

exhibits Comm as a tensor product of $\mathcal{O}$ and Comm.

Let $p_{\mathcal{C}} \colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. By applying Proposition E.4.2.3 to the bifunctor of ∞-operads $- \wedge -$ we obtain an induced ∞-operad $\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C})^{\otimes}$, and the forgetful functor $\mathrm{ev}_{\langle 1 \rangle} \colon \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \to \mathcal{C}$ can by Proposition E.4.2.3 (5) be upgraded to a morphism of ∞-operads.

[10] See [HA, 2.3.4.1] for a definition. It means that $\mathcal{O}$ is a unital ∞-operad and that the underlying ∞-category $\mathcal{O}$ is a contractible ∞-groupoid.
[11] See [HA, 2.2.5.1] for $- \wedge -$.

Then there is a commutative diagram as follows.

$$\begin{tikzcd} & \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C})\big) \arrow[dl, "\mathrm{ev}_{\circ}"', "\simeq"] \arrow[dr, "\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\langle 1\rangle})"] & \\ \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[rr, "p_{\mathcal{O}}^*"'] & & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \end{tikzcd} \tag{E.4}$$

Furthermore, the forgetful functor $\mathrm{ev}_{\circ}$ *is an equivalence. In particular, if* $p_{\mathcal{O}} = \mathrm{id}_{\mathsf{Fin}_*}$*, then* $\mathrm{Alg}_{\mathsf{Comm}}(\mathrm{ev}_{\langle 1\rangle})$ *is homotopic to* $\mathrm{ev}_{\langle 1\rangle}$ *and an equivalence.* ♡

Proof. Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. What we have to show for the first part of the claim is that the functor

$$\Phi_1\colon\ \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \to \mathrm{BiFunc}(\mathcal{O}, \mathsf{Comm}; \mathcal{C})$$

from Proposition E.5.0.2 is an equivalence. Note that by Proposition E.5.0.1, the functor

$$\Phi_2\colon\ \mathrm{BiFunc}(\mathcal{O}, \mathsf{Comm}; \mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}))$$

is an equivalence. We consider the following diagram of commutative squares that summarizes the situation.

$$\begin{tikzcd} \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[r] \arrow[d, "\Phi_1"'] & \mathrm{Fun}\big(\mathsf{Fin}_*, \mathcal{C}^{\otimes}\big) \arrow[d, "(\alpha)^*"] \\ \mathrm{BiFunc}(\mathcal{O}, \mathsf{Comm}; \mathcal{C}) \arrow[r] \arrow[d, "\Phi_2"', "\simeq"] & \mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathsf{Fin}_*, \mathcal{C}^{\otimes}\big) \arrow[d, "\widehat{(-)}"] \\ \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C})\big) \arrow[r] \arrow[d, "\mathrm{ev}_{\circ}"'] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathsf{Fin}_*, \mathcal{C}^{\otimes}\big)\Big) \arrow[d, "\mathrm{ev}_{\circ}"] \\ \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[r] & \mathrm{Fun}\big(\mathsf{Fin}_*, \mathcal{C}^{\otimes}\big) \end{tikzcd} \tag{$*$}$$

Define $\Phi'_{\mathcal{C}}$ to be the left vertical composition $\Phi'_{\mathcal{C}} := \mathrm{ev}_{\circ} \circ \Phi_2 \circ \Phi_1$. As the ∞-operad $\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C})^{\otimes}$ is cocartesian by [HA, 3.2.4.10], we can apply [HA, 2.4.3.9], which states that the forgetful functor $\mathrm{ev}_{\circ}$ is an equivalence. To show that α exhibits Comm as a tensor product of $\mathcal{O}$ and Comm it thus suffices to show that $\Phi'_{\mathcal{C}}$ is an equivalence.

Using naturality of $\widehat{(-)}$ we can identify the right vertical composition with precomposition with the following functor.

$$\mathsf{Fin}_* \xrightarrow{\mathrm{const}_{\circ} \times \mathrm{id}_{\mathsf{Fin}_*}} \mathcal{O}^{\otimes} \times \mathsf{Fin}_* \xrightarrow{p_{\mathcal{O}} \times \mathrm{id}_{\mathsf{Fin}_*}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{-\wedge-} \mathsf{Fin}_*$$

This functor is naturally equivalent to $\mathrm{id}_{\mathsf{Fin}_*}$, so we conclude that the vertical composition on the right in diagram $(*)$ is naturally equivalent to the identity.

Diagram ($*$) is natural in $\mathcal{C}$[12], so the morphism of ∞-operads $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ induces a commutative cubes as follows.

$$\begin{tikzcd}
 & \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[rr] \arrow[dl] \arrow[dd, "\Phi'_{\mathcal{C}}"'] & & \mathrm{Fun}\big(\mathsf{Fin}_*, \mathcal{C}^{\otimes}\big) \arrow[dl] \arrow[dd, "\mathrm{id}"] \\
\mathrm{Alg}_{\mathsf{Comm}}(\mathsf{Comm}) \arrow[rr] \arrow[dd, "\Phi'_{\mathsf{Comm}}"'] & & \mathrm{Fun}(\mathsf{Fin}_*, \mathsf{Fin}_*) \arrow[dd, "\mathrm{id}" near start] & \\
 & \mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[rr] \arrow[dl] & & \mathrm{Fun}\big(\mathsf{Fin}_*, \mathcal{C}^{\otimes}\big) \arrow[dl] \\
\mathrm{Alg}_{\mathsf{Comm}}(\mathsf{Comm}) \arrow[rr] & & \mathrm{Fun}(\mathsf{Fin}_*, \mathsf{Fin}_*) &
\end{tikzcd}$$

Note that the functor $\mathrm{Alg}_{\mathsf{Comm}}(\mathsf{Comm}) \to \mathrm{Fun}(\mathsf{Fin}_*, \mathsf{Fin}_*)$ can be identified with the inclusion of $\{\mathrm{id}_{\mathsf{Fin}_*}\}$, from which it also follows that Φ'_{Comm} can be identified with the identity. Passing to the induced functors from $\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C})$ into the pullbacks of the top and bottom squares we conclude that there is a commutative squares as indicated below

$$\begin{tikzcd}
\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[r] \arrow[d, "\Phi'_{\mathcal{C}}"'] & \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathsf{Fin}_*, \mathcal{C}^{\otimes}\big) \arrow[d, "\mathrm{id}"] \\
\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[r] & \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathsf{Fin}_*, \mathcal{C}^{\otimes}\big)
\end{tikzcd}$$

where the horizontal functors are the canonical inclusions. As both horizontal functors are by definition the same inclusion of a full subcategory, it follows[13] that $\Phi'_{\mathcal{C}}$ is homotopic to the identity functor and hence an equivalence.

It remains to show that there exists a commutative diagram (E.4). For this we can proceed very analogously. As we now know that Φ_1 in diagram ($*$) is an equivalence, it suffices to construct a homotopy between $p^*_{\mathcal{O}} \circ \mathrm{ev}_{\mathfrak{o}} \circ \Phi_2 \circ \Phi_1$ and $\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\langle 1\rangle}) \circ \Phi_2 \circ \Phi_1$. Completely analogously to the arguments above, this time using that the compositions

$$\mathcal{O}^{\otimes} \xrightarrow{p_{\mathcal{O}}} \mathsf{Fin}_* \xrightarrow{\mathrm{const}_{\mathfrak{o}} \times \mathrm{id}_{\mathsf{Fin}_*}} \mathcal{O}^{\otimes} \times \mathsf{Fin}_* \xrightarrow{p_{\mathcal{O}} \times \mathrm{id}_{\mathsf{Fin}_*}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{-\wedge-} \mathsf{Fin}_*$$

and

$$\mathcal{O}^{\otimes} \xrightarrow{\mathrm{id}_{\mathcal{O}^{\otimes}} \times \mathrm{const}_{\langle 1\rangle}} \mathcal{O}^{\otimes} \times \mathsf{Fin}_* \xrightarrow{p_{\mathcal{O}} \times \mathrm{id}_{\mathsf{Fin}_*}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{-\wedge-} \mathsf{Fin}_*$$

are both naturally equivalent to $p_{\mathcal{O}}$, one can obtain commutative diagrams

$$\begin{tikzcd}
\mathrm{Alg}_{\mathsf{Comm}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathsf{Fin}_*, \mathcal{C}^{\otimes}\big) \arrow[d, "p^*_{\mathcal{O}}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r] & \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes}\big)
\end{tikzcd}$$

12 One can check that the two squares involving Φ_1 and Φ_2 are natural in $\mathcal{C}$ by going through their definitions. This is also discussed in Remark F.3.0.4 below.

13 See Proposition B.4.4.1 and Proposition B.4.3.1.

for both $p^*_{\mathcal{O}} \circ \mathrm{ev}_{\mathfrak{o}} \circ \Phi_2 \circ \Phi_1$ as well as $\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\langle 1\rangle}) \circ \Phi_2 \circ \Phi_1$ as the left vertical functor. We thus obtain a homotopy between $p^*_{\mathcal{O}} \circ \mathrm{ev}_{\mathfrak{o}} \circ \Phi_2 \circ \Phi_1$ and $\mathrm{Alg}_{\mathcal{O}}(\mathrm{ev}_{\langle 1\rangle}) \circ \Phi_2 \circ \Phi_1$ by using that the bottom horizontal functor is the inclusion of a full subcategory and applying Proposition B.4.4.1 and Proposition B.4.3.1. □

E.7. Colimits and free algebras

In this section we discuss (operadic) colimits and free algebras, as well as compatibility of functors

$$\mathrm{Alg}_{\mathcal{O}}(F)\colon\ \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$$

induced by a symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, with free algebras and colimits.

We start in Section E.7.1 by discussing operadic colimits, which will be an ingredient for the later sections. In Section E.7.2 we then discuss free algebras, and in analogy we will also briefly show that induced functors on ∞-categories of left modules preserve free *modules* in Section E.7.4. In Section E.7.3 we provide a result for $\mathrm{Alg}_{\mathcal{O}}(F)$ preserving small colimits.

E.7.1. Operadic colimits

In this section we discuss some helpful results regarding operadic colimit diagrams. Section E.7.1.1 covers a criterion that simplifies checking whether certain types of diagrams in a symmetric monoidal ∞-category are operadic colimit diagrams, and Section E.7.1.2 applies this to show that colimit-preserving symmetric monoidal functors also preserve operadic colimits. Both statements as well as their proofs are essentially taken from [GH15, A.2.9][14].

E.7.1.1. A criterion for operadic colimits

We record the following proposition whose proof is essentially given in the proof of [GH15, A.2.9].

Proposition E.7.1.1 ([GH15, A.2.9])**.** *Let $q\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category that is compatible with small colimits in the sense of [HA, 3.1.1.18] and let $p\colon K^{\triangleright} \to \mathcal{C}^{\otimes}$ be a diagram such that $q \circ p$ is the constant functor with value $\langle i\rangle$. Let $m\colon \langle i\rangle \to \langle 1\rangle$ be the unique active morphism.*

Then the following two conditions are equivalent.

(1) p is an operadic q-colimit diagram[15].

(2) The composition

$$K^{\triangleright} \xrightarrow{p} \mathcal{C}^{\otimes}_{\langle i\rangle} \xrightarrow{m_!} \mathcal{C} \tag{E.5}$$

[14] The paper [GH15] is however concerned with the theory of non-symmetric ∞-operads, rather than the symmetric ∞-operads used in [HA], which is why we do not merely cite [GH15, A.2.9].

[15] See [HA, 3.1.1.2] for the definition.

is a colimit diagram. ♡

Proof. By [HA, 3.1.1.16] the condition (1) is equivalent to the following condition.

(3) For every object Y of $\mathcal{C}^{\otimes}$ the composition

$$K^{\triangleright} \xrightarrow{p} \mathcal{C}^{\otimes}_{\langle i \rangle} \xrightarrow{-\oplus Y} \mathcal{C}^{\otimes}_{\langle i \rangle \oplus q(Y)} \xrightarrow{m'_!} \mathcal{C} \tag{$*$}$$

is a colimit diagram, where $m' \colon \langle i \rangle \oplus q(Y) \to \langle 1 \rangle$ is the unique active morphism.

Note that given an object Y of $\mathcal{C}^{\otimes}$, we can write the unique active morphism

$$m' \colon \langle i \rangle \oplus q(Y) \to \langle 1 \rangle$$

as the composition

$$m' = \mu \circ \left(m \oplus m''\right)$$

with $m'' \colon q(Y) \to \langle 1 \rangle$ and $\mu \colon \langle 2 \rangle \to \langle 1 \rangle$ the unique active morphisms. By [HA, 2.2.4.8], we can identify $(m \oplus m'')_!$ with $m_! \oplus m''_!$, so that the composition in ($*$) can be identified with

$$K^{\triangleright} \xrightarrow{p} \mathcal{C}^{\otimes}_{\langle i \rangle} \xrightarrow{-\oplus Y} \mathcal{C}^{\otimes}_{\langle i \rangle \oplus q(Y)} \xrightarrow{m_! \oplus m''_!} \mathcal{C}^{\otimes}_{\langle 2 \rangle} \xrightarrow{\mu_!} \mathcal{C}$$

which can be further identified, using the functoriality of $\oplus$, with the composition

$$K^{\triangleright} \xrightarrow{p} \mathcal{C}^{\otimes}_{\langle i \rangle} \xrightarrow{m_!} \mathcal{C}^{\otimes}_{\langle 1 \rangle} \xrightarrow{-\oplus m''_!(Y)} \mathcal{C}^{\otimes}_{\langle 2 \rangle} \xrightarrow{\mu_!} \mathcal{C}$$

which finally can be identified with the following composition.

$$K^{\triangleright} \xrightarrow{p} \mathcal{C}^{\otimes}_{\langle i \rangle} \xrightarrow{m_!} \mathcal{C} \xrightarrow{-\otimes m''_!(Y)} \mathcal{C} \tag{$**$}$$

As we assumed that the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits, ($**$) is a colimit diagram for all objects Y of $\mathcal{C}^{\otimes}$ if and only if (E.5) is a colimit diagram[16]. □

E.7.1.2. Symmetric monoidal functors and operadic colimits

The following statement is given in [GH15, A.2.9] with the same proof as given below.

Proposition E.7.1.2 ([GH15, A.2.9])**.** *Let $q \colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ and $q' \colon \mathcal{C}'^{\otimes} \to \mathsf{Fin}_*$ be symmetric monoidal ∞-categories that are compatible with small colimits in the sense of [HA, 3.1.1.18] and let $F^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{C}'^{\otimes}$ be a symmetric monoidal functor such that F preserves colimits.*

Let $p \colon K^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ be a operadic q-colimit diagram. Then $F^{\otimes} \circ p$ is a operadic q'-colimit diagram. ♡

[16]The composition (E.5) can be identified with ($**$) in the special case of $Y = \mathbb{1}_{\mathcal{C}}$.

Proof. Let $p_0 = q \circ p$ and let r_0 be the constant functor $K^{\triangleright} \to \mathsf{Fin}_*$ with image $p_0(\infty)$[17]. Then there is a unique natural transformation $\alpha_0 \colon p_0 \to r_0$. By [HTT, 3.1.2.1] we can lift this natural transformation to a natural transformation $\alpha \colon p \to r$ of functors $K^{\triangleright} \to \mathcal{C}^{\otimes}$ such that for each object k of K the morphism $\alpha_k \colon p(k) \to r(k)$ is q-cocartesian.

Note that by construction of α_0 the functor α factors through $\mathcal{C}^{\otimes}_{\mathrm{act}}$. Furthermore, α_∞ is q-cocartesian and lies over the equivalence $\mathrm{id}_{p_0(\infty)}$ and is thus an equivalence by [HTT, 2.4.1.5]. Hence all the assumptions for [HA, 3.1.1.15 (2)] are satisfied and we can conclude that as p is an operadic q-colimit diagram, so is r.

As $F^{\otimes}$ maps q-cocartesian morphisms to q'-cocartesian morphisms and preserves equivalences, we can apply [HA, 3.1.1.15 (2)] also to $F^{\otimes} \circ \alpha$ to conclude that $F \circ p$ is an operadic q'-colimit diagram if and only if $F \circ r$ is, so it now suffices to show that $F \circ r$ is an operadic q'-colimit diagram.

Let $m \colon p_0(\infty) \to \langle 1 \rangle$ be the unique active morphism. Then by Proposition E.7.1.1 the composite

$$K^{\triangleright} \xrightarrow{r} \mathcal{C}^{\otimes}_{p_0(\infty)} \xrightarrow{m_!} \mathcal{C} \tag{$*$}$$

is a colimit diagram, and it suffices to show that

$$K^{\triangleright} \xrightarrow{r} \mathcal{C}^{\otimes}_{p_0(\infty)} \xrightarrow{F^{\otimes}} \mathcal{C}'^{\otimes}_{p_0(\infty)} \xrightarrow{m_!} \mathcal{C}' \tag{$**$}$$

is a colimit diagram.

But as F is symmetric monoidal, composition ($**$) can be identified with

$$K^{\triangleright} \xrightarrow{r} \mathcal{C}^{\otimes}_{p_0(\infty)} \xrightarrow{m_!} \mathcal{C} \xrightarrow{F} \mathcal{C}'$$

so that this is a colimit diagram follows from ($*$) being a colimit diagram and F preserving colimit diagrams by assumption. □

E.7.2. Free algebras

In this section we discuss free algebras; existence of free algebras in Section E.7.2.1 and compatibility of induced functors on ∞-categories of algebras with free algebras in Section E.7.2.2.

E.7.2.1. Detection of free algebras

Let $\mathcal{C}$ be a symmetric monoidal ∞-category, $\mathcal{O}$ an ∞-operad, and X an object of the underlying ∞-category of $\mathcal{O}$. We can then ask whether the forgetful functor

$$\mathrm{ev}_X \colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$$

has a left adjoint, i. e. a free algebra functor[18]. In a more general setting, [HA, 3.1.3.4] shows existence of a free algebra functor, under some assumptions. However, those assumptions, requiring existence of certain operadic colimit diagrams, are not a priori easy

[17] ∞ denotes the cone point of $K^{\triangleright}$.

[18] See [HA, 3.1].

to verify[19]. In the next proposition we thus provide easier to check conditions for $\mathcal{C}$ in the case that $\mathcal{O}$ is either Assoc or $\mathbb{E}_0$ that imply the existence of free algebras, and discuss descriptions of the free algebra generated by a an object of $\mathcal{C}$.

Proposition E.7.2.1 ([HA, 4.1.1.18 and 4.1.1.19])**.** *Let $q\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category. Let $\mathcal{O}$ be either Assoc or $\mathbb{E}_0$. Furthermore, assume the following.*

- *If $\mathcal{O} = \mathsf{Assoc}$, assume that $\mathcal{C}$ admits countable coproducts and that the tensor product preserves countable coproducts in each variable.*
- *If $\mathcal{O} = \mathbb{E}_0$, assume that $\mathcal{C}$ admits finite coproducts and that the tensor product preserves finite coproducts in each variable.*

Then the forgetful functor $\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}}}$ and for every object X of $\mathcal{C}$, the unit

$$X \to \mathrm{ev}_{\langle 1\rangle}\Big(\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}}}(X)\Big)$$

of the adjunction exhibits $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}}}(X)$ as a q-free $\mathcal{O}$-algebra generated by X[20].

Let X be an object of $\mathcal{C}$, let A be an object of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, and let $f\colon X \to \mathrm{ev}_{\langle 1\rangle}(A)$ be a morphism in $\mathcal{C}$. Then the following are equivalent.

(1) f exhibits A as a q-free $\mathcal{O}$-algebra generated by X.

(2) The morphism

$$\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}}}(X) \to A$$

that is adjoint to f is an equivalence in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.

(3)
- *If $\mathcal{O} = \mathsf{Assoc}$: The composition*

$$\coprod_{n\geq 0} X^{\otimes n} \xrightarrow{\coprod_{n\geq 0} f^{\otimes n}} \coprod_{n\geq 0} \mathrm{ev}_{\langle 1\rangle}(A)^{\otimes n} \to \mathrm{ev}_{\langle 1\rangle}(A)$$

is an equivalence, where the morphisms $\mathrm{ev}_{\langle 1\rangle}(A)^{\otimes n} \to \mathrm{ev}_{\langle 1\rangle}(A)$ are those associated to the evaluation of A at an active morphism $\langle n\rangle \to \langle 1\rangle$ in $\mathsf{Assoc}^{\otimes}$[21].

- *If $\mathcal{O} = \mathbb{E}_0$: The composition*

$$\mathbb{1} \amalg X \xrightarrow{\mathrm{id}_{\mathbb{1}} \amalg f} \mathbb{1} \amalg \mathrm{ev}_{\langle 1\rangle}(A) \xrightarrow{i \amalg \mathrm{id}} \mathrm{ev}_{\langle 1\rangle}(A)$$

is an equivalence, where i is the morphism associated to the evaluation of A at the unique morphism $\langle 0\rangle \to \langle 1\rangle$ in $(\mathbb{E}_0)^{\otimes}$. ♡

[19] Unless much stronger assumptions are available, such as the symmetric monoidal structure on $\mathcal{C}$ being compatible with small colimits. See [HA, 3.1.3.5].

[20] See [HA, 3.1.3.1 and 3.1.3.12] for a definition.

[21] Which active morphism is chosen does not change whether the composition is an equivalence or not.

Proof. For $\mathcal{O} = \mathsf{Assoc}$, this is precisely [HA, 4.1.1.18][22], albeit under stronger assumptions regarding what colimits $\mathcal{C}$ needs to be admit and its tensor product needs to be compatible with. That countable coproducts suffice is remarked in [HA, 4.4.1.19]. This follows by tracing through the proof of [HA, 4.1.1.18], where one is ultimately led to [HA, 3.1.3.4], where one needs to ensure that one can construct certain operadic q-colimit diagrams. One then notes that in the specific situation we need to apply this the diagram category is equivalent to $\coprod_{n\geq 0} \mathcal{P}(n)$, where $\mathcal{P}(n)$ are the spaces defined in [HA, 3.1.3.9]. For Assoc these spaces can easily seen to be contractible[23], so colimits indexed by this diagram category are countable coproducts.

The proof for $\mathcal{O} = \mathbb{E}_0$ is completely analogous; the relevant $\mathcal{P}(n)$ are empty for $n > 1$ rather than contractible. □

E.7.2.2. Symmetric monoidal functors and free algebras

Given a symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, an ∞-operad $\mathcal{O}$, and an object X of the underlying ∞-category of $\mathcal{O}$, the induced functor on ∞-categories of algebras

$$\mathrm{Alg}_{\mathcal{O}}(F)\colon\ \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$$

is compatible with the respective forgetful functors ev_X. The next proposition gives conditions for $\mathcal{C}$, $\mathcal{D}$, and F such that $\mathrm{Alg}_{\mathcal{O}}(F)$ is also compatible with the respective free algebra functors.

Proposition E.7.2.2. *Let $\alpha^{\otimes}\colon \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ be a morphism of ∞-operads, $q_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ and $q_{\mathcal{D}}\colon \mathcal{D}^{\otimes} \to \mathsf{Fin}_*$ symmetric monoidal ∞-categories, and $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a symmetric monoidal functor.*

Assume one of the following sets of assumptions.

(1)
- *$\mathcal{C}$ and $\mathcal{D}$ admit small colimits.*
- *The tensor product functors of $\mathcal{C}$ and $\mathcal{D}$ preserve small colimits separately in each variable.*
- *$F\colon \mathcal{C} \to \mathcal{D}$ preserves small colimits.*

(2)
- *$\mathcal{O}^{\otimes} = \mathsf{Triv}^{\otimes}$ and $\mathcal{O}'^{\otimes} = \mathsf{Assoc}$.*
- *$\mathcal{C}$ and $\mathcal{D}$ admit countable coproducts.*
- *The tensor product functors of $\mathcal{C}$ and $\mathcal{D}$ preserve countable coproducts separately in each variable.*
- *$F\colon \mathcal{C} \to \mathcal{D}$ preserves countable coproducts.*

(3)
- *$\mathcal{O}^{\otimes} = \mathsf{Triv}^{\otimes}$ and $\mathcal{O}'^{\otimes} = \mathbb{E}_0$.*
- *$\mathcal{C}$ and $\mathcal{D}$ admit finite coproducts.*

[22] The proof can be found above the statement.
[23] See [HA, above 4.1.1.18].

- *The tensor product functors of* $\mathcal{C}$ *and* $\mathcal{D}$ *preserve finite coproducts separately in each variable.*
- $F\colon \mathcal{C} \to \mathcal{D}$ *preserves finite coproducts.*

Then the following commutative diagram induced by F *(where the two horizontal functors are the forgetful functors given by precomposition with* α*)*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \arrow[r, "U_{\mathcal{C}}"] \arrow[d, "\mathrm{Alg}_{\mathcal{O}'}(F)"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[d, "\mathrm{Alg}_{\mathcal{O}}(F)"] \\ \mathrm{Alg}_{\mathcal{O}'}(\mathcal{D}) \arrow[r, "U_{\mathcal{D}}"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) \end{tikzcd} \tag{E.6}$$

is left adjointable[24]*, i. e.* $U_{\mathcal{C}}$ *and* $U_{\mathcal{D}}$ *have left adjoints* $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}$ *and* $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{D})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})}$*, and the associated push-pull transformation*

$$\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{D})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})} \circ \mathrm{Alg}_{\mathcal{O}}(F) \to \mathrm{Alg}_{\mathcal{O}'}(F) \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}$$

is a natural equivalence. ♡

Proof. By [HA, 3.1.3.5] in case (1) and Proposition E.7.2.1 in cases (2) and (3), the left adjoints exist and for A an object of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ the unit

$$\eta^{\mathcal{C}}_A\colon A \to U_{\mathcal{C}}\Big(\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(A)\Big)$$

of the adjunction exhibits $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(A)$ as the free $\mathcal{O}'$-algebra generated by A, and completely analogously for the other adjunction, whose unit we denote by $\eta^{\mathcal{D}}$.

Let A be an object in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. We have to show[25] that the morphism

$$\Big(\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{D})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})} \circ \mathrm{Alg}_{\mathcal{O}}(F)\Big)(A) \to \Big(\mathrm{Alg}_{\mathcal{O}'}(F) \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}\Big)(A)$$

that is adjoint to the following composition[26]

$$\begin{aligned} &\big(\mathrm{Alg}_{\mathcal{O}}(F)\big)(A) \\ \xrightarrow{\mathrm{Alg}_{\mathcal{O}}(F)\left(\eta^{\mathcal{C}}_A\right)} &\Big(\mathrm{Alg}_{\mathcal{O}}(F) \circ U_{\mathcal{C}} \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}\Big)(A) \\ \xrightarrow{\simeq} &\Big(U_{\mathcal{D}} \circ \mathrm{Alg}_{\mathcal{O}'}(F) \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}\Big)(A) \end{aligned}$$

is an equivalence (see [HTT, beginning of 7.3.1]).

[24] See [HTT, 7.3.1.1] for the definition.

[25] By Proposition A.3.2.1 a natural transformation is a natural equivalence if and only if it is a pointwise equivalence.

[26] The equivalence used is to be the one obtained from the equivalence $\mathrm{Alg}_{\mathcal{O}}(F) \circ U_{\mathcal{C}} \simeq U_{\mathcal{D}} \circ \mathrm{Alg}_{\mathcal{O}'}(F)$ encoded in the commutative diagram (E.6).

But by definition of $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{D})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})}$ (see [HA, 3.1.3.5 and 3.1.3.1] in case (1) and see Proposition E.7.2.1 in cases (2) and (3)), the former morphism is an equivalence if and only if the latter morphism exhibits $\left(\mathrm{Alg}_{\mathcal{O}'}(F) \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}\right)(A)$ as a $q_{\mathcal{D}}$-free $\mathcal{O}'$-algebra generated by $\big(\mathrm{Alg}_{\mathcal{O}}(F)\big)(A)$ – so this is what we need to show.

Similarly, by definition of $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}$, the morphism

$$\eta^{\mathcal{C}}_A \colon A \to \left(U_{\mathcal{C}} \circ \mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}\right)(A)$$

exhibits $\mathrm{Free}^{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(A)$ as a $q_{\mathcal{C}}$-free $\mathcal{O}'$-algebra generated by A.

Proof in case (1): Unpacking the definitions of free algebras (see [HA, 3.1.3.1]) one sees that the claim boils down to showing that $F^{\otimes}$ preserves certain operadic colimit diagrams, so the claim follows from Proposition E.7.1.2.

Proof in cases (2) and (3): In these cases we can use the criteria from Proposition E.7.2.1 and thus the claim follows from F being symmetric monoidal and preserving countable/finite colimits. □

E.7.3. Induced functors on Alg and colimits

In the following proposition we show that a colimit preserving symmetric monoidal functor induces a colimit preserving functor on ∞-categories of algebras.

Proposition E.7.3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories and $F\colon \mathcal{C} \to \mathcal{D}$ a symmetric monoidal functor. Assume that $\mathcal{C}$ and $\mathcal{D}$ admit all small colimits, that the tensor product functors of $\mathcal{C}$ and $\mathcal{D}$ preserve small colimits separately in each variable, and that F preserves small colimits.*

Let $\mathcal{O}$ be an ∞-operad. Then $\mathrm{Alg}_{\mathcal{O}}(F)$ preserves small colimits as well. ♡

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})"] \arrow[d, "U_{\mathcal{C}}"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) \arrow[d, "U_{\mathcal{D}}"] \\ \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \arrow[r, "F_*"'] & \mathrm{Fun}(\mathcal{O}, \mathcal{D}) \end{tikzcd} \qquad (*)$$

where $U_{\mathcal{C}}$ and $U_{\mathcal{D}}$ are the forgetful functors.

To show that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})$ preserves colimits it suffices by combining [HTT, 4.2.3.12] with [HA, 1.3.3.10 (2)] to show that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})$ preserves sifted colimits as well as coproducts.

By [HA, 3.2.3.1][27] together with [HTT, 5.1.2.3 (2)], the two vertical functors in diagram ($*$) detects sifted colimits. As F preserves all small colimits by assumption, we obtain with [HTT, 5.1.2.3 (2)] that the bottom horizontal functor in diagram ($*$) preserves all small, so in particular all sifted, colimits. We can thus conclude that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})$ preserves sifted colimits.

[27] Which is applicable to our situation by Proposition E.2.0.2.

It then follows from the proof of [HA, 3.2.3.3][27] that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})$ also preserves coproducts if the composition with the left adjoint $\mathrm{Free}_{\mathcal{C}}$ of $U_{\mathcal{C}}$ does. But by Proposition E.7.2.2 there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\mathrm{Alg}_{\mathcal{O}}(\mathcal{F})"] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) \\
\mathrm{Fun}(\mathcal{O}, \mathcal{C}) \arrow[u, "\mathrm{Free}_{\mathcal{C}}"] \arrow[r, "F_*"'] & \mathrm{Fun}(\mathcal{O}, \mathcal{D}) \arrow[u, "\mathrm{Free}_{\mathcal{D}}"']
\end{tikzcd}$$

where $\mathrm{Free}_{\mathcal{D}}$ is the left adjoint of $U_{\mathcal{D}}$. That the composition from the bottom left to the top right in this diagram preserves coproducts now follows immediately from F_* preserving small colimits as mentioned above and $\mathrm{Free}_{\mathcal{D}}$ preserving colimits as a left adjoint [HTT, 5.2.3.5]. □

E.7.4. Free modules

Similarly to Proposition E.7.2.2, which dealt with compatibility of induced functors on ∞-categories of algebras with free algebras, the next propositions discusses compatibility of induced functors on ∞-categories of left modules with free modules.

Proposition E.7.4.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor of monoidal ∞-categories and let R be an (associative) algebra in $\mathcal{C}$. Then the commutative diagram*

$$\begin{tikzcd}
\mathrm{LMod}_R(\mathcal{C}) \arrow[r, "\mathrm{ev}_{\mathfrak{m}}"] \arrow[d, "\mathrm{LMod}_R(F)"'] & \mathcal{C} \arrow[d, "F"] \\
\mathrm{LMod}_{F(R)}(\mathcal{D}) \arrow[r, "\mathrm{ev}_{\mathfrak{m}}"'] & \mathcal{D}
\end{tikzcd} \tag{E.7}$$

induced by F is left adjointable in the sense of [HTT, 7.3.1.1], i. e. the associated push-pull transformation

$$\mathrm{Free}_{\mathcal{D}} \circ F \to \mathrm{LMod}_R(F) \circ \mathrm{Free}_{\mathcal{C}}$$

is an equivalence, where $\mathrm{Free}_{\mathcal{C}}$ and $\mathrm{Free}_{\mathcal{D}}$ are the free module functors for $\mathcal{C}$ and $\mathcal{D}$, respectively (see [HA, 4.2.4.8]).

In other words, F preserves free left-R-modules. The analogous statement is true for right-R-modules. ♡

Proof. Let X be an object of $\mathcal{C}$. By Proposition A.3.2.1 it suffices to show that the push-pull morphism

$$(\mathrm{Free}_{\mathcal{D}} \circ F)(X) \to \big(\mathrm{LMod}_R(F) \circ \mathrm{Free}_{\mathcal{C}}\big)(X)$$

is an equivalence, and as $\mathrm{ev}_{\mathfrak{m}}$ is conservative by [HA, 4.2.3.3], we actually only need to show that $\mathrm{ev}_{\mathfrak{m}}$ of that morphism is an equivalence.

Consider the following commutative diagram that will be explained below.

$$\begin{tikzcd}
F(R)\otimes F(X) \arrow[r] \arrow[dd] & F(R)\otimes(\mathrm{ev}_{\mathfrak{m}}\circ \mathrm{Free}_{\mathcal{D}}\circ F)(X) \arrow[r] \arrow[d] & (\mathrm{ev}_{\mathfrak{m}}\circ \mathrm{Free}_{\mathcal{D}}\circ F)(X) \arrow[d] \\
 & F(R)\otimes\big(\mathrm{ev}_{\mathfrak{m}}\circ \mathrm{LMod}_R(F)\circ \mathrm{Free}_{\mathcal{C}}\big)(X) \arrow[r] \arrow[d] & \big(\mathrm{ev}_{\mathfrak{m}}\circ \mathrm{LMod}_R(F)\circ \mathrm{Free}_{\mathcal{C}}\big)(X) \arrow[d] \\
F(R)\otimes F(X) \arrow[r] \arrow[d] & F(R)\otimes F\big((\mathrm{ev}_{\mathfrak{m}}\circ \mathrm{Free}_{\mathcal{C}})(X)\big) \arrow[d] & F\big((\mathrm{ev}_{\mathfrak{m}}\circ \mathrm{Free}_{\mathcal{C}})(X)\big) \arrow[d] \\
F(R\otimes X) \arrow[r] & F\big(R\otimes(\mathrm{ev}_{\mathfrak{m}}\circ \mathrm{Free}_{\mathcal{C}})(X)\big) \arrow[r] & F\big((\mathrm{ev}_{\mathfrak{m}})\circ \mathrm{Free}_{\mathcal{C}}(X)\big)
\end{tikzcd}$$

The left horizontal morphisms are induced by the units of the adjunctions $\mathrm{Free}_{\mathcal{D}} \dashv \mathrm{ev}_{\mathfrak{m}}$ and $\mathrm{Free}_{\mathcal{C}} \dashv \mathrm{ev}_{\mathfrak{m}}$, and the right horizontal morphisms are (induced by) the action morphism of the respective modules. The top vertical morphisms on the left and the bottom vertical morphism on the right are the identity morphisms, and the bottom vertical morphism in the left and middle column are the equivalences arising from monoidality of F. In the middle and right column, the top vertical morphism is induced by the push-pull-transformation, and the middle vertical morphisms arise are the equivalences that arise from commutativity of (E.7).

The composition of the top two horizontal morphisms is an equivalence by the definition of free modules [HA, 4.2.4.1], and so is the composition of the bottom two horizontal morphisms. The two left vertical morphisms as well as the bottom and middle vertical morphism on the right are equivalences as well, so it follows that the vertical morphism at the top right is an equivalence, which is what needed to be shown. □

E.8. Relative tensor products

Let C be a monoidal category and R, S, and T associative algebras in C. If M is an R-S-bimodule and N an S-T-bimodule, then we can form the relative tensor product of M with N over S, denoted by $M \otimes_S N$, which yields an R-T-bimodule.

This construction is generalized to the ∞-categorical setting in [HA, 4.4][28], and can be (very) roughly summarized as follows. If $\mathcal{C}$ is a monoidal ∞-category that is compatible with $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits, R an associative algebra in $\mathcal{C}$, M a right-R-module, and N a left-R-module, then there exists a simplicial object in $\mathcal{C}$ denoted by $\mathrm{Bar}_R(M, N)$ that

[28]Unfortunately there seems to be a mistake in the definition of $\mathsf{Tens}^{\otimes}$ in [HA, 4.4.1.1]. For morphisms one should additionally require for any element j of $\langle n' \rangle^{\circ}$ such that $c'_{-}(j) \neq c'_{+}(j)$ that the preimage of j under α is non-empty. One can think of it like this: Any nontrivial step from $c'_{-}(j)$ to $c'_{+}(j)$ needs to come from a step in the preimages.

The same mistake occurs in the description [HA, 4.3.1.5] of the ∞-operad encoding bimodules. Here one needs to make the same correction. Without this correction algebras over this operad would not consist of triples (R, M, S) with R and S associative algebras and M an R-S-bimodule, but such triples together with an additional unit morphism $\mathbb{1} \to M$ for M, encoded by the morphism from the unique object $\emptyset$ over $\langle 0 \rangle$ to $\mathfrak{m}$.

is given in level n by[29] $M \otimes R^{\otimes n} \otimes N$ and has structure morphisms constructed from the unit morphism of R, the multiplication of R, and the action of R on M and N. The relative tensor product $M \otimes_R N$ is then the geometric realization of $\mathrm{Bar}_R(M, N)$. See [HA, 4.4.2.8].

In this section we will record some properties of relative tensor products that we will need.

Proposition E.8.0.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal ∞-categories and $F\colon \mathcal{C} \to \mathcal{D}$ a monoidal functor. Assume that $\mathcal{C}$ and $\mathcal{D}$ admit $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits, their tensor product functors commute with $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits in each variable separately, and F preserves $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits.*

Then F preserves relative tensor products. ♡

Remark E.8.0.2. Let us clarify what the statement of Proposition E.8.0.1 actually means at a more concrete or technical level. Let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Assoc}^{\otimes}$ and $p_{\mathcal{D}}\colon \mathcal{D}^{\otimes} \to \mathsf{Assoc}^{\otimes}$ be the cocartesian fibrations of ∞-operads that exhibit $\mathcal{C}$ and $\mathcal{D}$ as monoidal ∞-categories. Suppose we have given a morphism of generalized ∞-operads φ fitting into the following commutative diagram

$$\begin{tikzcd} & \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "p_{\mathcal{C}}"] & & \mathcal{D}^{\otimes} \arrow[dl, "p_{\mathcal{D}}"] \\ \mathsf{Tens}_{\succ}^{\otimes} \arrow[ur, "\varphi"] \arrow[rr] & & \mathsf{Assoc}^{\otimes} & \end{tikzcd}$$

where the bottom horizontal functor is the forgetful functor. Then the statement of Proposition E.8.0.1 is that if φ is an operadic $p_{\mathcal{C}}$-colimit diagram, then $F^{\otimes} \circ \varphi$ is an operadic $p_{\mathcal{D}}$-colimit diagram, see [HA, 4.4.2.3].

From this the various other, perhaps more concrete, formulations of what it means for a monoidal functor to preserve relative tensor products follow. For example we then have a commutative square

$$\begin{tikzcd} \mathrm{BiMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BiMod}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{BiMod}(\mathcal{D}) \times_{\mathrm{Alg}(\mathcal{D})} \mathrm{BiMod}(\mathcal{D}) \arrow[d] \\ \mathrm{BiMod}(\mathcal{C}) \arrow[r] & \mathrm{BiMod}(\mathcal{D}) \end{tikzcd}$$

where the horizontal functors are those induced by F and the vertical functors are the relative tensor product functors of [HA, 4.4.2.11]. ♢

Proof of Proposition E.8.0.1. We use the notation and setup from Remark E.8.0.2. Let the restriction of φ to $\mathsf{Tens}_{[2]}^{\otimes}$ correspond to a quintuple (R, M, S, N, T), with R, S, and

[29] That this is really how the bar construction looks like in level n can be seen by digging through and unpacking the definition [HA, 4.4.2.7], but it is a bit tedious.

T associative algebras in $\mathcal{C}$, with M an R,S-bialgebra, and N an S,T-bialgebra, see [HA, 4.4.2.2]. Similarly, let the restriction to $\mathsf{Tens}^{\otimes}_{[1]}$ correspond to a triple (R', X, T') with R' and T' associative algebras and X an R',T'-bialgebra.

By [HA, 4.4.2.8], the morphisms $R \to R'$ and $T \to T'$ induced by φ are equivalences[30] and the comparison morphism

$$\left|\mathrm{Bar}_S(M, N)\right| \to \mathrm{ev}_{\mathfrak{m}}(X)$$

is an equivalence. What we have to show is that then the morphisms $F(R) \to F(R')$ and $F(T) \to F(T')$ induced by $F^{\otimes} \circ \varphi$ are equivalences and that the comparison morphism

$$\left|\mathrm{Bar}_{F(S)}(F(M), F(N))\right| \to F(\mathrm{ev}_{\mathfrak{m}}(X)) \tag{$*$}$$

is an equivalence.

The former is clear because these morphisms are just given by F applied to the analogous morphisms $R \to R'$ and $T \to T'$ in $\mathcal{C}$.

As $F^{\otimes}$ maps $p_{\mathcal{C}}$-cocartesian morphisms to $p_{\mathcal{D}}$-cocartesian morphisms, it follows from the definition that

$$\mathrm{Bar}_{F(S)}(F(M), F(N)) \simeq F \circ \mathrm{Bar}_S(M, N)$$

see [HA, 4.4.2.7]. That ($*$) is an equivalence now follows from combining this with F preserving $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits by assumption. □

Proposition E.8.0.3. *Let $\mathcal{C}$ be a cocartesian symmetric monoidal structure[31] such that the underlying ∞-category of $\mathcal{C}$ admits $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits as well as pushouts*

Then the tensor product of $\mathcal{C}$ is compatible with $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits as well as pushouts in the sense of [HA, 3.1.1.18].

Let R, S, and T be associative algebras in $\mathcal{C}$. Let $f\colon M \to M'$ be a morphism in $\mathrm{BiMod}_{R,S}(\mathcal{C})$ and $g\colon N \to N'$ a morphism in $\mathrm{BiMod}_{S,T}(\mathcal{C})$. We obtain a commutative diagram

$$\begin{tikzcd}
M \otimes_S N \arrow[r, "\mathrm{id}_M \otimes_{\mathrm{id}_S} g"] \arrow[d, "f \otimes_{\mathrm{id}_S} \mathrm{id}_N"'] & M \otimes_S N' \arrow[d, "f \otimes_{\mathrm{id}_S} \mathrm{id}_{N'}"] \\
M' \otimes_S N \arrow[r, "\mathrm{id}_{M'} \otimes_{\mathrm{id}_S} g"'] & M' \otimes_S N'
\end{tikzcd}$$

in $\mathrm{BiMod}_{R,T}(\mathcal{C})$. Then this diagram is a pushout square. ♡

Proof. We first show that the symmetric monoidal structure on $\mathcal{C}$ is compatible with pushouts and $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits. So let X be an object of $\mathcal{C}$. Let $\mathcal{I}$ be either $\mathbf{\Delta}^{\mathrm{op}}$ or $\Lambda^2_0 = (\bullet \leftarrow \bullet \to \bullet)$ and $F\colon \mathcal{I} \to \mathcal{C}$ a functor. It suffices to show that the canonical comparison morphism

$$\mathrm{colim}(X \amalg F) \to X \amalg \mathrm{colim}\, F$$

[30] Condition (i) boils down to this, as Assoc is reduced, see [HA, 4.4.2.6].

[31] See [HA, 2.4.0.1] for a definition.

is an equivalence. As colimits commute with colimits [HTT, 5.5.2.3] this morphism factors as an equivalence $\operatorname{colim}(X \amalg F) \simeq (\operatorname{colim}\operatorname{const}_X) \amalg (\operatorname{colim} F)$ and the canonical morphism $(\operatorname{colim}\operatorname{const}_X) \amalg (\operatorname{colim} F) \to X \amalg \operatorname{colim} F$. It thus suffices to show that $(\operatorname{colim}\operatorname{const}_X) \to X$ is an equivalence, which follows from [HTT, 4.4.4.10], as $\mathcal{I}$ is weakly contractible[32].

We can now apply [HA, 4.3.3.9] to conclude that pushouts are detected by the forgetful functor $\mathrm{ev}_{\mathfrak{m}}\colon \mathrm{BiMod}_{R,T}(\mathcal{C}) \to \mathcal{C}$, so combining this with the description of relative tensor products from [HA, 4.4.2.8] it suffices to show that the commutative diagram

$$\begin{tikzcd}
\left|\mathrm{Bar}_S(M,N)\right| \arrow[r, "\left|\mathrm{Bar}_{\mathrm{id}_S}(\mathrm{id}_M,g)\right|"] \arrow[d, "\left|\mathrm{Bar}_{\mathrm{id}_S}(f,\mathrm{id}_N)\right|"'] & \left|\mathrm{Bar}_S(M,N')\right| \arrow[d, "\left|\mathrm{Bar}_{\mathrm{id}_S}(f,\mathrm{id}_{N'})\right|"] \\
\left|\mathrm{Bar}_S(M',N)\right| \arrow[r, "\left|\mathrm{Bar}_{\mathrm{id}_S}(\mathrm{id}_{M'},g)\right|"'] & \left|\mathrm{Bar}_S(M',N')\right|
\end{tikzcd}$$

is a pushout square in $\mathcal{C}$.

Using compatibility of colimits with colimits again it suffices to show for every $n \geq 0$ that the commutative square

$$\begin{tikzcd}
M \amalg \left(\coprod_{i=1}^n S\right) \amalg N \arrow[r, "\mathrm{id}_M \amalg \mathrm{id} \amalg g"] \arrow[d, "f \amalg \mathrm{id} \amalg \mathrm{id}_N"'] & M \amalg \left(\coprod_{i=1}^n S\right) \amalg N' \arrow[d, "f \amalg \mathrm{id} \amalg \mathrm{id}_{N'}"] \\
M' \amalg \left(\coprod_{i=1}^n S\right) \amalg N \arrow[r, "\mathrm{id}_{M'} \amalg \mathrm{id} \amalg g"'] & M' \amalg \left(\coprod_{i=1}^n S\right) \amalg N'
\end{tikzcd}$$

is a pushout square in $\mathcal{C}$, which yet again follows from colimits commuting with colimits, as this is evidently a coproduct of pushout diagrams. □

Construction E.8.0.4. Let $\mathcal{C}$ be a symmetric monoidal ∞-category, and assume that the underlying ∞-category admits $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits, and that the tensor product functor preserves $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits separately in each variable.

Let $f\colon R \to S$ and $g\colon R \to T$ be morphisms in $\mathrm{CAlg}(\mathcal{C})$. We can upgrade f and g to morphisms in right-R-modules and left-R-modules in $\mathrm{CAlg}(\mathcal{C})$, as we now explain for g, the case for f being completely analogous.

By [HA, 3.2.4.7] the induced symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is cocartesian, so by [HA, 2.4.3.9] the forgetful functor

$$\mathrm{ev}_{\mathfrak{a}}\colon \mathrm{Alg}\big(\mathrm{CAlg}(\mathcal{C})\big) \to \mathrm{CAlg}(\mathcal{C})$$

is an equivalence, and so we can upgrade g to a morphism $\overline{g}$ in $\mathrm{Alg}\big(\mathrm{CAlg}(\mathcal{C})\big)$ with $\mathrm{ev}_{\mathfrak{a}}(\overline{g}) \simeq g$.

By applying the section $\mathrm{Alg}\big(\mathrm{CAlg}(\mathcal{C})\big) \to \mathrm{LMod}\big(\mathrm{CAlg}(\mathcal{C})\big)$ discussed in [HA, 4.2.1.17] we obtain a morphism $\widetilde{g}\colon (R,R) \to (T,T)$ in $\mathrm{LMod}(\mathrm{CAlg}(\mathcal{C}))$ with $\mathrm{ev}_{\mathfrak{a}}(\widetilde{g}) \simeq \overline{g}$ and

[32]This means that the ∞-groupoid completion of $\mathcal{I}$ is contractible as a space.

$\mathrm{ev}_{\mathfrak{m}}(\widetilde{g}) \simeq g$. The forgetful functor $\mathrm{LMod}(\mathrm{CAlg}(\mathcal{C})) \to \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C}))$ is a cartesian fibration by [HA, 4.2.3.2] and a cartesian lift of $\overline{g}$ with target (T, T) lies over an equivalence in $\mathrm{CAlg}(\mathcal{C})$. This cartesian lift can be interpreted as the restriction of the T-action on T to R along $\overline{g}$. We obtain an induced morphism of left-R-modules $g' \colon R \to T$ with $\mathrm{ev}_{\mathfrak{m}}(g') \simeq g$.

By [HA, 3.2.3.2] the ∞-category $\mathrm{CAlg}(\mathcal{C})$ admits $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits, and as the forgetful functor $\mathrm{ev}_{\langle 1 \rangle} \colon \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ is both symmetric monoidal by Proposition E.4.2.3 (5) as well as preserves and detects $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits by [HA, 3.2.3.2], it follows that the induced tensor product on $\mathrm{CAlg}(\mathcal{C})$ is compatible with $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits as well.

We thus obtain a commutative diagram in $\mathrm{CAlg}(\mathcal{C})$ as follows

$$\begin{tikzcd}
R \arrow[rrr, "g"] \arrow[dd, "f"'] \arrow[dr, dash, "\simeq"] & & & T \arrow[d, dash, "\simeq"] \\
& R \otimes_R R \arrow[rr, "\mathrm{id}_R \otimes_{\mathrm{id}_R} g'"] \arrow[d, "f' \otimes_{\mathrm{id}_R} \mathrm{id}_R"'] & & R \otimes_R T \arrow[d, "f' \otimes_{\mathrm{id}_R} \mathrm{id}_T"] \\
S \arrow[r, dash, "\simeq"'] & S \otimes_R R \arrow[rr, "\mathrm{id}_S \otimes_{\mathrm{id}_R} g'"'] & & S \otimes_R T
\end{tikzcd} \tag{E.8}$$

where the unlabeled equivalences are the unitality equivalences of the relative tensor product discussed in [HA, 4.4.3.16], see also [HA, 4.4.3.18]. ♢

Proposition E.8.0.5. *Assume we are in the situation of Construction E.8.0.4, and that $\mathcal{C}$ additionally admits small colimits and that the tensor product preserves small colimits separately in each variable.*

Then the commutative square

$$\begin{tikzcd}
R \arrow[r, "g"] \arrow[d, "f"'] & T \arrow[d] \\
S \arrow[r] & S \otimes_R T
\end{tikzcd}$$

from (E.8) *is a pushout square in* $\mathrm{CAlg}(\mathcal{C})$. ♡

Proof. It suffices to show that the smaller square on the lower right in diagram (E.8) is a pushout square.

Note that by [HA, 3.2.3.3] $\mathrm{CAlg}(\mathcal{C})$ again admits small colimits. We can thus apply Proposition E.8.0.3, which shows the claim. □

Appendix F.

Cartesian symmetric monoidal ∞-categories

In this appendix we collect some results relating to cartesian symmetric monoidal ∞-categories. In Section F.1 we discuss how cocartesian fibrations whose fibers are compatible with products in the sense of Definition C.2.0.1 interact with cartesian symmetric monoidal structures. The short section Section F.2 describes limits in ∞-categories of monoids. The main part of this section is Section F.3, in which we discuss how to relate $\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C})$, $\mathrm{Mon}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C})$, $\mathrm{Mon}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}))$, and $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}))$, where $\mathcal{C}$ is an ∞-category admitting finite products that is equipped with the cartesian symmetric monoidal structure, and $\mathcal{O}$ and $\mathcal{O}'$ are ∞-operads.

F.1. Cocartesian fibrations and cartesian symmetric monoidal structures

Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration whose fibers are compatible with products in the sense of Definition C.2.0.1, and let $\pi_{\mathcal{D}}\colon \mathcal{D}^{\times} \to \mathcal{D}$ be the cartesian structure on the cartesian symmetric monoidal structure on $\mathcal{C}$ (see [HA, 2.4.1]). By Proposition C.2.0.3, p preserves products. The goal of this section is to show that the induced functor $p^{\times}: \mathcal{C}^{\times} \to \mathcal{D}^{\times}$ can be obtained as a pullback of p along $\pi_{\mathcal{D}}$. Before we can prove this, we first show the following statement regarding how cocartesian morphisms interact with weak cartesian structures.

Proposition F.1.0.1. *Let $q\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category and $\pi\colon \mathcal{C}^{\otimes} \to \mathcal{D}$ a weak cartesian structure*[1] *on $\mathcal{C}^{\otimes}$.*

Let $C \simeq C_1 \oplus \cdots \oplus C_n$ be an object of $\mathcal{C}_{\langle n\rangle}$ with C_i an object of $\mathcal{C}$ for $1 \leq i \leq n$. Let $\varphi\colon \langle n\rangle \to \langle m\rangle$ be a morphism in Fin_* and let $f: C \to C'$ be a q-cocartesian lift of φ.

[1] See [HA, 2.4.1.1] for a definition

Then there exists a commutative diagram

$$\begin{tikzcd}
\pi(C) \arrow[r, "\pi(f)"] \arrow[d, "\simeq"'] & \pi(C') \arrow[d, "\simeq"] \\
\prod\limits_{1\leq i\leq n} \pi(C_i) \arrow[r] & \prod\limits_{\substack{1\leq i\leq n,\\ \varphi(i)\neq *}} \pi(C_i)
\end{tikzcd}$$

where the bottom horizontal morphism is the projection to the subproduct, the right vertical morphism is an equivalence, and the left vertical morphism is induced by the canonical morphisms $\pi(C) \to \pi(C_i)$ (which are induced by inert morphisms lying over ρ^i), and thus an equivalence as π is a lax cartesian structure. ♡

Proof. We first consider the case in which φ is inert. Then we can identify f with the following canonical projection morphism.

$$\bigoplus_{1\leq i\leq n} C_i \to \bigoplus_{\substack{1\leq i\leq n,\\ \varphi(i)\neq *}} C_i$$

Let

$$g_j\colon \bigoplus_{\substack{1\leq i\leq n,\\ \varphi(i)\neq *}} C_i \to C_j$$

be the canonical projection morphism for $1 \leq j \leq n$ with $\varphi(j) \neq *$ and define h_j similarly to be the projection $\bigoplus_{1\leq i\leq n} C_i \to C_j$ for $1 \leq j \leq n$. That π is lax cartesian means that the morphism

$$\pi\left(\bigoplus_{1\leq i\leq n} C_i\right) \xrightarrow{\prod_{1\leq i\leq n} h_i} \prod_{1\leq i\leq n} \pi(C_i)$$

is an equivalence, and similarly for $\bigoplus_{1\leq i\leq n, \varphi(i)\neq *} C_i$. The claim now follows from the fact that for $1 \leq i \leq n$ with $\varphi(i) \neq *$ the composition $g_i \circ f$ can be identified with h_i.

Let us now consider the general case. Let $g\colon C' \to C''$ be a q-cocartesian lift of the active morphism $\langle m\rangle \to \langle 1\rangle$. As π is a weak cartesian structure, $\pi(g)$ is an equivalence. It thus suffices to consider the case where $m = 1$. We can factor φ as a composition $\varphi = \alpha \circ \beta$ where β is inert and α active (see [HA, 2.1.2.2]). Lifting β and α to a commuting triangle $f \simeq g \circ h$ of q-cocartesian morphisms, with h a lift of β and g a lift of α, we can again use the fact that π is a weak cartesian structure (and that $m = 1$) to conclude that $\pi(g)$ is an equivalence. We are thus reduced to the case of inert morphisms, which we have already proven. □

Proposition F.1.0.2. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration whose fibers are compatible with products in the sense of Definition C.2.0.1, and let $\pi_{\mathcal{C}}\colon \mathcal{C}^{\times} \to \mathcal{C}$ and $\pi_{\mathcal{D}}\colon \mathcal{D}^{\times} \to \mathcal{D}$ be the cartesian structures on the cartesian symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, respectively (see [HA, 2.4.1.5 (5)]).*

Then the square induced via [HA, 2.4.1.8 and 2.4.1.6] by the product preserving functor p (see Proposition C.2.0.3)

$$\begin{tikzcd} \mathcal{C}^\times \arrow[r, "\pi_\mathcal{C}"] \arrow[d, "p^\times"'] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D}^\times \arrow[r, "\pi_\mathcal{D}"'] & \mathcal{D} \end{tikzcd}$$

is a pullback in $\mathcal{C}\mathrm{at}_\infty$. ♡

Proof. Consider the following commutative diagram, where the square is a pullback square.

$$\begin{tikzcd} \mathcal{C}^\times \arrow[dr, dashed, "\theta^\otimes"] \arrow[drr, bend left, "\pi_\mathcal{C}"] \arrow[ddr, bend right, "p^\times"'] & & \\ & \mathcal{C}^\otimes \arrow[r, "\pi"] \arrow[d, "p^\otimes"'] & \mathcal{C} \arrow[d, "p"] \\ & \mathcal{D}^\times \arrow[r, "\pi_\mathcal{D}"'] & \mathcal{D} \end{tikzcd}$$

It suffices to show that $\theta^\otimes$ is an equivalence.

The ∞-category $\mathcal{D}^\times$ comes with a cocartesian fibration, which we will denote by $q\colon \mathcal{D}^\times \to \mathsf{Fin}_*$, that makes $\mathcal{D}^\times$ into a symmetric monoidal ∞-category with underlying ∞-category $\mathcal{D}$ (see [HA, 2.4.1.5 (4)]). With this we can now state the three claims through which the proof will proceed:

(A) $p^\otimes$ is a cocartesian fibration of ∞-operads[2].

It follows from (A) that the functor $q \circ p^\otimes\colon \mathcal{C}^\otimes \to \mathsf{Fin}_*$ upgrades $\mathcal{C}^\otimes$ to a symmetric monoidal ∞-category. Note that by construction $p^\times\colon \mathcal{C}^\times \to \mathcal{D}^\times$ arises as a symmetric monoidal functor between symmetric monoidal ∞-categories, so in particular $p^\times$ can be lifted to a functor over Fin_*. It then follows that also $\theta^\otimes$ can be lifted to a functor over Fin_*. This gives meaning to the next claim.

(B) The functor $\theta^\otimes$ can be upgraded to a symmetric monoidal functor.

(C) The functor $\theta = \theta^\otimes_{\langle 1\rangle}$ is an equivalence.

Once we have proven these three claims, the statement follows immediately from [HA, 2.1.3.8], which states that as a symmetric monoidal functor (by (B)), $\theta^\otimes$ is already an equivalence if θ is is an equivalence (which it is by (C)).

Proof of (A): As p is a cocartesian fibration we can apply Proposition C.1.1.1 and conclude that $p^\otimes$ is also a cocartesian fibration. We will use [HA, 2.1.2.12] to show that $p^\otimes$ is even a cocartesian fibration of ∞-operads. So let $D \simeq D_1 \oplus \cdots \oplus D_n$ be an object

[2] See [HA, 2.1.2.13] for the definition.

in $\mathcal{D}^{\times}_{\langle n\rangle}$ with D_i objects in $\mathcal{D}$ for $1 \leq i \leq n$, and let $f^i\colon D \to D_i$ for $1 \leq i \leq n$ be the canonical inert morphisms. We have to show that the induced morphism on fibers

$$\mathcal{C}^{\otimes}_{D} \xrightarrow{\prod_{1\leq i\leq n} f^i_!} \prod_{1\leq i\leq n} \mathcal{C}^{\otimes}_{D_i} \tag{F.1}$$

is an equivalence of ∞-categories. The fiber of $p^{\otimes}$ over some object D' can be identified with the fiber of p over $\pi_{\mathcal{D}}(D')$, and it follows from the description of $p^{\otimes}$-cocartesian morphisms in Proposition C.1.1.1 that this identification is compatible with the respective induced morphisms on fibers. We can thus identify functor (F.1) with the following functor.

$$\mathcal{C}_{\pi_{\mathcal{D}}(D)} \xrightarrow{\prod_{1\leq i\leq n} \pi_{\mathcal{D}}(f^i)_!} \prod_{1\leq i\leq n} \mathcal{C}_{\pi_{\mathcal{D}}(D_i)} \tag{F.2}$$

As $\pi_{\mathcal{D}}$ is a lax cartesian structure (see [HA, 2.4.1.1]) we can identify $\pi_{\mathcal{D}}(D)$ with the product $\prod_{1\leq i\leq n} \pi_{\mathcal{D}}(D_i)$ and the morphisms $\pi_{\mathcal{D}}(f^j)\colon \pi_{\mathcal{D}}(D) \to \pi_{\mathcal{D}}(D_j)$ for $1 \leq j \leq n$ with the projection pr_j. We can thus identify functor (F.2) with the following functor.

$$\mathcal{C}_{\prod_{1\leq i\leq n} \pi_{\mathcal{D}}(D_i)} \xrightarrow{\prod_{1\leq i\leq n} \mathrm{pr}_{i!}} \prod_{1\leq i\leq n} \mathcal{C}_{\pi_{\mathcal{D}}(D_i)}$$

But the cocartesian fibration p has by assumption fibers compatible with products, and this means exactly that functors of this form are equivalences.

Proof of (B): Let f be a $q \circ p^{\otimes} \circ \theta^{\otimes}$-cocartesian morphism in $\mathcal{C}^{\times}$. Then we have to show that $\theta^{\otimes}(f)$ is $q \circ p^{\otimes}$-cocartesian. As $p^{\times}$ is symmetric monoidal, the morphism $p^{\times}(f) = p^{\otimes}(\theta^{\otimes}(f))$ is q-cocartesian, so by [HTT, 2.4.1.3 (3)] it suffices to show that $\theta^{\otimes}(f)$ is $p^{\otimes}$-cocartesian. Applying Proposition C.1.1.1 we are further reduced to showing that $\pi(\theta^{\otimes}(f)) = \pi_{\mathcal{C}}(f)$ is p-cocartesian. As $\pi_{\mathcal{C}}$ is a weak cartesian structure, Proposition F.1.0.1 shows that $\pi_{\mathcal{C}}(f)$ is a projection from a product to a factor, and by the description of products in $\mathcal{C}$ given in Proposition C.2.0.3, projection morphisms in $\mathcal{C}$ are p-cocartesian.

Proof of (C): Consider the commuting diagram

$$\begin{tikzcd}
\mathcal{C}^{\times}_{\langle 1\rangle} \arrow[r] \arrow[d, "\theta"'] & \mathcal{C}^{\times} \arrow[d, "\theta^{\otimes}"'] \arrow[dr, bend left, "\pi_{\mathcal{C}}"] & \\
\mathcal{C}^{\otimes}_{\langle 1\rangle} \arrow[r] \arrow[d] & \mathcal{C}^{\otimes} \arrow[r, "\pi"] \arrow[d, "p^{\otimes}"'] & \mathcal{C} \arrow[d, "p"] \\
\mathcal{D}^{\times}_{\langle 1\rangle} \arrow[r] \arrow[d] & \mathcal{D}^{\times} \arrow[r, "\pi_{\mathcal{D}}"'] \arrow[d, "q"'] & \mathcal{D} \\
\{\langle 1\rangle\} \arrow[r] & \mathsf{Fin}_* &
\end{tikzcd}$$

where the horizontal functors on the left are all the respective inclusions, and the vertical functors on the left are the functors induced by vertical functors in the middle. All squares in the diagram are pullback squares. As $\pi_{\mathcal{D}}$ is a cartesian structure, the composition $\mathcal{D}^{\times}_{\langle 1\rangle} \to \mathcal{D}$ in the third row is an equivalence. As the two squares in the middle row are pullbacks (and hence so is the outer commuting rectangle in the middle row) it follows that the composition $\mathcal{C}^{\otimes}_{\langle 1\rangle} \to \mathcal{C}$ in the second row is an equivalence as well. As $\pi_{\mathcal{C}}$ is a cartesian structure, the composition $\mathcal{C}^{\times}_{\langle 1\rangle} \to \mathcal{C}$ at the top is also an equivalence. It follows that θ must also be an equivalence. □

Remark F.1.0.3. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration whose fibers are compatible with products in the sense of Definition C.2.0.1. Then combining Proposition F.1.0.2 with Proposition C.1.1.1 we obtain another, independent, proof of Proposition C.2.0.6. ♢

F.2. Monoids and limits

In this short section we briefly discuss limits in ∞-categories of monoids.

Proposition F.2.0.1. *Let $\mathcal{O}$ be an ∞-operad and $\mathcal{C}$ an ∞-category.*

Let $\mathcal{I}$ be a small ∞-category and assume that $\mathcal{C}$ admits $\mathcal{I}$-indexed limits. Then $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ *(for a definition see [HA, 2.4.2.1]) admits $\mathcal{I}$-indexed limits as well, and they are preserved and detected by the inclusion functor*

$$\iota\colon \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\big)$$

as well as the composition

$$\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\iota} \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\big) \xrightarrow{j^*} \mathrm{Fun}(\mathcal{O}, \mathcal{C})$$

where $j\colon \mathcal{O} = \mathcal{O}^{\otimes}_{\langle 1\rangle} \to \mathcal{O}^{\otimes}$ is the inclusion. ♡

Proof. As ι is the inclusion of a full subcategory, it follows from [HTT, 1.2.13.7] that to show that $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ admits $\mathcal{I}$-indexed limits and that ι preserves and detects them it suffices to show that $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ is closed under $\mathcal{I}$-indexed limits in $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C})$. But this follows immediately from the definition [HA, 2.4.2.1] in combination with the fact that limits in functor categories are computed pointwise [HTT, 5.1.2.3], and that limits commute with limits [HTT, 5.5.2.3].

For the composition $j^* \circ \iota$, note that there is a commutative diagram as follows

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[rr, "\simeq"] \arrow[d] & & \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \arrow[d, "\iota"] \\
\mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}^{\otimes}, \mathcal{C}^{\times}\big) \arrow[r] \arrow[d] & \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}^{\times}\big) \arrow[r, "\pi_*"] & \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathcal{C}\big) \arrow[d, "j^*"] \\
\mathrm{Fun}(\mathcal{O}, \mathcal{C}) \arrow[rr, "\mathrm{id}"'] & & \mathrm{Fun}(\mathcal{O}, \mathcal{C})
\end{tikzcd}$$

where the unlabeled functors are the obvious forgetful functors or inclusions, and the top horizontal functor is an equivalence by [HA, 2.4.2.5]. That $j^* \circ \iota$ preserves and detects $\mathcal{I}$-indexed limits now follows from [HA, 3.2.2.4] in combination with [HTT, 5.1.2.3] and Proposition E.2.0.2. □

F.3. Cartesian symmetric monoidal ∞-categories and iterating Mon and Alg

Let $\mathcal{C}$ be an ∞-category admitting finite products and let $\mathcal{O}$ and $\mathcal{O}'$ be two ∞-operads. Then $\mathcal{C}$ can be upgraded to a symmetric monoidal ∞-category with the cartesian symmetric monoidal structure $\mathcal{C}^\times$ (see [HA, 2.4.1.5]). We can then consider the ∞-category of $\mathcal{O} \otimes \mathcal{O}'$-algebras in $\mathcal{C}^\times$, denoted by $\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C})$. By [HA, 2.4.2.5] this ∞-category is equivalent to an ∞-category that can be constructed without invoking $\mathcal{C}^\times$, namely the ∞-category of $\mathcal{O} \otimes \mathcal{O}'$-monoids $\mathrm{Mon}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C})$.

On the other hand, the cartesian symmetric monoidal structure $\mathcal{C}^\times$ induces a symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$, and there is an equivalence

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})\big)$$

as we saw in Section E.5. One would expect that the induced symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ is again cartesian so that we can identify $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^\otimes$ with $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^\times$ and hence with $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})^\times$, so that we ultimately obtain further equivalences such as

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})\big) \simeq \mathrm{Mon}_{\mathcal{O}}\big(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})\big)$$

in $\mathcal{C}\mathrm{at}_\infty$.

In this section we will show that this is indeed the case, and describe the steps involved in these types of equivalences in detail, as we will need to know not only that such equivalences exist but also concrete descriptions of the corresponding objects under those equivalences.

Construction F.3.0.1. Let $p_{\mathcal{O}'}\colon \mathcal{O}'^\otimes \to \mathsf{Fin}_*$ be an ∞-operad, let $p_{\mathcal{C}}\colon \mathcal{C}^\otimes \to N\mathsf{Fin}_*$ a symmetric monoidal ∞-category, and let $\pi\colon \mathcal{C}^\otimes \to \mathcal{C}$ be a cartesian structure[3].

There is a bifunctor of ∞-operads

$$f\colon \mathsf{Fin}_* \times \mathcal{O}'^\otimes \xrightarrow{\mathrm{id}_{\mathsf{Fin}_*}\times p_{\mathcal{O}'}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{\wedge} \mathsf{Fin}_*$$

where $\wedge$ is the bifunctor of ∞-operads defined in [HA, 2.2.5.1].

Consider the functor[4][5]

$$q\colon\ \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^\otimes \xrightarrow{\iota_{\mathrm{Alg}}} \mathrm{Fun}\big(\mathcal{O}'^\otimes, \mathcal{C}^\otimes\big) \times_{\mathrm{Fun}(\mathcal{O}'^\otimes,\mathsf{Fin}_*)} \mathsf{Fin}_* \xrightarrow{\mathrm{pr}_2} \mathsf{Fin}_* \qquad \text{(F.3)}$$

[3]See [HA, 2.4.1.1] for the definition.
[4]We write $\mathrm{Alg}_{\mathcal{O}'}$ instead of $\mathrm{Alg}_{\mathcal{O}'/\mathsf{Fin}_*}$.
[5]One should be careful not to confuse the functor $\mathsf{Fin}_* \to \mathrm{Fun}(\mathcal{O}'^\otimes, \mathsf{Fin}_*)$ appearing in the pullback

defined as in Proposition E.4.1.5, which by Proposition E.4.1.5 and [HA, 3.2.4.2 and 3.2.4.3 (3)] defines a symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$.

Finally, define $\widetilde{\pi}'$ as the following composition.

$$\widetilde{\pi}'\colon \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \xrightarrow{\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}} \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \xrightarrow{\pi_*} \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\big) \qquad \diamondsuit$$

Proposition F.3.0.2. *In the situation of Construction F.3.0.1, the functor $\widetilde{\pi}'$ factors through $\mathrm{Mon}_{\mathcal{O}'^{\otimes}}(\mathcal{C})$, i. e. there exists a functor $\widetilde{\pi}$ fitting into a commuting diagram*

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \arrow[rr, "\widetilde{\pi}"] \arrow[dr, "\widetilde{\pi}'"'] & & \mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}) \arrow[dl] \\
& \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\big) &
\end{tikzcd}$$

where the functor $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}) \to \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\big)$ is the canonical inclusion[6]*.*

Furthermore, $\widetilde{\pi}$ is a cartesian structure on $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$. ♡

Proof. Let A be an object of $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$, lying over $\langle n \rangle$, i. e. $q(A) = \langle n \rangle$. What we have to show is that the functor

$$\widetilde{\pi}'(A) = \pi \circ \big(\mathrm{pr}_1(\iota_{\mathrm{Alg}}(A))\big)\colon \mathcal{O}'^{\otimes} \xrightarrow{\mathrm{pr}_1(\iota_{\mathrm{Alg}}(A))} \mathcal{C}^{\otimes} \xrightarrow{\pi} \mathcal{C}$$

is an $\mathcal{O}'$-monoid. For ease of notation we will write $A' \coloneqq \mathrm{pr}_1(\iota_{\mathrm{Alg}}(A))$.

So let $X \simeq X_1 \oplus \cdots \oplus X_m$ be an object of $\mathcal{O}'^{\otimes}_{\langle m \rangle}$, with X_i objects of $\mathcal{O}'$ for $1 \leq i \leq m$. For $1 \leq i \leq m$, let $g_i\colon X \to X_i$ be an inert morphism lying over $\rho^i\colon \langle m \rangle \to \langle 1 \rangle$. We have to show that then

$$\pi\big(A'(X)\big) \xrightarrow{\prod_{1 \leq i \leq m} \pi\left(A'(g_i)\right)} \prod_{1 \leq i \leq m} \pi\big(A'(X_i)\big) \qquad (*)$$

is an equivalence in $\mathcal{C}$.

By definition, $A'\colon \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ preserves inert morphisms, so the morphisms $A'(g_i)$ are inert morphisms in $\mathcal{C}^{\otimes}$. Furthermore, for $1 \leq i \leq m$ we have

$$\begin{aligned}
p_{\mathcal{C}}\big(A'(g_i)\big) &= p_{\mathcal{C}}\Big(\mathrm{pr}_1\big(\iota_{\mathrm{Alg}}(A)\big)(g_i)\Big) \\
&= \Big(\big(p_{\mathcal{C}_*} \circ \mathrm{pr}_1\big)\big(\iota_{\mathrm{Alg}}(A)\big)\Big)(g_i) \\
&= \Big(\Big(\widehat{f} \circ \mathrm{pr}_2\Big)\big(\iota_{\mathrm{Alg}}(A)\big)\Big)(g_i)
\end{aligned}$$

with the inclusion of the constant functors. Instead this functor is the one adjoint to the composition

$$\mathsf{Fin}_* \times \mathcal{O}'^{\otimes} \xrightarrow{\mathrm{id} \times p_{\mathcal{O}'}} \mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{\wedge} \mathsf{Fin}_*$$

In particular, this means that the functors $\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ one obtains from objects of $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}$ by projecting to the first factor are generally *not* functors over Fin_*, so even though they preserve inert morphisms we can not interpret them as maps of ∞-operads.

[6] $\mathrm{Mon}_{\mathcal{O}'^{\otimes}}(\mathcal{C})$ is defined as a full subcategory of $\mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\big)$, see [HA, 2.4.2.1]

$$
\begin{aligned}
&= f\left(\mathrm{id}_{(\mathrm{pr}_2 \circ \iota_{\mathrm{Alg}})(A)}, g_i\right)\\
&= f\left(\mathrm{id}_{q(A)}, g_i\right)\\
&= f\left(\mathrm{id}_{\langle n\rangle}, g_i\right)\\
&= \mathrm{id}_{\langle n\rangle} \wedge p_{\mathcal{O}'}(g_i)\\
&= \mathrm{id}_{\langle n\rangle} \wedge \rho^i
\end{aligned}
$$

where $\widehat{f}\colon \mathsf{Fin}_* \to \mathrm{Fun}\left(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*\right)$ is the adjoint of f and thus the functor occurring in the pullback in (F.3). So for $1 \leq i \leq m$ the morphism $A'(g_i)$ in $\mathcal{C}^{\otimes}$ is a $p_{\mathcal{C}}$-cocartesian lift of $\mathrm{id}_{\langle n\rangle} \wedge \rho^i$.

Let Y_i be an object in $\mathcal{C}$ for each element i in $\left(\langle n\rangle \wedge \langle m\rangle\right)^{\circ}$ such that there is an equivalence

$$A'(X) \simeq \bigoplus_{i \in \left(\langle n\rangle \wedge \langle m\rangle\right)^{\circ}} Y_i$$

in $\mathcal{C}^{\otimes}_{\langle n\rangle \wedge \langle m\rangle}$. Applying Proposition F.1.0.1 we have an identification

$$\pi(A'(X)) \simeq \prod_{i \in \left(\langle n\rangle \wedge \langle m\rangle\right)^{\circ}} \pi(Y_i)$$

such that for each $1 \leq j \leq m$ the morphism $\pi(A'(g_j))$ corresponds to the following projection to the subfactor.

$$\prod_{i \in \left(\langle n\rangle \wedge \langle m\rangle\right)^{\circ}} \pi(Y_i) \to \prod_{\substack{i \in \left(\langle n\rangle \wedge \langle m\rangle\right)^{\circ},\\ \left(\mathrm{id}_{\langle n\rangle} \wedge \rho^j\right)(i) \neq *}} \pi(Y_i)$$

As $\langle m\rangle^{\circ}$ can be written as the disjoint union $\bigcup_{1 \leq j \leq m} \left\{\, i \in \langle m\rangle^{\circ} \;\middle|\; \rho^j(i) \neq * \,\right\}$ it follows that we also have a decomposition of $\left(\langle n\rangle \wedge \langle m\rangle\right)^{\circ}$ as a disjoint union as follows

$$\left(\langle n\rangle \wedge \langle m\rangle\right)^{\circ} = \bigcup_{1 \leq j \leq m} \left\{\, i \in \left(\langle n\rangle \wedge \langle m\rangle\right)^{\circ} \;\middle|\; \left(\mathrm{id}_{\langle n\rangle} \wedge \rho^i\right)(i) \neq * \,\right\}$$

which implies that the morphism $(*)$ is an equivalence, and $\widetilde{\pi}'$ thus factors over $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$.

It remains to show that $\widetilde{\pi}$ is a cartesian structure. We start by showing that $\widetilde{\pi}$ is a lax cartesian structure. So let A_i be objects of $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ for $1 \leq i \leq n$, and let $g_i\colon A \coloneqq A_1 \oplus \cdots \oplus A_n \to A_i$ be an inert lift of ρ^i for each $1 \leq i \leq n$. We have to show that

$$\widetilde{\pi}(A) \xrightarrow{\prod_{1 \leq i \leq n} \widetilde{\pi}(g_i)} \prod_{1 \leq i \leq n} \widetilde{\pi}(A_i) \tag{$**$}$$

is an equivalence in $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$. As the inclusion $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C})$ is fully faithful and equivalences in functor categories are detected pointwise (see Proposition A.3.2.1), it suffices to check that for every $m \geq 0$ and every object X of $\mathcal{O}'^{\otimes}_{\langle m\rangle}$ evaluation at X of morphism $(**)$ is an equivalence in $\mathcal{C}$. As by Proposition F.2.0.1 the inclusion $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C})$ preserves products, and as products in functor categories are detected pointwise [HA, 5.1.2.3] we can thus identify the evaluation at X of the morphism $(**)$ with the morphism

$$\widetilde{\pi}'(A)(X) \xrightarrow{\prod_{1\leq i\leq n}\left(\widetilde{\pi}'(g_i)(X)\right)} \prod_{1\leq i\leq n} \widetilde{\pi}'(A_i)(X)$$

in $\mathcal{C}$, which by using the definition of $\widetilde{\pi}$ is the following morphism

$$\pi\Big(\big(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\big)(A)(X)\Big) \xrightarrow{\prod_{1\leq i\leq n}\left(\pi(h_i)\right)} \prod_{1\leq i\leq n} \pi\Big(\big(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\big)(A_i)(X)\Big) \qquad (***)$$

where we use the notation $h_i \coloneqq \big(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\big)(g_i)(X)$.

Let $1 \leq j \leq n$. By assumption, g_i is q-cocartesian, which by [HA, 3.2.4.3 (4)] implies that h_j is $p_{\mathcal{C}}$-cocartesian. Unwrapping the definition completely analogously to when we showed that $\widetilde{\pi}'$ factors over monoids we find that $p_{\mathcal{C}}(h_i) = \rho^i \wedge \mathrm{id}_{\langle m\rangle}$. That $(***)$ is an equivalence can now be shown completely analogously to before.

We next need to show that $\widetilde{\pi}$ is in fact a weak cartesian structure. So assume that $g\colon A \to A'$ is a q-cocartesian morphism lying over the active morphism $\alpha\colon \langle n\rangle \to \langle 1\rangle$. We have to show that $\widetilde{\pi}(g)$ is an equivalence in $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$. Similarly to before it suffices to check that for each $m \geq 0$ and object $X \in \mathcal{O}'^{\otimes}_{\langle m\rangle}$ the morphism $\pi(h)$ is an equivalence, where $h \coloneqq \big(\mathrm{pr}_1 \circ \iota_{\mathrm{Alg}}\big)(g)(X)$. Also analogously to the case above, we find that h is a $p_{\mathcal{C}}$-cocartesian lift of $\alpha \wedge \mathrm{id}_{\langle m\rangle}$, which is an active morphism as α is active. That $\pi(h)$ is an equivalence now follows from Proposition F.1.0.1.

Finally, it remains to show that the weak cartesian structure $\widetilde{\pi}$ is a cartesian structure. Consider the following commutative diagram, where the two top squares and the square on the right are pullback squares.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}_{\langle 1\rangle} \arrow[r, "k"] \arrow[d, "j"'] & \mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[r] \arrow[d, "r"] & \{\langle 1\rangle\} \arrow[d] \\
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \arrow[r, "\iota_{\mathrm{Alg}}"'] \arrow[dd, "\widetilde{\pi}"'] & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*)} \mathsf{Fin}_* \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \mathsf{Fin}_* \arrow[d, "\widehat{f}"] \\
 & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[r, "p_{\mathcal{C}*}"'] \arrow[d, "\pi_*"] & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathsf{Fin}_*\big) \\
\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C}) \arrow[r] & \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\big) &
\end{tikzcd}$$

The ∞-category of functors $\mathrm{Fun}_{\mathsf{Fin}_*}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)$ over Fin_* is to be taken with respect to $p_{\mathcal{O}'}$ and $p_{\mathcal{C}}$ – this description uses that as $\langle 1\rangle \wedge -$ is naturally isomorphic to the identity functor on Fin_* we can identify $\widehat{f}(\langle 1\rangle)$ with $p_{\mathcal{O}'}$.

What we need to show is that $\widetilde{\pi} \circ j$ is an equivalence. As ι_{Alg} is the inclusion of the full subcategory of objects A such that $\mathrm{pr}_1(A)$ preserves inert morphisms, we can apply Proposition B.5.2.1 to conclude that k is the inclusion of the full subcategory of objects A such that $(\mathrm{pr}_1 \circ r)(A)$ preserves inert morphisms. This implies that the composite $\widetilde{\pi} \circ j$ can be identified with the functor $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \to \mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$ that is an equivalence by [HA, 2.4.2.5]. □

Proposition F.3.0.3. *Let $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ and $p'_{\mathcal{O}}\colon \mathcal{O}'^{\otimes} \to \mathsf{Fin}_*$ be ∞-operads, let $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category, and let $\pi : \mathcal{C}^{\otimes} \to \mathcal{C}$ be a cartesian structure. Let $F\colon \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor of ∞-operads (see [HA, 2.2.5.3]).*

Then there is a commutative diagram as follows such Ψ, Φ_2, Φ_3 and Ψ' are equivalences. If F exhibits $\mathcal{O}''^{\otimes}$ as a tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$, then Φ_1 is an equivalence as well.

$$\begin{tikzcd}
\mathrm{Mon}_{\mathcal{O}''}(\mathcal{C}) \arrow[r] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{C}\big) \\
\mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \arrow[r] \arrow[u, "\Psi", "\simeq"'] \arrow[d, "\Phi_1"'] & \mathrm{Fun}\big(\mathcal{O}''^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[u, "\pi_*"'] \arrow[d, "F^*"] \\
\mathrm{BiFunc}\big(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\otimes}\big) \arrow[r] \arrow[d, "\Phi_2"', "\simeq"] & \mathrm{Fun}\big(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big) \arrow[d, "\widehat{(-)}"] \\
\mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})\big) \arrow[r] \arrow[d, "\Phi_3"', "\simeq"] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)\Big) \arrow[d, "(\pi_*)_*"] \\
\mathrm{Mon}_{\mathcal{O}}\big(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})\big) \arrow[r] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\big)\Big) \\
\mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})\big) \arrow[r] \arrow[u, "\Psi'", "\simeq"'] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\big)^{\times}\Big) \arrow[u, "\big(\pi_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathcal{C})}\big)_*"']
\end{tikzcd} \tag{F.4}$$

The symmetric monoidal ∞-category $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$ appearing on the bottom left carries the cartesian symmetric monoidal structure $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})^{\times}$ (see [HA, 2.4.1.5]) and $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ appearing on the left in the middle row carries the symmetric monoidal structure from Construction F.3.0.1. The horizontal functors are all the respective canonical functors that combine the various inclusions and forgetful functors or projections. The functor $\widehat{-}$ sends a functor G to its adjoint, which we denote by $\widehat{G}$. ♡

Proof. The existence of equivalences Ψ and Ψ' making the topmost and bottommost square of (F.4) commute is shown in [HA, 2.4.2.5].

Construction of Φ_1 and Φ_2 fitting into the diagram was handled in Proposition E.5.0.2 and Proposition E.5.0.1.

We are left to construct Φ_3. Proposition F.3.0.2 provides us with a cartesian structure

$$\widetilde{\pi}\colon\ \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes} \to \mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$$

Applying [HA, 2.4.2.5] we obtain Composition with $\widetilde{\pi}$ then induces an equivalence Φ_3 as in the following commuting diagram by

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})\big) \arrow[r, dashed, "\Phi_3", "\simeq"'] \arrow[d] & \mathrm{Mon}_{\mathcal{O}}\big(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})\big) \arrow[d] \\
\mathrm{Fun}_{\mathsf{Fin}_*}\Big(\mathcal{O}^{\otimes}, \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}\Big) \arrow[r, "\widetilde{\pi}_* \circ \mathrm{pr}"'] \arrow[d] & \mathrm{Fun}\big(\mathcal{O}^{\otimes}, \mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})\big) \arrow[d] \\
\mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\big)\Big) \arrow[r, "(\pi_*)_*"'] & \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Fun}\big(\mathcal{O}'^{\otimes}, \mathcal{C}\big)\Big)
\end{tikzcd}$$

where pr denotes the forgetful functor

$$\mathrm{Fun}_{\mathsf{Fin}_*}\Big(\mathcal{O}^{\otimes}, \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}\Big) \to \mathrm{Fun}\Big(\mathcal{O}^{\otimes}, \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})^{\otimes}\Big)$$

and the vertical functors are the canonical functors constructed the various forgetful functors, inclusions, and projections. The bottom square commutes by definition of $\widetilde{\pi}$, see Construction F.3.0.1. □

Remark F.3.0.4. The right column of (F.4) is covariantly functorial in $\mathcal{C}^{\otimes}$ (together with its cartesian structure) and contravariantly functorial in F.

Let

$$\begin{tikzcd}
\mathcal{O} \times \mathcal{O}' \arrow[r, "F"] \arrow[d, "\alpha^{\otimes} \times \beta^{\otimes}"'] & \mathcal{O}'' \arrow[d, "\gamma^{\otimes}"] \\
\mathcal{U} \times \mathcal{U}' \arrow[r, "G"'] & \mathcal{U}''
\end{tikzcd}$$

be a commutative diagram of functors over Fin_* with $\alpha^{\otimes}$, $\beta^{\otimes}$, and $\gamma^{\otimes}$ morphisms of ∞-operads, and F and G bifunctors of ∞-operads.

Let

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[r, "H^{\otimes}"] \arrow[d, "\pi_{\mathcal{C}}"'] & \mathcal{D}^{\otimes} \arrow[d, "\pi_{\mathcal{D}}"] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{D}
\end{tikzcd}$$

be a commutative diagram of ∞-categories with $H^{\otimes}$ a symmetric monoidal functor of symmetric monoidal ∞-categories and $\pi_{\mathcal{C}}$ and $\pi_{\mathcal{D}}$ cartesian structures.

Then the induced commutative diagram on the right column of (F.4) restricts to a commutative diagram as follows.

$$
\begin{tikzcd}
\mathrm{Mon}_{\mathcal{U}''}(\mathcal{C}) \arrow[rr, "\mathrm{Mon}_\gamma(H)"] & & \mathrm{Mon}_{\mathcal{O}''}(\mathcal{D}) \\
\mathrm{Alg}_{\mathcal{U}''}(\mathcal{C}) \arrow[rr, "\mathrm{Alg}_\gamma(H)"] \arrow[u, "\Psi", "\simeq"'] \arrow[d, "\Phi_1"'] & & \mathrm{Alg}_{\mathcal{O}''}(\mathcal{D}) \arrow[u, "\simeq", "\Psi"'] \arrow[d, "\Phi_1"] \\
\mathrm{BiFunc}\big(\mathcal{U}^\otimes, \mathcal{U}'^\otimes; \mathcal{C}^\otimes\big) \arrow[rr, "\mathrm{BiFunc}(\alpha^\otimes, \beta^\otimes; H^\otimes)"] \arrow[d, "\Phi_2"', "\simeq"] & & \mathrm{BiFunc}\big(\mathcal{O}^\otimes, \mathcal{O}'^\otimes; \mathcal{D}^\otimes\big) \arrow[d, "\simeq"', "\Phi_2"] \\
\mathrm{Alg}_{\mathcal{U}}\big(\mathrm{Alg}_{\mathcal{U}'}(\mathcal{C})\big) \arrow[rr, "\mathrm{Alg}_\alpha(\mathrm{Alg}_\beta(H))"] \arrow[d, "\Phi_3"', "\simeq"] & & \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{D})\big) \arrow[d, "\simeq"', "\Phi_3"] \\
\mathrm{Mon}_{\mathcal{U}}\big(\mathrm{Mon}_{\mathcal{U}'}(\mathcal{C})\big) \arrow[rr, "\mathrm{Mon}_\alpha(\mathrm{Mon}_\beta(H))"] & & \mathrm{Mon}_{\mathcal{O}}\big(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{D})\big) \\
\mathrm{Alg}_{\mathcal{U}}\big(\mathrm{Mon}_{\mathcal{U}'}(\mathcal{C})\big) \arrow[rr, "\mathrm{Alg}_\alpha(\mathrm{Mon}_\beta(H))"] \arrow[u, "\Psi'", "\simeq"'] & & \mathrm{Alg}_{\mathcal{O}}\big(\mathrm{Mon}_{\mathcal{O}'}(\mathcal{D})\big) \arrow[u, "\simeq", "\Psi'"']
\end{tikzcd}
$$

One could argue for this by considering the individual constructions, or one could use that the first, third, fourth, and fifth horizontal functor in (F.4) are monomorphisms[7][8] and apply the uniqueness part of Proposition B.4.3.1. This also implies compatibility with compositions.

Additionally, note that construction of Φ_1 and Φ_2 does not need the assumption that $\mathcal{C}$ carries a cartesian symmetric monoidal structure[9], so if we only consider the part of the above diagram involving Φ_1 and Φ_2, then we can drop this assumption. ♢

[7]That we only need those horizontal functors to be monomorphisms is because they are the "targets" in the diagram.

[8]The first and third horizontal functors are by definition fully faithful, so monomorphisms by Proposition B.4.4.1. The third and fourth horizontal functors are equivalent, so the fourth one is a monomorphism as well. Finally, the fifth horizontal functor is a monomorphism by a combination of the definitions, Proposition B.4.4.1, Proposition B.5.1.1, and Proposition B.1.2.1.

[9]See Proposition E.5.0.2 and Proposition E.5.0.1.

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